Long medians and long angle bisectors

SADI ABU-SAYMEH, YAQEEN AL-MOMANI, MOWAFFAQ HAJJA, MOSTAFA HAYAJNEH

1. *Introduction and preliminaries*

Following Euler, we denote the side lengths and angles of a triangle ABC by a, b, c, A, B, C in the standard order. Any line segment joining a vertex of ABC to any point on the opposite side line will be called a *cevian*, and a cevian AA' of length t will be called *long*, *strictly long*, or *balanced* according as $t \geqslant a$, $t > a$ or $t = a$. If A' lies strictly between B and C, AA' is called an *internal cevian*. This convention regarding cevians is not universal, and it is, for example, in a heavy contrast with that in [1, p. 73].

In [2], we found conditions for a triangle to have 0, 1, 2 or 3 long altitudes, and some related properties. In this Article, we consider medians and (internal) angle bisectors, starting with some relevant theorems. For brevity, we shall refer to the median emanating from vertex A as the A-median, and so on for other vertices and other cevians.

1.1 *Stewart's theorem*

Referring to Figure 1, Stewart's theorem states that if ABC is a triangle, and if D is a point on side BC with $DB = x$ and $DC = y$, then

$$AD^2 = \frac{xb^2 + yc^2}{x + y} - xy = \frac{xb^2 + yc^2}{a} - xy. \tag{1}$$

This can be proved by letting $\theta = \angle ADC = 180° - \angle ADB$, applying the cosine rule to triangles ADB and ADC, and eliminating $\cos\theta$.

The special case when D is the midpoint of BC is known as Apollonius' theorem. In this case, we have $AB^2 + AC^2 = 2(AD^2 + DC^2)$, or equivalently

$$b^2 + c^2 = 2U^2 + \frac{a^2}{2}, \text{ i.e. } U^2 = \frac{2b^2 + 2c^2 - a^2}{4}, \tag{2}$$

where $U = AD$.

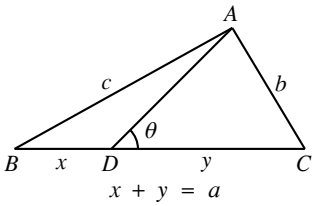


FIGURE 1:
Illustrating Stewart's theorem

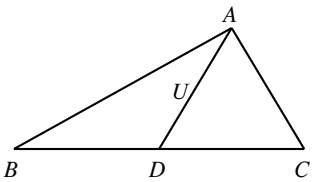


FIGURE 2:
Illustrating Apollonius' theorem

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

1.2 *The length of an angle bisector*

Let ABC be a triangle, and let AD be the angle bisector of A. By the angle bisector theorem, we have

$$\frac{BD}{CD} = \frac{c}{b}, \text{ and hence } BD = \frac{ca}{b+c}, CD = \frac{ba}{b+c}. \tag{3}$$

Applying Stewart's theorem, we obtain

$$AD^2 = \frac{cab^2}{a(c+b)} + \frac{bac^2}{a(c+b)} - \left(\frac{ca}{c+b}\right)\left(\frac{ba}{c+b}\right) = cb - \frac{cba^2}{(c+b)^2}. \tag{4}$$

This formula appears in many textbooks; for example, see [3, Exercise 3, p. 15].

2. *Non-existence of triangles with three long medians or three long angle bisectors*

It is proved in [2, Theorem 2] that no triangle can have three long altitudes. We can now see that this holds also for medians and angle bisectors.

Thus let U, V and W be the lengths of the medians AA', BB' and CC' of triangle ABC, and let u, v and w be the lengths of the angle bisectors AA^*, BB^* and CC^*. Then it follows from (2) by adding that

$$U^2 + V^2 + W^2 = \frac{3\left(a^2+b^2+c^2\right)}{4} < a^2 + b^2 + c^2. \tag{5}$$

Thus the inequalities $U \geqslant a$, $V \geqslant b$ and $W \geqslant c$ cannot hold simultaneously, and hence the three medians cannot all be long.

Similarly, it follows from (4) that

$$u^2 < bc,\ v^2 < ca,\ w^2 < ab,$$

and therefore $u^2 + v^2 + w^2 < bc + ca + ab$. But

$$0 \leqslant (b-c)^2 + (c-a)^2 + (a-b)^2 = 2\left[\left(a^2+b^2+c^2\right) - (bc+ca+ab)\right].$$

Therefore $bc + ca + ab \leqslant a^2 + b^2 + c^2$, and hence

$$u^2 + v^2 + w^2 < bc + ca + ab \leqslant a^2 + b^2 + c^2,$$

making it impossible for ABC to have three long angle bisectors.

Combining these results, we obtain the following theorem.

Theorem 1: No triangle can have three long altitudes, three long medians, or three long angle bisectors.

It also follows from (2) and (4) that

$$U < \sqrt{\frac{b^2+c^2}{2}} \text{ and that } u < \sqrt{bc}.$$

Thus the median of a triangle is less than the root quadratic mean of the two surrounding sides, and the angle bisector is less than the geometric mean of the surrounding sides. Letting $\mathcal{M}_r$, $r \in \mathbb{R}$, $r \neq 0$, be the rth Hölder's mean

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

defined for $x, y > 0$ by

$$\mathcal{M}_r(x, y) = \left(\frac{x^r + y^r}{2}\right)^{\frac{1}{r}},$$

we see, by taking logarithms and applying L'Hôpital's rule, that $\lim_{r \to 0} \mathcal{M}_r(x, y) = \sqrt{xy}$, and thus we define $\mathcal{M}_0$ by

$$\mathcal{M}_0(x, y) = \sqrt{xy}, \text{ the geometric mean.}$$

We now summarise the previous observations in the following theorem.

Theorem 2: In triangle ABC, let U and u be the lengths of the A-median and A-angle bisector, respectively. Then

$$U < \mathcal{M}_2(b, c), \text{ the root-quadratic mean of } b \text{ and } c \tag{6}$$

$$u < \mathcal{M}_0(b, c), \text{ the geometric mean of } b \text{ and } c. \tag{7}$$

In general, one defines $\mathcal{M}_r$ for any number n of positive numbers $a_1, \ldots, a_n$ by

$$\mathcal{M}_r(a_1, \ldots, a_n) = \left(\frac{a_1^r + \ldots + a_n^r}{n}\right)^{\frac{1}{r}}. \tag{8}$$

It is a very useful fact that $\mathcal{M}_r$ increases with r. Thus

$$s > r \Rightarrow \mathcal{M}_s(a_1, \ldots, a_n) \geqslant \mathcal{M}_r(a_1, \ldots, a_n), \tag{9}$$

with equality if, and only if, $a_1 = \ldots = a_n$.

Proofs of this appear in many books on inequalities, e.g. [4, Theorem 10.5, pp. 111-112].

Finally, it must come as pleasant news to know that both Theorems 1 and 2 above can be strengthened considerably, as is done later in Sections 4 and 5.

3. *The relative positions and lengths of the median, the angle bisector, and the altitude*

Figure 3 below shows a triangle ABC in which $c > b$ (and thus $C > B$), median AD, angle bisector AE, and altitude AH. We shall show that B, D, E and H appear in this order, as shown in Figure 3.

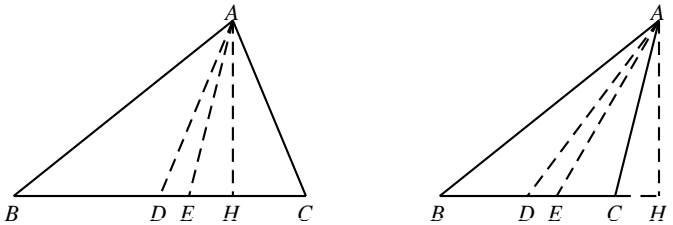


FIGURE 3: Relative positions of the median AD, the angle bisector AE, and the altitude AH

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

By the scissors' lemma applied to ADB and ADC, we conclude that $\angle ADB > \angle ADC$, and hence $\angle ADB$ is obtuse. Thus D lies between B and H. By the angle bisector theorem, $BE > EC$, and therefore E is between D and C. Also,

$$\angle AEC = B + \frac{A}{2} = \frac{B}{2} + \frac{A}{2} + \frac{B}{2} < \frac{B}{2} + \frac{A}{2} + \frac{C}{2} = 90°.$$

Hence $\angle AEC$ is acute, and thus E lies between H and D. So B, D, E and H appear in this order. It also follows that $AD > AE$, because $\angle AED$ is obtuse.

4. *Improving Theorem* 1

The seemingly great news announced in Theorem 1 is only seemingly so. In fact, we shall see below that the statements made about the altitudes, medians, and angle bisectors hold for *all* cevians, even if they are not concurrent.

We start with the very simple observation that if AA' is an internal cevian in triangle ABC, then either $AA' < AB$ or $AA' < AC$. To see this, either $\angle AA'B \geqslant 90°$ or $\angle AA'C \geqslant 90°$. Then clearly AA' is strictly smaller than the longer of the sides AB and AC.

We now use this to prove the following stronger form of Theorem 1.

Theorem 3: No three internal cevians AA', BB' and CC' in triangle ABC can be all long. More specifically, no cevian emanating from a vertex that holds a largest angle can be long.

Proof: We may assume that $a \geqslant b \geqslant c$. Since $b \geqslant c$, it follows from the observation above that $AA' < b$. But $c \leqslant b \leqslant a$. Therefore $AA' < a$, and thus AA' is not a long cevian.

At first sight, Theorem 3 does not seem to imply that the three altitudes of a triangle cannot be all long, since altitudes are not necessarily internal. However, one may always replace altitudes by internal, necessarily longer, cevians, and then apply Theorem 3 to the resulting cevians.

Let ABC be a triangle in which $a \geqslant b \geqslant c$. While no internal A-cevian can be long, the length of a B-cevian can be greater than, equal to, or smaller than b. This is because $BA \leqslant b$ and $BC \geqslant b$. Drawing a diagram, one may form the impression that all internal C-cevians are long. The following theorem shows this is false.

Theorem 4: Let ABC be a triangle in which $a \geqslant b \geqslant c$. Then all C-cevians are long if, and only if,

(i) $3a^2b^2 - a^4 - b^4 \geqslant 0$,

or equivalently, $\frac{1}{\phi} \leqslant \frac{a}{b} \leqslant \phi$, where ϕ is the golden ratio,

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

(ii) $\dfrac{(a^2+b^2)-2\sqrt{3a^2b^2-a^4-b^4}}{5} \leqslant c^2 \leqslant \dfrac{(a^2+b^2)+2\sqrt{3a^2b^2-a^4-b^4}}{5}$,
or equivalently, $|5c^2 - a^2 - b^2| \leqslant 2\sqrt{3a^2b^2 - a^4 - b^4}$.

Proof:

All C-cevians are long
$\Leftrightarrow$ the C-altitude h is long $\Leftrightarrow$ $hc \geqslant c^2 \Leftrightarrow ab\sin C \geqslant c^2$
$\Leftrightarrow$ $a^2b^2\left(1 - \left(\dfrac{a^2+b^2-c^2}{2ab}\right)^2\right) \geqslant c^4$, by the cosine rule
$\Leftrightarrow$ $5c^4 - 2c^2(a^2+b^2) + (a^2-b^2)^2 \leqslant 0$
$\Leftrightarrow$ (i) $f(x) = 5x^2 - 2x(a^2+b^2) + (a^2-b^2)^2$ has non-negative discriminant, and
(ii) $\dfrac{(a^2+b^2)-2\sqrt{3a^2b^2-a^4-b^4}}{5} \leqslant c^2 \leqslant \dfrac{(a^2+b^2)+2\sqrt{3a^2b^2-a^4-b^4}}{5}$.

The rest follows from the fact that the discriminant Δ of f is given by

$$\begin{aligned}\Delta &= 4(a^2+b^2)^2 - 20(a^2-b^2)^2 = 16\left[-a^4 - b^4 + 3b^2a^2\right] \\ &= -16b^4(x^2 - x - 1)(x^2 + x - 1), \text{ where } x = \frac{a}{b} \\ &= -16b^4(x-\phi)\left(x - \frac{1}{\phi}\right)(x+\phi)\left(x + \frac{1}{\phi}\right),\end{aligned}$$

and that

$$\Delta \geqslant 0 \Leftrightarrow \frac{1}{\phi} \leqslant \frac{a}{b} \leqslant \phi.$$

5. *Improving Theorem* 2

An Apollonius-type inequality states that if ABC is a triangle, and if D is the midpoint of BC, and if U is the length of AD, then

$$U < \frac{b+c}{2} = \mathcal{M}_1(b, c). \tag{10}$$

To see this, let W be the point on the extension of AD for which $DW = AD$, as shown in Figure 4 (i). In parallelogram $ABWC$,

$$AD = \frac{1}{2}AW < \frac{1}{2}(AB + BW) = \frac{1}{2}(AB + AC).$$

as desired.

A Stewart-type inequality that is stronger than the previous one states that if ABC is a triangle, and if D is a point on BC with $BD = s$, $CD = t$, $AD = x$, as shown in Figure 4 (ii), then

$$x < \frac{sb + tc}{s + t}. \tag{11}$$

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

This again follows by drawing a line from B that is parallel to AC and that meets AD produced at W. Triangles BDW and CDA are easily seen to be similar by AAA, and therefore

$$\frac{BD}{CD} = \frac{DW}{DA} = \frac{BW}{CA}, \text{ whence } DW = \frac{sx}{t} \text{ and } BW = \frac{sb}{t}.$$

The triangle inequality $AW < AB + BW$ now implies that

$$x + \frac{sx}{t} < c + \frac{sb}{t}, \text{ and thus } x < \frac{sb + tc}{s + t}.$$

as desired.

Taking AD to be the bisector of A, and letting $u = AD$, it follows from (11) and the angle bisector theorem that

$$u < \frac{bc}{b + c} + \frac{bc}{b + c} = \frac{2bc}{b + c} = \mathcal{M}_{-1}(b, c). \tag{12}$$

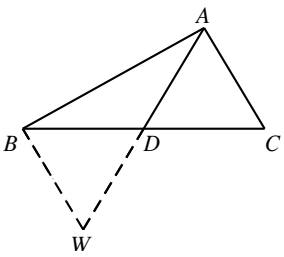


FIGURE 4(i): Illustrating the Apollonius-type inequality

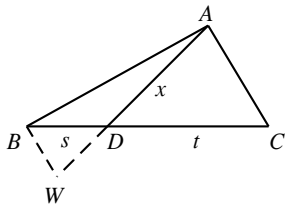


FIGURE 4(ii): Illustrating the Stewart-type inequality

Since $\mathcal{M}_r$ increases with r, it follows that the inequalities (10) and (12) are an improvement over Theorem 2. We record (10) and (12) in the following theorem, and we also show that they cannot be improved upon within the Hölder family, and even within the family of *all* means.

Theorem 5: In triangle ABC, let U and u be the lengths of the A-median and A-angle bisector, respectively. Then

$$U < \mathcal{M}_1(b, c), \text{ the arithmetic mean} \tag{13}$$

$$u < \mathcal{M}_{-1}(b, c), \text{ the harmonic mean.} \tag{14}$$

Furthermore, the indices 1 and -1 of $\mathcal{M}$ in (13) and (14) are the best possible in the sense that they cannot be improved.

Proof: Starting with any two positive numbers b and c, say with $c \geqslant b$, we consider the set of all triangles ABC whose side lengths are a, b and c in the standard manner. Thus a ranges between $c - b$ and $c + b$. Let $U = U(a)$ and $u = u(a)$ be the lengths of the A-median and A-angle bisector. Then

$$U^2 = \frac{2b^2 + 2c^2 - a^2}{4}, \qquad u^2 = bc - \frac{cba^2}{(c + b)^2}.$$

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

Taking the supremum over $a \in (c - b, c + b)$, and observing that both U and u are decreasing with a, we see that

$$\sup U^2 = \frac{2b^2 + 2c^2 - (c - b)^2}{4} = \left(\frac{b + c}{2}\right)^2 = (\mathcal{M}_1(b, c))^2,$$

$$\sup u^2 = bc - \frac{cb(c - b)^2}{(c + b)^2} = \left(\frac{2cb}{c + b}\right)^2 = (\mathcal{M}_{-1}(b, c))^2.$$

Thus $\sup U = \mathcal{M}_1$ and $\sup u = \mathcal{M}_{-1}$, and the proof is complete.

6. *Triangles in which none, one, or two of the medians are long*

By Theorem 3, no triangle can have three long medians (or in fact any three internal cevians), and therefore the number of long medians in any triangle is 0, 1, or 2. In this section, we characterise those triangles ABC in which exactly i medians are long for $i = 0, 1, 2$. This characterisation is given in Theorem 8 below. The proof uses Theorem 7, which in turn uses Theorem 6. It also uses the fact that

$$CC' \text{ is a long median } \Leftrightarrow \frac{2a^2 + 2b^2 - c^2}{4} \geqslant c^2 \Leftrightarrow 2a^2 + 2b^2 - 5c^2 \geqslant 0,$$

and similarly for other medians.

Theorem 6: In triangle ABC, let BB' and CC' be the B- and C-medians, and BB^* and CC^* be B- and C-angle bisectors. Then

$$\text{(i) } BB' > CC' \Leftrightarrow c > b, \qquad \text{(ii) } CC^* < BB^* \Leftrightarrow c > b. \quad (15)$$

Proof: The proof follows from the following obvious implications:

$$CC' < BB' \Leftrightarrow \frac{2a^2 + 2b^2 - c^2}{4} < \frac{2a^2 + 2c^2 - b^2}{4}$$

$$\Leftrightarrow 3b^2 < 3c^2 \Leftrightarrow b < c;$$

$$CC^* < BB^* \Leftrightarrow ab - \frac{abc^2}{(a + b)^2} < ac - \frac{acb^2}{(a + c)^2}$$

$$\Leftrightarrow \frac{a(a+b+c)(b-c)\left(a^3 + 3abc + a^2c + a^2b + b^2c + c^2b\right)}{(a+b)^2(a+c)^2} < 0$$

$$\Leftrightarrow b < c.$$

Notice that Theorem 6 (ii) is a stronger form of the classical Steiner-Lehmus theorem given by $BB^* = CC^* \Leftrightarrow c = b$. Proofs of the stronger form (15) and other stronger forms can be found in [5, 6], and other references. The proof above is adapted from that in [3].

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

Theorem 7: Let ABC be a triangle in which $a \geqslant b \geqslant c$, and let AA', BB', CC' be its medians. Then

$$\text{(i)} \quad AA' \text{ cannot be long,} \tag{16}$$

$$\text{(ii)} \quad AA' \leqslant BB' \leqslant CC', \tag{17}$$

$$\text{(iii)} \quad \text{if } BB' \text{ is long, then so is } CC'. \tag{18}$$

Proof: Part (i) follows from Theorem 3, while Part (ii) follows from (15). Part (iii) follows from the implication $BB' \geqslant b \Rightarrow CC' \geqslant BB' \geqslant b \geqslant c$.

Theorem 8: Let ABC be a triangle in which $a \geqslant b \geqslant c$. Then

(i) ABC has no long medians
$\Leftrightarrow 5c^2 - 2(a^2 + b^2) > 0$;

(ii) ABC has exactly one long median (necessarily from C)
$\Leftrightarrow 5c^2 - 2(a^2 + b^2) \leqslant 0$ and $2c^2 + 2a^2 - 5b^2 < 0$;

(iii) ABC has exactly two long medians (necessarily from B and C)
$\Leftrightarrow 2c^2 + 2a^2 - 5b^2 \geqslant 0$.

Proof: Using (16), (17), (18), we see that

$$\begin{aligned} ABC \text{ has no long medians} &\Leftrightarrow CC' \text{ is not long} \\ &\Leftrightarrow 5c^2 - 2a^2 - 2b^2 > 0; \\ ABC \text{ has exactly two long medians} &\Leftrightarrow BB' \text{ is long} \\ &\Leftrightarrow 2a^2 + 2c^2 - 5b^2 \geqslant 0. \end{aligned}$$

The question about the percentage of triangles having exactly 0, 1 or 2 long medians is addressed in another project.

We end this section by the following interesting property.

Theorem 9: If a triangle ABC has no long medians, then it is acute.

Proof: Suppose that ABC has no long medians. Then the B- and C-medians are not long, and hence

$$2a^2 + 2b^2 < 5c^2 \text{ and } 2a^2 + 2c^2 < 5b^2.$$

Adding, we obtain $a^2 < b^2 + c^2$, i.e. A is acute. Similarly all angles are acute.

7. *Balanced medians and balanced angle bisectors*

We now characterise those triangles which have two balanced medians or two balanced angle bisectors. By Theorem 3, no triangle can, of course, have three balanced medians or three balanced angle bisectors.

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

Theorem 10: In triangle ABC, let BB' and CC' be medians, and let BB^* and CC^* be angle bisectors. Then

$$BB' \text{ and } CC' \text{ are balanced} \quad \Leftrightarrow \quad b = c \text{ and } \cos A = \frac{1}{4}, \tag{19}$$

$$BB^* \text{ and } CC^* \text{ are balanced} \quad \Leftrightarrow \quad b = c \text{ and } A = \frac{3\pi}{7}. \tag{20}$$

Proof: Suppose that the two medians BB' and CC' have lengths V and W, and that they are both balanced. Thus

$$b = V \text{ and } c = W. \tag{21}$$

To show that $b = c$, we assume that $c > b$, and we reach a contradiction. In fact,

$$\begin{aligned} c > b \quad &\Rightarrow \quad V > W, \text{ by Theorem 6} \\ &\Rightarrow \quad b > c, \text{ by the assumption (21),} \end{aligned}$$

a contradiction. Thus $b = c$.

Under the assumption $b = c$, BB' and CC' are balanced if, and only if, $V = W = b = c$. By the cosine rule applied to BAB', this is equivalent to

$$\cos A = \frac{1^2 + 2^2 - 2^2}{2 \times 1 \times 2} = \frac{1}{4}.$$

Suppose now that the two angle bisectors BB^* and CC^* have lengths v and w, and that they are both balanced. Thus

$$b = v \text{ and } c = w. \tag{22}$$

To show that $b = c$, we assume that $c > b$, and we reach a contradiction. In fact,

$$\begin{aligned} c > b \Rightarrow v > w, &\text{ by Theorem 6} \\ \Rightarrow b > c, &\text{ by the assumption (22),} \end{aligned}$$

a contradiction. Thus $b = c$, and of course $B = C$.

Under the assumption $B = C\,(= 2x, \text{ say})$, we have

$$\begin{aligned} BB^* \text{ and } CC^* \text{are balanced} \quad &\Leftrightarrow \quad v = b \text{ and } w = c \quad \Leftrightarrow \quad v = c \\ &\Leftrightarrow \quad \angle BB^*A = A \quad \Leftrightarrow \quad 3x = \pi - 4x \\ &\Leftrightarrow \quad x = \frac{\pi}{7} \quad \Leftrightarrow \quad A = \frac{3\pi}{7}, B = C = \frac{2\pi}{7}. \end{aligned}$$

If ABC is a triangle in which $b = c$, and $A = 3\pi/7$, and if BB^* is an angle bisector, then the angles of BCB^* are given by $\pi/7$, $2\pi/7$ and $4\pi/7$. Triangles whose angles are these angles have been encountered earlier in the literature in several contexts, where they are called *heptagonal* or *Archimedean* triangles, and where many of their interesting properties are established; see [7, 8, 9, 10, 11].

The proof of Theorem 10 also yields the following theorem for isosceles triangles.

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

Theorem 11: Let ABC be an isosceles triangle with $b = c$. With the same notation as in Theorem 10, we have

$$BB' \text{ and } CC' \text{ are long} \iff \cos A \leqslant \frac{1}{4};$$

$$BB^* \text{ and } CC^* \text{ are long} \iff A \geqslant \frac{3\pi}{7};$$

$$AA' \text{ is long} \iff AA^* \text{ is long} \iff \cos A \geqslant \frac{3}{5};$$

$$ABC \text{ has no long medians} \iff \frac{1}{4} < \cos A < \frac{3}{5};$$

$$ABC \text{ has no long angle bisectors} \iff \cos^{-1}\frac{3}{5} < A < \frac{3\pi}{7}.$$

Proof: Referring to Figure 5, the theorem follows from the following equivalences.

$$\begin{aligned} BB^* \text{ and } CC^* \text{ are long} &\iff BB^* \geqslant AB \iff A \geqslant 3x \\ &\iff \pi - 4x \geqslant 3x \\ &\iff x \leqslant \frac{\pi}{7} \iff A = \pi - 4x \geqslant \frac{3\pi}{7}; \end{aligned}$$

$$\begin{aligned} AA' \text{ is long} &\iff AA' \geqslant a \iff b^2 - \frac{a^2}{4} \geqslant a^2 \\ &\iff \frac{a}{b} \leqslant \frac{2}{\sqrt{5}} \iff \sin\frac{A}{2} = \frac{a}{2b} \leqslant \frac{1}{\sqrt{5}}; \\ &\iff \cos A = 1 - 2\sin^2\frac{A}{2} \geqslant \frac{3}{5}; \end{aligned}$$

$$\begin{aligned} ABC \text{ has no long medians} &\iff AA', BB', CC' \text{ are not long} \\ &\iff \frac{1}{4} < \cos A < \frac{3}{5}; \end{aligned}$$

$$\begin{aligned} ABC \text{ has no long angle bisectors} &\iff AA^*, BB^*, CC^* \text{are not long} \\ &\iff \cos^{-1}\frac{3}{5} < A < \frac{3\pi}{7}. \end{aligned}$$

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

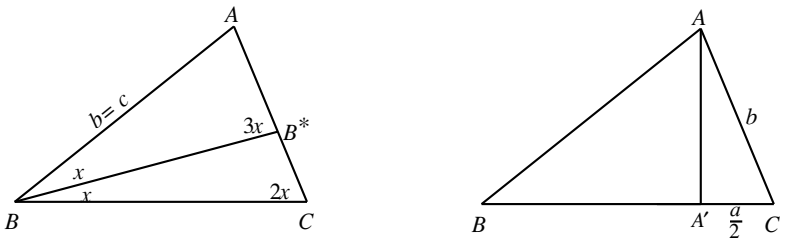


FIGURE 5: Illustrating Theorem 11

8. *Limitations on b, c so that a triangle ABC with AB = c, AC = b and with a long median* (*or angle bisector*) *at A exists*

Let $b, c > 0$ be given. In [2, Theorem 1], it is proved that there exists a triangle ABC with $AB = c$ and $AC = b$ and with a long A-altitude if, and only if,

$$\frac{1}{\phi} < \frac{b}{c} < \phi, \tag{23}$$

where ϕ is the golden ratio. In the next theorem, we prove similar results for medians and angle bisectors. We remark here that [2, Theorem 1] follows also from the proof of Theorem 4 of this Article.

Theorem 12: Let $b, c > 0$ be given. Then the following hold.

$\exists$ a triangle ABC with $AB = c$ and $AC = b$ and with a long A-median

$$\Leftrightarrow \frac{1}{3} < \frac{b}{c} < 3, \tag{24}$$

$\exists$ a triangle ABC with $AB = c$ and $AC = b$ and with a long A-angle bisector

$$\Leftrightarrow -1 + \sqrt{2} < \frac{b}{c} < 1 + \sqrt{2}. \tag{25}$$

Proof: It is clear that there exists a triangle ABC with $AB = c$ and $AC = b$ and with a long A-median A if, and only if,

$$\exists\, a \text{ such that } |b - c| < a < b + c \text{ and } 5a^2 \leqslant 2b^2 + 2c^2. \tag{26}$$

Since the function $f(x) = 5x^2$ is strictly increasing on the interval $\left[|b - c|, b + c\right]$, it follows that

$$(26) \Leftrightarrow f\left(|b - c|\right) < 2b^2 + 2c^2 \Leftrightarrow 5|b - c|^2 < 2b^2 + 2c^2$$
$$\Leftrightarrow 3b^2 + 3c^2 - 10bc < 0 \Leftrightarrow \frac{1}{3} < \frac{b}{c} < 3,$$

as desired.

Similarly, there exists a triangle ABC with $AB = c$ and $AC = b$ and with a long A-angle bisector if, and only if, there exists a such that

$$|b - c| < a < b + c \text{ and } bc - \frac{bca^2}{(b + c)^2} \geqslant a^2,$$

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

or equivalently, if, and only if,

$$\exists\, a \text{ such that } |b-c| < a < b+c \text{ and } a^2\left[(b+c)^2+bc\right] \leqslant bc(b+c)^2. \quad (27)$$

Since the function $g(x) = x^2\left[(b+c)^2+bc\right]$ is strictly increasing on $\left[|b-c|,\, b+c\right]$, it follows that

$$\begin{aligned}
(27) &\Leftrightarrow g\left(|b-c|\right) < bc(b+c)^2 \\
&\Leftrightarrow (b-c)^2\left[b^2+3bc+c^2\right] < bc(b+c)^2 \\
&\Leftrightarrow b^4 - 6b^2c^2 + c^4 < 0 \\
&\Leftrightarrow 3 - 2\sqrt{2} < \left(\frac{b}{c}\right)^2 < 3 + 2\sqrt{2} \\
&\Leftrightarrow -1 + \sqrt{2} < \frac{b}{c} < 1 + \sqrt{2},
\end{aligned}$$

as desired.

We end this Article by mentioning a remarkable property of the medians of any triangle, namely that they always serve as the side lengths of another triangle (see [12, §473, p. 282]). This contrasts heavily with the case with angle bisectors, which not only do not necessarily serve as the side lengths of a triangle, but also can be *any* positive numbers. As proved in [13, 14, 15, 16], given any three positive numbers, there exists a triangle whose angle bisectors have these numbers as lengths. In this context, it may be interesting to find nice-looking conditions on the side lengths of a triangle that are necessary and sufficient for its angle bisectors to form a triangle.

References

1. G. Leversha, *The geometry of the triangle*, UKMT (2013).
2. S. Abu-Saymeh, M. Hajja, and M. Hayajneh, Long altitudes and an unexpected appearance of the golden ratio, *Math. Gaz*. **102** (November 2018) pp. 523-527.
3. P. Yiu, *Euclidean geometry, preliminary version*, Department of Mathematics, Florida Atlantic University (Fall 1998).
4. Z. Cvetkovski, *Inequalities: theorems, techniques and selected problems*, Springer (2012).
5. M. Hajja, Stronger forms of the Steiner-Lehmus theorem, *Forum Geom*. **8** (2008) pp. 157-161.
6. S. Abu-Saymeh and M. Hajja, More variations of the Steiner-Lehmus theme, *Math. Gaz*. **103** (March 2019) pp. 1-11.
7. A. Altinaş, Some collinearities in the heptagonal triangle, *Forum Geom*. **16** (2016) pp. 249-256.
8. P. Yiu, Heptagonal triangle and their companies, *Forum Geom*. **9** (2009) pp. 125-148.

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press

9. L. Bankoff and J. Garfunkel, The heptagonal triangle, *Math. Mag*. **46** (1973) No. 1, pp. 7-19.
10. E. J. F. Primrose, An interesting triangle, *Math. Gaz*. **63** (March1979) pp. 42-43.
11. C. F. Parry, Steiner-Lehmus and the Feuerbach triangles, *Math. Gaz*. **79** (July 1995) pp. 275-285.
12. R. A. Johnson, *Advanced Euclidean geometry*, Dover Publications (1929).
13. P. Mironescu and L. Panaitopol, The existence of a triangle with prescribed angle bisector lengths, *Amer. Math. Monthly* **101** (1994) pp. 58-60.
14. G. Dinca and J. Mawhin, A constructive fixed point approach to the existence of a triangle with prescribed angle bisector lengths, Bull. *Belg. Math. Soc. Simon Stevin*, **17** (2010) pp. 333-341.
15. S. F. Osinkin, On the existence of a triangle with prescribed bisector lengths, *Forum Geom*. **16** (2016) pp. 399-405.
16. G. Heindl, How to compute a triangle with prescribed lengths of its angle bisectors, *Forum Geom*. **16** (2016) pp. 407-414.

SADI ABU-SAYMEH
2271 Barrowcliffe Dr., Concord, NC 28027, USA
e-mail: *ssaymeh@yahoo.com*
YAQEEN AL-MOMANI
Mathematics – Ajloun National University
Ajloun – Jordan
e-mail: *yaqeenal-momany@yahoo.com*
MOWAFFAQ HAJJA
Mathematics – Philadelphia University
Amman – Jordan
e-mail: *mowhajja@yahoo.com*
MOSTAFA HAYAJNEH
Mathematics – Yarmouk University Jordan
e-mail: *hayaj86@yahoo.com*

10.1017/mag.2021.106
© The Mathematical Association 2021

https://doi.org/10.1017/mag.2021.106 Published online by Cambridge University Press