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ON DEFINABLE SKOLEM FUNCTIONS IN WEAKLY O-MINIMAL NONVALUATIONAL STRUCTURES

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Abstract. We prove that all known examples of weakly o-minimal nonvaluational structures have no definable Skolem functions. We show, however, that such structures eliminate imaginaries up to definable families of cuts. Along the way we give some new examples of weakly o-minimal nonvaluational structures.

§1. Introduction. A fundamental application of o-minimal cell decomposition is the fact that o-minimal expansions of groups admit definable choice, implying—among others—the existence of atomic models, elimination of imaginaries and curve selection.

In the present paper we study the analogous properties in the context of weakly o-minimal structures. Recall that a structure $\mathcal{M} = \langle M, <, \ldots \rangle$ is *weakly o-minimal* if $<$ is a dense linear order and every definable subset of M is a finite union of convex sets. Weakly o-minimal structures were introduced by Cherlin-Dickmann [3] in order to study the model theoretic properties of real closed rings. They were later also used in Wilkie's proof of the o-minimality of real exponential field [15], as well as in van den Dries' study of Hausdorff limits [11]. Macpherson-Marker-Steinhorn [6], followed-up by Wencel [12, 14], began a systematic study of weakly o-minimal groups and fields, revealing many similarities with the o-minimal setting.

An important dichotomy between valuational structures—those admitting a definable proper nontrivial convex subgroup—and nonvaluational ones arose, supported by good evidence that the latter structures resemble o-minimal structures more closely than what the former ones do. For example, strong monotonicity and strong cell decomposition theorems were proved for nonvaluational structures. As a weakly o-minimal expansion of an ordered group cannot admit a definable choice function for the cosets of a nontrivial proper convex sub-group, our study is immediately restricted to the nonvaluational case.

The first results of the present note unravel a discrepancy between the weakly o-minimal nonvaluational setting and the strictly o-minimal setting: we show that in all known (to us) weakly o-minimal nonvaluational expansions of ordered groups definable Skolem functions do not exist. We conjecture that, in fact, no strictly weakly o-minimal expansion of an ordered group admits definable Skolem functions.

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The nonexistence of definable Skolem functions is proved, essentially, by contradicting (a possible generalisation of) curve selection in *o-minimal traces* (see Definitions 1.1 and 1.3 below). On the positive side, however, we prove that, using different techniques, elimination of imaginaries—to a certain extent—can still be obtained.

Weakly o-minimal structures arise naturally as expansions of o-minimal structures by externally definable sets [1]. Among those, *dense pairs* give rise to nonvaluational structures, motivating the following definition:

DEFINITION 1.1. A structure $\mathcal{M}$ is an *o-minimal trace* if there exists an o-minimal expansion of a group, $\mathcal{N}_0$, in a language $\mathcal{L}_0$ such that $M \subsetneq N_0$ is dense in N_0, $\mathcal{M}|\mathcal{L}_0 \prec \mathcal{N}_0$ and $\mathcal{M}$ is the structure induced on M from $\mathcal{N}_0$.

In other words, $\mathcal{M}$ is an o-minimal trace if there exists a dense pair $(\mathcal{N}_0, \mathcal{M}_0)$ of o-minimal structures (see [10] for details) and $\mathcal{M}$ is $\mathcal{M}_0$ expanded by all $\mathcal{N}_0$-definable sets. In particular no o-minimal trace is o-minimal. To the best of our knowledge all known examples of weakly o-minimal expansions of ordered groups are o-minimal traces. In particular, in Section 2 we prove:

PROPOSITION 1.2. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, $\mathcal{M}'$ any expansion of $\mathcal{M}$ by cuts. Then $\mathcal{M}'$ is nonvaluational if and only if it is an o-minimal trace.*

Section 3 is dedicated to the construction of an example of a weakly o-minimal nonvaluational ordered group that is not an o-minimal trace (or even a reduct of an o-minimal trace).

The key definition to our analysis of definable Skolem functions is the following.

DEFINITION 1.3. A weakly o-minimal structure $\mathcal{M}$ *has no external limits* if for every definable $f : (a, b) \to M$ where $a, b \in M \cup \{\pm\infty\}$, the limits $\lim_{t\to a^+} f(t)$ or $\lim_{t\to a^+} f(t)$ exist in $M \cup \{\pm\infty\}$.

We recall that by the previously mentioned *strong monotomicity* (or see [12, Lemma 1.3]) the above limits always exists in the Dedekind completion of M, justifying the above terminology. Of course, all o-minimal structures have no external limits, but beyond that we only know:

PROPOSITION 1.4. *O-minimal traces have no external limits.*

The connection with definable Skolem functions is established by the following proposition.

PROPOSITION 1.5. *A weakly o-minimal nonvaluational expansion of an ordered group is o-minimal if and only if it has no external limits and, after naming a nonzero constant, admits definable Skolem functions.*

From this we conclude that o-minimal traces do not have definable Skolem functions. In fact, since having no external limits is preserved under elementary equivalence and under the passage to ordered reducts, we obtain a stronger result:

THEOREM 1.6. *If $\mathcal{M}$ is elementarily equivalent to an ordered group reduct*[1] *of an o-minimal trace, then $\mathcal{M}$ has no definable Skolem functions.*

The following statement is appealing and appears to be open.

[1] This means that the order and the group structure are preserved in the reduct.

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CONJECTURE 1.7. *No strictly weakly o-minimal nonvaluational expansion of an ordered group has definable Skolem functions.*

This conjecture in particular would imply that no strictly weakly o-minimal structure has definable choice. The above conjecture would follow from an affirmative answer to the next question, which also appears to be open.

QUESTION 1.8. *Is every weakly o-minimal nonvaluational structure elementarily equivalent to a reduct of an o-minimal trace*?

A better understanding of the notion of external limits and generalisations of the theory of dense pairs are also relevant topics. We postpone them for future work.

Section 4 of our note is dedicated to the study of atomic models and elimination of imaginaries in the weakly o-minimal nonvaluational setting. We start with a hands-on proof of:

PROPOSITION 1.9. *Let $\mathcal{M}$ be an ordered group reduct of an o-minimal trace. Then* $\mathrm{Th}(\mathcal{M})$ *does not have atomic models over arbitrary sets.*

On the positive side, we show:

THEOREM 1.10. *Let $\mathcal{M}$ be a weakly o-minimal nonvaluational structure. Then $\mathcal{M}$ eliminates imaginaries up to $\emptyset$-definable families of definable cuts.*

We note that while this paper was being revised, an independent work Laskowski-Shaw [5] appeared, studying weakly o-minimal expansions of ordered groups by a single cut. They show that the structure has definable Skolem functions if and only if it is valuational.

§2. No definable Skolem functions. The present section is dedicated to proving that ordered group reducts of o-minimal traces have no definable Skolem funcitions. For convenience we will assume, throughout, that all groups are equipped with a constant $1 > 0$. We start with a few preliminaries.

As usual, if $\mathcal{M}$ is an ordered structure, a *cut* in $\mathcal{M}$ is a partition $M = C \cup D$, where $C < D$. A cut (C, D) is *definable* if C is. As a matter of convention, throughout this paper we will assume that if (C, D) is a cut in M then C has no maximum. With these conventions we let $\overline{M}$ be the set of all $\mathcal{M}$-definable (with parameters) cuts ordered by $(C, D) < (C', D')$ if $C' \cap D \neq 0$. Throughout we will implicitly identify M with a subset of $\overline{M}$ via $a \mapsto ((-\infty, a), [a, \infty))$.

Following the terminology of [7], a cut (C, D) is *rational* if D has a minimal element, and *irrational* otherwise. If $\mathcal{M}$ expands an ordered group a cut (C, D) is *nonvaluational* if $\inf\{y - x : x \in C, y \in D\}$ exists (in the sense of the structure $\mathcal{M}$, of course). As pointed out to us by M. Kamensky, if the group is p-divisible for some p then such an infimum, if it exists, must be 0. As a canonical example, if G is a definable proper, nontrivial, convex subgroup of M then the formula $x > G$ defines a *valuational* cut. Indeed, this is the typical case, as shows the following easy observation:

REMARK 2.1. Let $\mathcal{M}$ be a weakly o-minimal expansion of a group. Then $\mathcal{M}$ is nonvaluational if and only if no definable cut is valuational.

2.1. Nonvaluational expansions of o-minimal groups. As already mentioned, it follows from [1] that any expansion of an o-minimal structure by externally definable sets is weakly o-minimal. Since any set of cuts over an o-minimal structure is realised

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in some elementary extension, any expansion of an o-minimal structure by cuts is weakly o-minimal.

We start by sharpening Remark 2.1. Towards that end we remind that in [7, Lemma 2.2] it is shown that if $\mathcal{M}$ is o-minimal, and a realizes an irrational cut over M then no $b \in \mathcal{M}(a) \setminus \mathcal{M}$ realizes a rational cut over $\mathcal{M}$. The following lemma is an analogue for nonvaluational cuts. It follows from [13, Theorem 2.11], but we provide a succinct proof.

LEMMA 2.2. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Let (C, D) be an irrational nonvaluational cut over M and $C < a < D$ any realization. Then $\mathcal{M}(a)$ does not realize any valuational cuts over M.*

PROOF. Let $b \in \mathcal{M}(a)$ be any element. Then there exists an $\mathcal{M}$-definable function f such that $b = f(a)$. By o-minimality and the fact that $a \notin M$ (because (C, D) is an irrational cut) there is an $\mathcal{M}$-definable interval I containing a such that f is continuous and strictly monotone or constant on I. If f is constant on I then $b \in M$ and there is nothing to prove. So we may assume without loss of generality that f is strictly increasing. Restricting I, if needed, we may also assume that I is closed and bounded. So f is uniformly continuous on I.

By [7, Lemma 2.2] the type $p \in S_1(M)$ of a positive infinitesimal element is not realized in $\mathcal{M}(a)$. It follows, since (C, D) is nonvaluational, that for any $c \in C(\mathcal{M}(a))$, $d \in D(\mathcal{M}(a))$ and $0 < \delta \in \mathcal{M}(a)$ there are $c < c' < d' < d$ with $c' \in C(\mathcal{M})$ and $d' \in D(\mathcal{M})$ with $d' - c' < \delta$. Given any $0 < \epsilon \in \mathcal{M}(a)$ take $0 < \epsilon' < \epsilon$ with $\epsilon' \in \mathcal{M}$ and let $\delta \in M$ be such that $|f(x) - f(y)| < \epsilon'$ for any $x, y \in I$ with $|x - y| < \delta$. So

$$\inf\{f(d) - f(c) : c \in C(\mathcal{M}(a)), d \in D(\mathcal{M}(a))\} = 0.$$

Thus, b realizes a nonvaluational cut over $\mathcal{M}$. Since $b \in \mathcal{M}(a)$ is arbitrary, this finishes the proof of the lemma. ⊣

We also need the following observation.[2]

PROPOSITION 2.3. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group and $\mathcal{C} = \{(C_i, D_i)\}_{i \in I}$ a collection of irrational nonvaluational cuts in $\mathcal{M}$, where I is an index set. Then there exists $\mathcal{M} \prec \mathcal{N}$ such that M is dense in N and $\mathcal{N}$ realizes all cuts in $\mathcal{C}$.*

PROOF. Let $\{(C_i, D_i)\}_{i \in I}$ be a collection of cuts as in the assumption, and $p \in S_1(M)$ the type of a positive infinitesimal. We construct $\mathcal{N}$ by induction as follows. For $i = 1$ let $\mathcal{M}_1 := \mathcal{M}(a_1)$ where $a_1 \models (C_1, D_1)$ is any realization. By [7, Lemma 2.2] p is not realized in $\mathcal{M}_1$. Let $a < b \in M_1$ be any elements. Then there exists $r \in M$ such that $0 < r < b - a$ (otherwise $b - a \models p$). By the previous lemma a realises a nonvaluational cut, so there exists $a' \in M$ such that $a' + \frac{r}{2} > a$. Thus $a' + \frac{r}{2} \in (a, b) \cap M$. Since a, b were arbitrary this shows that M is dense in M_1.

Assume now that for all $j < i$ we have constructed $\mathcal{M}_j$ such that (C_j, D_j) is realized in $\mathcal{M}_{j+1}$ and such that M is dense in M_j (so in particular, $\mathcal{M}_j$ does not realise p). If i is a successor ordinal we let $\mathcal{M}_{i,0} = \mathcal{M}_{i-1}$ and if i is limit we let $\mathcal{M}_{i,0} := \bigcup_{j<i} \mathcal{M}_i$. Note that as density is preserved under passing to the limit, by

[2] We thank Y. Peterzil for pointing out this formulation.

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induction, M is dense in $M_{i,0}$. Finally, if (C_i, D_i) is realized in $\mathcal{M}_{i,0}$ set $\mathcal{M}_i = \mathcal{M}_{i,0}$. Otherwise set $\mathcal{M}_i := \mathcal{M}_{i,0}(a_i)$ where $a_i \models (C_i, D_i)$ is any realization.

We prove that $\mathcal{M}_i$ does not realize p. If $\mathcal{M}_i = \mathcal{M}_{i,0}$ this follows from the induction hypothesis. So we assume that this is not the case. Thus $(C_i(\mathcal{M}_{i,0}), D_i(\mathcal{M}_{i,0}))$ defines a cut in $\mathcal{M}_{i,0}$, and as by induction M is dense in $M_{i,0}$ this cut is still nonvaluational. Thus, applying [7, Lemma 2.2] again the desired conclusion follows. As in the induction base, it follows that M is dense in M_i.

Setting $\mathcal{N} := \bigcup_{i\in I} \mathcal{M}_i$, by construction, $\mathcal{N}$ realizes all $(C_i, D_i)_{i\in I}$ and, by induction M is dense in N, as required. ⊣

From the above result we can deduce the following which, though not needed in this paper, may be of independent interest:

COROLLARY 2.4. *Let $\mathcal{M}$ be an o-minimal expansion of a group. Then there exists a maximal elementary extension $\mathcal{M} \prec \mathcal{N}$ such that M is dense in N.*

PROOF. Let $(C_i, D_i)_{i\in I}$ enumerate all nonvaluational irrational cuts in M. Let $\mathcal{N}$ be the structure provided by the previous proposition with respect to this collection of cuts.

Let $\mathcal{N}_1 \succ \mathcal{N}$ be any proper extension and $a \in N_1 \setminus N$. Let $p := \mathrm{tp}(a/N)$. We may assume that N is dense in N_1 (otherwise M is certainly not dense in N_1 and we have nothing to prove). So p is nonvaluational. Since M is dense in N it is also nonvaluational in M. If p is irrational over N it is irrational over M, and so realised in N, which is impossible. So p is rational, and there exists $a' \in N$ such that p is an infinitesimal near a'. But then, say, $(a' - a) \cap N = \emptyset$, so N_1 is not dense in M. ⊣

Returning to our main argument we can now deduce Proposition 1.2.

PROPOSITION 2.5. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Then any expansion of $\mathcal{M}$ by nonvaluational cuts is an o-minimal trace.*

PROOF. Let $\widetilde{\mathcal{M}}$ be the expansion of $\mathcal{M}$ by unary predicates $\{C_i\}_{i\in I}$ interpreted as distinct irrational nonvaluational cuts in M. We have to show that there exists an elementary extension $\mathcal{M} \prec \mathcal{N}$ such that M is dense in N and $\widetilde{\mathcal{M}}$ is precisely the structure induced on $\mathcal{M}$ by all externally definable subsets from $\mathcal{N}$.

Let $\mathcal{N}$ be as in Proposition 2.3, realizing all C_i. Then $\mathcal{M} \prec \mathcal{N}$ and M is dense in N, so $(\mathcal{N}, \mathcal{M})$ is a dense pair. By [10, Theorem 2] the structure induced on $\mathcal{M}$ in the pair $(\mathcal{N}, \mathcal{M})$ is precisely the expansion of $\mathcal{M}$ by unary predicates for all cuts realized in $\mathcal{N}$. Thus, by construction, we get that $\widetilde{\mathcal{M}}$ is a reduct of the structure induced on $\mathcal{M}$ from $(\mathcal{N}, \mathcal{M})$. So it remains to show that any cut over M definable in $(\mathcal{N}, \mathcal{M})$ is definable in $\widetilde{\mathcal{M}}$.

So let $a \in N$ be any element. We have to show the $(-\infty, a) \cap M$ is definable in $\widetilde{\mathcal{M}}$. By construction there are $a_1, \ldots, a_n$ realizing the cuts $C_{i_1}, \ldots, C_{i_n}$ and an $\mathcal{M}$-definable continuous function, f, such that $f(a_1, \ldots, a_n) = a$. Choose $a_1, \ldots, a_n$ and f so that n is minimal possible. For every $\eta \in \{-1, 1\}^n$ say that f is of type η at a point $\bar{c} \in N^n$ if

$$f_i(x_i) := f(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_n)$$

is strictly monotone at c_i and for all $1 \leq i \leq n$ and $f_i(x_i)$ is increasing at c_i if and only if $\eta(i) = 1$. By the minimality of n there is some $\eta \in \{-1, 1\}^n$ such that f is of type η at $(a_1, \ldots, a_n)$. In particular, the set F_η of points $\bar{x}$ such that f is of type η at x is $\mathcal{M}$-definable with nonempty interior. Let $L_i := C_i$ if $\eta(i) = -1$ and $L_i := D_i$

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(the complement of C_i) otherwise, then $L := \prod_{i=1}^{n} L_i \cap F_\eta$ has nonempty interior and definable in $\widetilde{M}$, and for any $\bar{x} \in L$ we have that $f(\bar{x}) < f(a_1, \ldots, a_n) = a$. Since M is dense in N and $L(N)$ is not empty also $L(M)$ is not empty. Thus

$$x \in (-\infty, a) \iff \exists y (y \in L \wedge x < f(y))$$

and the right hand side is $\widetilde{M}$-definable. ⊣

2.2. No external limits. The key to proving Proposition 1.5 is the fact that o-minimal traces do not have external limits (Proposition 1.4). We will need the following fact (which follows from [4, Lemma 1.3]).

FACT 2.6. *Let $\mathcal{M} \prec \mathcal{N}$ be o-minimal structures. If $f : N \to N$ is a definable function such that $f(a) \in M$ for all $a \in M$, then there are finitely many intervals $\{I_i\}_{i=1}^{k}$ in N, and functions $f_i : I_i \to N$ definable over M, such that $M \setminus \bigcup_{i=1}^{k} (I_i \cap M)$ is finite and $f|I_i = f_i$.*

PROOF. Let $\mathcal{D}$ be the collection of all intervals D_i provided by [4, Lemma 1.3]. So $N \setminus \bigcup_{D \in \mathcal{D}} D$ is finite. Let $\mathcal{I} \subseteq \mathcal{D}$ be the sub-collection of those intervals in $\mathcal{D}$ that have infinite intersection with M. Since $f(M) \subseteq M$, by [4, Lemma 1.3(ii)], for every $I_i \in \mathcal{I}$, there is f_i definable over M, such that $f|I_i = f_i$. ⊣

We can now prove a slightly stronger statement than Proposition 1.4.

LEMMA 2.7. *Let $\mathcal{M} \prec \mathcal{N}$ be o-minimal structures. Let $\mathcal{M}_1$ be the structure induced on $\mathcal{M}$ by all $\mathcal{N}$-definable sets. Then $\mathcal{M}_1$ has no external limits.*

PROOF. If $a < b$ are elements in M and $f : (a, b) \cap M \to M$ is an $\mathcal{M}_1$-definable function then by [1] there exists an $\mathcal{N}$-definable function $F : (a, b) \to N$ such that $F|(a, b) \cap M = f$. Extend F to an $\mathcal{N}$-definable map $\hat{F} : N \to N$ such that $\hat{F}(M) \subseteq M$ (by letting, for example, $\hat{F}(x) = 0$, if $x \notin (a, b)$). By Fact 2.6, there is an interval $I \subseteq N$ containing some end-segment $(c, b) \cap M$ of $(a, b) \cap M$, and a function $G : I \to N$ definable over M, such that $\hat{F}|I = G$. Since $\mathcal{M} \prec \mathcal{N}$, the restriction of G to M is an $\mathcal{M}$-definable function $g : I \cap M \to M$. Moreover, since f and g agree on $(c, b) \cap I \cap M$, we have $\lim_{x \to b} f(x) = \lim_{x \to b} g(x)$. By o-minimality of $\mathcal{M}$ we know that $\lim_{x \to b} g(x) \in M \cup \{\pm\infty\}$, proving the claim. ⊣

As a special case, we obtain Proposition 1.4.

COROLLARY 2.8. *Any expansion of an o-minimal group by externally definable sets has no external limits. In particular*:

(1) *Every o-minimal trace has no external limits.*
(2) *There are weakly o-minimal valuational structures, e.g., RCVF, with no external limits.*

REMARK 2.9. Corollary 2.8(2) answers negatively a question of Peterzil (private communication) who asked whether having no external limits is characteristic of nonvaluational structures.

There is an explicit connection between definable nonvaluational cuts, definable Skolem functions and no external limits. For an existential formula $\varphi(x) := \exists y \psi(x, y)$, a function f is called a *Skolem function for* φ, if for all x with $\exists y \psi(x, y)$, we have $\psi(x, f(x))$.

LEMMA 2.10. *Let $\mathcal{M}$ be a weakly o-minimal expansion of an ordered group, C an $\mathcal{M}$-definable irrational nonvaluational cut such that $0 \in C$. Then the formula*

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$$\varphi_C(x) := \exists y(x > 0 \wedge y \in C \wedge x + y \notin C)$$

has a definable Skolem function only if $\mathcal{M}$ has external limits.

PROOF. Because C is nonvaluational, $\mathcal{M} \models \varphi_C(a)$ for all $a > 0$. Thus, if $f : (0, \infty) \to M$ is a definable Skolem function for $\varphi_C(x)$ we must have $\lim_{x \to 0} f(x) = \sup(C)$. Because C is irrational, if f is definable it witnesses that $\mathcal{M}$ has external limits. ⊣

We have thus proved Proposition 1.5.

PROPOSITION 2.11. *Let $\mathcal{M}$ be a weakly o-minimal nonvaluational expansion of an ordered group. Then $\mathcal{M}$ is o-minimal if and only if it has no external limits and $\mathcal{M}$ admits definable Skolem functions.*

PROOF. The left-to-right direction is well-known (see, for example, [12, Chapter 6]. In the other direction, if $\mathcal{M}$ is not o-minimal it has at least one definable (irrational) nonvaluational cut, C. Since by assumption $\mathcal{M}$ has no external limits, by Lemma 2.10 $\mathcal{M}$ has no definable Skolem functions. ⊣

In the context of o-minimal traces we can give a more precise statement, Theorem 1.6.

COROLLARY 2.12. *If $\mathcal{M}$ is elementarily equivalent to an ordered group reduct of an o-minimal trace then $\mathcal{M}$ has definable Skolem functions if and only if it is o-minimal.*

PROOF. By Corollary 2.8 o-minimal traces have no external limits. Since both having no external limits and being weakly o-minimal nonvaluational are elementary and preserved under reducts $\mathcal{M}$ has no external limits, the result follows from Proposition 2.11. ⊣

For reducts of o-minimal traces we can show an even stronger result:

PROPOSITION 2.13. *Let $\mathcal{M}$ be an ordered group reduct of an o-minimal trace. Then no expansion of $\mathcal{M}$ by externally definable sets has definable Skolem functions, unless $\mathcal{M}$ is o-minimal.*

Before proceeding to the proof of the proposition we need some handle over externally definable sets in the context of (reducts of) o-minimal traces. First, we need some terminology.

DEFINITION 2.14. An o-minimal structure $\widetilde{\mathcal{M}}$ in a signature $\mathcal{L}$ *witnesses that $\mathcal{M}$ is an o-minimal trace*, if $(\widetilde{\mathcal{M}}, \mathcal{M}|\mathcal{L})$ is a dense pair of o-minimal structures and $\mathcal{M}$ is the structure induced on M from $\widetilde{\mathcal{M}}$.

LEMMA 2.15. *Let $\mathcal{M}$ be an o-minimal trace witnessed by $\widetilde{\mathcal{M}}$ in the signature $\mathcal{L}$. Then any expansion of $\mathcal{M}$ by externally definable sets is definable in the (nondense) pair $(\widetilde{\mathcal{N}}, \mathcal{M}|\mathcal{L})$ where $\widetilde{\mathcal{N}} \succ \widetilde{\mathcal{M}}$ is saturated.*

PROOF. Let $(\widetilde{\mathcal{N}_1}, \widetilde{\mathcal{N}_0}) \succ (\widetilde{\mathcal{M}}, \mathcal{M}|\mathcal{L})$ be saturated. So the induced structure on N_0 in the pair is a saturated model of $\mathrm{Th}(\mathcal{M})$. Let us denote this structure $\mathcal{N}$.

By [1] every definable set in $\mathcal{N}$ is of the form $D \cap N_0^k$ for some $\widetilde{\mathcal{N}_1}$-definable set D. So any externally definable set in $\mathcal{M}$ is of the form $D \cap M^k$ for some $\widetilde{\mathcal{N}_1}$-definable set D. This is what we needed. ⊣

The above lemma generalises automatically to the case where $\mathcal{M}$ is an ordered group reduct of an o-minimal trace. So we are reduced to proving the following statement.

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LEMMA 2.16. *Let $\mathcal{M} \prec \mathcal{N}$ be o-minimal expansions of groups. Assume that there exists $c \in N \setminus M$ realising a nonvaluational cut C over M. Then no expansion of $\mathcal{M}$ by $\mathcal{N}$-definable sets has a definable Skolem function for the formula*

$$\varphi_C(x) := \exists y(x > 0 \wedge y \in C \wedge x + y \notin C)$$

PROOF. By Lemma 2.7 and Proposition 2.10. ⊣

Combining all the above observations we get a proof of Proposition 2.13:

PROOF OF PROPOSITION 2.13. Assume $\mathcal{M}$ is not o-minimal. Being nonvaluational it admits at least one definable nonvaluational irrational cut, C. By Lemma 2.15 any expansion of $\mathcal{M}$ by externally definable sets is a reduct of the structure induced on M in some o-minimal pair $(\mathcal{N}_0, \mathcal{M}_0)$. Since the structure induced on M in the pair is an expansion of $\mathcal{M}$ it admits at least one definable irrational nonvaluational cut. So by Lemma 2.16 this expansion has no definable Skolem functions. ⊣

REMARK 2.17. There are good reasons to believe that Proposition 2.13 extends to structures elementarily equivalent to such reducts. It seems, however, that to prove such a result more sophisticated techniques are required, going beyond the scope of the present paper.

We point out that the assumption in Proposition 2.11 of $\mathcal{M}$ expanding an o-minimal ordered group is necessary.

EXAMPLE 2.18. Let $\mathcal{R}$ be the structure obtained by appending two real closed fields one "on top" of the other. More precisely, the language is given by $(\leq, R_1, R_2, +_1, \cdot_1, +_2, \cdot_2)$ and the theory of $\mathcal{R}$ is axiomatised by:

(1) R_1, R_2 are unary predicates such that $(\forall x)(R_1(x) \leftrightarrow \neg R_2(x))$ and $(\forall x, y)(R_1(x) \wedge R_2(y) \rightarrow x < y)$.
(2) $+_i, \cdot_i$ are ternary relations supported only on triples of elements in R_i. They are graphs of functions on their domains, and R_i is a real closed field with respect to these operations.
(3) $\leq$ is an order relation compatible with the field ordering of R_1 and R_2 together with (1) above.

It follows immediately from quantifier elimination for real closed fields, that the above theory is complete and has quantifier elimination (after adding constants for 0, 1 in both fields, and relation symbols for the inverse function in both fields). Thus $\mathcal{R}$ is weakly o-minimal, and the only definable cut in $\mathcal{R}$ not realized in $\mathcal{R}$ is $(R_1(R), R_2(R))$. However, $\mathcal{R}$ does have external limits. Take the function $x \mapsto x^{-1}$ in the field structure on R_1 on the interval $(0, 1)$. Clearly its limit, as $x \to 0^+$ is the element in the Dedekind completion of $\mathcal{R}$ that realizes the cut $(R_1(R), R_2(R))$, and hence not in R.

As a note, the boolean algebra of definable subsets of R^n (any n) is the boolean algebra generated by sets of the form $S_1 \times S_2$ where $S_i \subseteq R_i^{n_i}$ are semi-algebraic sets with $n_1 + n_2 = n$ and closing under the natural action of $\mathrm{Sym}(n)$. It follows that $\mathcal{R}$ has definable Skolem functions.

§3. A new example. Our initial approach to proving Conjecture 1.7 was to verify whether all weakly o-minimal nonvaluational structures are o-minimal traces. As it turns out, the class of o-minimal traces is not closed under taking ordered group reducts. The present section is dedicated to an example, $\mathbb{Q}_{vs}^{\pi}$, of a weakly o-minimal

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expansion of the ordered group of rational numbers which is not a reduct of an o-minimal trace. However, $\mathbb{Q}^{\pi}_{vs}$ is elementarily equivalent to a reduct, $\mathbb{R}^{\pi}_{\text{alg}}$, of an o-minimal trace. This shows that the class of reducts of o-minimal traces is not elementary. We do not know the answer to the analogous question for o-minimal traces.

We first construct $\mathbb{R}^{\pi}_{\text{alg}}$. Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_{\text{alg}}$ the real closure of $\mathbb{Q}$. Then $(\mathbb{R}, \mathbb{R}_{\text{alg}})$ is a dense pair. Let $\widetilde{\mathbb{R}_{\text{alg}}}$ be the structure on $\mathbb{R}_{\text{alg}}$ induced from $\mathbb{R}$. Let

$$P = \{(x, y) \in \mathbb{R}^2_{\text{alg}} : y < \pi x\}$$

and

$$\mathbb{R}^{\pi}_{\text{alg}} = (\mathbb{R}_{\text{alg}}, \leq, 0, 1, +, P).$$

The structure $\mathbb{R}^{\pi}_{\text{alg}}$ is a weakly o-minimal nonvaluational expansion of an ordered group with no external limits, being a reduct of $\widetilde{\mathbb{R}_{\text{alg}}}$, which has these properties by virtue of being an o-minimal trace. Our first goal is to prove a quantifier elimination result for the theory of $\mathbb{R}^{\pi}_{\text{alg}}$ (Proposition 3.3 below).

LEMMA 3.1. *For all $\alpha \in \mathbb{Q}(\pi)$ the relation $\alpha x < y$ is $\emptyset$-definable in $\mathbb{R}^{\pi}_{\text{alg}}$. Moreover, for $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}(\pi)$ linearly independent over $\mathbb{Q}$ the relation $\sum_{i=1}^{n} \alpha_i x_i < 0$ is $\emptyset$-definable.*

PROOF. Abusing terminology, we will say that $\alpha \in \mathbb{Q}(\pi)$ is definable in $\mathbb{R}^{\pi}_{\text{alg}}$ if the relation $\alpha x < y$ is. We show that if $\alpha, \beta \in \mathbb{Q}(\pi)$ are definable in $\mathbb{R}^{\pi}_{\text{alg}}$ then so is $\alpha + \beta$ and that π^n is definable for all $n \in \mathbb{Z}$. Indeed, if α, β are definable by S_α, S_β then $\alpha + \beta$ is defined by

$$(\exists z_1, z_2)(S_\alpha(x, z_1) \wedge S_\beta(x, z_2) \wedge y > z_1 + z_2).$$

If $x \mapsto \pi^n x$ is definable by $P_n(x, y)$ then $x \mapsto \pi^{n+1} x$ is defined by

$$(\exists z)(P_n(x, z) \wedge P_1(z, y).$$

So to conclude the first part of the lemma it remains only to note that $P_{-1}(x, y)$ is given by $P_1(y, x)$.

For the second part of the lemma we will show that if $n > 1$ and $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ then:

$$\sum_{i=1}^{n} \alpha_i x_i < 0 \iff (\forall x'_1, \ldots, x'_n) \left(\bigwedge_{i=1}^{n} x'_i < \alpha_i x \to \sum_{i=1^n} x'_i < 0 \right)$$

unless $x_i = 0$ for all i. The left-to-right direction is clear, so we have to show the other implication. The assumption implies $\sum_{i=1}^{n} \alpha_i x_i \leq 0$, so we only have to check that equality cannot hold. First, observe that since the x_i are not all 0, we may assume—by induction on n—that the x_i are linearly independent over $\mathbb{Q}$. Indeed, if $x_1 = \sum_{i=2}^{n} q_i x_i$ for $q_i \in \mathbb{Q}$ we get

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 \sum_{i=2}^{n} q_i x_i + \sum_{i=2}^{n} \alpha_i x_i = \sum_{n=2}^{n} (q_i \alpha_1 + \alpha_i) x_i$$

and as $\{(q_i \alpha_1 + \alpha_i)\}_{i=2}^{n}$ are still independent over $\mathbb{Q}$ the claim follows. Now α_i are polynomials in π with rational coefficients and x_i are real algebraic numbers, so we can write $\sum_{i=1}^{n} \alpha_i x_i = \sum_{i=0}^{k} \beta_i \pi^i = 0$ where β_i are $\mathbb{Q}$-linear combinations of

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$x_1, \ldots, x_n$. So $\beta_i = 0$ for all i if and only if the α_i are all 0, which is impossible, since they are linearly independent. ⊣

The above lemma can be restated as follows:

COROLLARY 3.2. *Let $\mathcal{L}^\pi$ be the language of ordered $\mathbb{Q}$-vector spaces expanded by n-ary predicates for the relations $C_{\bar{\alpha}}(\bar{x}) := \sum_{i=1}^n \alpha_i x_i < 0$ for all $n \in \mathbb{N}$ and $\alpha_i \in \mathbb{Q}(\pi)$. Then $C_{\bar{\alpha}}(\bar{x})$ is definable in $\mathbb{R}^\pi_{\text{alg}}$ and $C_{\bar{\alpha}}(\bar{r}) \equiv C_{\bar{\beta}}(\bar{s})$ if and only if $\sum \alpha_i \otimes r_i = \sum \beta_j \otimes s_j$ as elements of $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}(\pi)$.*

Let T_0 be the $\mathcal{L}^\pi$-theory of ordered $\mathbb{Q}$-vector spaces expressing the conclusion of the previous corollary. By construction $\mathbb{R}^\pi_{\text{alg}} \models T_0$.

PROPOSITION 3.3. *The theory T_0 has quantifier elimination.*

PROOF. Let $\mathcal{Q}_1, \mathcal{Q}_2 \models T_0$ be saturated of the same cardinality. We will show that if $A_i \subseteq \mathcal{Q}_i$ are small divisible subgroups, and $f : A_1 \to A_2$ is a partial $\mathcal{L}^\pi$-isomorphism, then f can be extended to any $a \in \mathcal{Q}_1$.

Since f is an $\mathcal{L}^\pi$-isomorphism $\mathcal{Q}_i \models T_0$ we can extend f to an isomorphism of $A_1 \otimes \mathbb{Q}(\pi)$ with $A_2 \otimes \mathbb{Q}(\pi)$. Identify a with $a \otimes 1$. By quantifier elimination in the theory of ordered divisible abeian groups and since $A_1 \otimes \mathbb{Q}(\pi)$ is a divisible abelian sub-group, $\text{tp}(a/A_1 \otimes \mathbb{Q}(\pi))$ is determined by the cut it realises. Thus, it will suffice to show that the same cut over $A_2 \otimes \mathbb{Q}(\pi)$ is realised in $\mathcal{Q}_2 \otimes \mathbb{Q}(\pi)$. But since $\mathcal{Q}_2$ is a saturated model of T_0 we automatically get that $\mathcal{Q}_2 \otimes \mathbb{Q}(\pi)$ is saturated (for $\leq$), as required. ⊣

We are now ready to present our main example. Let

$$F = \{(x, y) \in \mathbb{Q}^2 : y < \pi x\}$$

and

$$\mathbb{Q}^\pi_{vs} = (\mathbb{Q}, \leq, 0, 1, +, F).$$

Clearly, $\mathbb{Q}^\pi_{vs} \models T_0$, so by quantifier elimination (Proposition 3.3) $\mathbb{Q}^\pi_{vs} \equiv \mathbb{R}^\pi_{\text{alg}}$. In particular, since $\mathbb{R}^\pi_{\text{alg}}$ is weakly o-minimal and nonvaluational, so is $\mathbb{Q}^\pi_{vs}$.

THEOREM 3.4. *$\mathbb{Q}^\pi_{vs}$ is not a reduct of an o-minimal trace.*

PROOF. Assume towards a contradiction that there exists an o-minimal structure $\mathcal{Q}$ with universe $\mathbb{Q}$, and $\mathcal{R} \succ \mathcal{Q}$ such that $(\mathcal{R}, \mathcal{Q})$ is a dense pair, and the structure induced on $\mathcal{Q}$ from $\mathcal{R}$ expands $\mathbb{Q}^\pi_{vs}$. In particular, R is a subset of $\mathbb{R}$. Moreover, $\mathbb{Q}(\pi) \subseteq R$, since the definable cut $((-\infty, \pi), (\pi, +\infty))$ in $\mathbb{Q}^\pi_{vs}$ is realized in R, by [10, Theorem 2].

The desired conclusion now follows from [8] as follows. First, by Theorem A thereof, either $\mathcal{R}$ is linearly bounded or there is a definable binary operation $\cdot$ such that $(R, \leq, +, \cdot)$ is a real closed field. Since $\mathcal{Q} \prec \mathcal{R}$, in the latter case we would have a binary operation $\cdot_{\mathcal{Q}}$ definable in $\mathcal{Q}$ making $(\mathbb{Q}, \leq, +, \cdot_{\mathcal{Q}})$ into a real closed field. But that is impossible, because $(\mathbb{Q}, +)$ is the standard addition on $\mathbb{Q}$, and therefore there exists at most one field structure (definable or not) expanding it, and $(\mathbb{Q}, +, \cdot)$ is not real closed.

So we are reduced to the linearly bounded case. The closure K of $F = \{(x, y) \in \mathbb{Q}^2 : y < \pi x\}$ in R^2 is the graph of the map $x \mapsto \pi x : R \to R$. Clearly, F is definable in the pair (R, Q), and hence, by [10, Corollary 4.9], K is definable in R. Hence $x \mapsto \pi x : R \to R$ is a definable endomorphism of $(R, +)$, and by [8, Theorem B],

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it is $\emptyset$-definable. Since $\mathcal{Q} \prec \mathcal{R}$, the restriction of this map to $\mathbb{Q}$ has image in $\mathbb{Q}$, a contradiction. ⊣

The above proof actually shows more:

COROLLARY 3.5. *If $\mathcal{Q} \equiv \mathbb{Q}^{\pi}_{vs}$ then $\mathcal{Q}$ is not an o-minimal trace. In particular, $\mathbb{R}^{\pi}_{\text{alg}}$ is not an o-minimal trace.*

REMARK 3.6. We have shown that $\mathbb{R}^{\pi}_{\text{alg}}$ witnesses the fact that the class of o-minimal traces is not closed under reducts, and $\mathbb{Q}^{\pi}_{vs}$ witnesses that the class of reducts of o-minimal traces is not closed under elementary equivalence.

§4. Atomic models and elimination of imaginaries. In the o-minimal context definable Skolem functions have two main applications. The first, is a simple proof of the existence of atomic models, and in its stronger form of *definable choice* it implies elimination of imaginaries.

For o-minimal structures both properties can be proved under fairly general assumptions even for structures not supporting definable Skolem functions. In the present section we investigate these two properties in the weakly o-minimal nonvaluational case. We show that all known examples of such structures do not have atomic models. We then discuss elimination of imaginaries, obtaining—using the new machinery developed in [2]—some positive results.

The obstacle to the existence of atomic models is simple: if $\mathcal{M}$ is strictly weakly o-minimal and nonvaluational there exists a definable cut C that is irrational over $\mathcal{M}$, i.e., $\mathcal{M}$ is dense at C. But this need not be the case over arbitrary sets. In the case of o-minimal traces, our control over the definable closure operator, allows us to construct such examples:

PROPOSITION 4.1. *Let $\mathcal{M}_0$ be a saturated o-minimal expansion of a group in the signature $\mathcal{L}_0$. Let $\text{tp}(c/\mathcal{M}_0)$ be that of an irrational nonvaluational cut, and $\mathcal{M}$ the expansion of $\mathcal{M}_0$ by the externally definable cut $x < c$. Then $\text{Th}(\mathcal{M})$ does not have atomic models over arbitrary sets.*

PROOF. Let $A \subseteq M$ be a small set. We note that by weak o-minimality (and the group structure) the only types isolated over A are algebraic. So it will suffice to find $A \subseteq M$ such that $\text{dcl}_{\mathcal{M}}(A)$ is not an elementary substructure.

Choose any small $\mathcal{N}_0 \subseteq \mathcal{M}_0$ such that $\text{tp}(c/N_0)$ is an irrational nonvaluational cut. This can be done as follows: choose $\{a_i\}_{i\in\omega} \subseteq M$ inductively by $a_i \models \text{tp}(c/A_{i-1})$, where $A_i = \text{dcl}_{\mathcal{M}_0}(A_{i-1}a_i)$ and $A_{-1} = \emptyset$. Then set $\mathcal{N}_0 := \bigcup_{i\in\omega} A_i$. Saturation of $\mathcal{M}_0$ assures that this construction can be carried out.

Now let $a \in \mathcal{M}$ be such that $\text{tp}_{\mathcal{L}_0}(a/N_0) = \text{tp}_{\mathcal{L}_0}(c/N_0)$. Then, by construction $\mathcal{N}_0$ is dense in $\mathcal{N}_0(a)$ and a realises an irrational cut over N_0. It follows that $\text{tp}(c/\mathcal{N}_0 a)$ is a rational cut. To show this it will suffice to prove that $|c - a| < b$ for all $b \in \mathcal{N}_0(a)$. But because N_0 is dense in $\mathcal{N}_0$ it will suffice to check the same thing for $b \in N_0$. This is now immediate from the choice of a and the fact that c realises an irrational nonvaluational cut over $\mathcal{N}_0$.

It follows that $\mathcal{N}_0(a) \not\equiv \mathcal{M}$ (because C—the externally definable set $x < c$—is a rational cut over $\mathcal{N}_0(a)$ but not over M). Finally, since, e.g., by [10] $\mathcal{M}$ is precisely the expansion of $\mathcal{M}_0$ by cuts defined by $\mathcal{M}_0(c)$ over M_0 we get that $\text{dcl}_{\mathcal{M}}(\mathcal{N}_0(a)) \subseteq \mathcal{N}_0(ac) \cap M$. Since $\dim_{\mathcal{M}_0}(c/M) = 1$ it follows that $\text{dcl}_{\mathcal{M}}(\mathcal{N}_0(a)) = \mathcal{N}_0(a)$, with the desired conclusion. ⊣

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We point out that the key to the above proof is the fact that $\mathrm{dcl}_{\mathcal{M}}(\mathcal{N}_0(a)) \subseteq \mathcal{N}_0(ac) \cap M$. Indeed, the exact same proof would work if $\mathcal{M}$ were the expansion of $\mathcal{M}_0$ by any number of distinct nonvaluational irrational cuts. It would also work if $\mathcal{M}$ were any ordered group reduct of such an expansion. More precisely:

COROLLARY 4.2. *Let $\mathcal{M}$ be an ordered group reduct of an o-minimal trace, which is not o-minimal. Then*:

- *there is $A \subseteq M$ such that $\mathrm{dcl}_{\mathcal{M}}(A)$ is not an elementary substructure.*
- *$\mathrm{Th}(\mathcal{M})$ does not have atomic models over arbitrary sets.*

As we do not know of any example of a weakly o-minimal nonvaluational structure whose theory is not that of an ordered group reduct of an o-minimal trace it is natural to ask:

QUESTION 4.3. *Assume that T is a weakly o-minimal nonvaluational theory expanding the theory of ordered group. Assume that T has atomic models (i.e., over any set). Is T o-minimal*?

We now turn to the question of elimination of imaginaries. To the best of our knowledge, the only place the problem is addressed is [12, Theorem 6.3]. As our previous proposition shows, the assumptions in the main part of Wencel's result are not met in all known examples.[3] Here we take a different approach. To start we need:

DEFINITION 4.4. Let $\mathcal{M}$ be a weakly o-minimal expansion of an ordered group. Let $\overline{M}$ be the set of $\mathcal{M}$-definable cuts, and $\mathcal{M}_0^*$ the structure with universe $\overline{M}$ and whose atomic sets are $\mathrm{Cl}_{\overline{M}}(D)$ for all D $\mathcal{M}$-definable over $\emptyset$.

It is not hard to see that $\mathcal{M}_0^*$ has the same definable sets as the "canonical o-minimal extension" defined in [12, Section 3]; we will not be using this fact. We only need the following statement taken from the M.Sc. thesis of E. Bar Yehuda, [2].

FACT 4.5. *Let $\mathcal{M}$ be a weakly o-minimal nonvaluational expansion of an ordered group, $C \subseteq M^{n+m}$ any set $\mathcal{M}$-definable over $\emptyset$. Then there exists a set $\tilde{C}$, $\mathcal{M}_0^*$-definable over $\emptyset$, such that for all $a \in M^n$ the set*

$$\tilde{C}_a := \{b \in \overline{M}^m : (a,b) \in \tilde{C}\} = \mathrm{Cl}_{\overline{M}}(C_a),$$

where $C_a := \{b \in M^m : (a,b) \in C\}$.

We can now state the result:

THEOREM 4.6. *Let $\mathcal{M}$ be a weakly o-minimal nonvaluational expansion of an ordered group. Then $\mathcal{M}_0^*$ eliminates imaginaries for $\mathcal{M}$. I.e., for any equivalence relation E, $\emptyset$-definable in $\mathcal{M}$ there exists a function f_E, $\emptyset$-definable in $\mathcal{M}_0^*$ such that for all $x, y \in \mathrm{dom}(E)$ we have $f(x) = f(y)$ if and only if $E(x,y)$.*

PROOF. Let $\overline{E}$ be as provided by Fact 4.5. Since $\mathcal{M}$ expands a group so does $\mathcal{M}_0^*$. So $\mathcal{M}_0^*$ has definable choice (possibly after naming one positive constant). So there is a $\emptyset$-definable function $f : \mathrm{dom}(\overline{E}) \to \mathrm{dom}(\overline{E})$ such that $f(x) = f(y)$ if and only if $\overline{E}_x = \overline{E}_y$. Now, if $x, y \in \mathrm{dom}(E)$ and $\models E(x,y)$ then $E_x = E_y$, implying that $\mathrm{Cl}_{\overline{M}} E_x = \mathrm{Cl}_{\overline{M}} E_y$, so that, by the construction of $\overline{E}$ and f we get that $f(x) = f(y)$ So it remains to check that if $x, y \in \mathrm{dom}(E)$ and $\overline{E}_x = \overline{E}_y$ then

[3] Moreover, Wencel's proof seems to use [12, Proposition 6.2]. The justification for this usage is unclear to us.

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$E(x, y)$. Towards that end, it will suffice to show that $E_x \cap E_y \neq \emptyset$ (because E is an equivalence relation). Since $\overline{E}_x = \mathrm{Cl}_{\overline{M}} E_x$ it is enough to prove:

Claim: If $C, D \subseteq M^n$ are such that $\mathrm{Cl}_{\overline{M}}(C) = \mathrm{Cl}_{\overline{M}}(D)$ then $C \cap D \neq \emptyset$.

Let C, D be as in the claim. It is standard to check that

$$\dim_{\mathcal{M}_0^*}(\mathrm{Cl}_{\overline{M}}(C)) = \dim_{\mathcal{M}}(C) > \dim(\mathrm{Cl}_M(C) \setminus C) \geq \dim_{\mathcal{M}}(\partial C). \qquad (*)$$

So the assumptions imply that $\dim_{\mathcal{M}}(C) = \dim_{\mathcal{M}}(D)$. It will therefore be enough to prove the weaker claim that if

$$\dim_{\overline{M}}(\mathrm{Cl}_{\overline{M}}(C) \cap \mathrm{Cl}_{\overline{M}}(D)) = \dim_{\mathcal{M}}(C) = \dim_{\mathcal{M}}(D) \qquad (**)$$

then $C \cap D \neq \emptyset$.

In what follows, we make use of Wencel's strong cell decomposition theorem (see [12] for definitions and the theorem). Fix strong cell decompositions $\mathcal{C}$ and $\mathcal{D}$ of C and D respectively. Since $\mathrm{Cl}_{\overline{M}}(C) = \bigcup\{\mathrm{Cl}_{\overline{M}}(C_i) : C_i \in \mathcal{C}\}$, and similarly for $\mathcal{D}$, there must be cells $C_i \in \mathcal{C}$ and $D_j \in \mathcal{D}$ satisfying $(**)$ above. Thus, we are reduced to proving the lemma under the assumption that C, D are strong cells.

Now the proof of the claim for strong cells is an easy induction. If C, D are 0-cells, there is nothing to show. If C, D are 1-cells then $\mathrm{Cl}_{\overline{M}}(C)$ and $\mathrm{Cl}_{\overline{M}}(D)$ are closed intervals, and their intersection, by assumption, is also an interval. Since M is dense in $\overline{M}$, we get that $C \cap D$ is an interval in M.

Essentially the same proof works if C, D are open cells—then the assumption implies that the intersection of their closures contains an open set, and by density of M in $\overline{M}$ and $(*)$ we get the desired conclusion.

In general, C, D are graphs of definable functions. Let π_C be a projection such that $\dim_{\mathcal{M}}(\pi_C(C)) = \dim(C)$. So, since C is a cell, π_C is injective on C and $\pi_C(C)$ is an open cell. On a set of small co-dimension of $\mathrm{Cl}_{\overline{M}}(C)$ the projection $\pi(C)$ is an injection so there exists a set of full dimension in $\mathrm{Cl}_{\overline{M}}(C) \cap \mathrm{Cl}_{\overline{M}}(D)$ such that π_C is an injection. Restricting to the image of that set under π_C the claim now follows by induction. ⊣

The above, implies in particular that if $\mathcal{M}$ is a structure as above, and for any $\mathcal{M}$-$\emptyset$-definable function $f : M^n \to \overline{M}$ there exists an $\mathcal{M}$-$\emptyset$-definable function $F : M^n \to M^m$ (some m) such that $F(x) = F(y)$ if and only if $f(x) = f(y)$ then $\mathcal{M}$ eliminates imaginaries. Indeed, if E is an $\mathcal{M}$-$\emptyset$-definable equivalence relation on M^n then by the theorem there exists a $\emptyset$-definable (in $\mathcal{M}_0^*$) function $f_E : \overline{M}^n \to \overline{M}$ such that $f_E(x) = f_E(y)$ if and only if $E(x, y)$ for $x, y \in \mathrm{dom}(E)$. Since the structure induced on M from $\mathcal{M}_0^*$ is $\mathcal{M}$, the restriction $f|M^n$ is $\mathcal{M}$-definable. By [2] it is $\mathcal{M}$-$\emptyset$-definable. So the function $\tilde{f}_E$ eliminates the imaginary M^n/E.

In view of the above it seems natural to identify the pair $(\mathcal{M}_0^*, \mathcal{M})$ with $\mathcal{M}^{eq}$, suggesting that this may well be the right context for studying weakly o-minimal nonvaluational structures.

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