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Some Refined Eigenvalue Perturbation Bounds for Two-by-Two Block Hermitian Matrices

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Abstract. We consider eigenvalue perturbation bounds for Hermitian matrices, which are associated with problems arising in various computational science and engineering applications. New bounds are discussed that are sharper than some existing ones, including the well-known Weyl bound. Two numerical examples are investigated, to illustrate our theoretical presentation.

AMS subject classifications: 65F10

Key words: Eigenvalue perturbation, Hermitian block matrix, Weyl bound.

1. Introduction

We consider a Hermitian matrix

$$\mathscr{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^H & A_{22} \end{pmatrix}, \tag{1.1}$$

where $A_{11} \in \mathscr{C}^{p\times p}$ and $A_{22} \in \mathscr{C}^{k\times k}$. In particular, a matrix $\mathscr{A}$ with block form (1.1) is also said to be an $n \times n$ *generalised saddle point matrix*, or a *saddle point matrix* if A_{22} is a null matrix. Generalised saddle point systems have a wide variety of applications in computational science and engineering (e.g. see Ref. [3] and references therein), and the spectrum of the generalised saddle point matrix plays an important role in solving the generalised saddle system with Krylov subspace methods. We also note that some other authors have discussed the spectral analysis of saddle point matrices — e.g. see Refs. [1,7] and references therein.

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We will assume a perturbation of $\mathscr{A}$ expressed as

$$\mathscr{E} = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^H & E_{22} \end{pmatrix}, \tag{1.2}$$

where $E_{11} \in \mathscr{C}^{p\times p}$ and $E_{22} \in \mathscr{C}^{k\times k}$ are Hermitian, and write $\widetilde{\mathscr{A}} = \mathscr{A} + \mathscr{E}$. The spectrum of any matrix M is denoted by $\sigma(M)$; and throughout this article we assume that both $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ are Hermitian, and that their eigenvalues are arranged in the increasing order.

A well-known perturbation bound for Hermitian matrices is as follows. Let λ_i and $\widetilde{\lambda}_i$ be the ith smallest eigenvalues of Hermitian matrices $\mathscr{A}$ and $\widetilde{\mathscr{A}}$, respectively. Then the Weyl bound

$$|\lambda_i - \widetilde{\lambda}_i| \le ||\mathscr{E}||_2 \tag{1.3}$$

is optimal without any restriction on the perturbation $\mathscr{E}$, and clearly it is independent of the structure of the matrices $\mathscr{A}$ and $\widetilde{\mathscr{A}}$. When a block diagonal matrix undergoes an off-diagonal perturbation (i.e. A_{12}, E_{11} and E_{22} are all zero), Mathias [10] provided another bound

$$|\lambda_i - \widetilde{\lambda}_i| \le \frac{||E_{12}||_2^2}{\eta_i}, \tag{1.4}$$

where

$$\eta_i = \min_{\lambda\in\sigma(A_{22})} |\lambda_i - \lambda|, \tag{1.5}$$

which was subsequently improved by Li & Li [8]. Recently, Nakatsukasa [11] presented an eigenvalue bound for a structured perturbation of the Hermitian 2×2 block matrix (1.1). Thus if λ_i and $\widetilde{\lambda}_i$ are the respective ith eigenvalues of $\mathscr{A}$ and $\widetilde{\mathscr{A}}$, then one has

$$|\lambda_i - \widetilde{\lambda}_i| \le ||E_{11}||_2 + 2||E_{12}||_2\tau_i + ||E_{22}||_2\tau_i^2 \tag{1.6}$$

provided

$$\tau_i = \frac{||E||_2 + ||E_{12}||_2}{\min_{\lambda\in\sigma(A_{22})}|\lambda_i - \lambda| - 2||\mathscr{E}||_2} > 0\,.$$

It is evident that this eigenvalue perturbation bound depends on the block structure of $\mathscr{A}$. Remark 1 in Ref. [11] suggests that $\tau_i < 1$ is necessary for a bound that is sharper than the Weyl bound (1.3), and Li *et al.* [9] improved the bound (1.6) as follows:

$$\begin{aligned} |\lambda_i - \widetilde{\lambda}_i| \le & ||E_{11}||_2 + \min\left\{1, 2\widetilde{\omega}(\delta_i^{(2)})\right\}||E_{12}||_2 \\ & + \max\left\{||E_{22}||_2 - ||E_{11}||_2, 0\right\}\omega(\delta_i^{(2)})\widetilde{\omega}(\delta_i^{(2)}), \end{aligned} \tag{1.7}$$

or

$$\begin{aligned} |\lambda_i - \widetilde{\lambda}_i| \le & ||E_{22}||_2 + \min\left\{1, 2\widetilde{\omega}(\delta_i^{(1)})\right\}||E_{12}||_2 \\ & + \max\left\{||E_{11}||_2 - ||E_{22}||_2, 0\right\}\omega(\delta_i^{(1)})\widetilde{\omega}(\delta_i^{(1)}), \end{aligned} \tag{1.8}$$

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where

$$\omega(x)=\begin{cases}\frac{||A_{12}||_2+||E_{12}||_2}{\sqrt{x^2+(||A_{12}||+||E_{12}||_2)^2}}, & x>0,\\ 1, & x\le 0,\end{cases} \tag{1.9}$$

$$\widetilde{\omega}(x)=\begin{cases}\frac{2||A_{12}||_2+||E_{12}||_2}{\sqrt{x^2+(||A_{12}||+||E_{12}||_2)^2}\sqrt{x^2+||A_{12}||_2^2}}, & x>0,\\ 1, & x\le 0,\end{cases} \tag{1.10}$$

and

$$\delta_i^{(k)}=\min_{\lambda\in\sigma(A_{kk})}|\lambda_i-\lambda|-\Big(||\mathscr{E}||_2+||E_{kk}||_2\Big),\quad k=1,2. \tag{1.11}$$

Also quite recently, Cheng *et al.* [4] gave another bound — viz. if $\delta_i^{(1)}$ and $\delta_i^{(2)}$ are positive for any i, then

$$|\lambda_i-\widetilde{\lambda}_i|\le||E_{11}||_2\omega(\delta_i^{(1)})^2+2\omega(\delta_i^{(1)})\omega(\delta_i^{(2)})||E_{12}||_2+||E_{22}||_2\omega(\delta_i^{(2)})^2. \tag{1.12}$$

We note that this bound (1.12) can be extended to the case that $\delta_i^{(k)}\le 0$ [9], and it also improves the bound in (1.6).

Here we combine the techniques in Refs. [4, 9, 11] to give some much sharper bounds. In Section 2, a perturbation bound for eigenvalues of Hermitian 2×2 block matrices is obtained (Theorem 2.1). In Section 3, we use this improved bound and the reducibility of a matrix to permute the perturbation matrix to a 2×2 block diagonal matrix by rows and columns, and derive an improving the Weyl bound (Theorem 3.2). Finally, numerical examples presented in Section 4 illustrate our theoretical results.

2. Sharper Bounds for 2×2 Block Matrices

In this section, we provide some sharper bounds. We write

$$\gamma_{st}=\omega(\delta_i^{(s)})\widetilde{\omega}(\delta_i^{(t)}),\quad s,t=1,2, \tag{2.1}$$

where $\omega(\delta_i^{(s)})$ and $\widetilde{\omega}(\delta_i^{(t)})$ are given by (1.9) and (1.10), respectively. It follows from Ref. [9] that $\widetilde{\omega}(x)\le\omega(x)\le 1$ for any $x\in\mathscr{R}$, hence $0\le\gamma_{st}\le 1$.

Let $(\lambda_i(t),x(t))$ be the ith smallest eigenpair such that

$$(\mathscr{A}+t\mathscr{E})x(t)=\lambda_i(t)x(t),\qquad ||x(t)||_2=1;$$

and let $x(t)=(x_1(t)^H,x_2(t)^H)^H$, where $x_1(t)\in\mathscr{C}^p$ and $x_2(t)\in\mathscr{C}^k$. From the definition of $\delta_i^{(2)}$ and (2.6) of Ref. [9], we have

$$||x_2(t)||_2\le\omega(\delta_i^{(2)}); \tag{2.2}$$

and from (2.7) of Ref. [9],

$$\int_0^1||x_2(t)||_2dt\le\widetilde{\omega}(\delta_i^{(2)}). \tag{2.3}$$

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Similarly,

$$||x_1(t)||_2 \leq \omega(\delta_i^{(1)}) , \tag{2.4}$$

and

$$\int_0^1 ||x_1(t)||_2 dt \leq \widetilde{\omega}(\delta_i^{(1)}) . \tag{2.5}$$

Clearly,

$$\begin{aligned} 2||E_{12}||_2 \int_0^1 ||x_1(t)||_2 ||x_2(t)||_2 dt \leq & ||E_{12}||_2 \int_0^1 \left(||x_1(t)||_2^2 + ||x_2(t)||_2^2 \right) dt \\ = & ||E_{12}||_2 . \end{aligned}$$

It follows from inequalities (2.3) and (2.4) that

$$\begin{aligned} 2||E_{12}||_2 \int_0^1 ||x_1(t)||_2 ||x_2(t)||_2 dt \leq & 2||E_{12}||_2 \omega(\delta_i^{(1)}) \int_0^1 ||x_2(t)||_2 dt \\ \leq & 2\omega(\delta_i^{(1)}) \widetilde{\omega}(\delta_i^{(2)}) ||E_{12}||_2 \\ = & 2\gamma_{12} ||E_{12}||_2 , \end{aligned}$$

and similarly

$$\begin{aligned} 2||E_{12}||_2 \int_0^1 ||x_1(t)||_2 ||x_2(t)||_2 dt \leq & 2||E_{12}||_2 \omega(\delta_i^{(2)}) \int_0^1 ||x_1(t)||_2 dt \\ = & 2\omega(\delta_i^{(2)}) \widetilde{\omega}(\delta_i^{(1)}) ||E_{12}||_2 \\ = & 2\gamma_{21} ||E_{12}||_2 . \end{aligned}$$

Combining the above three inequalities, we obtain

$$2||E_{12}||_2 \int_0^1 ||x_1(t)||_2 ||x_2(t)||_2 dt \leq \min\left\{1, 2\gamma_{12}, 2\gamma_{21}\right\} ||E_{12}||_2 . \tag{2.6}$$

From the proof of Theorem 2.2 in Ref. [9],

$$\begin{aligned} |\lambda_i - \widetilde{\lambda}_i| \leq & ||E_{11}||_2 \int_0^1 ||x_1(t)||_2^2 dt + 2||E_{12}||_2 \int_0^1 ||x_1(t)||_2 ||x_2(t)|| dt \\ & + ||E_{22}||_2 \int_0^1 ||x_2(t)||_2^2 dt , \end{aligned} \tag{2.7}$$

which together with the inequalities (2.2)-(2.6) gives

$$|\lambda_i - \widetilde{\lambda}_i| \leq \gamma_{11} ||E_{11}||_2 + \gamma_{22} ||E_{22}||_2 + \min\left\{1, 2\gamma_{12}, 2\gamma_{21}\right\} ||E_{12}||_2 . \tag{2.8}$$

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Remark 2.1. When $E_{12}=0$, we have $\widetilde{\omega}(x)=\omega(x)$, so the bound (2.8) reduces to (1.12). For other cases, the bound (2.8) always improves (1.12) since $\widetilde{\omega}(x)\le\omega(x)\le 1$.

To further improve the above bounds, we distinguish the following two different cases. For the case $||E_{11}||_2\le||E_{22}||_2$, from (2.7) and $||x_1(t)||_2^2=1-||x_2(t)||_2^2$ we have

$$|\lambda_i-\widetilde{\lambda}_i|\le||E_{11}||_2+2||E_{12}||_2\int_0^1||x_1(t)||_2||x_2(t)||_2dt+\Big(||E_{22}||_2-||E_{11}||_2\Big)\int_0^1||x_2(t)||_2^2dt,$$

and from (2.2), (2.3) and (2.6)

$$|\lambda_i-\widetilde{\lambda}_i|\le(1-\gamma_{22})||E_{11}||_2+\gamma_{22}||E_{22}||_2+\min\Big\{1,2\gamma_{12},2\gamma_{21}\Big\}||E_{12}||_2\,.$$

For the case $||E_{11}||_2\ge||E_{22}||_2$, from (2.4)-(2.7) it likewise follows that

$$|\lambda_i-\widetilde{\lambda}_i|\le\gamma_{11}||E_{11}||_2+(1-\gamma_{11})||E_{22}||_2+\min\Big\{1,2\gamma_{12},2\gamma_{21}\Big\}||E_{12}||_2\,.$$

We thus have the following theorem:

Theorem 2.1. *Let λ_i and $\tilde{\lambda}_i$ be the ith eigenvalues of $\mathscr{A}$ and $\widetilde{\mathscr{A}}$, respectively. Then there is $s\in\{1,2\}$ such that*

$$|\lambda_i-\widetilde{\lambda}_i|\le\gamma_{ss}||E_{ss}||_2+\min\Big\{\gamma_{3-s,3-s},1-\gamma_{ss}\Big\}||E_{3-s,3-s}||_2+\min\Big\{1,2\gamma_{12},2\gamma_{21}\Big\}||E_{12}||_2\,. \quad (2.9)$$

Remark 2.2. The new bound (2.9) differs from the bounds in Ref. [9]. Indeed, by combining the techniques in Refs. [4, 9, 11], it always improves the bounds (1.7), (1.8) and (1.12).

Thus for $s=1$, the bound (2.9) can be rewritten

$$\begin{aligned}|\lambda_i-\widetilde{\lambda}_i|\le&\gamma_{11}||E_{11}||_2+\min\Big\{\gamma_{22},1-\gamma_{11}\Big\}||E_{22}||_2+\min\Big\{1,2\gamma_{12},2\gamma_{21}\Big\}||E_{12}||_2\\ =&\begin{cases}\gamma_{11}||E_{11}||_2+\gamma_{22}||E_{22}||_2+\alpha\,, & \gamma_{22}\le1-\gamma_{11}\,,\\ \gamma_{11}||E_{11}||_2+(1-\gamma_{11})||E_{22}||_2+\alpha\,, & \gamma_{22}>1-\gamma_{11}\,,\end{cases}\end{aligned} \quad (2.10)$$

where $\alpha=\min\{1,2\gamma_{12},2\gamma_{21}\}||E_{12}||_2$. Now if $||E_{11}||_2\le||E_{22}||_2$, from the definition of γ_{st} given by (2.1) the bound (1.7) reduces to

$$|\lambda_i-\widetilde{\lambda}_i|\le(1-\gamma_{22})||E_{11}||_2+\gamma_{22}||E_{22}||_2+\min\Big\{1,2\widetilde{\omega}(\delta_i^{(2)})\Big\}||E_{12}||_2\,, \quad (2.11)$$

and a simple computation shows that the bound in (2.10) is sharper than the one in (2.11). If $||E_{11}||_2>||E_{22}||_2$, then the bound (1.7) can be reduced to

$$|\lambda_i-\widetilde{\lambda}_i|\le||E_{11}||_2+\min\Big\{1,2\widetilde{\omega}(\delta_i^{(2)})\Big\}||E_{12}||_2\,. \quad (2.12)$$

It is easy to see that the bound in (2.10) is sharper than the one in (2.12), so the assertion holds for $s=1$. The proof for $s=2$ is similar, and it can likewise be shown that the bound in (2.9) is always sharper than those in (1.8) and (1.12).

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Corollary 2.1. *For the standard saddle point case corresponding to the matrices*

$$\mathscr{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & 0 \end{pmatrix}, \quad \mathscr{E} = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^T & 0 \end{pmatrix}, \tag{2.13}$$

if λ_i and $\widetilde{\lambda}_i$ denote the respective ith eigenvalues of $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ then

$$|\lambda_i - \widetilde{\lambda}_i| \leq \gamma_{11}||E_{11}||_2 + \min\left\{1, 2\gamma_{12}, 2\gamma_{21}\right\}||E_{12}||_2 \,. \tag{2.14}$$

Remark 2.3. It is easy to see that the bound in (2.14) is sharper than

$$|\lambda_i - \widetilde{\lambda}_i| \leq \gamma_{11}||E_{11}||_2 + \min\left\{1, 2\gamma_{12}, 2\tilde{\omega}(\delta_1^{(1)})\right\}||E_{12}||_2 \,,$$

which is the bound in (2.21) of Ref. [9].

3. A Sharper Bound for a General Matrix

We now consider improving the Weyl bound if some information is known in advance, based on the technique in Section 2. Let $\mathscr{A} \in \mathscr{C}^{n\times n}$ and its perturbed matrix $\widetilde{\mathscr{A}}$ be Hermitian. Assume that there exists a unitary matrix $\mathscr{U}$ such that

$$\mathscr{U}\mathscr{E}\mathscr{U}^H = \begin{pmatrix} E_{11} & 0 \\ 0 & E_{22} \end{pmatrix}, \tag{3.1}$$

where $E_{11} \in \mathscr{C}^{p\times p}$ and $E_{22} \in \mathscr{C}^{k\times k}$ are both Hermitian matrices. On partitioning $\mathscr{U}\mathscr{A}\mathscr{U}^H$ into the form

$$\mathscr{U}\mathscr{A}\mathscr{U}^H = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^H & A_{22} \end{pmatrix}$$

where $A_{11} \in \mathscr{C}^{p\times p}$ and $A_{22} \in \mathscr{C}^{k\times k}$, we have

$$\mathscr{U}\widetilde{\mathscr{A}}\mathscr{U}^H = \mathscr{U}\mathscr{A}\mathscr{U}^H + \mathscr{U}\mathscr{E}\mathscr{U}^H. \tag{3.2}$$

Since the spectra of Hermitian matrices $\mathscr{U}\widetilde{\mathscr{A}}\mathscr{U}^H$ and $\mathscr{U}\mathscr{A}\mathscr{U}^H$ are the same as those of $\widetilde{\mathscr{A}}$ and $\mathscr{A}$ respectively, we get the following theorem on applying Theorem 2.1 to (3.2).

Theorem 3.1. *Let $\mathscr{A} \in \mathscr{C}^{n\times n}$ and $\widetilde{\mathscr{A}}$ be Hermitian. If there exists a unitary matrix $\mathscr{U}$ such that $\mathscr{U}\mathscr{E}\mathscr{U}^H$ has the block form (3.1), then there is $s \in \{1, 2\}$ such that*

$$|\lambda_i - \widetilde{\lambda}_i| \leq \gamma_{ss}||E_{ss}||_2 + \min\left\{\gamma_{3-s,3-s}, 1-\gamma_{ss}\right\}||E_{3-s,3-s}||_2 \,. \tag{3.3}$$

Remark 3.1. Under the assumption of Theorem 3.1, the Weyl bound (1.3) reduces to

$$|\lambda_i - \widetilde{\lambda}_i| \leq \max\left\{||E_{11}||_2, ||E_{22}||_2\right\},$$

so the bound (3.3) is sharper.

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In order to improve the Weyl bound, we introduce the definition of a reducible matrix.

Definition 3.1. (cf. Ref. [6]) Let $\langle n\rangle = \{1,...,n\}$. An $n\times n$ matrix $A=(a_{ij})$ is said to be reducible if there exists a nonempty subset α of $\langle n\rangle$ such that

$$a_{ij}=0 \qquad \forall\, i\in\alpha\,,\ j\in\alpha'\,,$$

where $\alpha'=\langle n\rangle\backslash\alpha$. Otherwise, A is called irreducible.

Thus if α and β are subsets of $\langle n\rangle$, for $A\in\mathscr{C}^{n\times n}$ we denote by $A[\alpha|\beta]$ the submatrix where the rows and columns are indexed in α and β, respectively. For simplicity, we write $A[\alpha]=A[\alpha|\alpha]$ and note that if α or β is empty then $A[\alpha|\beta]$ is an empty matrix. Our bound is given as follows.

Theorem 3.2. *Let $\mathscr{A}\in\mathscr{C}^{n\times n}$ and $\widetilde{\mathscr{A}}$ be Hermitian. Then there exists a subset α of $\langle n\rangle$ such that $||\mathscr{E}[\alpha]||_2\le||\mathscr{E}[\alpha']||_2$; and*

$$|\lambda_i-\widetilde{\lambda}_i|\le\min\left\{\gamma_{11},(1-\gamma_{22})\right\}||\mathscr{E}[\alpha]||_2+\gamma_{22}||\mathscr{E}[\alpha']||_2\,, \tag{3.4}$$

where γ_{22} is given by (2.1) *with $A_{11}=\mathscr{A}[\alpha]$, $A_{12}=\mathscr{A}[\alpha|\alpha']$, $A_{22}=\mathscr{A}[\alpha']$ and $E_{12}=0$, $\delta_i^{(1)}=\min_{\lambda\in\sigma(\mathscr{A}[\alpha])}|\lambda_i-\lambda|-(||\mathscr{E}[\alpha]||_2+||\mathscr{E}[\alpha']||_2)$ and $\delta_i^{(2)}=\min_{\lambda\in\sigma(\mathscr{A}[\alpha'])}|\lambda_i-\lambda|-2||\mathscr{E}[\alpha']||_2$.*

Proof. Consider the perturbation $\mathscr{E}$.

(1) If $\mathscr{E}$ is irreducible, there is nothing to do because we can take α to be empty. Thus $||\mathscr{E}[\alpha']||_2=||\mathscr{E}||_2$, from (1.3) we have $\delta_i^{(2)}\le 0$ such that $\omega(\delta_i^{(2)})=\tilde{\omega}(\delta_i^{(2)})=1$ and so $\gamma_{22}=1$. Obviously, the bound (3.4) reduces to Weyl's bound.

(2) If $\mathscr{E}$ is reducible, then there exists a nonempty subset α of $\langle n\rangle$ such that

$$e_{ij}=0\,,\qquad i\in\alpha\,,j\in\alpha'\,.$$

Since $\mathscr{E}$ is Hermitian, there is a permutation matrix $\mathscr{P}$ such that

$$\mathscr{P}\mathscr{E}\mathscr{P}^H=\begin{pmatrix}E_{11}&0\\0&E_{22}\end{pmatrix}, \tag{3.5}$$

where $E_{11}=\mathscr{E}[\alpha]\in\mathscr{C}^{p\times p}$ and $E_{22}=\mathscr{E}[\alpha']\in\mathscr{C}^{k\times k}$. Without loss of generality, we may assume $||\mathscr{E}[\alpha]||_2\le||\mathscr{E}[\alpha']||_2$, for otherwise we may interchange the position of α and α'. Let us partition $\mathscr{P}\mathscr{A}\mathscr{P}^H$ into the following form:

$$\mathscr{P}\mathscr{A}\mathscr{P}^H=\begin{pmatrix}A_{11}&A_{12}\\A_{12}^H&A_{22}\end{pmatrix}, \tag{3.6}$$

where $A_{11}=\mathscr{A}[\alpha]\in\mathscr{C}^{p\times p}$ and $A_{22}=\mathscr{A}[\alpha']\in\mathscr{C}^{k\times k}$, $A_{12}=\mathscr{A}[\alpha|\alpha']$. Then the result follows immediately from Theorem 3.1. □

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Remark 3.2. If α is empty, from the proof of Theorem 3.2 the bound (3.4) reduces to the bound (1.3), so the bound (3.4) is no improvement to the Weyl bound. However, if α is nonempty then from (3.4) and $0 \leq \gamma_{22} \leq 1$ we have

$$\begin{aligned} |\lambda_i - \widetilde{\lambda}_i| \leq & ||\mathscr{E}[\alpha]||_2 + \Big(||\mathscr{E}[\alpha']||_2 - ||\mathscr{E}[\alpha]||_2\Big)\gamma_{22} \\ \leq & ||\mathscr{E}[\alpha]||_2 + \Big(||\mathscr{E}[\alpha']||_2 - ||\mathscr{E}[\alpha]||_2\Big) \\ = & ||\mathscr{E}[\alpha']||_2 \,, \end{aligned}$$

from which one may see that the bound (3.4) is always sharper than the Weyl bound (1.3).

4. Numerical Experiments

We now compare results for our proposed perturbation bound with results for other earlier perturbation bounds. Let PbWeyl, PbL1, PbL2 and PbC denote the perturbation bounds in (1.3), (1.7), (1.8) and (1.12), respectively; and let PbNew denote our new perturbation bound (2.9) or (3.4). By diag(·) and tridiag(·) we denote a point-diagonal and a point-tridiagonal matrix, and by Diag(·) and Tridiag(·) a block-diagonal and a block-tridiagonal matrix, respectively — cf. Ref. [2] for more details. In addition, the MATLAB commands "sprandsym" and "sprand" that we use throughout are:

- $sprandsym(S)$, a sparse symmetric random matrix with the same structure as S; and
- $sprand(m,n,d)$, an $m \times n$ sparse random matrix with approximately $d*m*n$ nonzero entries.

Example 4.1. Let us consider the following boundary value problem from fluid mechanics, involving the Stokes equation [5]:

$$\begin{cases} -\triangle \mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \,, \\ \operatorname{div} \mathbf{u} = 0 & \text{in} \quad \Omega \,, \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega \,, \end{cases} \tag{4.1}$$

where $\Omega = (-1,1) \times (-1,1)$ is a two-dimensional square domain with boundary $\partial\Omega$, $\triangle$ denotes the Laplacian, ∇ the gradient operator, and $\mathbf{f}$, $\mathbf{u}$ and p are respectively the external force, the fluid velocity and the fluid pressure.

From the Incompressible Flow Iterative Solution Software (IFISS) package in Ref. [5], we run the driver *stokes-testproblem* to the test problem "Channel domain" with default options other than discretisation form and grid size. A generalised saddle point (**GSP**) matrix with the form in (1.1) is obtained when we implement the $Q_1 - P_0$ stabilised discretisation. The test matrix is of size 3202×3202. Let $E_{11} = \varepsilon * sprandsym(A_{11})$, $E_{12} = \varepsilon * sprand(p,k,0.1)$ and $E_{22} = \varepsilon * sprandsym(A_{22})$. Then $\mathscr{E}$ is generated randomly with $\varepsilon = 1e-6$.

As shown in Fig. 1 and Table 1, PbL1-PbNew, PbL2-PbNew and PbC-PbNew are positive, and our new perturbation bound (2.9) is sharper than those in (1.7), (1.8) and (1.12).

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Table 1: Calculated differences between perturbation bounds in associated intervals for Example 4.1.

	$(-\infty,0)$	0	$(0,\infty)$
PbL1-PbNew(2.9)	0	656	2502
PbL2-PbNew(2.9)	0	1056	2146
PbC-PbNew(2.9)	0	0	3202
PbWeyl-PbNew(3.4)	1060	0	2142
PbL1-PbL2	2146	0	1056
PbL1-PbC	3176	0	26
PbL2-PbC	1056	0	2146
PbL1-PbWeyl	2142	0	1060
PbL2-PbWeyl	0	0	3202
PbC-PbWeyl	2141	0	1061

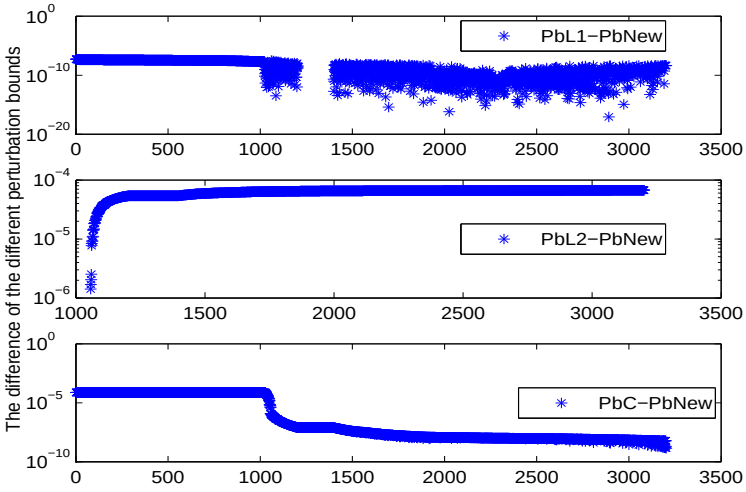


Figure 1: The difference between the new perturbation bound (2.9) and existing bounds for Example 4.1, where a logarithmic (base 10) scale is used on the Y-axis.

Indeed, the new perturbation bound outperforms the bounds in Ref. [9] for at least two-thirds of eigenvalues, and the bound in Ref. [4] for all eigenvalues. Fig. 2 and Table 1 reveal that our new bound is sharper than the Weyl bound for about two-thirds of the eigenvalues, but not for the rest of the eigenvalues. Table 1 also serves to compare the other perturbation bounds, where it can be seen that PbL1 and PbC denoting the respective bounds (1.7) and (1.12) improve the Weyl bound (1.3) for about two-thirds of eigenvalues. However, it is difficult to compare the bounds (1.7) and (1.12) from the data here.

Example 4.2. Let us consider another boundary value problem involving second-order partial differential equations [2]:

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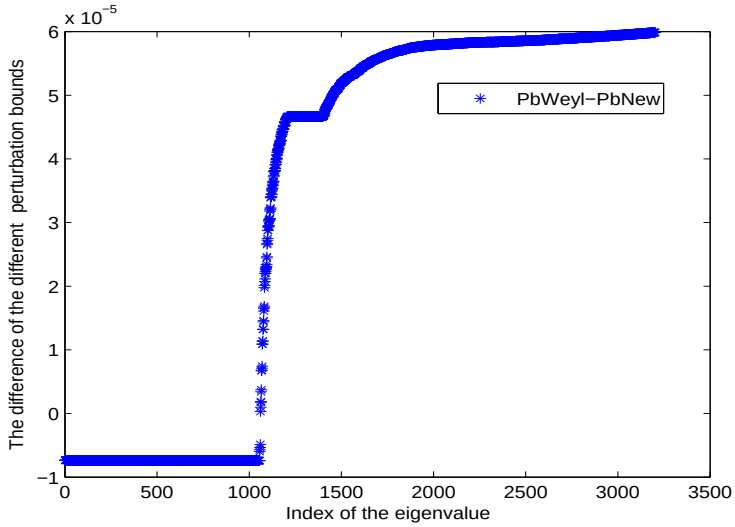


Figure 2: The difference between the new perturbation bound (2.9) and the Weyl bound (1.3) for Example 4.1.

$$\begin{cases} -\alpha\triangle\mathbf{u}-\nabla p=f(\xi,\eta) & \text{in} \quad \Omega\,, \\ \operatorname{div}\mathbf{u}-\sigma(\xi,\eta)p=g(\xi,\eta) & \text{in} \quad \Omega\,, \\ \mathbf{u}=0 & \text{on} \quad \partial\Omega\,, \\ p=0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{4.2}$$

where $\Omega=(0,1)\times(0,1)\subset\mathbb{R}^2$ is a square domain on the (ξ,η)-plane with boundary $\partial\Omega$, $\triangle$ denotes the Laplacian and ∇ is the gradient operator, α is a positive constant, $\mathbf{u}=(u_\xi,u_\eta)^T$, and $f(\xi,\eta)$ and $g(\xi,\eta)$ are bounded nonlinear functions.

If the partial differential equation (4.2) is discretised by central differences with uniform step size $h=1/(N+1)$, and the nodes in the domain are labeled by the natural ordering, we obtain the 3×3 block matrix (where here and below the H superscript denotes the Hermitian operation):

$$\mathscr{A}=\begin{pmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} \\ 0 & \hat{A}_{22} & \hat{A}_{23} \\ \hat{A}_{13}^H & \hat{A}_{23}^H & \hat{A}_{33} \end{pmatrix}, \tag{4.3}$$

involving the matrices $\hat{A}_{ij}\in\mathbb{R}^{N^2\times N^2}$ given by

$$\begin{aligned} &\hat{A}_{11}=\hat{A}_{22}=\text{ Tridiag}(-\alpha I,T,-\alpha I)\ \text{ with }\ T=\text{tridiag}(-\alpha,4\alpha,-\alpha),\\ &\hat{A}_{13}=\text{Diag}(S,S,\cdots,S)\ \text{ with }\ S=\frac{h}{2}\text{tridiag}(1,0,-1),\\ &\hat{A}_{23}=\frac{h}{2}\text{Tridiag}(I,0,-I),\ \ \hat{A}_{33}=\text{Diag}(T_1,T_2,\cdots,T_N)\ \text{ with}\\ &T_k=\text{diag}(\sigma_{1,k},\sigma_{2,k},\cdots,\sigma_{N,k}),\quad k=1,2,\cdots,N \end{aligned}$$

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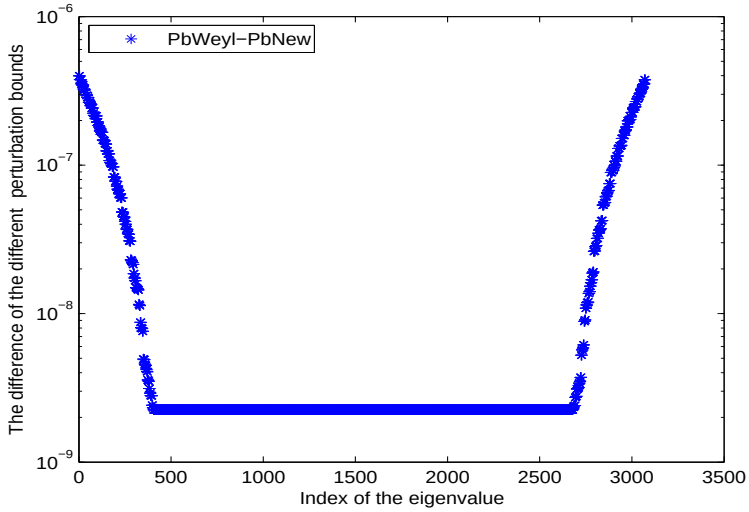


Figure 3: The difference between the new perturbation bound (3.4) and the Weyl bound (1.3) for Example 4.2, where a logarithmic (base 10) scale is used on the Y-axis.

$\sigma_{l,k}$ denotes the value of $\sigma(\xi,\eta)=h^2e^{\xi^2+\eta^2}$ at the grid point (lh,kh), and $\alpha=0.001$. The perturbation $\mathscr{E}$ is generated randomly with the following form

$$\mathscr{E}=\begin{pmatrix}\hat{E}_{11} & 0 & \hat{E}_{13}\\ 0 & \hat{E}_{22} & 0\\ \hat{E}_{13}^H & 0 & \hat{E}_{33}\end{pmatrix}, \tag{4.4}$$

where the matrix $\hat{E}_{ij}\in\mathbb{R}^{N^2\times N^2}$ for $i,j=1,2,3$, $\hat{E}_{ii}=\varepsilon*sprandsym(\hat{A}_{ii})$ for $i=1,2,3$, and $\hat{E}_{13}=\varepsilon*sprandsym(\hat{A}_{13})$ with $\varepsilon=1e-5$. It is easy to check that there is a permutation matrix

$$\mathscr{P}=\begin{pmatrix}0 & 0 & I\\ I & 0 & 0\\ 0 & I & 0\end{pmatrix}$$

such that $\mathscr{P}\mathscr{E}\mathscr{P}^H$ and $\mathscr{P}\mathscr{A}\mathscr{P}^H$ have the forms in (3.5) and (3.6) respectively, where

$$E_{11}=\begin{pmatrix}\hat{E}_{33} & \hat{E}_{13}^H\\ \hat{E}_{13} & \hat{E}_{11}\end{pmatrix},\quad E_{22}=\hat{E}_{2,2}\,,$$
$$A_{11}=\begin{pmatrix}\hat{A}_{33} & \hat{A}_{13}^H\\ \hat{A}_{13} & \hat{A}_{11}\end{pmatrix},\quad A_{12}=\begin{pmatrix}\hat{A}_{23}^H\\ 0\end{pmatrix},\quad A_{22}=\hat{A}_{22}\,.$$

Fig. 3 shows that our new perturbation bound (3.4) always outperforms the Weyl bound for this example.

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