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Putnam's resolving maps in dimension zero

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Abstract. A block code from an irreducible shift of finite type can be lifted canonically through resolving maps to a resolving map. There is an application to Markovian maps.

1. *Introduction*

Let $\mathcal{S}$ be the set of one-to-one almost everywhere factor maps between irreducible Smale spaces. Putnam [**16**] proved the following theorem: given π in $\mathcal{S}$, there are Smale spaces $\overline{X}$ and $\overline{Y}$, and maps α, β and $\widetilde{\pi}$ in $\mathcal{S}$, such that α and β are u-resolving, $\widetilde{\pi}$ is s-resolving and the following diagram commutes:

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\widetilde{\pi}} & \overline{Y} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{\pi} & Y \end{array}$$

Putnam's result was a surprise even in dimension zero, where resolving maps have been studied extensively. In the zero-dimensional case (where a Smale space is a shift of finite type (SFT) and a finitely presented system is a sofic shift), we reinterpret and extend Putnam's work. As a lot is known [**1, 3, 13, 14**] about resolving maps, this adds to our understanding of general factor maps. We hope this work will also be relevant to progress in positive dimension.

We view Putnam's construction in two stages: first construct the map β, then construct the diagram by a fiber product argument. For β, we construct a canonically associated map we call π_+, the futures cover of π. This map is itself a consequence of a more general construction, the full futures cover of π (see §3). The fiber product argument is carried out in §4, and associates a canonical lift to π. For these sections, the map π can be any block code from an irreducible SFT to a sofic shift, even an infinite to one code. In the infinite-to-one case, the lift $\widetilde{\pi}$ is only 'eresolving'—resolving in the sense of existence (§2)—but this is meaningful.

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In §5, we show that our construction is the unique (up to topological conjugacy) minimal construction—any other such lift must factor through it. This, and the generalization to finitely presented range systems, answers (in the zero-dimensional case) the two questions raised by Putnam [**16**].

The futures cover arises from an effort to simplify Putnam's construction. Where both the futures cover and the map β constructed by Putnam are defined, they and their associated diagrams are isomorphic (§6). The futures cover is also closely related to a canonical cover of Nasu (see §6). The three constructions (futures cover, Putnam cover, Nasu cover) are all isomorphic where they are all defined (for one-to-one almost everywhere maps between irreducible SFTs).

An unexpected spinoff of this investigation is some new information about infinite-to-one maps. A magic word construction very familiar for finite-to-one maps turns out to be meaningful (7.1) and of some use for infinite-to-one maps. We also learn that any factor map between irreducible SFTs can, after precomposition with a degree one resolving map, be made Markovian (7.2).

2. *Background*

In this section, we recall and slightly expand some background results and terminology. For further details, see [**8, 11**]. We will be concerned with subshifts (X, T) where X is a shift-invariant space of doubly infinite sequences over a finite alphabet, with the usual topology (relative of product of discrete, making X zero-dimensional compact metrizable) and T is the restriction of the shift map σ to X. (We use the left shift, $(\sigma x)_i = x_{i+1}$.) The alphabet for (X, T) may be denoted $\mathcal{A}(X)$ or $\mathcal{A}(T)$, and the set of words of length n occurring in points of X is denoted $\mathcal{W}_n(X)$ or $\mathcal{W}_n(T)$. A point x of a subshift (X, S) is *left transitive* (*right transitive*) if for every S-word W there exist negative (positive) integers $i \leq j$ such that $W = x[i, j]$. A point is *doubly transitive* if it is both left transitive and right transitive. Points x and x' in a subshift are *left asymptotic* if there exists n in $\mathbb{Z}$ such that $x(-\infty, n] = x'(-\infty, n]$. Systems (X, T) and (X', T') are *topologically conjugate* if there is a homeomorphism $\varphi : X \to X'$ such that $T'\varphi = \varphi T$. A SFT is a system (X, S) such that there exists n such that a point x is in X if and only if $x[i+1, i+n] \in \mathcal{W}_n(S)$ for every $i \in \mathbb{Z}$. A SFT is *1-step* if this length n can be chosen to be 2. An *edge SFT* is a 1-step SFT for which the alphabet is the set of edges in a non-degenerate directed graph $\mathcal{G}$ and EE' is an allowed word of length 2 if the terminal vertex of E equals the initial vertex E'. A SFT is *irreducible* if it has a dense forward orbit. A directed graph is *non-degenerate* if every vertex has at least one incoming edge and at least one outgoing edge. An edge SFT defined by a non-degenerate directed graph $\mathcal{G}$ is irreducible iff there is a path in $\mathcal{G}$ from any vertex to any other. An *irreducible component* of a SFT is a maximal irreducible subsystem. The entropy of a SFT is the maximum of the entropies of its irreducible components.

A block code is a map $\varphi : (X, S) \to (Y, T)$ between subshifts such that for some n there is a map $\varphi : \mathcal{W}_{2n+1}(S) \to \mathcal{A}(T)$ such that for all x in X and $i \in \mathbb{Z}$, $(\varphi x)_i = \varphi(x[i-n, i+n])$. When such an n exists, the block code has *range* n. If $n = 0$, then φ is a 1-block code. Every continuous shift commuting map (or code) between subshifts is given by a block code. If φ is a surjective block code, then it is a *factor map*.

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A factor map φ from an irreducible SFT onto a subshift Y has a well-defined *degree* d: the cardinality of the preimage of every double transitive point in Y is d. In this case we also say that φ is d-to-one almost everywhere. A surjective code respects entropy if it is finite-to-one. An irreducible SFT has no proper subsystem of equal entropy. A *sofic shift* is a subshift which is the image of a SFT under a block code. The SFTs are the zero-dimensional Smale spaces and the sofic shifts are the zero-dimensional finitely presented dynamical systems (see [**7, 16**]).

Let φ denote a factor map from a subshift (X, S) to a subshift (Y, T). φ is *right resolving* if it is a 1-block code with the following property: if $a_0 \in \mathcal{A}(X)$, with image symbol $b_0 \in \mathcal{A}(Y)$, and if b_0b_1 is a Y-word, then there exists at most one $a_1 \in \mathcal{A}(X)$ such that a_1 maps to b_1 and a_0a_1 is an X word. φ is *right closing*, or *u-resolving*, if for all left asymptotic x and $x' \in X$, $\varphi(x) = \varphi(x')$ implies that $x = x'$. (The terminology 'right closing' is standard in symbolic dynamics [**8, 11**]; Putnam's more general terminology 'u-resolving' means injective on unstable sets [**16**], which in this setting means right closing. We generally favor Putnam's terminology.)

Similarly, φ is *left resolving* if it is a 1-block code with the following property: if $a_1 \in \mathcal{A}(X)$, with image symbol $b_1 \in \mathcal{A}(Y)$, and if b_0b_1 is a Y-word, then there exists at most one $a_0 \in \mathcal{A}(X)$ such that a_0 maps to b_0 and a_0a_1 is an X word. Likewise, φ is *left closing*, or *s-resolving*, if for all right asymptotic x and $x' \in X$, $\varphi(x) = \varphi(x')$ implies $x = x'$. Here 's-resolving' means injective on stable sets [**16**]. We say that a map is resolving if it is left, right, u- or s-resolving. A resolving map is finite-to-one and therefore its domain and image are subshifts of equal entropy.

We say that φ is *right eresolving* if the right resolving condition holds, but with the uniqueness condition replaced by existence. That is, in the definition of right resolving, we replace 'at most one' with 'at least one'. We say that φ is u-eresolving if given x in X and $y \in Y$ such that $\varphi(x)$ is left asymptotic to y, there exists at least one $\overline{x}$ in X such that $\overline{x}$ is left asymptotic to x and $\varphi(\overline{x}) = y$. For block codes, u-eresolving has been called *right continuing* [**6**, §5]. The terms *left eresolving*, *left continuing* and s-eresolving are defined analogously.

In the following proposition, we record some known facts relating these notions. Obviously the left/s analogues likewise hold.

PROPOSITION 2.1. *Let φ be a block code from a subshift (X, S) onto a subshift (Y, T).*

(1) *If φ is right resolving, then φ is u-resolving (i.e. right closing).*
(2) *If φ is u-resolving, then there is a topological conjugacy ψ and a right resolving φ' such that φ' is ψ followed by φ.*
(3) *If φ is right eresolving, then φ is u-eresolving (i.e. right continuing).*
(4) *If φ is u-eresolving, then there is a topological conjugacy ψ and a right eresolving φ' such that φ' is ψ followed by φ.*
(5) *If (X, S) and (Y, T) are irreducible SFTs of equal entropy, then φ is u-resolving if and only if it is u-eresolving; if, in addition, these SFTs are 1-step and φ is 1-block, then φ is right resolving if and only if it is right eresolving.*
(6) *If φ is u-resolving and (X, S) is an irreducible SFT, then (Y, T) is an SFT if and only if φ is u-eresolving.*

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Proof. (1) and (3) are easy observations. (2) is proved in [**8**, Proposition 4.33]. (4) is proved in [**6**, Proposition 5.1]. (5) is partly the result [**11**, Proposition 8.2.2], and the rest of (5) and (6) are proved by a similar appeal to the Perron–Frobenius theory element [**11**, Theorem 4.4.7]. □

Remark 2.2. Resolving maps from irreducible SFTs onto strictly sofic shifts are important in the theory of sofic shifts, and such maps cannot be eresolving. On the other hand, eresolving maps between irreducible SFTs of unequal entropy are relevant to understanding Markovian codes [**6**]. In this paper we encounter both types of map in the same setting, so for clarity we have introduced the new term eresolving. (Caveat: the term 'resolving' has sometimes been used to mean u-resolving; or eresolving; or both resolving and eresolving.)

LEMMA 2.3. *Suppose that $\varphi : (X, S) \to (Y, T)$ is a u-resolving block code such that (X, S) is a SFT, (Y, T) is an irreducible SFT and (X_0, S_0) is an irreducible component of (X, S) of maximum entropy.*

Then any point of (X, S) which is left asymptotic to X_0 is contained in X_0.

Proof. This is [**8**, Lemma 5.1.4] in the case that φ is right resolving. The conclusion then follows from Proposition 2.1(2). □

'Magic words' are an important technical tool in the theory of codes from SFTs. Let $\varphi : (X, S) \to (Y, T)$ be a surjective one-block code. Above every Y-word $W = W[1, n]$ of length $|W| = n$ there is a set of X-words $W' = W'[1, n]$ of length n which by the coding rule are sent to W, i.e. $\varphi W' = W$. Given $1 \leq i \leq n$, set $d(W, i) = |\{W'_i : \varphi W' = W\}|$. We define the *resolving degree* $\delta(\varphi)$ of φ to be the minimum of $d(W, i)$ over all Y-words W and $1 \leq i \leq |W|$. W is a *magic word* if, for some i, $d(W, i) = \delta(\varphi)$. A *magic symbol* for φ is a magic word of length 1, that is, an element μ of $\mathcal{A}(Y)$ such that whenever W is a Y-word and $W_i = \mu$, then $d(W, i) = \delta(\varphi)$. (In other words, if μ occurs at a position in W, then 'above' W we see every symbol which maps to μ, and this set of symbols has cardinality $\delta(\varphi)$.) For an irreducible SFT (X, S) and finite-to-one block code $\varphi : (X, S) \to (Y, T)$, the proof that φ has a well-defined degree proceeds by showing every doubly transitive point has exactly $\delta(\varphi)$ preimages [**8, 11**].

We have defined magic words and symbols exactly as in [**8**, pp. 102, 113], except that we do not restrict to the case that φ is finite-to-one. In particular, our number $\delta(\varphi)$ is the number d in [**8**, p. 101]. Two block codes α, β are *topologically conjugate* if there are topological conjugacies ψ_1, ψ_2 such that $\psi_1\beta = \alpha\psi_2$. We introduce the term 'resolving degree' because it turns out to be a conjugacy invariant of infinite-to-one maps as well as finite-to-one maps (7.1). Likewise, the next lemma will be useful to us in the general (not just the finite-to-one) case.

LEMMA 2.4. *A surjective block code φ is topologically conjugate to a 1-block code with a magic symbol. If φ is u-resolving, then in addition this code can be required to be right resolving.*

Proof. The first statement is proved in [**8**, Proposition 4.3.2]. The restriction in the statement to finite-to-one codes is not needed in the proof. For the second statement,

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after passing to a topologically conjugate code, we may assume that φ is right resolving. The recoding construction of [**8**, Proposition 4.3.2] is easily adjusted to respect the right resolving property. □

The next lemma states that a u-resolving map from an irreducible SFT onto a sofic shift is (despite Proposition 2.1(6)) u-eresolving with respect to left transitive points in the image.

LEMMA 2.5. *Suppose that (X, S) is SFT and $\varphi : (X, S) \to (Y, T)$ is a u-resolving block code. Suppose that y is left transitive in Y, and x is a point in X such that φx is left asymptotic to y.*

Then there exists a point x' in X such that x' is left asymptotic to x and $\varphi x' = y$. Also, if φ is right resolving and $(\varphi x)(-\infty, 0] = y(-\infty, 0]$, then x' can be chosen such that $x'(-\infty, 0] = x(-\infty, 0]$.

Proof. After passing to a topologically conjugate map, we can assume without loss of generality that φ is right resolving with a magic symbol μ. Choose $n < 0$ such that $y_n = \mu$. As μ is a magic symbol, for all $k > 0$ there exists an X-word $a_n \cdots a_{n+k}$ such that $\varphi(a_n \cdots a_{n+k}) = y_n \cdots y_{n+k}$ and $a_n = x_n$. As φ is right resolving, the word $a_n \cdots a_{n+k}$ is uniquely determined by a_n and y. Define x' by setting $x'_i = x_i$ if $i \leq n$ and $x'_i = a_i$ if $i > n$. □

3. *The futures cover of a map*

To begin, let $\pi : (X, S) \to (Y, T)$ be a 1-block code from a 1-step SFT onto a sofic shift. We associate to π a surjective right resolving 1-block code $\pi_{++} : (Y_{++}, T_{++}) \to (Y, T)$, where (Y_{++}, T_{++}) is a 1-step SFT. Given $y \in Y$ and $m \in \mathbb{Z}$, define the following subset of $\mathcal{A}(S)$:

$$U_m(y) = \{x_m : x \in X \text{ and } \pi(x_j) = y_j \text{ if } j \leq m\}. \tag{3.1}$$

Define the alphabet of T_{++} to be

$$\mathcal{A}(T_{++}) = \{U_0(y) : y \in Y\} = \{U_m(y) : y \in Y \text{ and } m \in \mathbb{Z}\}.$$

Define the allowed T_{++}-words of length two (and thus determine the 1-step SFT T_{++}) as follows: UU' is allowed if $U \in \mathcal{A}(T_{++})$ and there exists j in $\mathcal{A}(T)$ such that

$$U' = \{i \in \mathcal{A}(S) : \pi(i) = j \text{ and } i \text{ follows some element of } U\}.$$

Note that, given U, j and U' as above, we do have U' in $\mathcal{A}(T_{++})$, since there exists a y in Y with $U_0(y) = U$ and $y_1 = j$, and then $U' = U_1(y)$. Similarly, we see that the set of all Y_{++}-words is

$$\{U_0(y)U_1(y) \cdots U_n(y) : n \geq 0, y \in Y\}.$$

The 1-block code $\pi_{++} : (Y_{++}, T_{++}) \to (Y, T)$ is defined by sending a Y_{++}-symbol U to the Y-symbol which is $\pi(i)$ for every i in U. The map π_{++} is clearly well-defined, surjective and right resolving.

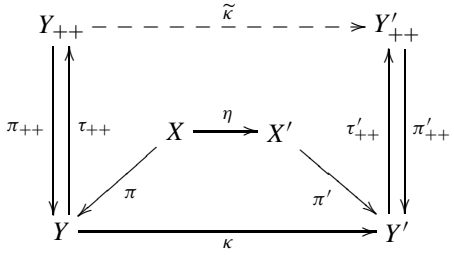
Finally, we associate to π_{++} the shift-commuting Borel injection $\tau_{++} : Y \hookrightarrow Y_{++}$ defined by setting $(\tau_{++}y)_i = U_i(y)$. The map τ_{++} is a section to π ($\pi\tau_{++}$ is the identity map on Y).

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Definition 3.2. The code π_{++} (and the SFT (Y_{++}, T_{++})) is called the *full futures cover* of π, and τ_{++} is called its canonical section.

The next result adapts arguments of Krieger [**10**].

PROPOSITION 3.3. *Suppose that $\pi : (X, S) \to (Y, T)$ and $\pi' : (X', S') \to (Y', T')$ are 1-block codes from 1-step SFTs onto sofic shifts. Let $\eta : (X, S) \to (X', S')$ and $\kappa : (Y, T) \to (Y', T')$ be topological conjugacies such that $\kappa\pi = \pi'\eta$. Let the full futures covers and sections for π and π' be as in Definition 3.2. Then there is a unique topological conjugacy $\widetilde{\kappa}$ such that the following diagram commutes.*



Proof. The images of the sections τ_{++} and τ'_{++} are dense, and between these images $\widetilde{\kappa}$ is determined. Thus the uniqueness claim is clear. Let N be a positive integer such that the block codes η, κ, η^{-1} and κ^{-1} have range at most N.

Given $y \in Y$ and $y' = \kappa y \in Y'$, we claim that $U_0(y')$ is the set of symbols $(\eta x)_0$ such that $x \in X$ and x satisfies the following two conditions:

(1) $x_i \in U_i(y),\ -3N \le i \le -N$;

(2) $x[-3N \cdots N]$ is an X-word which determines $(\kappa\pi x)[-2N, 0] = y'[-2N, 0]$.

To prove the claim, first suppose that $x' \in X'$ and $(\pi' x')(-\infty, 0] = y'(-\infty, 0]$. Set $x = \eta^{-1}x'$. Then $(x')_0 = (\eta x)_0$ and $(\kappa\pi x)(-\infty, 0] = y'(-\infty, 0]$. This shows $x[-3N, N]$ satisfies (2). As $y'(-\infty, 0]$ determines $(\kappa^{-1}y')(-\infty, -N]$ and $\kappa^{-1}y' = \pi x$, we have $x_i \in U_i(y)$ for $i \le -N$ and x satisfies (1).

Conversely, suppose that $x \in X$ satisfying (1) and (2). As $x_{-3N} \in U_{-3N}(y)$, there exists $\overline{x}$ in X such that $(\pi\overline{x})_i = y_i$ if $i \le -3N$, and $\overline{x}_{-3N} = x_{-3N}$. As X is a 1-step SFT, without loss of generality we choose x such that in addition to (1) and (2) we have $\overline{x}_i = x_i$, $i \le -3N$.

Now $(\kappa\pi x)(-\infty, -2N] = y'(-\infty, -2N]$ (because $(\pi x)_i = y_i$, $i \le -N$) and $(\kappa\pi x)[-2N, 0] = y'[-2N, 0]$ (by (2)). Since $(\kappa\pi x)(-\infty, 0] = y'(-\infty, 0]$, we have $(\eta x)_0 \in U_0(y')$. This completes the proof of the claim.

The claim shows that for any y in Y, the Y_{++}-word $(\tau y)[-3N, N]$ determines the Y'_{++} symbol $\tau y'[0]$. As the image of τ is dense in Y_{++}, this provides a function K from Y_{++}-words of length $4N + 1$ to $\mathcal{A}(Y'_{++})$. Define the block code $\widetilde{\kappa}$ on Y_{++} by setting $(\widetilde{\kappa}w)_0 = K(w[-3N, N])$. Clearly now $\widetilde{\kappa}\tau = \tau'\kappa$. It follows easily that $\widetilde{\kappa}$ maps Y_{++} onto Y'_{++} and $\kappa\pi_{++} = \pi'_{++}\widetilde{\kappa}$.

In the same way, construct a block code $\widetilde{\kappa^{-1}}$ from Y'_{++} to Y_{++}. The compositions of $\widetilde{\kappa}$ and $\widetilde{\kappa^{-1}}$ give the identity map on a dense set, hence everywhere. Therefore, $\widetilde{\kappa}$ is a conjugacy. □

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Given a block code $\pi : X \to Y$ from a SFT onto a sofic system, let φ be a conjugacy from X to a higher block presentation, $(\varphi x)_0 = x[-k \cdots k]$, where k is sufficiently large that the composition $\pi\varphi^{-1}$ is a 1-block map from a 1-step SFT, so its full futures cover $(\pi\varphi^{-1})_{++}$ has been defined. We also say that $(\pi\varphi^{-1})_{++}$ is *a full futures cover* of π and (suppressing k) denote it as π_{++}. More generally, we make the following definition.

Definition 3.4. If p is any u-resolving map from a SFT onto Y and there exists a conjugacy φ such that $p = \pi\varphi$, where π is a full futures cover as defined above, then we say that p is *a full futures cover of* π.

By Proposition 3.3, if p and p' are two different full futures covers of π, then there is a conjugacy φ such that $p' = p\varphi$. Thus, given π we have a distinguished class of u-resolving maps from SFTs onto Y.

Next we consider the special case which is our main interest.

Definition 3.5. Suppose that π is a one block code from a 1-step irreducible SFT (X, S) onto a sofic shift (Y, T). Let (Y_+, T_+) be the unique maximum entropy irreducible component of the SFT (Y_{++}, T_{++}), and let π_+ be the restriction of π_{++} to this component. We call π_+ (or (Y_+, T_+)) the *futures cover of* π. The restriction of τ_{++} to the left transitive points of Y is denoted τ_+.

The following proposition shows that the definition makes sense and lists some basic properties of the futures cover.

PROPOSITION 3.6. *Suppose that $\pi : (X, S) \to (Y, T)$ is a 1-block code from an irreducible 1-step SFT onto a sofic shift.*

(1) *The full futures cover of π has a unique irreducible component of maximal entropy (denoted Y_+).*
(2) *The futures cover π_+ is surjective right resolving with degree 1 and $\tau_+ : Y \hookrightarrow Y_+$.*
(3) *Y_+ is a 1-step SFT and the set of allowed Y_+-words is*

$$\{U_0(y)U_1(y)\cdots U_n(y) : n \geq 0,\ y \text{ is left transitive in } Y\}$$

(4) *After removal of the section maps τ_{++} and τ'_{++}, Proposition 3.3 remains true if each $++$ is replaced by $+$.*

Proof. The claim (1) and the rest of (2) do not depend on the conjugacy class of π_{++}; so, taking into account Proposition 3.3 and Lemma 2.4, to prove them we may assume that π has a magic symbol, μ. Let $\widetilde{\mu}$ denote $\pi^{-1}\{\mu\}$. If $\pi_{++}x = y$ and $y_n = \mu$, then x_n must equal $\widetilde{\mu}$. As π_{++} is right resolving, for any n in $\mathbb{Z}$ the conditions $y = \pi_{++}x$ and $x_n = \widetilde{\mu}$ determine $x[n, \infty)$. Consequently, if y is left transitive in Y, then y has a unique preimage under π_{++}. As π_{++} must map any irreducible component of Y_{++} of maximum entropy onto Y, it follows that there is only one such component. We let Y_+ denote this component and we let π_+ denote the restriction of π_{++} to Y_+.

The claim (2) is now clear.

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The image of τ_+ is a dense subset of Y_+, and we have already seen that the set of all Y_{++}-words is

$$\{U_0(y)U_1(y)\cdots U_n(y) : n \geq 0,\ y \in Y\},$$

i.e. it is the set of words $(\tau_{++}y)[0, n]$, $y \in Y$. This proves (3).

The existence claim of (4) follows from (1) and Proposition 3.3. The uniqueness claim holds because $\widetilde{\kappa}$ is uniquely determined on a dense set, the image of τ_+. □

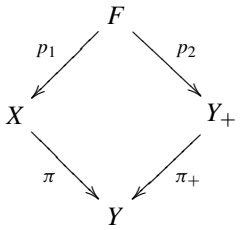
As with the full futures cover, we make a definition which gives us a canonical conjugacy class.

Definition 3.7. Given a block code π from an irreducible SFT onto a sofic shift, we say that a map p is *a futures cover of* π if it is conjugate to the restriction of a full futures cover of π to its unique irreducible component of maximum entropy.

4. *A resolving lift of a block code*

To begin, let π be a 1-block code from an irreducible 1-step SFT (X, S) onto a sofic shift (Y, T). Let π_+ be the futures cover $Y_+ \to Y$ of Definition 3.5. We will apply a fiber product construction to produce (in the zero-dimensional case) a generalization of Putnam's resolving lift.

The *fiber product* (also known as *fibered product* or *pull-back*) of $\pi : (X, S) \to (Y, T)$ and $\pi_+ : (Y_+, T_+) \to (Y, T)$ is the subsystem of $(X \times Y_+, S \times T_+)$ with domain $F = \{(x, w) : \pi(x) = \pi_+(w)\}$. The coordinate projections $p_1 : (x, w) \mapsto x$ and $p_2 : (x, w) \mapsto w$ give a commuting diagram of surjective maps



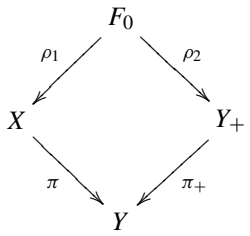
We view F as a subshift, with alphabet

$$\begin{aligned}\mathcal{A}(F) &= \{(x_j, w_j) : (x, w) \in F\} \\ &\subset \{(i, U) \in \mathcal{A}(X) \times A(Y_+) : \pi(i) = \pi_+(U)\}.\end{aligned}$$

The subshift F is a 1-step SFT, because X and Y_+ are 1-step SFTs, and π and π_+ are 1-block codes. As π_+ is finite-to-one, so is p_1; so an irreducible component of F has maximum entropy (which is equal to that of X) if and only if its image under p_1 is all of X. If there were two such components, then there would be a left transitive point x in X with two preimages in F, so the left transitive point πx would have two preimages under π_+ in Y_+, which is impossible. Thus, F has a unique irreducible component of maximum entropy in F, which we denote F_0; and the restriction ρ_1 of p_1 to F_0 has degree one. Let ρ_2 similarly denote the restriction of p_2 to F_0. We have a commuting diagram of

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surjective maps



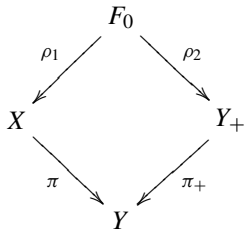
As π_+ is right resolving, so is p_1, and therefore so is ρ_1.

LEMMA 4.1. *We have* $\mathcal{A}(F_0) = \{(i, U) \in \mathcal{A}(X) \times \mathcal{A}(Y_+) : i \in U\}$.

Proof. First, suppose that $(i, U) \in \mathcal{A}(F_0)$. Then (i, U) must occur as z_0 in some point z of F_0 which maps (via $\pi\rho_1 = \pi_+\rho_2$) to a left transitive point y in Y. As $(\rho_1 z)_0 = i$ and $\pi(\rho_1 z) = y$, we have $i \in U_0(y)$. As $(\rho_2 z)_0 = U$ and $\pi_+(\rho_2 z) = y$ and y has a unique preimage under π_+, which is the sequence $(U_m(y))_{m\in\mathbb{Z}}$, we have $U = U_0(y)$. Thus, $i \in U$.

Next, suppose $U \in \mathcal{A}(Y_+)$ and $i \in U$. Then there exists a left transitive y in Y and an x in X such that $\pi(x_j) = y_j$ for $j \leq 0$, and $U_0(y) = U$ and $x_0 = i$. Thus, $(x, y)_i = (x_i, U_i(y))$ for $i \leq 0$, and for some $n \leq 0$ the sequence $(x, y)(-\infty, n]$ is a sequence from F_0. By Lemma 2.3, there are no paths exiting from F_0, so $(i, U) \in \mathcal{A}(F_0)$. □

PROPOSITION 4.2. *Let (X, S) be an irreducible 1-step SFT. Let $\pi : (X, S) \to (Y, T)$ be a surjective one-block code with a magic symbol. In the associated fiber product diagram*



all maps are surjective 1-block; ρ_1 and π_+ are right resolving; and ρ_2 is left eresolving. If π is finite-to-one, then ρ_2 is left resolving.

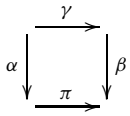
Proof. The claims before 'ρ_2 is left eresolving' are already established. Now suppose that $(i_1, U_1) \in \mathcal{A}(F_0)$ and U_0U_1 is a word of length two in Y_+. To show that ρ_2 is left eresolving, we must show there exists $(\overline{i_0}, \overline{U_0})$ in $\mathcal{A}(F_0)$ which can precede (i_1, U_1) and which maps to U_0. As $(i_1, U_1) \in \mathcal{A}(F_0)$, we have $i_1 \in U_1$; because U_1 follows U_0, we may choose i_0 in U_0 such that i_1 follows i_0 in X. As $i_0 \in U_0$, we have $(i_0, U_0) \in \mathcal{A}(F_0)$. Let $(\overline{i_0}, \overline{U_0}) = (i_0, U_0)$. We have shown ρ_2 is left eresolving.

The final claim follows from Proposition 2.1(5): a map between equal entropy irreducible SFTs is left resolving if and only if it is left eresolving. □

We can now assemble a presentation-invariant version of our zero-dimensional generalization of Putnam's theorem.

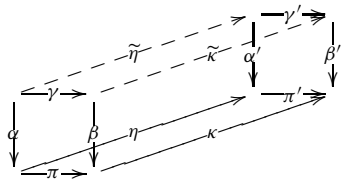
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THEOREM 4.3. *Suppose that $\pi : (X, S) \to (Y, T)$ is a block code from an irreducible SFT onto a sofic shift. Then there are surjective block codes α, β, γ from irreducible SFTs which give a commutative diagram*



such that α and β are one-to-one almost everywhere u-resolving and γ is s-eresolving. If π is finite-to-one, then γ is also s-resolving.

Such a diagram will be produced if β is a version of the futures cover π_+ for π and α, γ are the restrictions to the unique maximal entropy component of the projections from the fiber product of π and β. In this case, the constructed diagram is canonically associated to π: if $(\alpha, \beta, \gamma, \pi)$ and $(\alpha', \beta', \gamma', \pi')$ are two such diagrams and subshift conjugacies η, κ effect a conjugacy of π and π', then there will be a unique pair of subshift conjugacies $\widetilde{\eta}, \widetilde{\kappa}$ such that the following diagram commutes:



Proof. The conclusions of Theorem 4.3 are invariant under replacement of π by a topologically conjugate map. By Lemma 2.4, the map π is topologically conjugate to a map which satisfies the conditions of Proposition 4.2, and this proves the existence claims of Theorem 4.3. That the construction is canonical then follows from Proposition 3.6(4), because the fiber product construction is canonical for a given pair π, π_+. □

Remark 4.4. A (mostly) weaker version of Putnam's result in zero dimension was stated some time ago in [**9**, before Theorem A]. Given π bounded-to-one between irreducible SFTs, [**9**] provides a factorization $\pi\alpha = \beta\widetilde{\pi}$, with all maps between irreducible SFTs, with $\widetilde{\pi}$ left resolving and β right resolving. However, here the map α need not be in any way resolving, and the maps α, β need not have degree one.

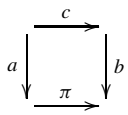
The overlooked resolving possibilities for α probably had to do, at least in part, with the nature of fiber products. In Proposition 4.2, if F_0 is replaced by the full fiber product $F(\pi, \pi_+)$, then the projection $F(\pi, \pi_+) \to Y_+$ will be u-resolving (s-resolving) if and only if π is already u-resolving (s-resolving). Thus, it is surprising (at least, without the guidance of Putnam's theorem) that the restriction to F_0 has any resolving property independent of π, and this is perhaps part of the reason that Putnam's theorem had no zero-dimensional precursor.

5. *The canonical mapping property*

In this section we restrict attention to maps in the category $\mathcal{C}$ of surjective finite-to-one block codes from irreducible SFTs to sofic shifts. For c_1, c_2 in $\mathcal{C}$, the composition c_1c_2 is left/right/u/s -resolving if and only if both c_1 and c_2 are [**5**, Propositions 4.10 and 4.11].

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Definition 5.1. Given $\pi \in \mathcal{C}$, we define $\mathcal{D}(\pi)$ to be the category of commutative diagrams $D = (a, b, c, \pi)$ of the form

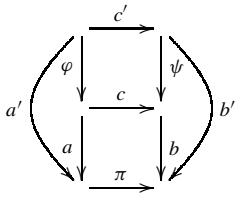


which satisfy the following conditions:

(1) a, b and c are in $\mathcal{C}$;
(2) a and b are u-resolving;
(3) c is s-resolving.

Note that the domains of a, π and c are irreducible SFTs.

For $D = (a, b, c, \pi)$ and $D' = (a', b', c', \pi')$ in $\mathcal{D}(\pi)$, a morphism $D' \to D$ is a pair (φ, ψ) such that $\{\varphi, \psi\} \subset \mathcal{C}$, and the following diagram commutes (i.e. D' factors through D).



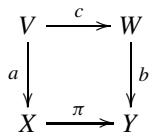
We may write such a morphism as $(\varphi, \psi) : D' \to D$. Note that φ and ψ here are forced to be u-resolving. Diagrams D, D' are isomorphic in this category (they are 'conjugate over π') if there are topological conjugacies φ, ψ such that $(\varphi, \psi) : D' \to D$. The *conjugacy class* of D is the set of all D' in $\mathcal{D}(\pi)$ which are conjugate over π to D. A diagram D is *minimal* in a subset $\mathcal{E}$ of $\mathcal{D}(\pi)$ if $D \in \mathcal{E}$ and $D' \to D$ for every D' in $\mathcal{E}$. For technical reasons we are particularly concerned with the following subcategory of $\mathcal{D}(\pi)$.

Definition 5.2. $\mathcal{D}^{(1)}(\pi)$ is the full subcategory of $\mathcal{D}(\pi)$ whose objects are the diagrams $D = (a, b, c, \pi)$ in $\mathcal{D}(\pi)$ such that the SFT $F(b, \pi)$ (the fiber product of b and π) has a unique irreducible component of maximum entropy.

The condition of Definition (5.2) is satisfied whenever the degree of b is 1. A 'full' subcategory of $\mathcal{D}(\pi)$ is simply one whose morphisms are all of the $\mathcal{D}(\pi)$ morphisms between its objects. When a fiber product $F(b, \pi)$ contains a unique irreducible component of maximum entropy, we denote that component as $F_0(b, \pi)$.

The next proposition shows that much of the structure of a diagram $D = (a, b, c, \pi)$ in $\mathcal{D}(\pi)$ is dictated by the pair (b, π).

PROPOSITION 5.3. *Suppose that D is the following diagram in* $\mathcal{D}(\pi)$.



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Then there is an irreducible component V' of maximum entropy in $F(\pi, b)$ and a constant-to-one map $\varphi : V \to V'$, such that the following diagram D'

$$\begin{array}{ccc} V' & \xrightarrow{p_2} & W \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{\pi} & Y \end{array}$$

is in $\mathcal{D}(\pi)$ and $(\varphi, \mathrm{Id}) : D \to D'$. Here p_1 and p_2 denote the restrictions to V' of the usual projections $(x, w) \mapsto x$, $(x, w) \mapsto w$ from $F(\pi, b)$ to X and W.

Proof. For $v \in V$, define $\varphi(v) = (av, cv)$. Then φ maps V into $F(\pi, b)$. As V is irreducible, the image of φ must be contained in an irreducible component V' of $F(\pi, b)$. Then $h(V) = h(\varphi V) \leq h(F(\pi, b)) = h(X) = h(V)$, so V' has full entropy in $F(\pi, b)$ and $\pi V = V'$.

As $p_1\varphi = a$ is u-resolving, so are p_1 and φ; because $p_2\varphi = c$ is s-resolving, so are p_2 and φ. The resolving properties for p_1 and p_2 show that $D' = (p_2, p_1, b, \pi)$ is in $\mathcal{D}(\pi)$. The map φ is constant-to-one because it is a u-resolving and s-resolving map between irreducible SFTs [**8**, Proposition 4.3.4]. Clearly $(\varphi, \mathrm{Id}) : D \to D'$. □

Given $\pi \in \mathcal{C}$, let D_π be the element of $\mathcal{D}(\pi)$ constructed as for Proposition 4.2, i.e. D_π is the diagram

$$\begin{array}{ccc} F_0 & \xrightarrow{\rho_2} & Y_+ \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle \pi_+} \\ X & \xrightarrow{\pi} & Y \end{array}$$

where $F_0 = F_0(\pi, \pi_+)$, all maps are 1-block codes, and the three SFTs F_0, Y_+ and X are irreducible 1-step. We characterize the conjugacy class of D_π with a mapping property (answering a question in [**16**]).

THEOREM 5.4. *Suppose that $\pi \in \mathcal{C}$.*

(1) *If $D \in \mathcal{D}^{(1)}(\pi)$, then there exists a unique morphism $(\varphi_1, \varphi_2) : D \to D_\pi$.*

(2) *D is minimal in $\mathcal{D}^{(1)}(\pi)$ if and only if D is conjugate over π to D_π.*

QUESTION 5.5. *Is Theorem 5.4(1) true with $\mathcal{D}(\pi)$ in place of $\mathcal{D}^{(1)}(\pi)$?*

Proof. To prove (1), suppose that we have $D \in \mathcal{D}(\pi)$ as below.

$$\begin{array}{ccc} V & \xrightarrow{c} & W \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{\pi} & Y \end{array}$$

Without loss of generality, we suppose that the SFTs are 1-step and the maps in the diagram are 1-block codes.

By Proposition 5.3, without loss of generality we may assume $V = F_0(\pi, b)$, with $(a, c) = (p_1, p_2)$ where p_1, p_2 are the projection maps from $F_0(\pi, b)$ to X and W.

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Again without loss of generality, by Proposition 2.1(2), we may suppose that b is right resolving. As a then arises from the fiber product construction, a is also right resolving. The map c is at this point still only s-resolving (i.e. left closing).

We will find $(\varphi, \psi) : D \to D_\pi$. We first give the rule for ψ. For w left transitive in W, define ψw by

$$(\psi w)_m = U_m(bw), \quad m \in \mathbb{Z},$$

where $U_m(y)$ was defined in (3.1). This is a shift-commuting rule and the left transitive points are dense; so, to show that the rule determines a block code, it suffices to show that the map $w \mapsto U_0(bw)$ is uniformly continuous on left transitive points w.

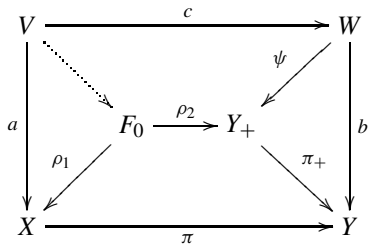
As $c : V \to W$ is s-eresolving (by Proposition 2.1(5)), we may choose $N \geq 0$ such that if $v \in V$ and $(cv)_i = (w')_i$ for $i \geq -N$, then there exists $\widetilde{v}$ in V such that $c\widetilde{v} = w'$ and $\widetilde{v}_i = v_i$ for $i \geq 0$ (see [**6**, Proposition 5.1]). Now, suppose that $\overline{w}$ and w are left transitive points in W, $\overline{w}[-N, 0] = w[-N, 0]$ and $i \in U_0(bw)$. We will show $i \in U_0(b\overline{w})$. For a point (biinfinite sequence) z, we use the notation $z_- = z(-\infty, 0]$ and $z_+ = z[0, +\infty)$.

As $i \in U_0(bw)$, we may choose x in X such that $x_0 = i$ and $(\pi x)_- = (bw)_-$. As bw is left transitive and b is right resolving and $(bw)_- = (\pi x)_-$, by Lemma 2.5 there is a w' in W such that $(w')_- = w_-$ and $bw' = \pi x$. Therefore, (x, w') is a point in the fiber product $F(\pi, b)$. As w' is left transitive, the point (x, w') must be in the irreducible component of maximum entropy, i.e. $(x, w') \in V$. Define w'' in W by $w''(-\infty, 0] = \overline{w}(-\infty, 0]$ and $w''[-N, \infty) = w'[-N, \infty)$. Note, that the overlapping definition is consistent. Now

$$(c(x, w'))[-N, \infty) = w''[-N, \infty),$$

so there exists $\widetilde{v} = (\widetilde{x}, \widetilde{w})$ such that $c\widetilde{v} = w''$ and $\widetilde{v}[0, \infty) = (x, w')[0, \infty)$. Therefore, $(\pi\widetilde{x})(-\infty, 0] = (b\overline{w})(-\infty, 0]$, $\widetilde{x}_0 \in U_0(b\overline{w})$ and $\widetilde{x}_0 = x_0$. We have shown that our rule gives a block code $\psi : W \to Y_+$.

We summarize our situation with the following commuting diagram



in which F_0 denotes $F_0(\pi, \pi_+)$ and $V = F_0(\pi, b)$. The dotted arrow $V \to F_0$ is the map φ we need but have not yet defined. The maps a, ψc factor through π, π_+ and therefore can be factored through $F(\pi, \pi_+)$ by some map $\varphi : V \to F(\pi, \pi_+)$. The image of φ must be F_0; φ defines the dotted arrow. Now $(\varphi, \psi) : D \to D_\pi$. There can be only one morphism $(\varphi, \psi) : D \to D_\pi$ because it is uniquely determined on the dense set of transitive points, because π_+ and ρ_1 are one-to-one on transitive points. This proves (1).

On to (2). It is obvious that a diagram conjugate over π to a minimal element of $\mathcal{D}^{(1)}(\pi)$ (such as D_π) will also be a minimal element of $\mathcal{D}^{(1)}(\pi)$. Now suppose that $D = (a, b, c, \pi)$ is a minimal element of $\mathcal{D}^{(1)}(\pi)$. We have $(\varphi_1, \psi_1) : D_\pi \to D$,

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which forces a and b to have degree 1. We also have $(\varphi_2, \psi_2) : D \to D_\pi$, so $(\varphi_1\varphi_2, \psi_1\psi_2) : D \to D$. By the uniqueness of the morphism, $\varphi_2\varphi_1$ and $\psi_2\psi_1$ must be identity maps, as must $\varphi_1\varphi_2$ and $\psi_1\psi_2$. Thus, D and D_π are conjugate over π. □

Remark 5.6. Suppose that $\pi \in \mathcal{C}$. The characterization of the conjugacy class of π_+ by a minimal mapping property is analogous to the characterization of the right Fisher cover of a sofic system by a minimal mapping property [**4**]. Similarly, the relation of the futures cover of π to the full futures cover is analogous to the relation of the Fisher cover to the Krieger cover [**10**].

6. *The maps of Nasu and Putnam*

Let $\pi : X \to Y$ be a block code from an irreducible SFT onto a sofic shift. Without loss of generality (after recoding if necessary), we take the code to be a 1-block and the SFT to be a 1-step. In Proposition 4.2, we associated to π a certain diagram $D_\pi = (\rho_1, \rho_2, \pi, \pi_+)$. In this section we explain the relationship of D_π to the construction of Putnam and the relation of π_+ to a canonical cover defined by Nasu.

Putnam. Given a surjective one-to-one almost everywhere morphism π between irreducible Smale spaces, Putnam constructed a commuting diagram $\widetilde{D} = (\alpha, \widetilde{\pi}, \beta, \pi)$,

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\widetilde{\pi}} & \overline{Y} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{\pi} & Y \end{array}$$

with all of the maps one-to-one almost everywhere surjective morphisms π between irreducible Smale spaces, with α and β u-resolving, and with $\widetilde{\pi}$ s-resolving.

Theorem 4.3 and Putnam's theorem overlap in the case that π is a surjective, one-to-one almost everywhere. block code between irreducible SFTs. We claim in this case that the two diagrams D and $\widetilde{D}$ are conjugate over π. To show this, we first find a conjugacy of subshifts $\psi : Y_+ \to \overline{Y}$ such that $\pi_+ = \beta\psi$. Without loss of generality, we may presume that π is a 1-block map and X is a 1-step SFT.

Putnam constructed $\overline{Y}$ as the completion in a certain metric δ_Y of the set $\pi(W)$ of points left asymptotic to the orbit of a chosen periodic point y_0, where y_0 has a unique preimage x_0 under π, and W is the set of points in X left asymptotic to the orbit of x_0. Our desired map ψ must agree with the natural injection ι from $(\pi_+)^{-1}(\pi(W))$ to $\pi(W)$. We claim that ι is uniformly continuous (with respect in the domain to the restriction of a metric giving the usual topology on Y_+, and with respect in the range to the metric δ_Y).

For brevity, we will not define all of the terms and facts that we use from the subtle construction of [**16**]. Let y and y' be in $\pi(W)$, with $z = \pi_+^{-1}(y)$ and $z' = \pi_+^{-1}(y')$ in Y_+. Now suppose that $N > 0$ and $z[-N, N] = z'[-N, N]$. By the definition of Y_+, we have $y[-N, N] = y'[-N, N]$ and moreover there exists a bijection $\nu : \pi^{-1}(y) \to \pi^{-1}(y')$ such that $(\nu w)[-N, N] = w[-N, N]$, for all w in $\pi^{-1}(y)$. Define a point y'' in Y by $y''(-\infty, \infty) = y(-\infty, -N]y'[-N + 1, \infty)$. Then for large enough N, the path (y, y'', y') is a rectangular path from y to y'' (see [**16**, Definition 2.13]), whose length is the sum $\mathrm{dist}(y, y'') + \mathrm{dist}(y'', y')$, which is at most 2^{-N+1}, and which is an upper bound

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for $\delta_Y^0(y, y')$ (see [**16**, Definition 2.14]). Then, for a number r with $0 < r < 1$ and the uniform constant $C = \max \delta_Y^0$, we get

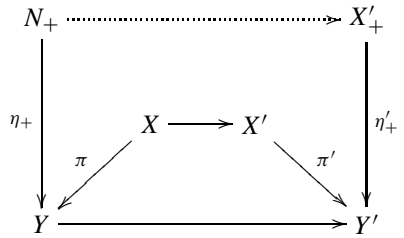
$$\delta_Y(y, y') \leq \sum_{k=0}^{N} r^k 2^{-N+k} + \sum_{k=N+1}^{\infty} C r^k.$$

This quantity goes to zero with N.

Thus ι is uniformly continuous, and extends to a continuous map $\psi : Y_+ \to \overline{Y}$. This map commutes with the shift on a dense set, hence everywhere, and is thus a block code. By Theorem 5.4 there is also a block code $\psi' : \overline{Y} \to Y_+$ such that $\beta = \pi_+ \psi'$. Clearly ψ and ψ' are inverse to each other, so ψ is a conjugacy and β is a version of π_+. Thus, by Proposition 5.3, the map α is the map ρ_1 preceded by some constant-to-one factor map; because the degree of α is one, the constant-to-one map is a conjugacy of subshifts. This completes the proof.

Nasu. In [**12**], given a 1-block code $\pi : X \to Y$ from an edge SFT onto a sofic system, Nasu associated to π a map he called the *induced right resolving cover* of π. (This cover appears as an important tool in [**9**].) This cover is a 1-block code we denote by $\eta_+ : N_+ \to Y$. Here N_+ is another edge SFT.

The given map π is presented by a labeled graph $\mathcal{G}$, where the unlabeled graph defines the edge SFT and the label (by symbols from $\mathcal{A}(Y)$) defines the one-block code. Nasu defined his cover by constructing another labeled graph, $\mathcal{G}_+$. For a subset V of vertices of $\mathcal{G}$ and a Y-word x, let $S_+(V, x)$ denote the set of terminal vertices $S_+(V, x)$ of the paths labeled x that go from some vertex in V. A *right compatible set* of $\mathcal{G}$ is a set $S_+(V, x)$ where V is a singleton. Nasu defined the vertex set of $\mathcal{G}_+$ to be the set of all maximal right compatible sets and their non-empty sucessors. For each vertex S in $\mathcal{G}_+$, a unique edge labeled a goes from W to $S_+(W, a)$ if $S_+(W, a)$ is non-empty. This defines the edge set of $\mathcal{G}_+$, and thereby the edge SFT N_+ and the 1-block map $\eta_+ : N_+ \to Y$. Clearly η_+ is surjective and right resolving. Nasu showed [**12**, Theorem 3.1] that η_+ is canonically associated to π, in the sense that any conjugacy from π to another 1-block code $\pi' : X' \to Y'$ (such a conjugacy by definition is given by conjugacies $X \to X'$, $Y \to Y'$ which intertwine π and π') lifts to an isomorphism of η_+ and η'_+:



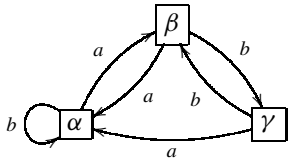
Nasu's cover η_+ (like the futures cover π_+) is well-defined even if the domain SFT is not irreducible or the map π is not finite-to-one. If X is irreducible, then both N_+ and Y_+ will be irreducible, and the degrees of π_+ and η_+ are well-defined. Here the map π_+ will always have degree one, but η_+ might have greater degree. For example, if π is itself right resolving, then clearly (as noted by Nasu [**12**, p.569]) η_+ will equal π.

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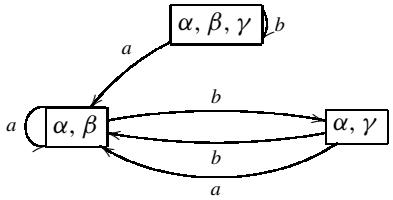
Now restrict to the case that X is irreducible and the degree of π is one. In this case, there is a conjugacy ψ between the domains of π_+ and η_+ such that $\eta_+ = \pi_+\psi$. To see this we simply note that in the degree one case, the symbols $U_0(y)$ in $\mathcal{A}(Y_+)$ are maximal compatible sets from the unique preimage symbol of a magic symbol for π. Thus, the identity map embeds Y_+ into N_+, and because the irreducible SFT N_+ has no proper subsystem of equal entropy, this embedding is surjective. In particular, our diagram D_π, produced using $F_0(\pi, \pi_+)$ will be conjugate over π to the diagram produced in the same way using $F_0(\pi, \eta_+)$. Each of the diagrams is another version of Putnam's diagram in dimension zero.

It is natural to ask whether Nasu's right resolving cover must factor through the futures cover. This is not always the case, as we see in the next example.

Example 6.1. Let π be the map given in [**2**, Example 1, p. 491]. This map π can be defined by the following labeled graph $\mathcal{G}$.



Forgetting the labels in the graph gives an unlabeled graph, with adjacency matrix $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$, which defines an irreducible edge SFT (X, S). The edge labeling defines a 1-block code π from (X, S) to a shift (Y, T), which is the full 2-shift on symbols a, b. A labeled graph $\mathcal{G}_{++}$ defining a full futures cover (Y_{++}, T_{++}) of π as an edge SFT, and the 1-block code π_{++}, can be computed as follows. Vertices of $\mathcal{G}_{++}$ are sets of vertices of $\mathcal{G}$. Begin with the vertex $v = \{\alpha, \beta, \gamma\}$. The set of $\mathcal{G}$-paths labeled a starting from v end in the set $\{\alpha, \beta\}$. Add this set as another vertex and draw an edge labeled a from v to v'. Continue in this fashion until reaching a graph such that no new vertices can be added. Reduce to the maximal subgraph such that each vertex has an incoming edge and an outgoing edge. This produces the following $\mathcal{G}_{++}$.



The futures cover (Y_+, T_+) is obtained by restricting this graph to the irreducible component of maximum entropy, which as an unlabeled graph has adjacency matrix $\left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)$. As π is right resolving, the associated right resolving cover η_+ of Nasu is equal to π. Consequently, in this example there is no factor map φ such that $\eta = \pi_+\varphi$, for the following reason. There is a fixed point x such that $\pi(x)$ is the fixed point b^∞, and φ could only map x to a fixed point y such that $\pi_+(y) = b^\infty$; but there is no such fixed point y in the domain of π_+.

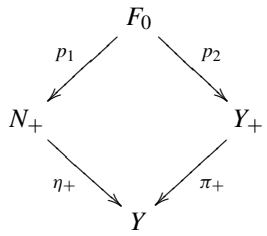
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7. *Resolving degree and Markovian maps*

In this section we point out that the resolving degree is an invariant of topology conjugacy for block codes from irreducible SFTs. (This is well known when the code is finite-to-one; then resolving degree equals degree.) We also give an application of Theorem 4.3 to Markovian codes.

PROPOSITION 7.1. *Suppose that π and π' are topologically conjugate maps from irreducible SFTs. Then they have the same resolving degree.*

Proof. Suppose that π is a 1-block code with a magic symbol μ. Let η_+ and π_+ be the Nasu and futures covers associated to π. Form the fiber product diagram



Here F_0 denotes the unique irreducible component of maximum entropy in the fiber product (it is unique because π_+ has degree 1). The projections p_1 and p_2 are surjective. Clearly $\{\mu\}$ is a magic symbol for the map p_2 and the cardinality of its preimage, which is the degree of p_2, is the cardinality of $\pi^{-1}(\mu)$. As η_+ and π_+ are canonically associated to π, so is the fiber product diagram and, in particular, the degree of p_2. Therefore, this number will be the same for π and π'. □

A surjective block code between irreducible SFTs is *Markovian* if any (equivalently every) fully supported ergodic Markov measure on the range space lifts to a fully supported ergodic Markov measure on the domain [**6, 15**]. From the current paper, we find that any block code between irreducible SFTs is in a certain sense not far from Markovian.

PROPOSITION 7.2. *Suppose that $\varphi : (X, S) \to (Y, T)$ is a surjective block code between irreducible SFTs. Then there is a u-resolving degree 1-block code α from an irreducible SFT onto (X, S) such that $\varphi\alpha$ is Markovian.*

Proof. By Theorem 4.3, there is an α of the desired form such that $\varphi\alpha$ equals an s-eresolving code followed by a u-resolving code, both between irreducible SFTs. Such codes (and their compositions) are Markovian [**6**]. □

QUESTION 7.3. *It is an open problem since* [**6**] *to provide a decision procedure to determine whether a surjective block code between irreducible SFTs is Markovian. Do the constructs around the futures cover provide clues?*

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