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A note on Ramanujan's arithmetical function $\tau(n)$. By Mr J. R. WILTON, Trinity College.

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1. The function $\tau(n)$, defined by the equation

$$f(x) = x\{(1-x)(1-x^2)(1-x^3)\dots\}^{24} = \Sigma\tau(n)\,x^n, \tag{1·1}$$

is considered by Ramanujan* in his memoir "On certain arithmetical functions"

The associated Dirichlet series

$$\rho(s) = \Sigma\tau(n)\,n^{-s} \tag{1·2}$$

converges when $\sigma = Rs > \sigma_0$, for a sufficiently large positive σ_0. Ramanujan conjectured the truth of the remarkable identity

$$\rho(s) = \Pi\left\{\frac{1}{1-\tau(p)\,p^{-s}+p^{11-2s}}\right\}, \tag{1·21}$$

the product being taken over all primes p; and (1·21) has since been established by Mordell†. It follows that, if

$$n = p_1^{a_1}p_2^{a_2}\dots p_r^{a_r},$$

and $\cos\theta_p = \frac{1}{2}p^{-\frac{11}{2}}\tau(p)$, then

$$\tau(n) = n^{\frac{11}{2}}\frac{\sin(a_1+1)\theta_{p_1}}{\sin\theta_{p_1}}\dots\frac{\sin(a_r+1)\theta_{p_r}}{\sin\theta_{p_r}},$$

and $\tau(n)\,\tau(n') = \tau(nn')$ if n is prime to n'.

Ramanujan further conjectured that

$$\tau(p) \leqslant 2p^{\frac{11}{2}}, \tag{1·3}$$

so that θ_p is real; from this, as he shows‡, it would follow that

$$\tau(n) \leqslant n^{\frac{11}{2}}d(n), \tag{1·31}$$

where $d(n)$ is the number of divisors of n, and that

$$\tau(n) \geqslant n^{\frac{11}{2}}$$

for an infinite number of values of n. But he succeeded in proving only that

$$\tau(n) = O(n^7) \quad \text{and} \quad \tau(n) = \Omega(n^5).$$

These inequalities were subsequently refined by Hardy§ to

* *Trans. Camb. Phil. Soc.* 22 (1916), 159–184 (*Collected papers*, No. 18).
† *Proc. Camb. Phil. Soc.* 19 (1920), 117–124 (communicated 1917).
‡ *Collected papers*, 153–154.
§ G. H. Hardy, "Note on Ramanujan's arithmetical function $\tau(n)$", *Proc. Camb. Phil. Soc.* 23 (1927), 675–680.

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(1·32), (1·33) $\tau(n) = \Omega(n^{11/2})$ and $\tau(n) = O(n^6)$.

Hardy also showed that there are positive constants A and B such that

$$(1\cdot4) \qquad An^{12} < \tau^2(1) + \tau^2(2) + \ldots + \tau^2(n) < Bn^{12} \qquad (n \geqslant 1),$$

so that the average order of $\tau(n)$ is precisely $O(n^{11/2})$.

2. In this note I am not concerned with the problems to which Ramanujan's conjecture (1·3) gives rise, but (i) I establish by a classical method* an expansion theorem (Theorem A) for $\Sigma(x-k)^\kappa \tau(k)$, where $\kappa > 0$, in a series of Bessel functions; and (ii) by a method practically identical with that by which Hardy† first proved the existence of an infinite number of roots of the ζ-function on the critical line, I show that Ramanujan's function $\rho(s)$ has an infinite number of roots on the line $\sigma = 6$. The proofs of these theorems are quite straightforward, but they are not in any sense elementary, for they employ Cauchy's theorem and the transformation equation of the modular function; in addition Theorem A, through Lemma 2, employs a process of Diophantine approximation due to Hardy and Littlewood‡.

3. THEOREM A. *If* (i) $x > 0$, (ii) $\kappa > 0$, *and* (iii) p *and* q *are co-prime integers,* $0 < p < q$, *and* $pp' + 1 \equiv 0 \pmod q$, *or if* (iii *a*) $p = p' = 0$ *and* $q = 1$, *then*

$$(3\cdot1) \qquad \sum_{k=1}^{x} \frac{(x-k)^\kappa}{\Gamma(\kappa+1)} \tau(k)\, e^{2\pi i k p/q} = \left(\frac{q}{2\pi}\right)^\kappa \sum_{k=1}^{\infty} \left(\frac{x}{k}\right)^{6+\frac{1}{2}\kappa} \tau(k)\, e^{2\pi i k p'/q}\, J_{12+\kappa}\left\{\frac{4\pi}{q}\sqrt{(kx)}\right\}.$$

The equation (3·1) is presumably true for $\kappa = 0$ (and perhaps for $\kappa > -\frac{1}{2}$) but this I have not proved.

For the proof I require two known theorems, which I state as lemmas.

Lemma 1. *If* (i) p *and* q *are co-prime integers,* $0 < p < q$, *and* $pp' + 1 \equiv 0 \pmod q$, *or if* $p = p' = 0$, $q = 1$, *and if* (ii)

$$(3\cdot2) \qquad x = e^{-y}, \quad Ry > 0, \quad y = \frac{2\pi z}{q} - \frac{2\pi i p}{q}, \quad \gamma = \frac{2\pi}{qz} - \frac{2\pi i p'}{q},$$

then

$$(3\cdot3) \qquad f(x) = f(e^{-y}) = z^{-12} f(e^{-\gamma}) = \left(\frac{2\pi}{q}\right)^{12} \left(y + 2\pi i \frac{p}{q}\right)^{-12} f(e^{-\gamma}),$$

where $f(x)$ *is defined by* (1·1).

* S. Wigert, "Sur quelques fonctions arithmétiques", *Acta Math.* 37 (1914), 113–140 (§ 4).

† *Comptes Rendus*, April 1914, 1012–4.

‡ *Acta Math.* 37 (1914), 193–238 (226).

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The equation (3·3) is an evident modification of the transformation equation of the modular function*.

Lemma 2. *On the circle* $x = re^{i\theta}$, $r < 1$,

$$f(x) = O\{(1-r)^{-6}\}, \tag{3·4}$$

uniformly in θ.

This lemma is established by Hardy†, and is used by him in his proof of (1·33).

It is a very simple matter to deduce from these two lemmas a proof of Theorem A in the case $\kappa > 2$; and since, by (1·33), the series (3·1) converges absolutely if $\kappa > \frac{3}{2}$, the theorem follows for $\kappa > \frac{3}{2}$. To complete the proof when $\frac{3}{2} \geqslant \kappa > 0$ a further lemma is necessary.

Lemma 3. *On the circle* $x = re^{i\theta}$, $r < 1$, *any partial sum of* $f(x)$

$$\sum_{k=1}^{n} \tau(k) x^k = O\left\{(1-r)^{-6} \log \frac{1}{1-r}\right\}, \tag{3·5}$$

uniformly in n *and* θ. *And, if* $r = 1$,

$$\sum_{k=1}^{n} \tau(k) e^{ik\theta} = O(n^6 \log n), \tag{3·51}$$

uniformly in θ.

The proof follows immediately from Lemma 2. For, if C be the circle $|z| = e^{-c/n}$ (c is chosen to be either 1 or 2),

$$\begin{aligned} s_n = \sum_{1}^{n} \tau(k) x^k &= \frac{1}{2\pi i} \int_C f(z) \left(\frac{x}{z^2} + \ldots + \frac{x^n}{z^{n+1}}\right) dz \\ &= \frac{x}{2\pi i} \int_C \frac{f(z)}{z} \frac{dz}{z-x} - \frac{x^{n+1}}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} \frac{dz}{z-x} \\ &= \delta f(x) - \frac{x^{n+1}}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} \frac{dz}{z-x}, \end{aligned}$$

where $\delta = 1$ if $|x| < |z|$, and $\delta = 0$ if $|x| > |z|$.

Let $|x| = r = e^{-\alpha}$, and choose $c = 1$ if $\alpha \geqslant 3/2n$, or if $0 \leqslant \alpha \leqslant 1/2n$, but $c = 2$ if $1/2n < \alpha < 3/2n$, so that in every case

$$|\alpha - (c/n)| \geqslant 1/2n.$$

Then

$$|s_n - \delta f(x)| \leqslant \frac{e^{-\alpha(n+1)}}{2\pi} \int_0^{2\pi} \frac{|f(z)|}{|z-x|} \frac{d\Im}{e^{-c}},$$

where $z = e^{-\frac{c}{n} + \Im i}$. Hence, by Lemma 2, if $x = e^{-\alpha+\theta i}$, there are positive constants A, B, C, H, such that

* Compare G. H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis", *Proc. London Math. Soc.* (2), 17 (1918), 75–115, Lemma 4·31. (Ramanujan's *Collected Papers*, No. 36.)

† *Proc. Camb. Phil. Soc.* 23 (1927), 675–680, Lemma 1.

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$$|s_n - \delta f(x)| < Ae^{-an}(1-e^{-\frac{c}{n}})^{-6}\int_0^{2\pi}\frac{d\vartheta}{|1-e^{a-\frac{c}{n}+i(\vartheta-\theta)}|}$$

$$< Bn^6 e^{-an}\int_0^{2\pi}\frac{d\vartheta}{|1-e^{a-\frac{c}{n}+i\vartheta}|}$$

Now, if $n \geqslant 2$,

$$\left(1-e^{a-\frac{c}{n}}\right)^2 > \left(a-\frac{c}{n}\right)^2\left(1+a-\frac{c}{n}\right) > \left(1-\frac{3}{2n}\right)\left(a-\frac{c}{n}\right)^2 \geqslant \frac{1}{16n^2},$$

and therefore

$$|1-e^{a-\frac{c}{n}+i\vartheta}|^2 = (1-e^{a-\frac{c}{n}})^2 + 4e^{a-\frac{c}{n}}\sin^2\frac{\vartheta}{2} > \frac{1}{16n^2} + \frac{1}{2}\sin^2\frac{\vartheta}{2},$$

so that

$$|s_n - \delta f(x)| < 4\sqrt{2}Bn^6 e^{-an}\int_0^{\frac{1}{2}\pi}\frac{d\vartheta}{\sqrt{\{1/(8n^2)+\sin^2\vartheta\}}}$$

$$< 16Bn^6 e^{-an}\int_0^{\frac{1}{2}\pi}\frac{d\vartheta}{\sqrt{(n^{-2}+\vartheta^2)}}$$

$$< Cn^6 e^{-an}\log n. \tag{3·52}$$

If $a = 0$, $\delta = 0$, and this proves (3·51). If $a > 0$, then (3·52) has, as n varies, a single maximum at $n = n_1$, where

$$a = \frac{1}{n_1}\left(6+\frac{1}{\log n_1}\right) \leqslant \frac{1}{n_1}\left(6+\frac{1}{\log 2}\right)$$

Hence, if $0 < a \leqslant \frac{1}{2}$, say,

$$|s_n| \leqslant |f(x)| + Cn_1^6 \log n_1 < Ha^{-6}\log(1/a),$$

which is equivalent to (3·5).

The proof of Theorem A is now a matter of a few lines. Consider the series on the right of (3·1), namely

$$\left(\frac{q}{2\pi}\right)^\kappa \sum_{k=1}^{\infty}\left(\frac{x}{k}\right)^{6+\frac{1}{2}\kappa}\tau(k)\,e^{2\pi i k p'/q}J_{12+\kappa}\left\{\frac{4\pi}{q}\sqrt{(kx)}\right\} \tag{3·6}$$

$$= \left(\frac{2\pi}{q}\right)^{12}\sum_{k=1}^{\infty}\tau(k)\,e^{2\pi i k p'/q}\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} t^{-13-\kappa}\,e^{xt-(4\pi^2 k/q^2 t)}\,dt$$

if $a > 0$, by Sonine's integral* for $J_\nu(z)$. Since, if

$$r = |\exp(-4\pi^2/q^2 t)| = \exp\{-4\pi^2 a/(q^2|t|^2)\},$$

$$1 - r \sim 4\pi^2 a/(q^2|t|^2) \text{ as } t \to \infty,$$

* G. N. Watson, *Theory of Bessel functions* (1922), 177 (8). It is at this stage that the proof follows the classical method of Wigert (*loc. cit.*).

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it follows from Lemma 3 by a known theorem* that the order of summation and integration may be changed if $\kappa > 0$; and (3·6) is equal to

$$(3\cdot61)\qquad \frac{e^{2\pi i x p/q}}{2\pi i}\int_{a-\infty i}^{a+\infty i} e^{xy}\left(\frac{2\pi}{q}\right)^{12}\left(y+2\pi i\frac{p}{q}\right)^{-13-\kappa}\sum_{1}^{\infty}\tau(k)\,e^{-k\gamma}\,dy,$$

on writing $y + 2\pi i(p/q)$ for t, where γ is defined by (3·2). Since $\Sigma\tau(k)\,e^{-k\gamma} = f(e^{-\gamma})$, it follows from Lemma 1 that (3·61) is equal to

$$\frac{e^{2\pi i x p/q}}{2\pi i}\int_{a-\infty i}^{a+\infty i} e^{xy}\left(y+2\pi i\frac{p}{q}\right)^{-1-\kappa} f(e^{-y})\,dy$$

$$= \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} y^{-1-\kappa}\,e^{xy}\sum_{1}^{\infty}\tau(k)\,e^{2\pi i k p/q}\,e^{-ky}\,dy$$

$$(3\cdot62)\qquad = \sum_{k=1}^{x}(x-k)^{\kappa}\,e^{2\pi i k p/q}\,\frac{\tau(k)}{2\pi i}\int_{c-\infty i}^{c+\infty i} y^{-1-\kappa}\,e^{y}\,dy$$

$$(3\cdot63)\qquad = \sum_{k=1}^{x}\frac{(x-k)^{\kappa}}{\Gamma(\kappa+1)}\,\tau(k)\,e^{2\pi i k p/q},$$

by Laplace's integral†,

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} t^{-z}\,e^{t}\,dt.$$

In (3·62) $c = a\,(x-k)$, and the change of order follows from the absolute convergence test‡. It is clear that the integrals for which $k \geqslant x$ are zero.

The equality of (3·6) and (3·63) when $\kappa > 0$ is the statement of Theorem A.

4. An application of Abel's lemma (partial summation) to (3·51) when $\theta = 0$ shows that Ramanujan's series (1·2) converges if $\sigma > 6$. More generally, the Fourier series

$$(4\cdot1)\qquad \sum_{n=1}^{\infty}\frac{\tau(n)}{n^{s}}\begin{matrix}\cos\\ \sin\end{matrix} n\theta$$

is uniformly convergent§ if $\sigma > 6$; and, more precisely, the Fourier series

$$(4\cdot2)\qquad \sum_{2}^{\infty}\frac{\tau(n)}{n^{6}\log^{2+\delta}n}\begin{matrix}\cos\\ \sin\end{matrix} n\theta$$

is uniformly convergent if $\delta > 0$.

* See, for example, E. W. Hobson, *Theory of functions of a real variable*, 2 (1926), 288–289.

† Whittaker and Watson, *Modern analysis* (1920), 246.

‡ Bromwich, *Theory of infinite series* (1908), 453, Theorem B.

§ Since $$\left\{\sum_{1}^{n}\tau(k)\right\}^{2} < n\sum_{1}^{n}\tau^{2}(k) < Bn^{13},$$
it follows that (4·1) is absolutely and uniformly convergent if $\sigma > 6\frac{1}{2}$.

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From Hardy's inequality (1·4) it follows by partial summation that

$$\Sigma\tau^2(n)/(n^{12}\log^{1+2\delta}n)$$

is convergent if $\delta > 0$, and by the Riesz-Fischer theorem

$$\sum_2^\infty \frac{\tau(n)}{n^6\log^{\frac{1}{2}+\delta}n}\begin{matrix}\cos\\ \sin\end{matrix} n\theta$$

is a Fourier series; hence by a theorem due to Kolmogoroff and Seliverstoff*

$$\sum_2^\infty \frac{\tau(n)}{n^6\log^{1+\delta}n}\begin{matrix}\cos\\ \sin\end{matrix} n\theta \tag{4·3}$$

is convergent almost everywhere.

In one respect (4·3) goes beyond (4·2), but in another respect it states a good deal less, inasmuch as there is an undetermined set of values of θ, of measure zero, for which the convergence of (4·3) is left undecided.

Proof that $\rho(s)$ *has roots on* $\sigma = 6$.

5. I consider a somewhat more general function than Ramanujan's $\rho(s)$, namely†

$$\rho(s;\ p,\ q) = \sum_{n=1}^\infty \frac{\tau(n)}{n^s}e^{2\pi i n p/q} \qquad (p,\ q) = 1. \tag{5·1}$$

But I am not able to prove that this function has roots on $\sigma = 6$ unless $p^2 \equiv 1 \pmod q$—which, of course, includes $\rho(s)$ when $q = 1$. We have, if $\sigma > 6$,

$$\left(\frac{q}{2\pi}\right)^s \Gamma(s)\,\rho(s;\ p,\ q) = \sum_1^\infty \tau(n)\,e^{2\pi i n p/q}\int_0^\infty x^{s-1}e^{-2\pi n x/q}\,dx$$

$$\begin{aligned}
&= \int_0^\infty x^{s-1} f\left\{\exp\left(-\frac{2\pi x}{q} + 2\pi i\frac{p}{q}\right)\right\}dx \qquad (5·2)\ddagger\\
&= \int_1^\infty x^{s-1} f\left\{\exp\left(-\frac{2\pi x}{q} + 2\pi i\frac{p}{q}\right)\right\}dx\\
&\qquad + \int_0^1 x^{s-13} f\left\{\exp\left(-\frac{2\pi}{qx} + 2\pi i\frac{p'}{q}\right)\right\}dx\\
&= \int_1^\infty \left[x^{s-1} f\left\{\exp\left(-\frac{2\pi x}{q} + 2\pi i\frac{p}{q}\right)\right\}\right.\\
&\qquad \left. + x^{11-s} f\left\{\exp\left(-\frac{2\pi x}{q} + 2\pi i\frac{p'}{q}\right)\right\}\right]dx,
\end{aligned}$$

* *Comptes Rendus*, 178 (1924), 303. Hobson, *loc. cit.*, 626.

† (p, q) denotes the greatest common factor of p and q.

‡ The change of order follows from Lemma 3 and the theorem quoted from Hobson, as in (3·61). If $\sigma > 7$, Lemma 2 and the absolute convergence test are sufficient.

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where $pp' \equiv -1 \pmod{q}$. It follows that $\rho(s;\ p,\ q)$ is an integral function satisfying the functional equation

$$(q/2\pi)^s \Gamma(s)\rho(s;\ p,\ q) = (q/2\pi)^{12-s}\Gamma(12-s)\rho(12-s;\ p',\ q),$$

so that it has simple zeros at $s = 0, -1, -2, \ldots$.

Let

$$R(t;\ p,\ q) = (q/2\pi)^{6+it}\Gamma(6+it)\rho(6+it;\ p,\ q) \tag{5·3}$$
$$= R(-t;\ p',\ q).$$

In (5·2) put $x = e^y$, $s = 6 + it$, then

$$R(t;\ p,\ q) = \int_{-\infty}^{\infty} \phi(y)\, e^{ity}\, dy, \tag{5·31}$$

where

$$\phi(y) = e^{6y} f\{\exp[(-2\pi e^y + 2\pi i p)/q]\}, \tag{5·32}$$

so that $\phi(y)$, with all its differential coefficients, vanishes at $+\infty$; and, by the transformation equation of the modular function, the same is true at $-\infty$. On integrating (5·31) by parts we have, therefore,

$$(-it)^n R(t;\ p,\ q)\, e^{(\alpha-i\beta)t} = \int_{-\infty}^{\infty} \phi^{(n)}(y)\, e^{-i\beta t}\, e^{it(y-i\alpha)}\, dy$$

$$= \int_{-\infty}^{\infty} \phi^{(n)}(y + i\alpha)\, e^{it(y-\beta)}\, dy, \tag{5·33}$$

by an evident contour integration, provided that

$$-\tfrac{1}{2}\pi < \alpha < \tfrac{1}{2}\pi.$$

Changing the signs of α, β and y, and writing p' for p, we have

$$(it)^n R(t;\ p',\ q)\, e^{-(\alpha-i\beta)t} = \int_{-\infty}^{\infty} \phi^{(n)}(y + i\alpha)\, e^{-it(y-\beta)}\, dy, \tag{5·34}$$

by the transformation equation. If the sum of (5·33) and (5·34) is integrated with regard to t from 0 to ∞, it follows from Fourier's integral theorem and (5·3) that

$$\int_{-\infty}^{\infty} t^n R(t;\ p,\ q)\, e^{(\alpha-i\beta)t}\, dt = 2\pi i^n \phi^{(n)}(\beta + i\alpha). \tag{5·35}$$

It is easy to see that, if

$$\beta = \log(h/k) \qquad (h,\ k) = 1,$$

the limit of the integral (5·35) as $\alpha \to \frac{1}{2}\pi$ is zero. For let

$$(pk - h,\ qk) = (pk - h,\ q) = l, \qquad qk = lm;$$

choose r so that

$$r\{(pk - h)/l\} + 1 \equiv 0 \pmod{m},$$

and let

$$z = (e^{i\alpha} - i)\, h/l, \text{ so that } Rz = (h/l)\cos\alpha > 0.$$

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Then

$$\left(\frac{ke^{-i\alpha}}{h}\right)^6 \phi(\beta+i\alpha) = f\left\{\exp\left(-\frac{2\pi h}{qk}e^{i\alpha}+2\pi i\frac{p}{q}\right)\right\}$$

$$= f\left\{\exp\left(-\frac{2\pi z}{m}+2\pi i\frac{(pk-h)/l}{m}\right)\right\} = z^{-12} f\left\{\exp\left(-\frac{2\pi}{mz}+2\pi i\frac{r}{m}\right)\right\};$$

$$\phi = \frac{-l^{12}}{\{2hk(1-\sin\alpha)\}^6} f\left[\exp\left\{-\frac{\pi l}{mh}\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)^{\frac{1}{2}}+\frac{\pi i}{mh}\cdot(2rh-l)\right\}\right],$$

where $\phi = \phi(\beta+i\alpha)$; it follows that, as $\alpha \to \frac{1}{2}\pi$, for every n

$$\phi^{(n)}\{\log(h/k)+i\alpha\} \to 0;$$

i.e., from (5·35),

$$\lim_{\alpha\to\frac{1}{2}\pi-0}\int_{-\infty}^{\infty} t^n R(t;\ p,\ q)\left(\frac{h}{k}\right)^{-it} e^{\alpha t}\,dt = 0. \tag{5·36}$$

Let $R_1(t)$ be the real part of $R(t;\ p,\ q)$,

$$R_1(t) = \tfrac{1}{2}\{R(t;\ p,\ q)+R(-t;\ q-p,\ q)\} = \tfrac{1}{2}\{R(t;\ p,\ q)+R(t;\ q-p',\ q)\}. \tag{5·4}$$

I now prove that $R_1(t)$ has an infinite number of real roots; from which it follows, by (5·3) and (5·4), that the function

$$\rho(s;\ p,\ q)+\rho(s;\ -p',\ q)$$

has an infinite number of roots on the line $\sigma = 6$. The only case presenting any interest is that in which $p' = -p$, i.e. $p^2 \equiv 1 \pmod q$, so that, for example, the two functions

$$\Sigma n^{-s}\tau(n)e^{2\pi i n/q} \quad \text{and} \quad \Sigma n^{-s}\tau(n)e^{-2\pi i n/q}$$

have an infinite number of roots on $\sigma = 6$.

From (5·36), if $h = k = 1$, we have

$$\int_{-\infty}^{\infty} t^n R_1(t)\,e^{\alpha t}\,dt = A(\alpha,\ n) = A, \tag{5·41}$$

say, where, for every n, $A \to 0$ as $\alpha \to \frac{1}{2}\pi$.

Suppose that $R_1(t)$ does not change sign when $|t| \geqslant T > 1$; and let

$$0 < \gamma \leqslant \alpha < \tfrac{1}{2}\pi.$$

We have

$$\int_{-T}^{T} t^n R_1(t)\,e^{\alpha t}\,dt = A - \int_{T}^{\infty} - \int_{-\infty}^{-T}$$

$$= A - \int_{T}^{\infty} t^n\{R_1(t)\,e^{\alpha t}+(-)^n R_1(-t)\,e^{-\alpha t}\}\,dt. \tag{5·42}$$

If $R_1(t)$ and $R_1(-t)$ are of the same sign when $t \geqslant T$, we choose n

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to be even; if they are of opposite signs we choose n to be odd; then, in either case,

$$\left|\int_T^\infty t^n \{R_1(t)\, e^{\alpha t} + (-)^n R_1(-t)\, e^{-\alpha t}\}\, dt\right| \geqslant T^n e^{\gamma T} \int_T^\infty |R_1(t)|\, dt$$

$$= T^n F(T), \tag{5·43}$$

where $F(T)$ is independent of α and of n. Choose $\alpha = \alpha(n)$ so that $|A| < \frac{1}{2}F(T)$; then from (5·42) and (5·43)

$$T^{-n}\left|\int_{-T}^{T}\right| > \tfrac{1}{2}F(T). \tag{5·44}$$

But, if

$$M(T) = \underset{-T \leqslant t \leqslant T}{\text{Max.}} |tR_1(t)|,$$

$$T^{-n}\left|\int_{-T}^{T}\right| \leqslant \frac{M(T)}{n}\, e^{\frac{1}{2}\pi T}, \tag{5·45}$$

which is inconsistent with (5·44) if n is sufficiently great. The contradiction shows that $R_1(t)$ must change sign infinitely often, and therefore has an infinite number of real roots of odd order.

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