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Homeomorphisms from Julia sets into connectedness loci

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Abstract. We construct a homeomorphism from the filled Julia set of a given polynomial into the connectedness locus of the family of all polynomials of higher degree by using intertwining surgery. A polynomial in the image of such a homeomorphism has a renormalization of 'capture' type and the inverse map is given by the 'marking' of the captured critical orbit.

1. *Introduction*

Milnor [**Mi**] studied real cubic polynomials and classified renormalizations (or hyperbolic components) of cubic polynomials into four cases: *adjacent*, *bitransitive*, *capture* and *disjoint cases*. Roughly speaking, a renormalization (in the complex case) is a polynomial-like restriction of some iterate of a given polynomial with connected filled Julia set, and these four cases are defined as follows. Let U be the domain of definition of a renormalization (or the immediate basin of an attracting periodic point).

Adjacent case: U contains both of the critical points (so the renormalization has degree three).

Bitransitive case: U contains one of the critical points and forward images of U contain the other critical point (so the renormalization has degree four).

Capture case: U contains one of the critical points and the other critical points are eventually mapped into the filled Julia set of the renormalization (so the renormalization has degree two).

Disjoint case: There exist two renormalizations (or attractive cycles). Hence, necessarily, they have degree two and the domain of definition of each renormalization (or the immediate basin of each attracting periodic cycle) contains a critical point.

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Capture and disjoint renormalizations have degree two and they seem rather easier to analyze than the others. Epstein and Yampolsky [**EY**] proved that there exist small copies of $M \times M$ (or $M_{p/q} \times M_{p/q}$) in the cubic connectedness locus by using the intertwining surgery technique, where M is the Mandelbrot set and $M_{p/q}$ is the small copy of M attached to the main cardioid and contained in the p/q-limb of M. A polynomial in such a small copy has two renormalizations (that is, disjoint case) and the straightening of them gives the homeomorphism.

Buff and Henriksen [**BH**] studied cubic one-parameter families of the form $\{\lambda z + az^2 + z^3 : a \in \mathbb{C}\}$ for $|\lambda| = 1$ and proved that there exist copies of the Mandelbrot set and copies of the filled Julia set of $\lambda z + z^2$ attached to them (intersecting at one point). Here a copy of the Mandelbrot set corresponds to parameters which has two renormalizations, one of which is hybrid equivalent to $\lambda z + z^2$. The straightening of the other renormalization gives the homeomorphism and such parameters are contained in a small copy of $M \times M$ obtained by Epstein and Yampolsky. A polynomial in a copy of the filled Julia set of $\lambda z + z^2$ has a renormalization of capture-type hybrid equivalent to $\lambda z + z^2$. The forward image of the captured critical point 'marks' the filled Julia set of the renormalization and this gives the homeomorphism.

More generally, a polynomial P of degree $d \geq 3$ has a *capture renormalization* if there exist $s > 0$ and topological disks U, U' such that:

(1) $P^s : U' \to U$ is a polynomial-like map with connected filled Julia set K;
(2) there exists a critical point ω of P such that $\bigcup_{n\geq 0} P^n(K)$ contains all the critical points of P except ω (ω can be a multiple critical point);
(3) $P^N(\omega) \in K$ for some $N > 0$.

The above ω is called the *captured critical point*. We can also consider the case when two or more critical points are captured, but for simplicity we only treat the one captured critical point case, so we adopt this definition. Let us denote by Cap_d the set of polynomials of degree d having capture renormalizations. By the straightening theorem [**DH2**], a capture renormalization is hybrid equivalent to some polynomial Q. Let $\psi : U \to \mathbb{C}$ denote a hybrid conjugacy. We define the *straightening map of capture renormalizations* by $T : \mathrm{Cap}_d \ni P \mapsto (Q, \psi(P^N(\omega))) \in \mathrm{Poly}_d \times \mathbb{C}$, where Poly_d is the set of polynomials of degree d and N is the smallest integer satisfying (3) of the definition above. Since ψ is uniquely determined on the filled Julia set, T is well defined. Let $\mathcal{C}_d \subset \mathrm{Poly}_d$ be the connectedness locus (the set of polynomials with connected Julia set) of degree d. Then, the image of T is contained in

$$\bigcup_{P\in\mathcal{C}_d} \{P\} \times K(P).$$

In this paper, we construct the local inverse of the map T and prove the following.

THEOREM 1.1. *Let P_0 be a polynomial with connected filled Julia set $K(P_0)$ of degree less than $d \geq 3$. Then there exists a homeomorphism $h_{P_0} : K(P_0) \to h_{P_0}(K(P_0)) \subset \mathcal{C}_d$ with $T \circ h_{P_0}(z) = (P_0, z)$ and $h_{P_0}(J(P_0)) \subset \partial\mathcal{C}_d$.*

See Theorem 3.1 for more detail (see also Figure 2). This theorem is an extension of the result of Buff and Henriksen. In the cubic case, the image $h_{P_0}(K(P_0))$ is attached to

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a copy of $M_{p/q} \times M_{p/q}$ obtained by Epstein and Yampolsky. We necessarily have the similar combinatorial condition in the higher-degree case. The image is attached to the set of intertwined polynomials obtained by Epstein and Yampolsky. The precise combinatorial condition is given in Theorem 3.1. Capture renormalizations with other combinatorics are discussed in §7.

It is likely that the image of such a homeomorphism is contained in a one-dimensional analytic subset. Furthermore, computer pictures suggest that such a homeomorphism can be extended to some neighborhood of the filled Julia set. See §7.6 for more details.

2. *Preliminaries*

2.1. *Böttcher coordinates and external rays.* Let Poly_d be the set of all monic centered polynomials $P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$ of degree $d \geq 2$ and $\mathrm{Poly}_d^0 \subset \mathrm{Poly}_d$ the set of all polynomials of the form $P(z) = z^d + c$. We can naturally identify $\mathrm{Poly}_d^0 \cong \mathbb{C}$. For $P \in \mathrm{Poly}_d$, let $J(P)$ be the Julia set and $K(P)$ be the filled Julia set of P. Let $C(P)$ be the set of critical points of P. Let $\mathcal{C}_d = \{P \in \mathrm{Poly}_d : J(P) \text{ is connected}\}$ be the *connectedness locus* of Poly_d and let $\mathcal{M}_d = \mathcal{C}_d \cap \mathrm{Poly}_d^0$ be the connectedness locus of Poly_d^0. The Mandelbrot set M is equal to $\mathcal{M}_2 = \mathcal{C}_2$.

Define the *Green's function* $g_P : \mathbb{C} \to [0, +\infty)$ for P by

$$g_P(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+(P^n(z)).$$

Then a vector field $\xi_P = \operatorname{grad} g_P / |\operatorname{grad} g_P|^2$ is meromorphic on $\mathbb{C} \setminus K(P)$ with poles at $C(g_P)$ (the set of critical points of g_P), which coincides with the backward orbit of all critical points of P outside $K(P)$. Let

$$\gamma_P = \bigcup_{x \in C(g_P)} W^s_{\xi_P}(x),$$

where $W^s_{\xi_P}(x)$ is the stable manifold of x for the flow defined by ξ_P. If $C(g_P)$ is empty (or, equivalently, $P \in \mathcal{C}_d$), then γ_P is empty.

The *Böttcher coordinate* φ_P is the conformal map which conjugates P and z^d near infinity. Since $(\varphi_P)_* \xi_P = \xi_{z^d}$, we can extend φ conformally with the help of the flows defined by ξ_P and ξ_{z^d}. More precisely, the following proposition holds (see [**DH1**] and [**BH**]).

PROPOSITION 2.1. *There exists a conformal map $\varphi_P : \mathbb{C} \setminus (K(P) \cup \gamma_P) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ (where $\mathbb{D}$ is the unit disk) such that*

$$\varphi_P(P(z)) = (\varphi_P(z))^d, \quad \log|\varphi_P| = g_P, \quad (\varphi_P)_* \xi_P = \xi_{z^d}$$

and

$$\varphi : \bigcup_{P \in \mathrm{Poly}_d} \{P\} \times (\mathbb{C} \setminus (K(P) \cup \gamma_P)) \to \mathbb{C} \setminus \overline{\mathbb{D}}$$

defined by $\varphi(P, z) = \varphi_P(z)$ is holomorphic.

Furthermore, the map $\Phi_d : \mathbb{C} \setminus \mathcal{M}_d \to \mathbb{C} \setminus \overline{\mathbb{D}}$ defined by $\Phi_d(c) = \varphi_{z^d+c}(c)$ is a conformal isomorphism tangent to the identity at infinity.

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Define the *Green's function* G_d for $\mathcal{M}_d$ by $G_d(c) = g_{z^d+c}(c)$. We have $G_d(c) = \log|\Phi_d(c)|$ if $c \notin \mathcal{M}_d$. The *external ray* $R_P(\theta)$ (respectively $\mathcal{R}_d(\theta)$) for P (respectively $\mathcal{M}_d$) with angle $\theta \in \mathbb{R}/\mathbb{Z}$ is defined by

$$R_P(\theta) = \varphi_P^{-1}(\{re^{2\pi i\theta} \mid r \in (1,\infty)\}),$$
$$\mathcal{R}_d(\theta) = \Phi_d^{-1}(\{re^{2\pi i\theta} \mid r \in (1,\infty)\}).$$

Since $(\phi_P)_*\xi_P = \xi_{z^d} = \bar{z}^{-1}$, the ray $R_P(\theta)$ is the flow line of ξ_P converging to infinity with angle θ.

Note that if the Julia set of P is not connected, φ_P^{-1} is defined only on $\{re^{2\pi i\theta} \mid r \in (r_0,\infty)\}$ for some $r_0 \geq 1$ in general. If $r_0 > 1$, we say that the ray *bifurcates* at $\lim_{r\to r_0}\varphi_P^{-1}(re^{2\pi i\theta})$ (the limit always exists and lies in $C(g_P)$). We say that an external ray $R_P(\theta)$ (respectively $\mathcal{R}_d(\theta)$) *lands* at $x \in J(P)$ (respectively $x \in \partial\mathcal{M}_d$) if the ray does not bifurcate (this is always true for Φ_d) and x is the limit of $\varphi_P^{-1}(re^{2\pi i\theta})$ (respectively $\Phi_d^{-1}(re^{2\pi i\theta})$) as $r \to 1$. The limit x is called the *landing point* of the ray and θ is called a *landing angle* for x. Every rational external ray (that is, θ is rational) lands at some point in $J(P)$ (respectively $\partial\mathcal{M}_d$) unless it bifurcates (see [**DH1**]). If θ is periodic by d (that is, $d^N\theta = \theta$ for some $N > 0$), then the landing point x of $R_P(\theta)$ is a repelling or parabolic periodic point. On the contrary, for $P \in \mathcal{C}_d$ and a repelling or parabolic periodic point x of P, landing angles of x are periodic of the same period. Hence, there exist finitely many landing angles for x.

If a repelling or parabolic periodic point x of $P \in \mathrm{Poly}_d$ has finitely many landing angles, then we can define the *(combinatorial) rotation number* of x as follows. Let s be the period of x and let sq be the period of the landing angles for x. Since P^s is a local diffeomorphism at x, the cyclic order of the landing rays for x is preserved by P^s. Therefore, there exist some $N \geq 1$, $p \geq 0$ and $q \geq 1$ such that the number of the landing angles for x is equal to Nq and P^s rotates the landing rays counterclockwise by Np. We define the combinatorial rotation number of x for P by p/q.

Consider $P(z) = z^d + c$ where c is the landing point of some rational parameter ray $\mathcal{R}_d(\theta)$ with $\theta \in \mathbb{Q}/\mathbb{Z}$. If the denominator of θ_0 is divisible by d, then P is subhyperbolic (that is, the critical point 0 is strictly preperiodic for P). On the contrary, if $P(z) = z^d + c$ is subhyperbolic, every landing angle θ of c for P is rational with denominator divisible by d and θ is also the landing angle of c for $\mathcal{M}_d$ [**DH1**]. Therefore, $d^{N+q}\theta \equiv d^N\theta \mod 1$ for some integer $N, q \geq 1$. Such a c is called a *Misiurewicz point* for $\mathcal{M}_d$.

2.2. *Invariant sectors.* Now we introduce the notion of invariant sectors. A *sector* S at a point x_0 is the closure of a Jordan domain whose boundary contains x_0, with two marked points $x_\pm$ other than x_0 on the boundary. We always assume that x_0, x_-, x_+ are ordered counterclockwise in this order. We call x_0 the *center* of S, $x_\pm$ the *corners* of S. The *circular arc* $\partial^0 S$ of S is the closure of the component of $\partial S \setminus \{x_-, x_+\}$ which does not contain x_0. The *radial arcs* $\partial^\pm S$ of S are the closures of the components of $\partial S \setminus \{x_0, x_\pm\}$ which do not contain $x_\mp$ respectively.

For sectors S_1, S_2, there exists a unique homeomorphism $h : S_1 \to S_2$ such that h is conformal in the interior and that h maps the center to the center, and the corners to

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the corners by the Riemann mapping theorem (since the boundaries are Jordan curves, a Riemann map extends continuously on the boundary). We call the map h the *isomorphism* between S_1 and S_2. We call a sector $\Delta_1 = \{\operatorname{Im} z \geq 0, |z| \leq 1\}$ centered at the origin and with corners ± 1 *the standard sector*. We say a family of sectors $\{S_\lambda\}_{\lambda\in\Lambda}$ is *continuous* on $\lambda \in \Lambda$ if the isomorphism $h_\lambda : \Delta_1 \to S_\lambda$ is continuous on $\lambda \in \Lambda$ with respect to the supremum norm (or equivalently, a map $(\lambda, z) \mapsto h_\lambda(z)$ is continuous on $\Lambda \times \Delta_1$).

LEMMA 2.2. *Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of sectors and let $\gamma_\lambda^\sigma : I^\sigma \to \partial^\sigma S_\lambda$ be homeomorphisms for $\sigma = 0, \pm$, where I^σ is a closed interval.*

Assume γ_λ^σ is continuous on $\Lambda \times I^\sigma$ for $\sigma = 0, \pm$ and there exists some σ_0 such that $\gamma_\lambda^{\sigma_0}$ is analytic on the interior of I^{σ_0} for any λ. Then the family $\{S_\lambda\}_{\lambda\in\Lambda}$ is continuous.

Proof. Let $h_\lambda : \overline{\mathbb{D}} \to S_\lambda$ be the Riemann mapping which maps 1 to the center and $\pm i$ to the corners. In other words, h_λ is the isomorphism from the sector $\overline{D}$ centered at 1 with corners $\pm i$ to S_λ. We need only show that h_λ is continuous on λ. Since $\gamma_\lambda^{\sigma_0}$ is real analytic, h_λ can be extended to a neighborhood U_λ of $\mathbb{D} \cup \gamma^{\sigma_0}(\operatorname{int} I^{\sigma_0})$.

Since the sectors are bounded by a continuous family of Jordan curves, it follows that $\{S_\lambda\}$ is locally bounded. Hence, we may assume it is bounded and $\{h_\lambda|_{\mathbb{D}}\}$ forms a normal family.

Assume there exists a sequence $\lambda_n \to \lambda_0$ which satisfies $\sup_{z\in\bar{\mathbb{D}}} |h_{\lambda_n}(z) - h_{\lambda_0}(z)| > 0$. By passing to a subsequence, we may assume $h_{\lambda_n}|_{\mathbb{D}} \to h \neq h_{\lambda_0}$, where h is a conformal map or a constant map.

On the contrary, there exists a neighborhood V^{σ_0} of the interior of I^{σ_0} such that $\psi_{\lambda_n} = h_{\lambda_n}^{-1} \circ \gamma_{\lambda_n}^{\sigma_0}$ can be extended conformally on V^{σ_0} for sufficiently large n by the reflection principle $\psi_{\lambda_n}(\bar{z}) = (\overline{\psi_{\lambda_n}(z)})^{-1}$. Since ψ_{λ_n} maps the interior of I^{σ_0} to a component of $\partial\mathbb{D} \setminus \{1, \pm i\}$, it follows that the derivative of ψ_{λ_n} at a given point in the interior of I^{σ_0} is bounded above. Therefore, ψ_{λ_n} also forms a normal family, and by passing to a further subsequence, we may assume ψ_{λ_n} converges to a conformal map or a constant map $\psi : V^{\sigma_0} \to \mathbb{C}$.

Therefore, we have $h \circ \psi = \gamma_{\lambda_0}$, where both maps are defined. Since γ_{λ_0} is not constant, h and ψ are non-constant conformal maps. Since $(\operatorname{int} S_{\lambda_n}, h_{\lambda_n}^{-1}(0))$ converges to $(\operatorname{int} S_{\lambda_0}, h^{-1}(0))$ in the Carathéodory topology, $h : \mathbb{D} \to \operatorname{int} S_{\lambda_0}$ is a Riemann map. By [**Po**, Theorem 9.11, Problem 9.3.7], h_{λ_n} converges uniformly to h on $\overline{\mathbb{D}}$. Therefore, h maps 1 and $\pm i$ to the center and the corners of S_{λ_0} respectively. Since h and h_{λ_0} are the isomorphisms of sectors $\overline{\mathbb{D}}$ and S_{λ_0}, they must coincide, which is a contradiction. □

Consider a tuple of sectors $(S(0), \ldots, S(q-1))$ at x. In this paper, we always assume $S(j) \cap S(j') = \{x\}$ if $j \neq j'$ and a tuple of sectors are ordered counterclockwise around the center. Let $S = \bigcup S(j)$. We can define the isomorphism between q-tuples of sectors similarly. Let

$$\Delta_q(j) = \{|z| \leq 1 : 2j\pi/q \leq \arg z \leq (2j+1)\pi/q\}$$

and we call a q-tuple of sectors $(\Delta_q(j))_{j=0,\ldots,q-1}$ *the standard q-tuple of sectors*.

Let f be a holomorphic germ at x with $f(x) = x$. We say $(S(0), \ldots, S(q-1))$ is *f-invariant* if there exists some neighborhood U of x and $f(S \cap U) = S \cap f(U)$. We consider a q-tuple of sectors only in the case when the rotation number of x is equal to p/q for some

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$p > 0$ prime to q. Then $(S(j) \cap U)/f^q \cong (S \cap U)/f$ is an annulus and we can define the *modulus* of an invariant tuple of sectors $(S(j))_{0 \le j \le q-1}$ (or an f^q-invariant sector $S(j)$) at x for f by

$$\mathrm{mod}_f(S(j))_{0 \le j \le q-1} = \mathrm{mod}_f S(j) = \mathrm{mod}((S(j) \cap U)/f^q).$$

Similarly, we can also define invariant sectors at a periodic orbit for a holomorphic map f.

Example. Let $R > 0$, $\eta > 0$ and $\theta_0 \in \mathbb{R}/\mathbb{Z}$. If $\mathcal{R}_d(\theta_0)$ lands at some point c, then

$$\mathcal{S}_d(\theta_0) = \overline{\Phi_d^{-1}(\{\exp(r + 2\pi i\theta) : 0 < r < R,\ |\theta - \theta_0| < \eta r\})}.$$

is a sector at c with corners $\Phi_d^{-1}(\exp(R + 2\pi i(\theta_0 \pm \eta R)))$. Similarly, if $R_P(\theta_0)$ lands at a point x, define a sector

$$S_P(\theta_0) = \overline{\varphi_P^{-1}(\{\exp(r + 2\pi i\theta) : 0 < r < R,\ |\theta - \theta_0| < \eta r\})}.$$

Assume x is a repelling or parabolic fixed point of P with rotation number p/q and let $\theta_0, \theta_1, \ldots, \theta_{q-1}$ be a periodic cycle of landing angles of x. Then $(S_P(\theta_0), \ldots, S_P(\theta_{q-1}))$ is a P-invariant tuple of sectors at x. The complementary sectors (the closures of components of $\{g_P \le R\} \setminus \bigcup S_P(\theta_j)$) also form an invariant tuple of sectors.

The following lemma is a special version of [**EY**, Lemma 2.16].

LEMMA 2.3. *Let x be a repelling fixed point of a polynomial f. Let p/q be a rotation number of x and $\theta_0, \ldots, \theta_{q-1}$ be a periodic cycle of landing angles of x. If $K(f)$ has no interior, then we have*

$$\inf_{\mathcal{S}} \mathrm{mod}_f \mathcal{S} = 0,$$

where the infimum is taken for all invariant q-tuples of sectors $\mathcal{S} = (S_0, \ldots, S_{q-1})$ with $\bigcup S_j \supset K(f)$ and $S_j \cap R_f(\theta_k) = \emptyset$ for all $0 \le j, k < q$.

Proof. Take a small Jordan neighborhood U of x such that U is a relatively compact subset of $f^q(U)$. Since $K(f)$ has no interior, we can choose $\mathcal{S}$ such that the distance between $\partial^+ S_0 \cap f^q(U) \setminus U$ and $\partial^- S_0 \cap f^q(U) \setminus U$ is arbitrarily small.

Therefore, the length of a path in S_0 which connects ∂U and $\partial f^q(U)$ with respect to the hyperbolic metric in S_1 is arbitrarily large. This implies that $\mathrm{mod}_f \mathcal{S} = \mathrm{mod}(S_0 \cap \overline{f^q(U)} \setminus U)/f^q$ is arbitrarily small. □

2.3. *Polynomial-like maps and intertwinings.* A *polynomial-like map* $f : U' \to U$ is a proper holomorphic map between topological disks such that U' is a relatively compact subset of U. The *filled Julia set* $K = K(f; U', U)$ of a polynomial-like map $f : U' \to U$ is defined by $K = \bigcap_n f^{-n}(U')$ and let $J(f; U', U) = \partial K$ be the *Julia set* of $f : U' \to U$. The straightening theorem [**DH2**] states that every polynomial-like map is hybrid equivalent to some polynomial of the same degree. Furthermore, if the filled Julia set is connected, then such a polynomial is unique up to affine conjugacy. (Two polynomial-like maps are *hybrid equivalent* if there exists a quasiconformal conjugacy which is defined in a neighborhood of the filled Julia set and whose complex dilatation is zero almost everywhere in the filled Julia set.)

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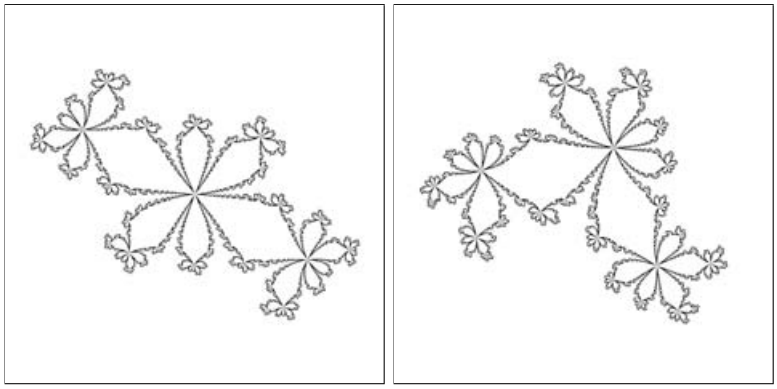


FIGURE 1. Intertwinings of the Douady rabbit $P(z) = z^2 - 0.122561\cdots + 0.744861\cdots i$ and itself at the α-fixed point.

Let x be a repelling fixed point of a polynomial-like map $f : U' \to U$ with rotation number p/q and let $(S(0), \ldots, S(q-1))$ be an f-invariant tuple of sectors at x. Since the sectors are ordered counterclockwise, it follows that $f(S(j) \cap V) = S(j+p) \cap f(V)$ for a small neighborhood V of x (indices are understood modulo q). We say $(S(0), \ldots, S(q-1))$ is *nice* if the circular arcs of $S(j)$ are contained in the boundary of U and $f(\partial^{\pm} S(j) \cap U') = \partial^{\pm} S(j+p)$.

Let $P_1 \in \mathcal{C}_{d_1}$ and $P_2 \in \mathcal{C}_{d_2}$. Assume α_j is a repelling or parabolic fixed point of P_j with rotation number p/q. We say a polynomial $Q \in \mathrm{Poly}_{d_1+d_2-1}$ is the *intertwining* of (P_1, α_1) and (P_2, α_2) if there exists a polynomial-like restriction $f : U_j' \to U_j$ $(j = 1, 2)$ of f such that:

- $f : U_j' \to U_j$ is hybrid equivalent to P_j;
- $K(f; U_1', U_1) \cap K(f; U_2', U2) = \{x\}$;
- a hybrid conjugacy $\psi_j : U_j \to \mathbb{C}$ satisfies that $\psi_1(\alpha_1) = \psi_2(\alpha_2) = x$.

Note that x is a repelling or parabolic fixed point with rotation number p/q. Examples of the intertwining are given in Figure 1 and Figure 3.

As we can see in Figure 1, intertwinings of (P_1, α_1) and (P_2, α_2) have combinatorial ambiguity (how their filled Julia sets are intertwined). In this paper, we always fix one set of combinatorial data (unless stated). Then, the intertwining is unique up to affine conjugacy [**I**, §5] (for intertwinings of two quadratic polynomials, uniqueness is first proved by Epstein and Yampolsky [**EY**]).

The existence of the intertwining at repelling fixed points is proved by Epstein and Yampolsky [**EY**], and [**H**] treats also some cases where at least one of α_1 and α_2 is parabolic. They use quasiconformal (or trans-quasiconformal) surgery (called *intertwining surgery*) to construct intertwinings. The surgery construction by Epstein and Yampolsky will be reviewed in §5 and used in the proof of our results.

3. *Statement of results*

THEOREM 3.1. *Let α_0 be a repelling fixed point of rotation number p/q for a polynomial P_0 of degree $d_0 \geq 2$ with connected filled Julia set. Let c_0 be a Misiurewicz point for*

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$\mathcal{M}_{d_1}$ and θ be a landing angle of c_0. Let $N, q \geq 1$ be the smallest integers which satisfy $d_1^{N+q}\theta \equiv d_1^N\theta$. Suppose $\alpha_1 = P_1^N(c_0) = P_1^{N+1}(0)$ is a repelling fixed point with rotation number p/q. Let $Q \in \mathcal{C}_d$ be the intertwining of (P_0, α_0) and (P_1, α_1) where $d = d_0 + d_1 - 1$.

Then there exists a homeomorphism $h_{P_0,P_1} : K(P_0) \to h_{P_0,P_1}(K(P_0)) \subset \mathcal{C}_d$ such that $h_{P_0,P_1}(\alpha_0) = Q$ and for $Q_x = h_{P_0,P_1}(x)$:

(1) *there exist some topological disks U_x and U'_x such that $Q_x : U'_x \to U_x$ is a polynomial-like map hybrid equivalent to P_0. Let $\psi_x : U_x \to \mathbb{C}$ be a hybrid conjugacy;*

(2) *there exists a unique critical point ω_x with multiplicity $d_1 - 1$ which does not lie in the filled Julia set K_x of a polynomial-like map $Q_x : U'_x \to U_x$. Moreover, we have $Q_x^{N+1}(\omega_x) \in K_x$ and $\psi_x \circ Q_x^{N+1}(\omega_x) = x$ (note that ψ_x is uniquely determined on K_x);*

(3) *$Q_x \in \partial\mathcal{C}_d$ if $x \in J(P_0)$.*

The conditions (1) and (2) imply that Q_x has a capture renormalization and $T(Q_x) = (P_0, x)$ where T is the straightening map in §1.

Figure 2 shows an example of homeomorphic images of a cubic Julia set in a one-dimensional slice of $\mathcal{C}_4$, and the Julia sets of P_0 and the intertwining of P_0 and P_1.

Theorem 3.1 can be applied only if P_0 has a repelling fixed point, but we can also show a similar result by applying intertwining surgery at a repelling periodic orbit instead of a fixed point (see §7.1 for more detail). Therefore, we can obtain Theorem 1.1 as a corollary of this theorem.

Remark. It is known that the set of polynomials of degree d having a polynomial-like restriction hybrid equivalent to P_0 forms an analytic set [**DH2**, §II.6]. Therefore, the image of h_{P_0,P_1} is contained in the intersection of such an analytic set and an algebraic subset

$$A = \{P \in \mathrm{Poly}_d : P \text{ has a critical point of multiplicity } d_1 - 1\} \subset \mathrm{Poly}_d.$$

Since $\dim A = (d-1) - (d_1 - 2) = d - d_1 + 1 = d_0$ and $\dim \mathrm{Poly}_{d_0} = d_0 - 1$, it is natural to expect that such an analytic set has codimension $d_0 - 1$ in A, that is, the image of h_{P_0,P_1} is contained in an analytic set of dimension one. It is easy to see this for some cases (for example, P_0 has $d_0 - 1$ non-repelling cycles, or $d_0 = 2, \dots$), but it is still open in general.

Furthermore, if the image is contained in a one-dimensional analytic set, we might ask whether h_{P_0,P_1} can be extended to an open neighborhood. See also §7.6.

4. *Outline of the proof*

Basically, our proof of Theorem 3.1 is based on intertwining surgery of P_0 and a polynomial $P \in \mathrm{Poly}_{d_1}^0$ near P_1 (not only the case $P \in \mathcal{M}_{d_1}$).

Intuitively, since $\alpha_1 = P_1^{N+1}(0)$ is the mating fixed point, the corresponding forward image of the critical point for a perturbation P of P_1 should move all of a given small neighborhood of the mated fixed point. In the surgery construction, we need to take invariant sectors for P outside the filled Julia set to place the dynamics of P_0 by surgery. For a good perturbation, $P^{N+1}(0)$ lies in such a sector, so we can consider the 'marking' by the corresponding point of $P^{N+1}(0)$ in the dynamical plane for P_0.

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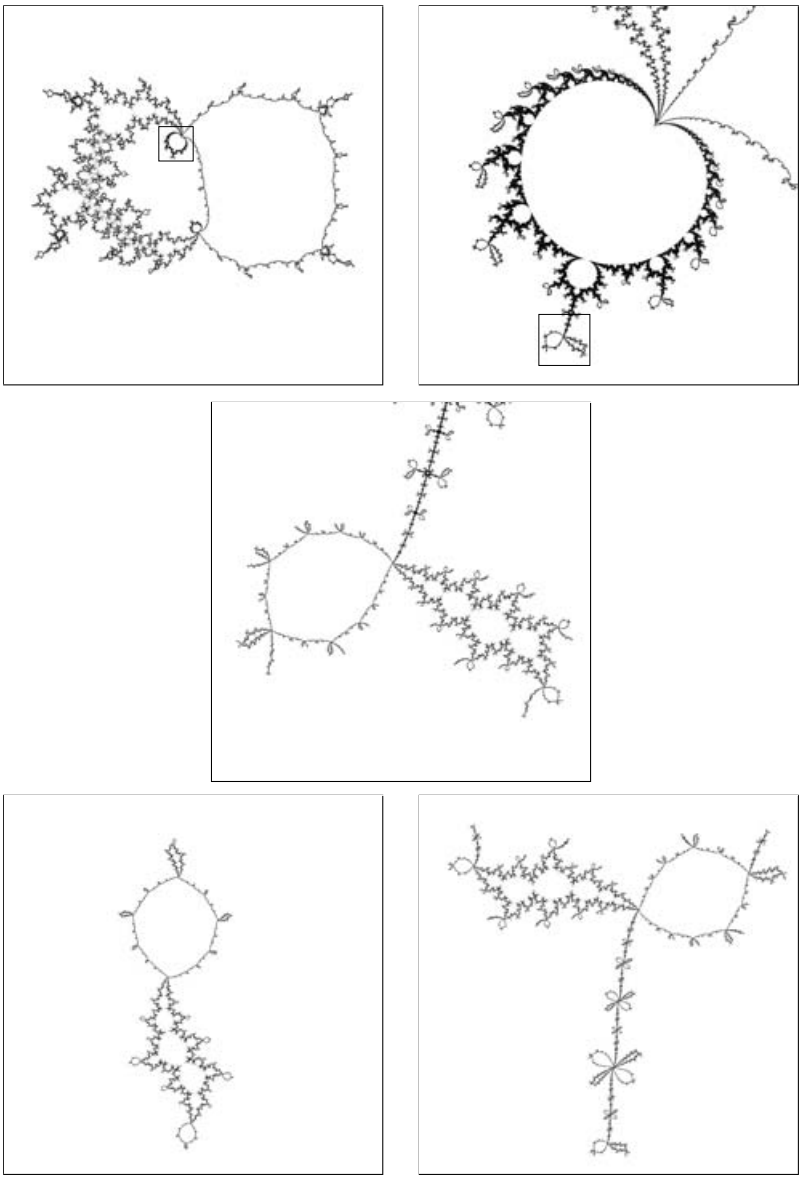


FIGURE 2. The first three pictures are the parameter space of quartic polynomials of the form $(z-1)^2((az+\lambda-2)z-1)+1$ having a superattracting fixed point and a fixed point of multiplier $\lambda = \exp(\pi i(\sqrt{5}-1))$ and its magnifications. We can see that the Julia set of a cubic polynomial $P_0(z) = z(\lambda + az + z^2)$ where $a = -(2\lambda-3)/(\lambda-2)$ (the fourth picture) is homeomorphically embedded in the third picture. The last picture is the Julia set of the intertwining of P_0 and P_1 in Theorem 3.1 for this copy of $J(P_0)$.

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Since we can take such sectors arbitrarily small, for such perturbations, $P^{N+1}(0)$ moves all of the sectors and the 'marking' seem to yield a surjection from sectors at P_1 to a neighborhood of $K(P_0) \setminus \{$the mating fixed point$\}$ defined by the 'marking' and a good intertwining surgery construction seems to yield a map ('intertwining map') from sectors at P_1 to a neighborhood of the intertwining of P_0 and P_1 in the parameter space Poly_d where $d = \deg P_0 + d_1 - 1$. We will see that the marking map is a homeomorphism on the inverse image of the filled Julia set and the intertwining map is continuous there and this proves Theorem 3.1.

The difficulty of the proof is non-uniqueness of an intertwining Q of P_0 and P since P may have disconnected Julia set. Surgery construction has many ambiguities and hence there does not exist a natural map $P \mapsto Q$ given by intertwining surgery with P_0. So we need to give a concrete construction in order not to have such ambiguities and in order to have good continuity results.

Ambiguities mainly occur when we take invariant sectors. Hence, we first construct a family of polynomial-like mappings $(f_\lambda : V'_\lambda \to V_\lambda)$ hybrid equivalent to polynomials near P_1, nicely parameterized and having some modulus estimates for invariant sectors. The construction of a family (f_λ) is given in §5.

We intertwine P_0 and f_λ to obtain a polynomial Q_λ. Then we can prove some partial continuity on λ and obtain the following theorem.

THEOREM 4.1. *Under the same assumptions as Theorem 3.1, there exist maps $\iota_{P_0} : \Delta_q \to \mathrm{Poly}_d$ (intertwining map) and $\mu_{P_0} : \Delta_q \to \mathbb{C}$ (marking map) such that for $Q_\lambda = \iota_{P_0}(\lambda)$:*

(i) *Q_0 is the intertwining of (P_0, α_0) and (P_1, α_1).*

(ii) *There exist topological disks U_λ and U'_λ such that $Q_\lambda : U'_\lambda \to U_\lambda$ is a polynomial-like map hybrid equivalent to P_0 by a hybrid conjugacy $\rho_\lambda : U_\lambda \to \mathbb{C}$.*

(iii) *There exists a unique critical point ω_λ outside U'_λ. The multiplicity of ω_λ is $d_1 - 1$. We have $Q_\lambda^{N+1}(\omega_\lambda) \in U_\lambda$ and $\rho_\lambda \circ Q_\lambda^{N+1}(\omega_\lambda) = \mu_{P_0}(\lambda)$.*

(iv) *Let $K = \mu_{P_0}^{-1}(K(P_0))$. Then $\mu_{P_0}|_K : K \to K(P_0)$ is a homeomorphism.*

(v) *If $\lambda \in K$ (respectively ∂K), then Q_λ lies in $\mathcal{C}_d$ (respectively $\partial\mathcal{C}_d$).*

(vi) *$\iota_{P_0}|_K : K \to \mathcal{C}_d$ is continuous.*

Proof of Theorem 3.1. We prove that $h_{P_0,P_1} = \iota_{P_0} \circ \mu_{P_0}^{-1}$ satisfies the theorem, where ι_{P_0} and μ_{P_0} are in Theorem 4.1. By Theorem 4.1(iv) and (vi), $h_{P_0,P_1} : K(P_0) \to \mathcal{C}_d$ is continuous. Thus, Q_x, U_x, U'_x, ψ_x and ω_x in Theorem 3.1 are given by Q_λ , U_λ, U'_λ, ρ_λ and ω_λ in Theorem 4.1 for $\lambda = \mu_{P_0}^{-1}(x)$, respectively.

By Theorem 4.1(i), $Q_0 = Q$ is the intertwining of (P_0, α_0) and (P_1, α_1). (1) follows from Theorem 4.1(ii). (2) follows from Theorem 4.1(iii). Since $\psi_x \circ Q_x^{N+1}(\omega_x) = \mu_{P_0}(\lambda) = x$, $Q_x^{N+1}(\omega_x)$ lies in $K_x = \psi_x^{-1}(K(P_0))$. (3) follows from Theorem 4.1(v).

Finally, we prove that $h_{P_0,P_1} : K(P_0) \to h_{P_0,P_1}(K(P_0))$ is a homeomorphism. By (1) and (2), $Q_x = h_{P_0,P_1}(x)$ for $x \in K(P_0)$ has a renormalization of capture type. Thus, we can consider the straightening map $T : h_{P_0,P_1}(K(P_0)) \to \mathcal{C}_{d_0} \times \mathbb{C}$ (for the definition, see §1). By definition and by (1) and (2), we have $T(Q_x) = (P_0, x)$. Therefore, $\pi_2 \circ T \circ h_{P_0,P_1}(x) = x$ and h_{P_0,P_1} is a bijection onto its image, where $\pi_2 : \mathrm{Poly}_d \times \mathbb{C} \to \mathbb{C}$ is the projection to the second coordinate. Since a continuous bijection from a compact set into a Hausdorff space is a homeomorphism, h_{P_0,P_1} is a homeomorphism. □

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5. *A family of polynomial-like mappings*

In this section, we construct a good family of polynomial-like maps $(f_\lambda : V'_\lambda \to V_\lambda)$. More precisely, we prove the following theorem.

THEOREM 5.1. *Let $P_1(z) = z^{d_1} + c_0$ be a polynomial of degree $d_1 \geq 2$. Suppose for some $N \geq 1$, $\alpha_1 = P_1^N(c_0) = P_1^{N+1}(0)$ is a repelling fixed point of P_1 with rotation number p/q.*

Then, for any $m > 0$ and any small $\varepsilon > 0$, there exists a continuous family $(f_\lambda : V'_\lambda \to V_\lambda)_{\lambda\in\Delta_q}$ of polynomial-like mappings of degree d_1 analytic except at $\lambda = 0$ such that $0 \in V'_\lambda$ is a repelling fixed point of f_λ with rotation number p/q, and there exists a continuous family of nice f_λ-invariant tuples of sectors $\{(S^k_{f_\lambda}(0), \ldots, S^k_{f_\lambda}(q-1))\}_{\lambda\in\Delta_q}$ at 0 for $k = 0, 1, 0\pm, 1\pm$ such that:

(1) *$(f_0, 0)$ is hybrid equivalent to (P_1, α_1).*

(2) *Let $S^k_{f_\lambda} = \bigcup_j S^k_{f_\lambda}(j)$. Then, $\bigcup_k S^k_{f_\lambda} = \overline{V_\lambda}$ and the interiors of sectors $S^k_{f_\lambda}(j)$ are mutually disjoint. Furthermore, we have $\operatorname{mod} S^0_{f_\lambda} = m$ and $\operatorname{mod} S^1_{f_\lambda} = \varepsilon$.*

(3) *For each $\lambda \in \Delta_q$, sectors $S^k_{f_\lambda}(j)$ are ordered counterclockwise around 0 as follows:*

$$S^{0-}_{f_\lambda}(0), S^0_{f_\lambda}(0), S^{0+}_{f_\lambda}(0), S^{1-}_{f_\lambda}(0), S^1_{f_\lambda}(0), S^{1+}_{f_\lambda}(0), S^{0-}_{f_\lambda}(1), \ldots,$$
$$S^{0+}_{f_\lambda}(q-1), S^{1-}_{f_\lambda}(q-1), S^1_{f_\lambda}(q-1), S^{1+}_{f_\lambda}(q-1), S^{0-}_{f_\lambda}(0),$$

that is, $\partial^+ S^{0-}_{f_\lambda}(0) = \partial^- S^0_{f_\lambda}(0)$ and so on.

(4) *$J(f_\lambda) \subset S^1_{f_\lambda}$ for any $\lambda \in \Delta_q$ and $J(f_\lambda)$ is totally disconnected if $\lambda \neq 0$.*

(5) *f_λ has only one critical point ω_{f_λ} (necessarily of multiplicity $d_1 - 1$). We have $f^k_\lambda(\omega_{f_\lambda}) \in S^1_{f_\lambda}$ for $0 \leq k \leq N$ and $f^{N+1}_\lambda(\omega_{f_\lambda}) \in S^0_{f_\lambda}$.*

(6) *The isomorphism $h^0_{f_\lambda} : \Delta_q \to S^0_{f_\lambda}$ satisfies that $(h^0_{f_\lambda})^{-1} \circ f^{N+1}_\lambda(\omega_{f_\lambda}) = \lambda$.*

We will find such a family (f_λ) in the space of polynomials of degree $d_1 + 1$. f_λ has two critical points of multiplicity $d_1 - 1$ (ω_{f_λ} in the theorem) and one, and an attracting periodic orbit of period q whose multiplier is close to 1.

For $0 < \mu < 1$, let $B(z) = z(z+\mu)/(1+\mu z)$ be a Blaschke product of degree two with two attracting fixed points 0 and infinity. The critical point of B inside the unit disk is $\omega_B = -(1-(1-\mu^2))/\mu < 0$ and the postcritical set is contained in the negative real axis.

LEMMA 5.2. *For any $m > 0$, B has a closed invariant sector $\mathcal{S}_\mu \subset \bar{\mathbb{D}}$ at a repelling fixed point 1 such that $\operatorname{mod}_B \mathcal{S}_\mu > m$ for μ sufficiently close to 1 and it does not intersect the grand orbit of the critical point ω_B.*

Proof. Let L be the linearizing coordinate at the origin, that is, L is defined and conformal near the origin, $L(0) = 0$ and $L(B(z)) = \mu L(z)$. For small $r, \varepsilon > 0$,

$$O_\mu = L^{-1}(\{|z| \leq r\} \setminus \{|\arg z - \pi| < \varepsilon\})$$

is an invariant sector at 0 with modulus $-2(\pi - \varepsilon)/\log\mu$, which tends to infinity as μ tends to 1.

Let $\hat{\mathcal{S}}_\mu$ be the closure of the component of $\bigcup_{n>0} B^{-n}(O_\mu)$ which contains O_μ. The interval $[0, 1]$ is contained in $\hat{\mathcal{S}}_\mu$ and since $O_\mu \setminus \{0\}$ is forward invariant and disjoint

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from the postcritical set, $\hat{\mathcal{S}}_\mu \setminus \{0\}$ is disjoint from the grand orbit of the postcritical set. Take a compact neighborhood U of 1 and $\mathcal{S}_\mu = U \cap \hat{\mathcal{S}}_\mu$ satisfies the lemma. □

Let $\mathcal{H}_{p/q}$ be the hyperbolic component of $\mathcal{M}_2$ attached to the main cardioid at $c_{p/q} = \frac{1}{2}e^{2\pi ip/q} - \frac{1}{4}e^{4\pi ip/q}$. ($z^2 + c_{p/q}$ is affinely conjugate to $e^{2\pi ip/q}z + z^2$ and has a parabolic fixed point of multiplier $e^{2\pi ip/q}$.) For $c \in \mathcal{H}_{p/q}$, $Q(z) = z^2 + c$ has a repelling fixed point α_Q with rotation number p/q and an attracting periodic point x_Q of period q whose immediate basin contains α_Q in its boundary. The multiplicity map $\lambda_{p/q} : \mathcal{H}_{p/q} \to \mathbb{D}$ defined by $\lambda_{p/q} = (Q^q)'(x_Q)$ is a conformal isomorphism [**DH1**].

Now, consider the case $c = \lambda_{p/q}^{-1}(\mu)$. By the Riemann mapping theorem and the reflection principle, Q^q restricted to the immediate basin of this periodic point is conformally conjugate to B restricted to $\mathbb{D}$. Therefore, this lemma implies that Q has an invariant tuple of closed invariant sectors $(S(0), \dots, S(q-1))$ at α_Q and $\mathrm{mod}_Q(S(j), \alpha_Q) > m$.

Let $P_1(z) = z^{d_1} + c_0$ be the polynomial in the assumption of Theorem 5.1. Since $P_1^N(c_0) = \alpha_1$, there exist exactly q rays landing at c_0. (Otherwise, we have at least $2q$ landing angles for α_1. Since there exists only one critical point, $\{P_1^n\}_{n>0}$ is normal on one of the sectors divided by the landing rays for α_1, this is a contradiction.) An example of the intertwining of (Q, α_Q) and (P_1, α_1) is given in Figure 3. Let $\theta_0 < \theta_1 < \cdots < \theta_{q-1} < \theta_0 + 1$ be the landing angles of c_0. Take a polynomial $P(z) = z^{d_1} + c$ with $c \in \mathcal{R}_{d_1}(\theta_{j_0})$ sufficiently close to c_0 for $0 \le j_0 < q$. Let α_P be the repelling fixed point of P which is the continuation of α_1 for P_1.

LEMMA 5.3. *If $\mu < 1$ is sufficiently close to 1, then we can construct an intertwining f of (P, α_P) and (Q, α_Q) such that $f^{N+1}(\omega)$ lies in the invariant sector for f which corresponds to $S(j_0)$ for Q, where ω is the critical point of f which corresponds to the critical point 0 for P.*

Remark. Since $J(P)$ is disconnected, an intertwining of (P, α_P) and (Q, α_Q) is not unique.

Proof. Our proof is based on the construction of the intertwining surgery technique given by Epstein and Yampolsky. See [**EY**] or §6 for details.

First, we briefly review the surgery construction of intertwinings by Epstein and Yampolsky [**EY**] in our context. Take a polynomial-like restriction $P : U_P' \to U_P$ and a nice invariant tuple of sectors $(S_P^k(0), \dots, S_P^k(q-1))$ at α_P for $k = 0, 1$, such that

$$\bigcup_j S_P^0(j) \supset K(P), \quad S_P^1(j) \cap K(P) = \{\alpha_P\}$$

and $S_P^0(j) \cap S_P^1(j') = \{\alpha_P\}$ for any j and j'. Similarly, take U_Q, U_Q' and $S_Q^k(j)$ for Q. We may assume $S_Q^0(j) \supset S(j)$ and $P^{N+1}(0) = P^N(c) \in S_P^1(j_0)$, because $d_1^N\theta_0, \dots, d_1^N\theta_{q-1}$ are the landing angles for α_P and $P^N(c) \in R_P(d^N\theta_{j_0})$ (so we need only take such that $S_Q^0(j) \cap R_P(d^N\theta_j)$ is non-empty). Then, we can construct a polynomial-like mapping $g : U_g' \to U_g$ hybrid conjugate to $P : U_P' \to U_P$ such that the invariant sectors $S_g^k(j)$ corresponding to $S_Q^k(j)$ satisfy

$$\mathrm{mod}_g\, S_g^0(j) = \mathrm{mod}_Q\, S_Q^1(j-1), \quad \mathrm{mod}_g\, S_g^1(j) = \mathrm{mod}_Q\, S_Q^0(j). \qquad (*)$$

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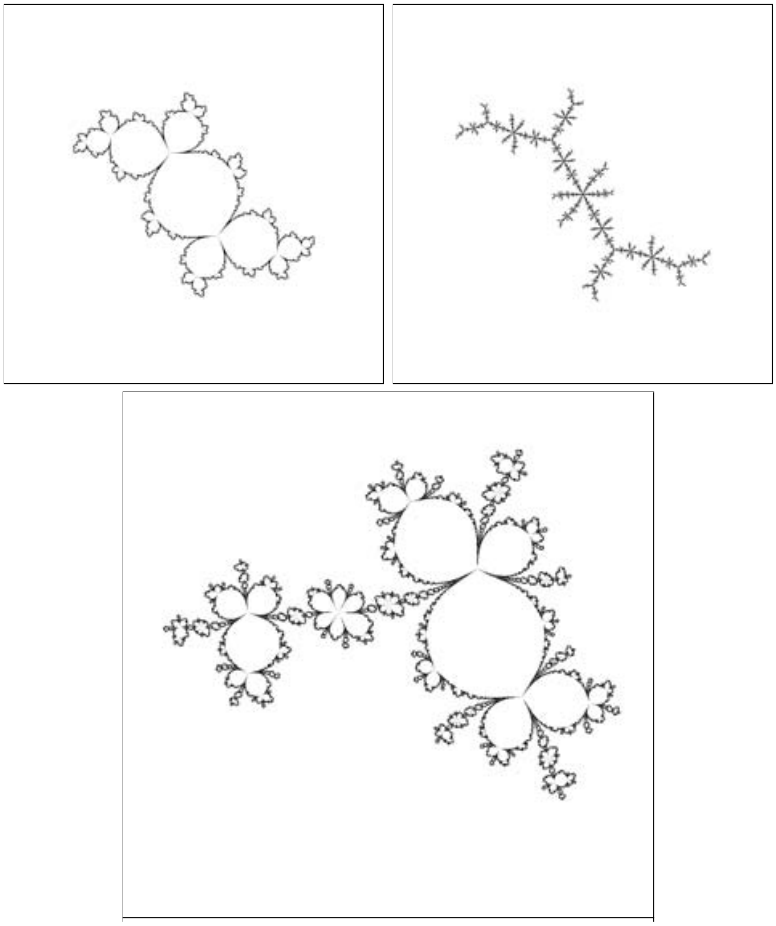


FIGURE 3. The intertwining of a fat rabbit $Q(z) = z^2 - 0.124\,974\cdots + 0.659\,133\ldots i$ and a Misiurewicz quadratic polynomial $P_1(z) = -0.101\,096\cdots + 0.956\,285\ldots i$. Q has an attracting periodic point of period 3 with multiplier 0.9. $P_1^4(0)$ is the α-fixed point of P_1.

Then, there exists a quasiconformal homeomorphism $\phi : U_g \to U_Q$ such that restrictions $\phi|_{S_g^0(j)} : S_g^0(j) \to S_Q^1(j-1)$ and $\phi|_{S_g^1(j)} : S_g^1(j) \to S_Q^0(j)$ are isomorphisms between sectors.

Remark. The equalities on moduli $(*)$ are necessary for a homeomorphism ϕ to be quasiconformal (see [**Bi**] or Lemma 6.1 for details). Since $\mathrm{mod}_P\, S_P^1(j)$ cannot be arbitrarily large, we cannot choose sectors so that these equalities hold for P and Q in general. This is why we deform P and use a polynomial-like map g instead of P itself. (Note that $\mathrm{mod}_Q\, S_Q^0(j) > \mathrm{mod}_B\, \mathcal{S}_\mu$ tends to infinity as μ tends to 1.)

On the contrary, since $J(P) = K(P)$ has no interior, we may assume $\mathrm{mod}_P\, S_P^0(j)$ is arbitrarily small by Lemma 2.3 and we can choose $S_Q^1(j)$ so that the above equalities hold. Thus, we do not need to deform Q here.

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Define a proper quasiregular map $\tilde{f} : U'_{\tilde{f}} \to U_{\tilde{f}} = U_P$ such that

$$\tilde{f} = \begin{cases} P & \text{on } S^0_P(j) \cap U'_P, \\ \phi^{-1} \circ g \circ \phi & \text{on } \phi^{-1}(S^0_g(j) \cap U'_Q), \end{cases}$$

and $U'_{\tilde{f}}$ is a relatively compact subset of $U_{\tilde{f}}$. Moreover, we can construct an $\tilde{f}$-invariant almost complex structure σ with bounded dilatation on $U_{\tilde{f}}$. By the measurable Riemann mapping theorem, $\tilde{f}$ is conjugate to a polynomial-like map, which is hybrid equivalent to a polynomial f by the straightening theorem.

To prove the lemma, take $A(z) = \lambda z$ ($\lambda > 0$) such that $\mathrm{mod}_A \Delta_1 = \mathrm{mod}_Q S^0_Q(j_0)$ and let $\tilde{\psi} : S^1_P(j_0) \to \Delta_1$ be a diffeomorphism which conjugates from P^q to A. Then, $\psi = h \circ \tilde{\psi} : S^1_P(j_0) \to S^0_Q(j_0)$ is a diffeomorphism and we have $(\psi \circ P)^* \sigma_0 = \psi^* \sigma_0$, where $h : \Delta_1 \to S^0_Q(j_0)$ is the isomorphism. Hence, the dilatation of ψ is bounded. If c is sufficiently close to c_0, then $P^{N+1}(0) \in S^1_P(j_0) \cap R_P(d^N \theta_{j_0})$ is sufficiently close to α_P and $[P^{N+1}(0)] \in S^1_P(j_0)/P^q$ is not close to the boundary. Therefore, $\psi(P^{N+1}(0))$ must lie in $S(j_0)$ if μ is sufficiently close to 1.

Define an almost complex structure σ on U_P by

$$\sigma = \begin{cases} (\psi \circ P^n)^* \sigma_0 & \text{on } P^{-n}(S^1_P(j_0)), \\ \sigma_0 & \text{otherwise}, \end{cases}$$

where σ_0 is the standard complex structure on $\mathbb{C}$. It is well defined because $(\psi \circ P)^* \sigma_0 = \psi^* \sigma_0$ on $S^1_P(j_0)$ by definition. Moreover, the dilatation of σ is bounded because P is holomorphic. Hence, by the measurable Riemann mapping theorem, there exists a quasiconformal map $\chi : U_P \to U_g$ such that $\chi^*(\sigma_0) = \sigma$. Then, $g = \chi \circ P \circ \chi^{-1} : U'_g = \chi(U_P) \to U_g$ is a polynomial-like map hybrid equivalent to P. Furthermore, we have $\mathrm{mod}_g \chi(S^1_P(j_0)) = \mathrm{mod}_Q S^0_Q(j_0)$.

Since $\phi|_{S^0_g(j_0)}$ and $\psi \circ \chi^{-1}|_{S^0_g(j_0)}$ are the isomorphisms of the sectors between $S^0_g(j_0)$ and $S^1_Q(j_0)$, they must be equal. Let ω_g be the critical point of g. Then,

$$\begin{aligned} \phi \circ g^{N+1}(\omega_g) &= \psi \circ \chi^{-1}(g^{N+1}(\omega_g)) \\ &= \psi \circ P^{N+1}(\chi^{-1}(\omega_g)) \\ &= \psi \circ P^{N+1}(0) \in S(j_0). \end{aligned}$$

□

Let Ψ be a hybrid conjugacy between Q and f defined near $K(Q)$. Note that Ψ is uniquely determined on $K(Q)$. Denote by $x = \Psi(x_Q)$ the attracting periodic point of period q with multiplier μ for f, and let $\alpha_f = \Psi(\alpha_Q)$ be the fixed point for f at which P and Q are intertwined.

LEMMA 5.4. *For $0 \le j_0 < q$, let f be a polynomial obtained in Lemma 5.3. There exists a homeomorphism $\Xi_{j_0} : S(j_0) \to \mathrm{Poly}_{d_1+1}$ into its image such that $\Xi_{j_0}(\alpha_Q) = P_1$ and for $f_w = \Xi_{j_0}(w)$.*

- *For $w \neq \alpha_Q$, f_w is quasiconformally conjugate to f. We can choose the conjugacy $\Phi_w : \mathbb{C} \to \mathbb{C}$ between f and f_w such that the Beltrami differential of Φ_w is supported only on the basin of attraction of x. Let $\omega_w = \Phi_w(\omega)$ be the critical point of f_w.*

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- *The multiplier of the attracting periodic point $x_w = \Phi_w(x)$ does not depend on w. In particular, f_w has a quadratic-like restriction hybrid equivalent to Q. Let Ψ_w be the hybrid conjugacy between f_w and Q.*
- *Let $S^0_{f_w}(j) = \Psi_w^{-1}(S(j))$. Then $\{(S^0_{f_w}(0), \dots, S^0_{f_w}(q-1))\}_{w \in S(j_0)}$ is a continuous family and*
$$f_w^{N+1}(\omega_w) \in S^0_{f_w}(j_0), \quad \Psi_w(f_w^{N+1}(\omega_w)) = w.$$

Proof. Let $S^0_f(j_0) = \Psi(S(j_0))$ be an invariant sector at α_f for f. We consider a quasiconformal deformation of f supported on the grand orbit

$$O = \bigcup_{n,m \geq 0} f^{-n}(f^m(S^0_f(j_0)))$$

of $S^0_f(j_0)$. Then O/f is an annulus with modulus

$$\begin{aligned} \operatorname{mod}(O/f) &= \operatorname{mod}(S(j_0)/Q) \\ &= -\frac{2(\pi - \varepsilon)}{\log \mu}, \end{aligned}$$

and it contains the grand orbit $\tilde{\omega}$ of ω (the critical point which corresponds to that of P).

Let $\tilde{S}_f(j_0) = \bigcup_{n \geq 0} f^{nq}(S^0_f(j_0)) \subset O$. Take an essential simple closed curve γ of O/f passing through $\tilde{\omega}$. Denote the components of $(O/f) \setminus \gamma$ by A_1 and A_2. By changing the moduli of A_1 and A_2, and by 'spinning' [**PT**, §1.1] along γ, we can deform f quasiconformally to another polynomial g such that $g^{N+1}(\omega_g)$ corresponds to an arbitrary point in the interior of $S(j_0)$ by the hybrid conjugacy between g and Q, where ω_g is the corresponding critical point of ω.

More precisely, let $\hat{S}(j_0) = \bigcup_{n>0} Q^n(S(j_0))$. Then $S(j_0)/Q \cong \hat{S}(j_0)/Q \cong O/f$ is conformally isomorphic to a round annulus $A_r = \{r < |z| < 1\}$ for some $0 < r < 1$. The projection $\hat{S}(j_0) \to \hat{S}(j_0)/Q \cong A_r$ is a universal covering. Let $\pi : O \to A_r$ be the composition of the natural projection to O/f and the conformal isomorphism. We may assume $\pi(\omega) = t_0 \in (r, 1)$. Let $\gamma = \{|z| = t_0\}$. For $t \in (r, 1)$ and $\theta \in \mathbb{R}$, consider a quasiconformal homeomorphism $h_{t,\theta} : A_r \to A_r$ given by

$$h_{t,\theta}(z) = \begin{cases} \exp\left(2\pi i\theta \dfrac{|z| - r}{t_0 - r}\right) \dfrac{(t-r)|z| - (t - t_0)r}{(t_0 - r)|z|} z & \text{for } |z| \leq t_0, \\ \exp\left(2\pi i\theta \dfrac{1 - |z|}{1 - t_0}\right) \dfrac{(1-t)|z| + (t - t_0)}{(1 - t_0)|z|} z & \text{for } |z| > t_0. \end{cases}$$

(The case $t = t_0$ corresponds to the 'spinning' deformation [**PT**].) Then $h_{t,\theta}(t_0) = te^{2\pi i\theta}$ and $h_{t,\theta}|_{\partial A_r}$ is the identity. Let $\sigma_{t,\theta} = \pi^* \circ h_{t,\theta}^* \sigma_0$ be the complex structure on O, where σ_0 is the standard complex structure. We can extend $\sigma_{t,\theta}$ by σ_0 on $\mathbb{C} \setminus O$. Clearly, $\sigma_{t,\theta}$ is f-invariant. By the measurable Riemann mapping theorem, there exists a quasiconformal map $\Phi_{t,\theta} : \mathbb{C} \to \mathbb{C}$ such that $\Phi_{t,\theta}^* \sigma_0 = \sigma_{t,\theta}$. Then the map $f_{t,\theta} = \Phi_{t,\theta} \circ f \circ \Phi_{t,\theta}^{-1}$ is a polynomial (we can normalize $\Phi_{t,\theta}$ such that $f_{t,\theta} \in \operatorname{Poly}_d$). Since $(t, \theta) \mapsto \sigma_{t,\theta}$ is continuous, $(t, \theta) \mapsto f_{t,\theta}$ is also continuous. Let $\omega_{f_{t,\theta}} = \Phi_{t,\theta}(\omega)$ be the critical point which corresponds to ω for f.

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CLAIM. *The periodic point $x_{t,\theta}$ for $f_{t,\theta}$ has the same multiplier as x for f (that is, $f'_{t,\theta}(x_{t,\theta}) = f'(x) = \mu$).*

The proof of the claim is similar to that of [**PT**, Lemma 2.1]. Roughly speaking, the deformation above preserves the complex structure of the quotient torus and the multiplier at a fixed point is determined by the complex structure of the quotient torus (a punctured neighborhood modulo the dynamics) and the homotopy class of its 'longitude' $[\gamma]$, where γ is a simple closed curve surrounding the fixed point. This fact is represented by the following elementary lemma (the proof is left to the reader).

LEMMA 5.5. *Let $\mu_1, \mu_2 \in \mathbb{C}$ with $0 < |\mu_1|, |\mu_2| < 1$. Let $T_j = \mathbb{C}^*/(z \sim \mu_1 z)$ be a complex torus ($j = 1, 2$). Denote by $[\gamma]_j$ the homotopy class in T_j of a simple closed curve γ surrounding 0.*

If there exists a conformal isomorphism $\tau : T_1 \to T_2$ such that $\tau_*([\gamma]_1) = [\gamma]_2$, then $\mu_1 = \mu_2$.

Proof of the claim. Let O_1 be a neighborhood of x. Then, the quotients

$$T = (O_1 \setminus \{x\})/f \ (\supset O/f) \quad \text{and} \quad T_{t,\theta} = (\Phi_{t,\theta}(O_1) \setminus \{x_{t,\theta}\})(\supset \Phi_{t,\theta}(O)/f_{t,\theta})$$

are complex tori. A quasiconformal conjugacy $\Phi_{t,\theta}$ between f and $f_{t,\theta}$ induces a quasiconformal homeomorphism between them.

Define a quasiconformal homeomorphism $\tau : T \to T_{t,\theta}$ by the following. Let $\tau([z]) = [\Phi_{t,\theta}]$ for $[z] \in T \setminus O/f$. On O/f, define τ by the following isomorphisms:

$$O/f \xrightarrow{\simeq} A_r \xrightarrow{h_{t,\theta}^{-1}} A_r \xrightarrow{\simeq} O/f \xrightarrow{\Phi_{t,\theta}} \Phi_{t,\theta}(O)/f_{t,\theta}.$$

It is easy to see that it does not depend on the choice of the conformal isomorphism $O/f \xrightarrow{\simeq} A_r$. Since $h_{t,\theta}$ is identity on the boundary, τ is continuous. By the definition of $\Phi_{t,\theta}$, it is easy to see τ is a conformal isomorphism. Since $\Phi_{t,\theta}$ and $h_{t,\theta}$ depend continuously on (t, θ), τ maps the homotopy class of a simple closed curve surrounding x to that of $x_{t,\theta}$.

Using Lemma 5.5 via respective linearizing coordinates at x and $x_{t,\theta}$, we have $\mu_{t,\theta} = \mu$. □

Since $f_{t,\theta}$ is a quasiconformal deformation of f, it also has a quadratic-like restriction near $\Phi_{t,\theta} \circ \Psi(K(Q))$, which corresponds to that of f hybrid equivalent to Q. It is hybrid equivalent to Q because the multiplier of $x_{t,\theta}$ is constant under the deformation. Let $\Psi_{t,\theta}$ be a hybrid conjugacy between Q and $f_{t,\theta}$. Then, $\tilde{h}_{t,\theta} = \Psi_{t,\theta}^{-1} \circ \Phi_{t,\theta} \circ \Psi$ is a quasiconformal self-conjugacy of Q such that

$$\begin{aligned} \tilde{h}_{t,\theta}^* \sigma_0 &= \Psi^* \circ \Phi_{t,\theta}^* \circ (\Psi_{t,\theta}^{-1})^* \sigma_0 \\ &= \Psi^* \circ \Phi_{t,\theta}^* \sigma_0 \\ &= \Psi^* \sigma_{t,\theta} \end{aligned}$$

on the interior of $K(Q)$, because Ψ and $\Psi_{t,\theta}$ are hybrid conjugacies and conformal in the interior of $K(Q)$. In particular, $\tilde{h}_{t,\theta}^* \sigma_0 = (h_{t,\theta} \circ \pi \circ \Psi)^* \sigma_0$ on $\hat{S}(j_0) \subset \Psi^{-1}(O)$.

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Namely, $h_{t,\theta} \circ \pi \circ \Psi \circ \tilde{h}_{t,\theta}^{-1} : \tilde{h}_{t,\theta}(\hat{S}(j_0)) \to A_r$ is conformal. Furthermore, since $\pi \circ \Phi : (\pi \circ \Phi)^{-1}(A_r) \to A_r$ is a covering map, $h_{t,\theta}$ naturally induces a piecewise smooth (hence quasiconformal) homeomorphism $\tilde{h}'_{t,\theta} : \mathbb{C} \to \mathbb{C}$ which is the identity outside $(\pi \circ \Phi)^{-1}(A_r)$ such that $\pi \circ \Phi \circ \tilde{h}'_{t,\theta} = h_{t,\theta} \circ \pi \circ \Phi$. Then, $\tilde{h}'_{t,\theta} \circ \tilde{h}_{t,\theta}^{-1} : \mathbb{C} \to \mathbb{C}$ is a conformal map which conjugates Q to Q itself. Thus, $\tilde{h}'_{t,\theta} = \tilde{h}_{t,\theta}$ and $\tilde{h}_{t,\theta}(\hat{S}(j_0)) = \hat{S}(j_0)$.

Now, let us consider the point $\zeta_{t,\theta} = \Psi_{t,\theta}^{-1} \circ f_{t,\theta}^{N+1}(\omega_{t,\theta}) = \tilde{h}_{t,\theta} \circ \Psi^{-1} \circ f^{N+1}(\omega) \in \hat{S}(j_0)$. Then,

$$\begin{aligned}
\pi \circ \Psi(\zeta_{t,\theta}) &= \pi \circ \Psi \circ \tilde{h}_{t,\theta} \circ \Psi^{-1} \circ f^{N+1}(\omega) \\
&= h_{t,\theta} \circ \pi \circ \Psi \circ \Psi^{-1} \circ f^{N+1}(\omega) \\
&= h_{t,\theta} \circ \pi \circ f^{N+1}(\omega) \\
&= h_{t,\theta}(t_0) \\
&= te^{2\pi i\theta}.
\end{aligned}$$

Since $\pi \circ \Psi : \hat{S}(j_0) \to A_r$ is a universal covering, this implies that $(t, \theta) \mapsto \zeta_{t,\theta}$ is a homeomorphism between $(r, 1) \times \mathcal{R}$ and $\hat{S}(j_0)$.

Hence, we can parameterize $f_{t,\theta}$ by $w \in \operatorname{int} S(j_0)$ and denote it by f_w. We also denote $\omega_{f_{t,\theta}}$ by ω_w and $\psi_{t,\theta}$ by ψ_w. By taking $S(j_0)$ a little bit smaller, we may assume that f_w is well defined for all $w \in S(j_0) \setminus \{\alpha_Q\}$.

It remains to prove that $\lim_{w \to \alpha_Q} f_w$ exists and equals P_1. The hybrid conjugacy Ψ_w between Q and f_w is conformal on the immediate basin of the attracting periodic orbit for Q and depends continuously with respect to w. In fact, such a conformal conjugacy on an attractive basin is unique and depends analytically for an analytic family of polynomials. Furthermore, Ψ_w extends continuously to the closure because f_w is hyperbolic and $J(f_w)$ is locally connected. Since $\Psi_w(\alpha_Q)$ is a fixed point of f_w, it depends continuously on w.

Let f_∞ be an accumulation point of f_w as $w \to \alpha_Q$. Then, Ψ_w converges to a conformal map Ψ_∞ conjugating f_∞ and Q, such that $f_\infty^{N+1}(\omega_\infty)$ is the fixed point $\Psi_\infty(\alpha_Q)$ of f_∞, where ω_∞ is the critical point which is the continuation of ω_w for f_w. Therefore, f_∞ must be equal to P_1. □

Simple calculation shows that we may assume a polynomial f_w has the form

$$f_w(z) = (z - a)^{d_1}(d_1 z - (d_1 + 1)b + a) + (-a)^{d_1}((d_1 + 1)b - a),$$

for some $a, b \in \mathbb{C}$ because f_w has exactly two critical points, whose multiplicities are $d_1 - 1$ and 1. Moreover, since f_w has a periodic point of period q with multiplicity μ, f_w is contained in a one-dimensional analytic family of polynomials of degree $d_1 + 1$. A map $\Xi^{-1} : f_w \mapsto \psi(f_w^{N+1}(\omega_w))$ is analytic on $\Xi^{-1}(S(j))$ except at $\Xi^{-1}(\alpha_Q) = P_1$. Therefore, Ξ is conformal except at α_Q.

Proof of Theorem 5.1. For each $w \in S$, take topological disks U_w and U'_w such that:

(1) $f_w : U'_w \to U_w$ is a polynomial-like map of degree d_1 ($\omega_w \in U'_w$ and the other critical point does not lie in U'_w);

(2) $S^0_{f_w}(j)$ is a nice invariant sector.

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Since $S^0_{f_w}(j) = \Psi_w^{-1}(S(j))$ and Ψ_w is holomorphic on the interior of $S(j) \subset K(Q)$, $\mathrm{mod}_{f_w} S^0_{f_w}(j) = \mathrm{mod}_Q S(j)$. By Lemma 5.2, we can choose μ so that this quantity is equal to m.

We can further assume that there exists a nice invariant tuple of sectors $(S^1_{f_w}(j))$ with modulus arbitrarily small such that it is continuous on w and $S^1_{f_w} = \bigcup_j S^1_{f_w}(j)$ contains $J(f_w : U'_w \to U_w)$ and ω_w. This can be done because we can take sectors bounded by a part of an equipotential curve $\{|\phi_{f_w}| = R\}$ and modified external rays that are of the form

$$\phi^{-1}_{f_w}(\{\exp(r + 2\pi i(\theta \pm \eta r)) : 0 < r < R\})$$

for some $\eta, R > 0$. By Lemma 2.3 and the continuity of the moduli on η, the modulus of $S^1_{f_w}(j)$ can be arbitrarily small for sufficiently small $R > 0$ and large η. Therefore, for each w, we can choose $R = R(w)$ and $\eta = \eta(w)$ so that $\mathrm{mod}_{f_w} S^1_{f_w}(j) = \varepsilon$. Furthermore, since $S^1_{f_w}(j)$ is continuous on w, R and η by Lemma 2.2, we can choose R and η so that they depend continuously on w, hence $S^1_{f_w}(j)$ is continuous on w.

Similarly, we can construct families of nice invariant tuples of sectors $(S^{k\pm}_{f_w}(0), \ldots, S^{k\pm}_{f_w}(q-1))_w$ whose radial boundaries are modified external rays or radial boundaries of $S^0_{f_w}(j)$ for some j, and circular boundaries depend continuously on w, satisfying (2) and (3) in Theorem 5.1. Then, they are also continuous on w by Lemma 2.2.

Take an isomorphism $\Delta_q \to S = \bigcup_j S(j)$ and parameterize by $\lambda \in \Delta_q$ instead of $w \in S$ and write f_λ and so on. This proves Theorem 5.1.

In fact, properties (1)–(3) follow from the construction above. (5) and (6) follow from Lemma 5.4. (Since $S^0_{f_w}(j_0) \subset \Psi^{-1}(K(Q))$, $\Psi_w|_{S(j_0)} : S(j_0) \to S^0_{f_w}(j_0)$ is the isomorphism. Thus, $h^0_{f_w}$ can be decomposed as $h^0_{f_w} : \Delta_q \to S \xrightarrow{\Psi_w} S^0_{f_w}$.) For $\lambda \neq 0$, we have $f_\lambda^{N+1}(\omega_{f_\lambda}) \in S^0_{f_\lambda}$ by (5). Since $J(f_\lambda : U'_\lambda \to U_\lambda) \subset S^1_{f_w}$ and (2), $S^0_{f_\lambda}$ does not intersect the Julia set. Therefore, $f_\lambda^{N+1}(\omega_{f_\lambda})$ (hence ω_{f_λ}) does not lie in the Julia set $J(f_\lambda : U'_\lambda \to U_\lambda)$. This proves (4). □

6. *Proof of Theorem 4.1*

Let $\theta_0 < \theta_1 < \cdots < \theta_{q-1} (< \theta_0 + 1)$ be the landing angles of α_0 for P_0. (If there exist more than q landing angles, we take one cycle of them.) Fix $R > 0$ and $\tilde{\eta} > \hat{\eta} > \eta > 0$. Let $D_{P_0} = \{G_{P_0}(z) < R\}$ and $D'_{P_0} = P_0^{-1}(D_{P_0})$. We define a nice invariant tuple of sectors $(S^k_{P_0}(j))$ $(k = 0, 1, 0\pm, 1\pm)$ for $P_0 : D'_{P_0} \to D_{P_0}$ as follows:

$$\begin{aligned}
S^1_{P_0}(j) &= \{\varphi^{-1}_{P_0}(\exp(r + 2\pi i\theta)) : |\theta - \theta_j| \le \eta r,\ 0 \le r \le R\},\\
S^{0-}_{P_0}(j) &= \{\varphi^{-1}_{P_0}(\exp(r + 2\pi i\theta)) : \hat{\eta} r \le (\theta - \theta_{j-1}) \le \tilde{\eta} r,\ 0 \le r \le R\},\\
S^{0+}_{P_0}(j) &= \{\varphi^{-1}_{P_0}(\exp(r + 2\pi i\theta)) : \hat{\eta} r \le -(\theta - \theta_j) \le \tilde{\eta} r,\ 0 \le r \le R\},\\
S^{1\pm}_{P_0}(j) &= \{\varphi^{-1}_{P_0}(\exp(r + 2\pi i\theta)) : \eta r \le \pm(\theta - \theta_j) \le \hat{\eta} r,\ 0 \le r \le R\},
\end{aligned}$$

and let $S^0_{P_0}(j)$ be the closure of the component of $D_{P_0} \setminus (S^{0-}_{P_0}(j) \cup S^{0+}_{P_0}(j))$ which does not intersect $S^1_{P_0}(j)$. Let $S^k_{P_0} = \bigcup_j S^k_{P_0}(j)$.

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Let $m = \mathrm{mod}_{P_0}(S^0_{P_0}, \alpha_0)$ and $\varepsilon = \mathrm{mod}_{P_0}(S^1_{P_0}, \alpha_0)$. Take $(f_\lambda : V'_\lambda \to V_\lambda)_{\lambda\in\Delta_q}$ in Theorem 5.1 for this m and ε. Let $h^k_{f_\lambda}$ and $S^k_{f_\lambda}(j)$ be in this theorem. Since $\{(S^k_{f_\lambda}(j))_j\}_\lambda$ is a continuous family, $h^k_{f_\lambda}$ depends continuously on λ.

Let $h^k_{P_0} : \Delta_q \to S^k_{P_0}$ denote the isomorphism for $k = 0, 1, 0\pm, 1\pm$. Define a homeomorphism $\psi_\lambda : S^0_{P_0} \cup S^1_{P_0} \to S^0_{f_\lambda} \cup S^1_{f_\lambda}$ by $\psi_\lambda = h^k_{f_\lambda} \circ (h^k_{P_0})^{-1}$ on $S^k_{P_0}$. We need to extend ψ_λ quasiconformally on D_{P_0}. For simplicity, we give the construction only the case $q = 1$ (so we have $S^k_{f_\lambda} = S^k_{f_\lambda}(0)$ and so on) and on $S^{0+}_{P_0}$. Take a diffeomorphism $\psi_\lambda : \partial^\pm S^{0\pm}_{P_0} \to \partial^\pm S^{0\pm}_{f_\lambda}$ which conjugates P_0 to f_λ and depends continuously on λ. Namely, $\psi_\lambda(P_0(z)) = f_\lambda(\psi_\lambda(z))$ for $z \in D'_{P_0} \cap \partial^\mp S^{0\pm}_{f_\lambda}$. Remark that $\partial^\mp S^{0\pm}_{P_0} = \partial^\pm S^{1\mp}_{P_0}$.

Since $\mathrm{mod}_{P_0}(S^k_{P_0}(j), \alpha_0) = \mathrm{mod}_{f_\lambda}(S^k_{f_\lambda}(j), 0)$ for $k = 0, 1$, we can extend ψ_λ quasiconformally on D_{P_0}. The proof here is essentially due to Bielefeld [**Bi**] (although the proof in [**Bi**] does not mention the continuity on parameters, it is trivial because of the continuity of sectors).

LEMMA 6.1. *There exists a quasiconformal extension $\psi_\lambda : S^{0+}_{P_0} \to S^{0+}_{f_\lambda}$ which depends continuously on λ such that the dilatation of ψ_λ is uniformly bounded on λ.*

Proof. Let L_{P_0} denote the inverse of the (normalized) linearizing coordinate for P_0 at α_{P_0}, that is, holomorphic map with $L_{P_0}(0) = \alpha_{P_0}$, $L'_{P_0}(0) = 1$ and $L_{P_0}(P'_0(\alpha_0)z) = P_0(L_{P_0}(z))$. It is well known that L_{P_0} can be extended to all of $\mathbb{C}$ by using the dynamics because α_{P_0} is repelling. Consider a holomorphic map

$$\tilde{L}_{P_0}(z) = L_{P_0} \circ \exp(z \log P'_0(\alpha_0)).$$

Then, it satisfies $\tilde{L}_{P_0}(T_1(z)) = P_0(\tilde{L}_{P_0}(z))$, where $T_1(z) = z + 1$ is the translation by 1.

For $k = 0, 0+$, let $\tilde{S}^k_{P_0}$ be a component of $\tilde{L}^{-1}_{P_0}(S^k_{P_0})$ such that the boundary of $\exp(\tilde{S}^k_{P_0})$ contains the origin. We can take the branch of $\log P'_0(\alpha_0)$ properly so that $\tilde{S}^k_{P_0}$ is invariant by T_1. We may also assume $\partial^+\tilde{S}^0_{P_0} = \partial^-\tilde{S}^{0+}_{P_0}$ where $\partial^\pm\tilde{S}^k_{P_0}$ is the upper or lower boundary of a (left) infinite strip $\tilde{S}^k_{P_0}$, that is, a component of $\tilde{L}^{-1}_{P_0}(\partial^\pm S^k_{P_0})$ which is a path accumulating to the left infinity ($\mathrm{Re}\, z \to -\infty$). See Figure 4. In other words, we consider $\tilde{S}^k_{P_0}$ as a sector at the left infinity and $\partial^\pm$ as the radial arcs of it. Note that $\tilde{L}_{P_0}$ has critical points and it is not an isomorphism between sectors $\tilde{S}^k_{P_0}$ and $S^k_{P_0}$ in general.

Similarly, we can take a holomorphic semiconjugacy $\tilde{L}_\lambda$ from T_1 to f_λ and strips $\tilde{S}^k_{f_\lambda}$. Since the linearizing coordinate depends continuously on parameters, $\tilde{L}_\lambda$ depends continuously on λ. Let $\tilde{\psi}_\lambda : \partial^\pm\tilde{S}^{0+}_{P_0} \to \partial^\pm\tilde{S}^{0+}_\lambda$ be a diffeomorphism given by $\psi_\lambda \circ \tilde{L}_{P_0} = \tilde{L}_\lambda \circ \tilde{\psi}_\lambda$. Then, we have $\tilde{\psi}_\lambda \circ T_1(z) = T_1 \circ \tilde{\psi}_\lambda(z)$ for $z \in T_1^{-1}(\partial^+\tilde{S}^{0+}_{P_0}) \subset \partial^+\tilde{S}^{0+}_{P_0}$. This implies that $\tilde{\psi}_\lambda$ is an *almost translation* on $\partial^+\tilde{S}^{0+}_{P_0}$, that is, $\tilde{\psi}_\lambda(z) - z$ is bounded and $\tilde{\psi}'_\lambda(z)$ is bounded from zero and infinity.

We also claim that $\tilde{\psi}_\lambda$ is an almost translation on $\partial^-\tilde{S}^{0+}_{P_0}$. Since the interiors of $\tilde{S}^0_{P_0}/T_1$ and $\tilde{S}^0_{f_\lambda}/T_1$ are annuli with the same moduli, they are conformally isomorphic to a round annulus $A = \{r \le |z| \le 1\}$ where $r = e^{-2\pi m}$ and $m = \mathrm{mod}_{P_0}(S^0_{P_0}, \alpha_0) = \mathrm{mod}_{f_\lambda}(S^0_{f_\lambda}, 0)$. Therefore, there exist isomorphisms of sectors $\kappa^0_{P_0} : \tilde{S}^0_{P_0} \to \hat{S}^0_{P_0}$ and $\kappa^0_\lambda : \tilde{S}^0_{f_\lambda} \to \hat{S}^0_{f_\lambda}$

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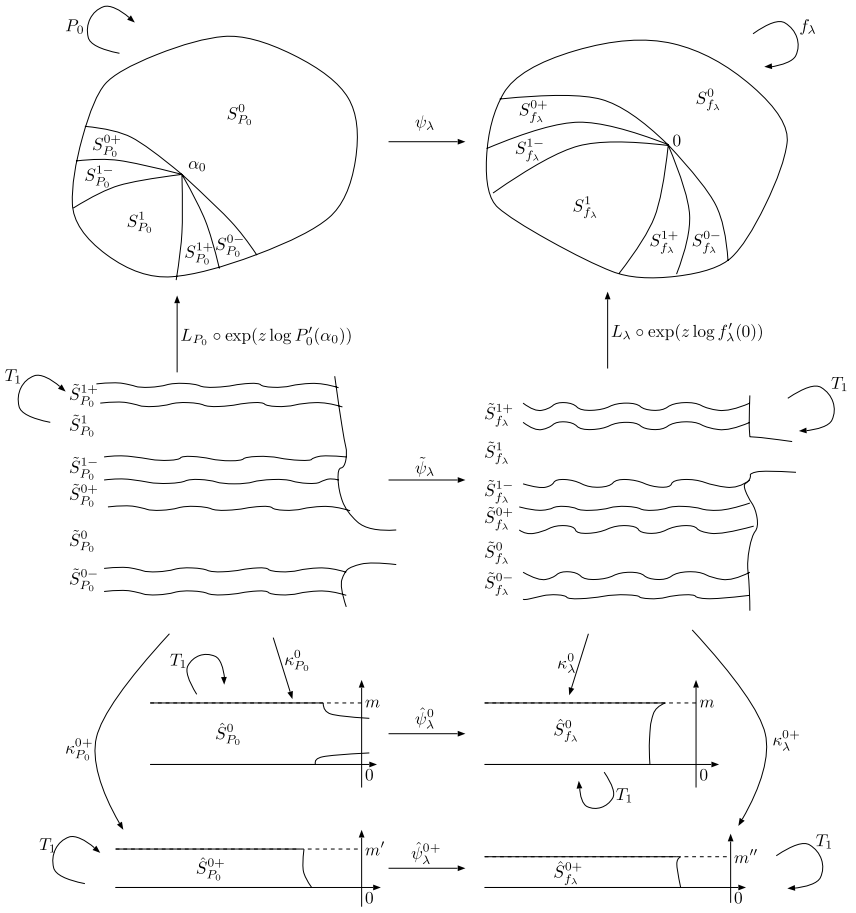


FIGURE 4. Construction of ψ_λ. Note that ψ_λ is a conjugacy only on $\partial^\pm S^{0\pm}_{P_0} = \partial^\mp S^{1\mp}_{P_1}$.

which conjugates from T_1 to itself (hence they are almost translations), where $\hat{S}^0_{P_0}$, $\hat{S}^0_{f_\lambda}$ are subsets of $H_m = \{0 \le \operatorname{Im} z \le m\}$ and they contain $H_m \cap \{\operatorname{Re} z < -M\}$ for some $M > 0$.

Let $\hat{\psi}^0_\lambda = \kappa^0_\lambda \circ \tilde{\psi}_\lambda \circ (\kappa^0_{P_0})^{-1} : \hat{S}^0_{P_0} \to \hat{S}^0_{f_\lambda}$. Then, it is the isomorphism of sectors. By the reflection principle, $\hat{\psi}^0_\lambda$ can be extended on $\{\operatorname{Re} z < -M\}$ and it has period $2mi$. So, $\exp((\pi/m)\hat{\psi}^0_\lambda((m/\pi)\log z))$ is a well-defined conformal map on a small punctured disk at 0. Hence, 0 is a removable singularity and it follows that $\hat{\psi}^0_\lambda$ is an almost translation. Since the composition of almost translations is also an almost translation, $\tilde{\psi}_\lambda$ is an almost translation.

Similarly, take isomorphisms $\kappa^{0+}_{P_0} : \tilde{S}^{0+}_{P_0} \to \hat{S}^{0+}_{P_0} \subset H_{m'} = \{0 \le \operatorname{Im} z \le m'\}$ and $\kappa^{0+}_\lambda : \tilde{S}^{0+}_{f_\lambda} \to \tilde{S}^{0+}_{f_\lambda} \subset H_{m''} = \{0 \le \operatorname{Im} z \le m''\}$ which give conjugacies from T_1 to itself, and consider an almost translation $\hat{\psi}^{0+}_\lambda = \kappa^{0+}_\lambda \circ \tilde{\psi}_\lambda \circ (\kappa^{0+}_{P_0})^{-1} : \partial^\pm \hat{S}^{0+}_{P_0} \to \partial^\pm \hat{S}^{0+}_{f_\lambda}$. Since $\partial^\pm \hat{S}^{0+}_{P_0}$ and $\partial^\pm \hat{S}^{0+}_{f_\lambda}$ are contained in horizontal straight lines, we can easily extend

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$\hat{\psi}_\lambda^{0+} : \kappa_{P_0}^{0+}(\tilde{S}_{P_0}^{0+}) \to \hat{S}_{f_\lambda}^{0+}$ by interpolating on $H_{m'} \cap \{\operatorname{Re} z < -M\}$. Namely, define $\hat{\psi}_\lambda^{0+}$ by

$$\hat{\psi}_\lambda^{0+}(x+iy) = \frac{y}{m'} \operatorname{Re}(\hat{\psi}_\lambda^{0+}(x)) + \left(1 - \frac{y}{m'}\right) \operatorname{Re}(\hat{\psi}_\lambda^{0+}(x+im')) + i\frac{m''}{m'}y$$

on $\operatorname{int} H_{m'} \cap \{\operatorname{Re} z < -M\}$ and extend it diffeomorphically on $\hat{S}_{P_0}^{0+}$. Then, simple calculation shows that $\hat{\psi}_\lambda^{0+}$ is an almost translation, so its dilatation is bounded above. It follows that ψ_λ can be extended quasiconformally on $S_{P_0}^{0+}$ by

$$\psi_\lambda = \tilde{L}_\lambda^{-1} \circ (\kappa_\lambda^{0+})^{-1} \circ \tilde{\psi}_\lambda^{0+} \circ \kappa_{P_0}^{0+} \circ \tilde{L}_{P_0}.$$

By the construction, it is easy to see ψ_λ depends continuously on λ and the dilatation of ψ_λ is uniformly bounded. □

Define $g_\lambda : U'_{g_\lambda} \to U_{g_\lambda}$ by

$$\begin{aligned}
U_{g_\lambda} &= D_{P_0}, \\
U'_{g_\lambda} &= ((S_{P_0}^0 \cup S_{P_0}^{0-} \cup S_{P_0}^{0+}) \cap D'_{P_0}) \cup \psi_\lambda^{-1}((S_{f_\lambda}^1 \cup S_{f_\lambda}^{1-} \cup S_{f_\lambda}^{1+}) \cap V'_\lambda), \\
g_\lambda(z) &= \begin{cases} P_0(z) & \text{for } z \in (S_{P_0}^0 \cup S_{P_0}^{0-} \cup S_{P_0}^{0+}) \cap D'_{P_0}, \\ \psi_\lambda^{-1} \circ f_\lambda \circ \psi_\lambda(z) & \text{for } z \in \psi_\lambda^{-1}((S_{f_\lambda}^1 \cup S_{f_\lambda}^{1-} \cup S_{f_\lambda}^{1+}) \cap V'_\lambda). \end{cases}
\end{aligned}$$

Then, g_λ is well defined and continuous (hence quasiregular) because ψ_λ gives a conjugacy between P_0 and f_λ on $\partial^\pm(S_{P_0}^{0\pm}) = \partial^\mp(S_{P_0}^{1\mp})$. Furthermore, g_λ depends continuously on λ. We want to construct an invariant almost complex structure σ on D_{P_0} so that g_λ is holomorphic (hence polynomial-like) with respect to σ. Then, we use the straightening theorem to obtain the polynomial with the desired property. However, to see the continuity easily, we do these two things at the same time.

Let $S_{g_\lambda}^k = S_{P_0}^k$. Fix $R_1 > 1$ and let $A = \{R_1^{1/d} \le |z| \le R_1\}$ be a round annulus. Take a quasiconformal diffeomorphism $\tau_\lambda : \mathbb{C} \setminus \mathbb{D}(R_1^{1/d}) \to \mathbb{C} \setminus U'_{g_\lambda}$ which maps A to $\overline{U_{g_\lambda} \setminus U'_{g_\lambda}}$ such that $g_\lambda \circ \tau(z) = \tau(z^d)$ on $\{|z| = R_1^{1/d}\}$, $(\lambda, z) \mapsto \tau_\lambda(z)$ is continuous and its dilatation is uniformly bounded on λ. Define g_λ on $\mathbb{C} \setminus U'_{g_\lambda}$ by $g_\lambda(z) = \tau_\lambda((\tau_\lambda^{-1}(z))^d)$. Then, $g_\lambda : \mathbb{C} \to \mathbb{C}$ is a quasiregular map of degree d and $(\lambda, z) \to g_\lambda(z)$ is continuous on $\Delta_q \times \mathbb{C}$.

Define an almost complex structure $\sigma = \sigma(\lambda)$ on $\mathbb{C}$ by

$$\sigma = \begin{cases} (\tau_\lambda^{-1})^*\sigma_0 & \text{if } z \notin U'_{g_\lambda}, \\ (g_\lambda^n)^*(\tau_\lambda^{-1})^*\sigma_0 & \text{if } g_\lambda^n(z) \notin U'_{g_\lambda}, \\ \sigma_0 & \text{otherwise,} \end{cases}$$

where σ_0 is the standard complex structure. By the definition of g_λ, it is well defined and invariant by g_λ.

LEMMA 6.2. *The dilatation of σ is uniformly bounded (independent of λ).*

Proof. By definition, σ has uniformly bounded dilatation outside U'_{g_λ}. Since g_λ is conformal except on $g_\lambda^{-1}(S_{g_\lambda}^{1\pm}) \cup (\mathbb{C} \setminus U'_{g_\lambda})$, the dilatation of σ is uniformly bounded outside $E = \bigcup g_\lambda^{-n}(S_{g_\lambda}^{1\pm})$.

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On $S^{1+}_{g_\lambda}$, the dilatation of $\sigma = (g_\lambda^n)^*(\tau_\lambda^{-1})^*\sigma_0 = (\psi_\lambda^{-1} \circ f_\lambda^n \circ \psi_\lambda)^*(\tau_\lambda^{-1})^*\sigma_0$ is uniformly bounded. Therefore, the dilatation of σ is uniformly bounded on $g_\lambda^{-1}(S^{1\pm}_{g_\lambda})$. Since g_λ is holomorphic on $E \setminus g_\lambda^{-1}(S^{1\pm}_{g_\lambda})$ and $g_\lambda^{-1}(S^{1\pm}_{g_\lambda})$ is forward invariant, it follows that the dilatation of σ is uniformly bounded on E. □

By the measurable Riemann mapping theorem, there exists a quasiconformal map $H_\lambda : \mathbb{C} \to \mathbb{C}$ such that $H_\lambda^*\sigma_0 = \sigma$. Therefore, we obtain a polynomial $Q_\lambda = H_\lambda \circ g_\lambda \circ H_\lambda^{-1}$. Let $\omega_\lambda = H_\lambda \circ \psi_\lambda^{-1}(\omega_{f_\lambda})$ be the critical point of Q_λ which corresponds to the critical point of f_λ and let $K_\lambda = H_\lambda(K(P_0))$ (the filled Julia set of a polynomial-like restriction hybrid equivalent to P_0).

LEMMA 6.3. *If $Q_{\lambda_0}^n(\omega_{\lambda_0}) \in K_{\lambda_0}$ for some $n > 0$, then a map $\lambda \mapsto Q_\lambda$ is continuous at λ_0.*

To prove this lemma, we need another lemma, which is the following.

LEMMA 6.4. *Suppose $Q_\lambda^n(\omega_\lambda) \in K_\lambda$ for some $n > 0$. Let P be a polynomial of degree d. If there exists a quasiconformal map $\Psi : \mathbb{C} \to \mathbb{C}$ such that $P \circ \Psi = \Psi \circ Q_\lambda$ and $\bar\partial\Psi \equiv 0$ almost everywhere on K_λ, then P and Q_λ are affinely conjugate.*

Remark. Similar results are proved in [**EY**, §5.3] and [**I**, Theorem 5.1]. Epstein and Yampolsky [**EY**] use the fact that the straightening map (defined between parameter spaces) of renormalizations is proper [**EY**, Proposition 5.4]. However, we do not know even the continuity of the straightening map of capture renormalizations ([**I**, Theorem 5.1] also treats a more subtle case).

Proof. We use the fact that the postcritical set $PC(Q_\lambda)$ attracts almost every point in the Julia set (see, for example, [**Mc**]). Since $PC(Q_\lambda) \cap J(Q_\lambda) \subset K_\lambda$ except at finitely many points, almost every point $z \in J(Q_\lambda)$ satisfies $d(Q_\lambda^n(z), K_\lambda) \to 0$ as $n \to \infty$. Let U, V be topological disks such that $Q_\lambda : U \to V$ is a polynomial-like map with filled Julia set K_λ. Then, for such $z \in J(Q_\lambda)$, we have $Q_\lambda^n(z) \in U$ for all sufficiently large n, hence $Q_\lambda^n(z) \in K_\lambda$.

Therefore, $J(Q_\lambda) \setminus \bigcup_n Q_\lambda^{-n}(K_\lambda)$ is of measure zero. On $Q_\lambda^{-n}(K_\lambda)$, we have

$$\begin{aligned}(\partial P^n \circ \Psi) \cdot \bar\partial\Psi &= \bar\partial(P^n \circ \Psi)\\ &= \bar\partial(\Psi \circ Q_\lambda^n)\\ &= 0 \quad \text{almost everywhere.}\end{aligned}$$

Thus, we have $\bar\partial\Psi \equiv 0$ almost everywhere on $J(Q)$. Therefore, Q_λ and P are hybrid equivalent by Ψ. Since Q_λ has connected Julia set, Q_λ and P are affinely conjugate by the straightening theorem. □

Proof of Lemma 6.3. Suppose $\lambda_k \to \lambda_0$ and Q_{λ_k} does not converge to Q_{λ_0}. By passing to a subsequence, we may assume $Q_{\lambda_k} \to P \neq Q_{\lambda_0}$. Furthermore, by Lemma 6.2, H_λ forms a normal family. Thus, we may further assume $H_{\lambda_k} \to H$ where H is a quasiconformal homeomorphism. Since $\bar\partial H_{\lambda_k} \equiv 0$ almost everywhere on $K(P_0)$, we have $\bar\partial H \equiv 0$ almost everywhere on $K(P_0)$.

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Let $\Psi = H \circ H_{\lambda_0}^{-1}$. Then, we have

$$\begin{aligned} P &= H \circ g_{\lambda_0} \circ H^{-1} \\ &= H \circ H_{\lambda_0}^{-1} \circ Q_{\lambda_0} \circ H_{\lambda_0} \circ H^{-1} \\ &= \Psi \circ Q_{\lambda_0} \circ \Psi^{-1}. \end{aligned}$$

By Lemma 6.4, we have $P = Q_{\lambda_0}$, and that is a contradiction. □

Let $\iota_{P_0}(\lambda) = Q_\lambda$. It is clear that Q_λ has a polynomial-like restriction $Q_\lambda : U'_\lambda \to U_\lambda$ hybrid equivalent to P_0. Let $\rho_\lambda : U_\lambda \to \mathbb{C}$ be a hybrid conjugacy between $Q_\lambda|_{U'_\lambda}$ and P_0. Notice that $\rho_\lambda = H_\lambda^{-1}$ on $K(Q_\lambda|_{U'_\lambda})$.

Let $\omega_\lambda = H_\lambda \circ \psi_\lambda^{-1}(\omega_{f_\lambda})$ denote the critical point of Q_λ which corresponds to the critical point ω_{f_λ} for f_λ. Define $\mu_{P_0}(\lambda) = \rho_\lambda \circ Q_\lambda^{N+1}(\omega_\lambda)$. If $Q_\lambda^{N+1}(\omega_\lambda) \in K_\lambda = \rho_\lambda^{-1}(K(P_0))$ (this condition is equivalent to $\mu_{P_0}(\lambda) \in K(P_0)$), then

$$\begin{aligned} \mu_{P_0}(\lambda) &= \rho_\lambda \circ Q_\lambda^{N+1}(\omega_\lambda) \\ &= H_\lambda^{-1} \circ Q_\lambda^{N+1}(H_\lambda \circ \psi_\lambda^{-1}(\omega_{f_\lambda})) \\ &= g_\lambda^{N+1}(\psi_\lambda^{-1}(\omega_{f_\lambda})) \\ &= \psi_\lambda^{-1} \circ f_\lambda^{N+1}(\omega_{f_\lambda}) \\ &= \psi_\lambda^{-1} \circ h_\lambda^0 \circ (h_\lambda^0)^{-1}(f_\lambda^N(\omega_{f_\lambda})) \\ &= \psi_\lambda^{-1} \circ h_\lambda^0(\lambda) \\ &= h_{P_0}^0(\lambda), \end{aligned}$$

since $g_\lambda^k(\psi_\lambda^{-1}(\omega_{f_\lambda})) \in S_{g_\lambda}^1$ by Theorem 5.1. Therefore, $\mu_{P_0}|_K : K \to K(P_0)$ is a homeomorphism where $K = \mu_{P_0}^{-1}(K(P_0)) = (h_{P_0}^0)^{-1}(K(P_0))$.

LEMMA 6.5. *If $Q_{\lambda_0}^{N+1}(\omega_{\lambda_0}) \in J = \partial K$, then $Q_\lambda \in \partial \mathcal{C}_d$.*

Proof. Let $z_0 = \mu_{P_0}(\lambda_0) = \rho_{\lambda_0} \circ Q_{\lambda_0}^{N+1}(\omega_{\lambda_0})$. Then, $z_0 \in J(P_0)$ because $\rho_{\lambda_0}(J) = J(P_0)$. Let $\mathcal{S} = \bigcup_{n\geq 0} P_0^{-n}(\tilde{S}_{P_0}^1)$, where $\tilde{S}_{P_0}^1 = S_{P_0}^1 \cup S_{P_0}^{1-} \cup S_{P_0}^{1+}$. Take a sequence $z_k \in D_{P_0} \setminus (\mathcal{S} \cup K(P_0))$ with $z_k \to z_0$. To do this, first take a sequence of repelling periodic points $z'_k \neq \alpha_{P_0}$ converging to z_0. It is easy to see that there exists a point on every external ray landing at z'_k, arbitrarily close to z'_k, such that z_k is not contained in $\mathcal{S}$. Therefore, we can take such a sequence $z_k \to z_0$.

Let $\lambda_k = (h_{P_0}^0)^{-1}(z_k)$. Then, $\lambda_k \to \lambda_0$ and $g_{\lambda_k}^{N+1}(\omega_{\lambda_k}) = z_k$. Let n_k be the smallest $n \geq 0$ such that $P_0^{n+1}(z_k) \notin D_{P_0}$ (such $n \geq 0$ exists because $P_0^n(z_k)$ escapes to infinity, since $z_k \notin K(P_0)$). Since $P_0^n(z_k) \notin \tilde{S}_{P_0}^1$, we have $g_{\lambda_k}^n(z_k) = P_0^n(z_k)$ for $n \leq n_k$. Thus, $g_{\lambda_k}^n(z_k) = g_{\lambda_k}^{n+N+1}(\omega_{\lambda_k})$ escapes to infinity; thus, the Julia set of g_{λ_k} is disconnected, and so is $J(Q_{\lambda_k})$, that is, $Q_{\lambda_k} \notin \mathcal{C}_d$.

By Lemma 6.3, $\lambda \to Q_\lambda$ is continuous at λ_0. Thus, $Q_{\lambda_k} \to Q_{\lambda_0}$ and we have proved the lemma. □

Therefore, we have proved Theorem 4.1.

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7. *Further studies*

7.1. *Gluing at repelling periodic orbits.* We can easily generalize Theorem 3.1 to the case where the gluing point of P_0 is not a fixed point but a periodic orbit. Let O_0 be a repelling periodic orbit of P_0 with period s and rotation number p/q. Instead of P_1 in the theorem, we consider a map $\tilde{P}_1 : \Sigma \times \mathbb{C} \to \Sigma \times \mathbb{C}$ defined by

$$\tilde{P}_1(k, z) = \begin{cases} (1, P_1(z)) & \text{when } k = 0, \\ (k+1, z) & \text{otherwise,} \end{cases}$$

where $\Sigma = \mathbb{Z}/s\mathbb{Z}$. If $\tilde{P}_1$ has a repelling periodic orbit of $\tilde{P}_1$ with period s and rotation number p/q (such a periodic orbit corresponds to a fixed point of P_1), then we can glue (P_0, O_0) and $(\tilde{P}_1, O_1)$. If $P_1(z) \in \mathrm{Poly}_d^0$ satisfies the assumption of Theorem 3.1, that is, if it is Misiurewicz and $\alpha_1 = P_1^{N+1}(0)$ is a repelling fixed point of rotation number p/q, then we can glue P_0 and $\tilde{P}_1$ at O_0 and $O_1 = \{(k, \alpha_1) : k \in \Sigma\}$ and obtain the same result as Theorem 3.1.

Similarly, we can also consider a map $\tilde{P}_0 : \Sigma \times \mathbb{C} \to \Sigma \times \mathbb{C}$ of the form $\tilde{P}_0(k, z) = (k+1, P_{0,k}(z))$ where $P_{0,k}$ is a polynomial for $k \in \Sigma$. If the Julia set of $\tilde{P}_0$ is fiberwise connected (that is, $J(P_{0,s-1} \circ \cdots \circ P_{0,0})$ is connected), we can glue $\tilde{P}_0$ and P_1 at respective periodic orbits of rotation number p/q of period s. Hence, we can obtain a polynomial Q having a renormalization of capture type of period s hybrid equivalent to $\tilde{P}_0$ and the same results as Theorem 3.1 hold.

More precisely, let $O_0 = \{(k, \alpha_{0,k})\}$ and $O_1 = \{\alpha_{1,k}\}$ be repelling periodic orbits of period s with rotation number p/q of $\tilde{P}_0$ and P_1, respectively. Assume $P_1^{N+1}(0) = \alpha_{1,0}$ and $P_1^N(0) \neq \alpha_{1,s-1}$. Then, we can glue $(\tilde{P}_0, O_0)$ and (P, O_P) near (P_1, O_1) (so that $(k, \alpha_{0,k})$ and $\alpha_{1,k}$ are identified for each $k \in \Sigma$) and show that there exists a homeomorphism $h_{\tilde{P}_0, P_1} : K(P_{0,s-1} \circ \cdots \circ P_{0,0}) \to \mathcal{C}_d$ such that the same condition as Theorem 3.1 holds, where $d = \sum \deg(P_{0,k} - 1) + \deg P_1$.

7.2. *Gluing at a parabolic point.* Assume the gluing fixed (or periodic) point α_0 of P_0 is parabolic. Then, since f_λ are (sub)hyperbolic, we can glue P_0 and f_λ together by intertwining surgery [**H**]. But we do not know whether the obtained map Q_λ in Theorem 4.1 depends continuously on λ or not. Thus, it is unknown whether Theorem 3.1 holds for such (P_0, α_0).

7.3. *Two or more captured critical points.* It is natural to ask: if two or more critical points are captured, then there exists a homeomorphism $K(P_0)^n \to \mathcal{C}_d$ into its image. Applying recursively Theorem 3.1, we can obtain such a bijection. However, by this way, we cannot derive its continuity.

To see its continuity, we have to glue three or more polynomials at once. More precisely, take mutually disjoint repelling periodic orbits O_n of period s_n with rotation number p_n/q_n $(n = 1, \ldots, N)$. Take a map $\tilde{P}_n : \mathbb{Z}/s_n\mathbb{Z} \times \mathbb{C} \to \mathbb{Z}/s_n\mathbb{Z} \times \mathbb{C}$ having the same property of $\tilde{P}_1$ in §7.1 and assume $\tilde{P}_n$ has a repelling periodic orbit $\tilde{O}_n$ of period s_n with rotation number p_n/q_n. Then, we can consider the intertwining of P_0 and $\tilde{P}_1, \ldots, \tilde{P}_N$, where P_0 and $\tilde{P}_n$ are intertwined at O_n and $\tilde{O}_n$. As in the proof of

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Theorem 3.1, we can consider intertwinings of P_0 and perturbations of $\tilde{P}_1, \ldots, \tilde{P}_N$ and obtain a homeomorphism $h_{P_0,\tilde{P}_1,\ldots,\tilde{P}_N} : K(P_0)^N \to \mathcal{C}_d$ where $d = d_0 + \tilde{d}_1 + \cdots + \tilde{d}_N - N$ and $\tilde{d}_n = \deg \tilde{P}_n^{s_n}$ (note that $\tilde{d}_n = \deg(\tilde{P}_n|_{\{0\}\times\mathbb{C}})$ is well defined).

7.4. *Other combinatorics.* There exist polynomials having renormalizations of capture type with other combinatorics than those that appear in Theorem 3.1, even in the case when the renormalization of such a polynomial has period one and contains all but one critical point.

In fact, the polynomial Q in Theorem 3.1 always satisfies the following:

(1) there is a repelling fixed (or periodic) point α of rotation number p/q at which $2q$ (or more) rays land;
(2) the rays landing at α divide the plane into $2q$ (or more) regions and exactly q regions do not intersect the filled Julia set K of the renormalization;
(3) the union U of these q regions contains exactly one critical point ω;
(4) there exists an $N > 0$ such that $Q^n(\omega) \in U$ for $n < N$ and $Q^N(\omega) \in K$.

But, for example, there exists a polynomial which satisfies the above conditions except for the last one. Namely, let N be the smallest integer satisfying $Q^N(\omega) \in K$. Then, $Q^n(\omega) \notin U$ for some $n < N$. It corresponds to a parameter λ in Theorem 4.1 such that there exist $0 = M_1 < N_1 < M_2 < N_2 < \cdots < M_J < N_J$ $(J > 1)$ such that g_λ in §6 satisfies that

$$g_\lambda^n(\omega_{g_\lambda}) \in \begin{cases} S^1_{g_\lambda} \setminus K & \text{for } M_j < n \le N_j, \\ S^0_{g_\lambda} \setminus K & \text{for } N_j < n \le M_{j+1}, \end{cases}$$

and $g_\lambda^{N_J+1}(\omega_{g_\lambda}) \in K$. For such a case, it seems that similar results to Theorem 3.1 also hold. However, our parameterization method does not work well and the proof has not yet been done.

In the case of capture renormalization of period one, we believe that all possible combinatorics are described above.

CONJECTURE 7.1. Let P be a polynomial having a capture renormalization of period one. Assume there exists exactly one captured critical point (counted without multiplicity) and there exists no other critical point outside the filled Julia set K of the renormalization.

Then, there exists a periodic point in K whose landing rays separate the captured critical point from K.

It is also interesting to study combinatorics of capture renormalizations of period greater than one or capture renormalizations having two or more captured critical points. Rational lamination (see [**T, K**]) seems to play an important role in studying such combinatorial properties.

7.5. *(Dis)continuity with respect to* (P_0, λ). Although our theorem is valid only in the case when the gluing point is repelling, Buff and Henriksen [**BH**] proved the same result in the case $P_0(z) = \lambda z + z^2$ for $|\lambda| = 1$ and the gluing point is the unique fixed point 0. In particular, our theorem is valid also in the case $P_0(z) = z + z^2$ (or equivalently,

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$P_0(z) = z^2 + 1/4$) gluing at the parabolic fixed point. Therefore, we have the following corollary.

COROLLARY 7.2. *Let*

$$\mathcal{MK}_2 = \bigcup_{c \in \mathcal{M}_2} \{c\} \times K(z^2 + c),$$

and P_1 be a quadratic polynomial of the form $z^2 + c$ such that $P_1^{N+1}(0)$ is equal to the beta fixed point of P_1.

Then, the map $h : \mathcal{MK}_2 \to \mathcal{C}_3$ defined by $h(c, x) = h_{z^2+c, P_1}(x)$ is well defined and a bijection into its image. A restriction $K(z^2 + c) \ni z \mapsto h(c, z) \in \mathcal{C}_3$ is a homeomorphism into its image for each $c \in \mathcal{M}_2$.

It is well known that $\pi_1 \circ h^{-1}$ is also continuous where $\pi_1(c, z) = c$ [**DH2**]. But, it seems that h is discontinuous because of parabolic implosion (discontinuity of filled Julia set at parabolic parameters). Note that our proof does not imply such a continuity on (P_0, λ) because Q_λ in §6 does not depend continuously on λ in general [**DH2**].

On the contrary, $h^{-1} = T$ is the straightening map of capture renormalizations (see §1). Even when $d_0 = 2$, the hybrid conjugacy ψ_P is discontinuous on P. Indeed, to have ψ_P continuous, it is necessary that P and P_0 are conformally conjugate near the respective filled Julia set, but elementary calculation shows that it is impossible. However, it does not mean that h^{-1} is discontinuous, and this problem is still open.

7.6. *Dimension of inverse images of the straightening map.* As remarked after Theorem 3.1, for $P_0 \in \mathcal{C}_{d_0}$ and a Misiurewicz polynomial $P_1 \in \mathcal{M}_{d_1}$, the image of a map h_{P_0, P_1} in Theorem 3.1 is contained in an analytic subset, which is of the form

$$A(d_1, P_0) = \{P \in \mathrm{Poly}_d : P \text{ has a critical point of multiplicity } d_1 - 1 \text{ and a polynomial-like restriction hybrid equivalent to } P_0\},$$

where $d = d_0 + d_1 - 1$.

CONJECTURE 7.3. For any $P_0 \in \mathrm{Poly}_{d_0}$, $A(d_1, P_0)$ has codimension $d_1 + d_0 - 2$ in Poly_d (hence has dimension one). For any Misiurewicz polynomial $P_1 \in \mathcal{M}_{d_1}$, $h_{P_0, P_1} : K(P_0) \to A(d_1, P_0)$ has a quasiconformal extension on a neighborhood of $K(P_0)$.

See [**BH**] for an example.

More generally, we conjecture the following.

CONJECTURE 7.4. Let Λ be an analytic family of polynomials (or rational maps) of degree d and $2 \le d' < d$. Let $P_0 \in \mathcal{C}_{d'}$. Then, the set of polynomials having a polynomial-like restriction hybrid equivalent to P_0 forms an analytic subset of Λ of codimension $d' - 1$.

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