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Carrots for dessert

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Abstract. We formulate and prove a precise statement of asymptotic shrinking of 'carrot-fields' around the Mandelbrot set **M**. This phenomenon was suggested in a seminal text on polynomial-like mappings by Douady and Hubbard [On the dynamics of polynomial-like mappings. *Ann. Sci. Éc. Norm. Supér.* **t.18** (1985), 287–343]. This is helpful for understanding how copies of **M** sit in the bifurcation loci of families of rational maps.

1. *Introduction*

For a general analytic family of rational maps $(f_t)_{t\in X}$, the *bifurcation locus* is the set of parameters $t \in X$ such that the Julia set does not move continuously (in the Hausdorff topology) over any neighborhood of t. For instance, the bifurcation locus of the quadratic family $(Q_c(z) = z^2 + c)_{c\in\mathbb{C}}$ is the boundary of the *Mandelbrot set* **M** (i.e., $\mathbf{M}\backslash\mathrm{int}(\mathbf{M})$), where **M** is defined as the set of parameters $c \in \mathbb{C}$ such that the Julia set $J(Q_c)$ is connected (see Figure 1).

McMullen demonstrated in [**McM**] the following universality property of **M**: the boundaries of small Mandelbrot sets are dense in the bifurcation locus of *any* non-trivial holomorphic family of rational maps. A *small* Mandelbrot set (or a *copy of* **M**) is the image of **M** by a homeomorphism h that preserves the dynamics: some restriction of an iterate of $h(c)$ and Q_c are topologically conjugated on their Julia sets. This result of McMullen motivates the study of how those copies are embedded in the bifurcation locus of a general family.

In their pioneering paper on polynomial-like mappings [**DH2**], Douady and Hubbard exhibited for the first time such an image of **M** interlocked in the bifurcation locus of a particular family of rational maps (see Figure 2). In their paper, they developed the theory of what they called 'polynomial-like mappings'. This theory allows us to recognize copies of **M** in the parameter space of analytic families of rational maps. Moreover, they suggest that the copies of **M** come with some additional structure. Namely, in

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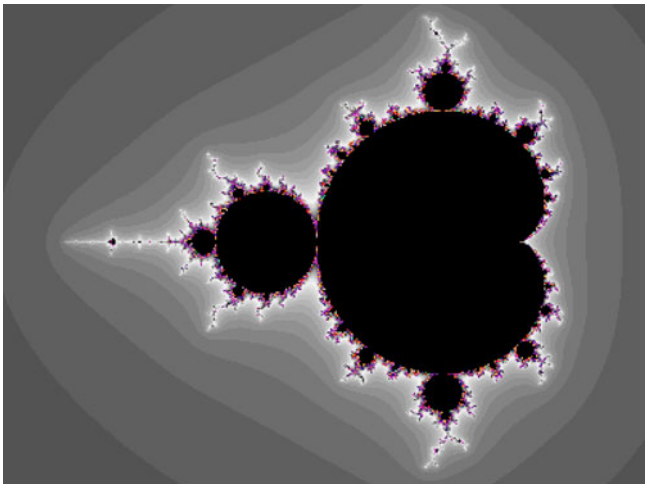


FIGURE 1. Mandelbrot set.

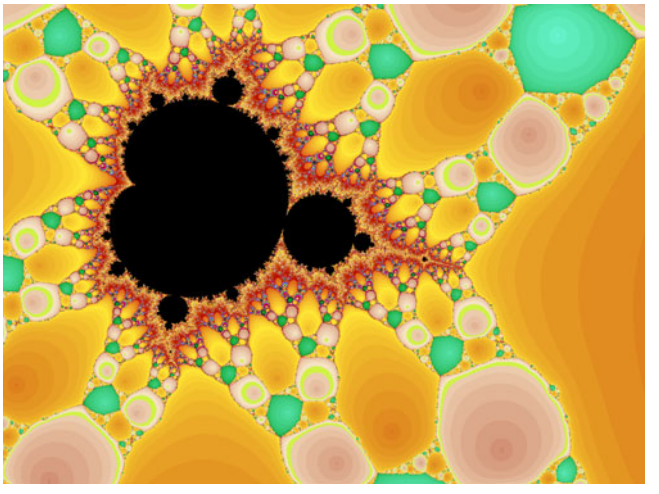


FIGURE 2. Based on Figure 16 of 'carrots for dessert' in [**DH2**]; carrots are in orange.

their particular example, they *observe in the picture* (Figure 16 of the section 'Carrots for dessert' in [**DH2**]) that there are connected components of the complement of the bifurcation locus which are attached to the tips of the copy, and that their diameters tend to 0 (see Figure 2). Note that those components are hyperbolic components and that it is not known, even for the quadratic family, that the diameter of hyperbolic components tends to 0. Douady and Hubbard called them 'carrots'. They also present the notion of carrot fields in the setting of polynomial-like mappings.

In the present paper, we give a reformulation of the Douady–Hubbard definition of 'carrots' and of 'carrot fields' around **M**. We prove in Theorem 2 a general statement concerning the asymptotic shrinking of 'carrot fields' around **M**.

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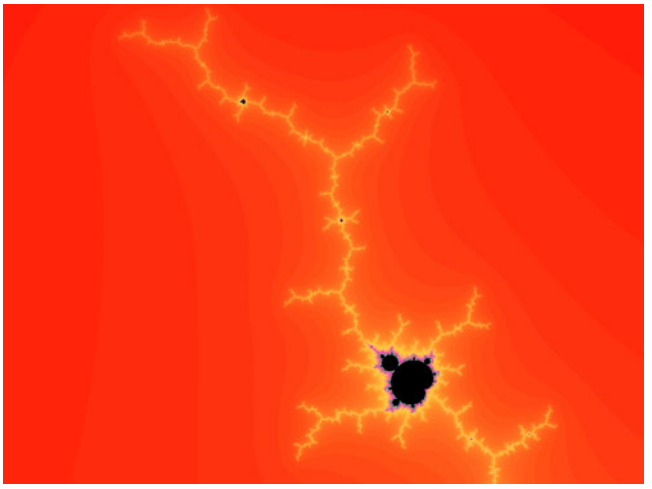


FIGURE 3. A Mandelbrot copy in **M**.

We give in Theorem 5 the simplest possible application, namely to the copies of **M** inside **M** (see Figure 3). We prove that, if $\mathbf{M}_0$ is a copy of **M** in **M**, only finitely many connected components of $\mathbf{M}\backslash\mathbf{M}_0$ have diameter greater than some $\epsilon > 0$. To prove this, we interpret any component of $\mathbf{M}\backslash\mathbf{M}_0$ as a carrot relative to $\mathbf{M}_0$. We also prove that Theorem 2 is a necessary condition for the Mandelbrot set to be locally connected (the famous MLC—Mandelbrot is Locally Connected—conjecture).

Our Theorem 2 provides a tool that can be applied to many holomorphic families. For instance, a forthcomming paper [**PR3**] describes application to the case of the parabolic Mandelbrot set, $\mathbf{M_1}$.

The proof uses two fundamental tools: the generalized Yoccoz–Levin inequality and the combination of two kinds of puzzle in the parameter plane (the Yoccoz puzzle together with the puzzle defined by the carrots).

2. *Statement and results*

2.1. *Carrots for* $Q_0 : z \mapsto z^2$. Let z_1, z_2 be two points of $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \Re e(z) > 0\}$ such that the Euclidean closed triangle Δ supported by $0, z_1, z_2$ is disjoint from $\Delta + 2i$. Then the *carrot* $\Delta_0 = \exp(\Delta)$ is homeomorphic to Δ. For any $n \in \mathbb{N}$, any $p \in \mathbb{N}^*$ with $2 \wedge p = 1$, we can consider the *carrot* $\Delta_{p/2^n}$ which is the connected component of $Q_0^{-n}(\Delta_0)$ containing the point $e^{2i\pi p/2^n}$ of the unit circle $\mathbf{S}^1$. The carrots are all homeomorphic images of Δ.

2.2. *Carrots for* **M**. Douady and Hubbard gave in [**DH1**] a dynamical parameterization of the complement of **M**, denoted by $\Phi : \mathbb{C}\backslash\mathbf{M} \to \mathbb{C}\backslash\overline{\mathbb{D}}$, which is the conformal representation tangent to the identity at ∞. It has the property that its inverse $\Psi = \Phi^{-1}$ admits a limit $\gamma(p/q)$ on every ray of rational angle p/q (i.e., $\gamma(p/q) = \lim_{r \to 1^+} \Psi(re^{2i\pi p/q})$) and, moreover, in any Stolz angle based at $e^{2i\pi p/q}$. Therefore, one can take the image $\Psi(\Delta_{p/2^n})$ which is a compact set in $\mathbb{C}$.

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Definition 1. The *dyadic carrot field* $\Delta^{\mathbf{M}}$ *around* $\mathbf{M}$ generated by Δ is the set consisting of all the 'carrots' $\Delta^{\mathbf{M}}_{p/2^n} := \Psi(\Delta_{p/2^n})$, where $p/2^n \in \mathbb{Q}$.

A first remark is that $\Delta^{\mathbf{M}}_{p/2^n}$ is homeomorphic to Δ. With this terminology, the theorem of shrinking of dyadic carrots of $\mathbf{M}$ is the following theorem.

THEOREM 2. *For any dyadic carrot field* $\Delta^{\mathbf{M}}$ *of* $\mathbf{M}$,

$$\lim_{n\to\infty} \sup_p \operatorname{diam}(\Delta^{\mathbf{M}}_{p/2^n}) = 0.$$

2.3. *Corollary for a copy* $\mathbf{M}_0$ *of* $\mathbf{M}$. Let $\mathbf{M_0}$ be a copy of $\mathbf{M}$ and denote by $\chi_{\mathbf{M_0}}$ the homeomorphism from $\mathbf{M}_0$ to $\mathbf{M}$. The *dyadic tips* $\gamma_0(p/2^n)$ of $\mathbf{M_0}$ are the images of the dyadic tips $\gamma(p/2^n)$ of $\mathbf{M}$ by $\chi^{-1}_{\mathbf{M_0}}$. The *dyadic tip* $\gamma(p/2^n)$ of $\mathbf{M}$ is the limit $\gamma(p/2^n)$ of Ψ, at the point $e^{2i\pi p/2^n}$ in $\mathbf{S}^1$.

Definition 3. Denote by $\Delta^{\mathbf{M_0}}_{p/2^n}$ the connected component of $\mathbf{M}\backslash\mathbf{M_0}$ containing the tip $\gamma_0(p/2^n)$ in its closure and disjoint from $\mathbf{M_0}$.

We call $\Delta^{\mathbf{M_0}}_{p/2^n}$ the $p/2^n$ carrot of $\mathbf{M_0}$ or the $p/2^n$ dyadic decoration of $\mathbf{M_0}$, and denote by $\Delta^{\mathbf{M_0}}$, or the *relative carrot field of* $\mathbf{M_0}$, the set of all carrots of $\mathbf{M_0}$.

THEOREM 4. (Douady–Hubbard, Yoccoz) *For any copy* $\mathbf{M_0}$ *of* $\mathbf{M}$ *in* $\mathbf{M}$:

$$\mathbf{M} = \mathbf{M_0} \cup \bigcup_{n\geq 0} \bigcup_{p/2^n\in\mathbb{Q}} \Delta^{\mathbf{M_0}}_{p/2^n}.$$

The shrinking decorations theorem for strict copies $\mathbf{M_0}$ of $\mathbf{M}$ in $\mathbf{M}$ can then be stated as in the following theorem.

THEOREM 5. *For any strict copy* $\mathbf{M_0}$ *of* $\mathbf{M}$ *in* $\mathbf{M}$,

$$\lim_{n\to\infty} \sup_p \operatorname{diam}(\Delta^{\mathbf{M_0}}_{p/2^n}) = 0.$$

The two Theorems 2 and 5 have very similar proofs, the proof of the first being slightly more complicated. We shall detail the proof of the first and sketch the difference to the proof of the second.

Dzmitry Dudko presents a different and independent proof of the shrinking decorations theorem for strict copies $\mathbf{M_0}$ of $\mathbf{M}$ in $\mathbf{M}$ in [**Du**]. His statement includes more generally strict copies of the Multibrot set inside the Multibrot set of the same degree. The proof we give here would also easily extend to the Multibrot case. In a forthcomming paper [**PR4**], we shall treat the more general case of Mandelbrot-like families.

3. *Framework*

3.1. *Independence on* Δ. The degenerate case where the triangle Δ is defined by $z_1 = z_2 > 0$ is in fact the only one we need to consider. Let us call the carrot a stick when the triangle is degenerate. We shall prove the following theorem.

THEOREM 6. *For the dyadic stick field of* $\mathbf{M}$ *generated by* $\Delta = [0, 1]$,

$$\lim_{n\to\infty} \sup_p \operatorname{diam}(\Delta^{\mathbf{M}}_{p/2^n}) = 0.$$

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LEMMA 7. *Theorem 6 implies Theorem 2.*

Proof. We shall prove in fact that the result is independent of the non-trivial triangle chosen. Let Δ^1 and Δ^2 be any two possibly degenerate triangles in $\mathbb{H}_+ \cup \{0\}$. Then there exists $\delta > 0$ such that $\Delta^1 \setminus \{0\}$ is contained in a hyperbolic δ-neighborhood of $\Delta^2 \setminus \{0\}$ in $\mathbb{H}_+$ and *vice versa*. As $\exp : \mathbb{H}_+ \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$, $Q_0 : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\Phi : \mathbb{C} \setminus \mathbf{M} \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ are local hyperbolic isometries, the same statement holds with the same δ for each $\Delta^{j\mathbf{M}}_{p/2^n}$, $j = 1, 2$. Thus, by elementary estimates on hyberbolic metrics, there exists $k = k(\delta) > 1$ such that, for any reduced dyadic $\theta = p/2^n$,

$$\frac{1}{k} \leq \frac{\operatorname{diam}(\Delta^{1\mathbf{M}}_\theta)}{\operatorname{diam}(\Delta^{2\mathbf{M}}_\theta)} \leq k,$$

where $\operatorname{diam}(\cdot)$ denotes Euclidean diameter.

This follows from the following inequality (see [**Po**]), where $K \subset \widehat{\mathbb{C}}$ is a compact set and B_H denotes the ball for the hyperbolic metric in $\widehat{\mathbb{C}} \setminus K$:

$$\operatorname{diam}(B_H(z, d)) \leq C(d) \operatorname{dist}(z, K).$$

Indeed, if D is the diameter of $\Delta^{1\mathbf{M}}_\theta$, then $\operatorname{dist}(z, K) \leq D$ for $z \in \Delta^{1\mathbf{M}}_\theta$ and there exists $z' \in \Delta^{2\mathbf{M}}_\theta$ such that $z \in B_H(z', d)$, so that $|z - z'| \leq DC(d)$. Hence, $\operatorname{diam} \Delta^{2\mathbf{M}}_\theta \leq (C(d) + 1)D$. Computing the reverse inequality yields the comparison.

To prove Theorem 2, it suffices to consider a particular stick field, say the field for $z_1 = z_2 = 1$, which is Theorem 6. □

The choice of this particular stick or degenerate carrot field allows us to avoid discussions on the position of the stick (or carrot) with respect to the parameter rays (see §3.4.1).

3.2. *Argument on the toy example.* One of the main arguments in the proof will be that for any strict copy M' of $\mathbf{M}$ in $\mathbf{M}$, the stick field defined by $\Delta^{\mathbf{M}}_{p/2^n} = \Psi(\Delta_{p/2^n})$ does not touch M'. This turns out to allow us to surround a potentially large stick $\Delta^{\mathbf{M}}_{p/2^n}$ by an annulus, which is disjoint from M' and whose modulus is bounded uniformly from below.

We can visualize this argument first on the trivial toy example. We replace Δ_0 that was defined by the interval $[0, 1]$ simply by the compact set $T_0 = \exp([1/2, 1]) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. This completely trivializes the problem by considerations on the comparison of hyperbolic and Euclidean distance similar to those above. In this simpler case, T_0 has a finite hyperbolic diameter, diam, in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Thus, any of its iterated preimages $T_{p/2^n}$ under Q_0 have the same finite hyperbolic diameter and so do the images $T^{\mathbf{M}}_{p/2^n} = \Psi(T_{p/2^n})$. Hence, the Euclidean diameter of any $T^{\mathbf{M}}_{p/2^n}$ is bounded uniformly from above by a universal constant $k = k(\text{diam})$ times the Euclidean distance between $T^{\mathbf{M}}_{p/2^n}$ and $\mathbf{M}$. Since the latter tends to zero uniformly as $n \to \infty$, we have, in the toy case,

$$\lim_{n \to \infty} \operatorname{diam}(T_{p/2^n}) = 0.$$

Note that a uniform bound on the hyperbolic diameter is equivalent to a uniform bound on the moduli of the annuli separating $\mathbf{M}$ and $T^{\mathbf{M}}_{p/2^n}$.

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With this in mind, let us proceed to the proof of Theorem 2. Then, as mentioned above, Theorem 5 will follow by using essentially the same proof; we shall point out the differences.

3.3. *Proof of Theorem 2.* We prove the following result.

PROPOSITION 8. *For the dyadic stick field of* $\mathbf{M}$ *generated by* $\Delta = [0, 1]$, *let* $\{\Delta_i = \Delta^{\mathbf{M}}_{p_i/2^{r_i}}\}_{i\in\mathbb{N}}$ *be any sequence of sticks with* $r_{i+1} > r_i$ *and with roots* c_i *tending to* c_∞. *Then,*

$$\operatorname{diam}(\Delta_i) \underset{i\to\infty}{\longrightarrow} 0.$$

Remark 9. Theorem 2 is an easy corollary of this proposition by compactness of the Mandelbrot set. (The details are left to the reader.)

Sketch of the proof of Proposition 8. Note that, necessarily, $c_\infty \in \partial\mathbf{M}$.

(1) First of all, the Yoccoz–Levin parameter inequality (Theorem 17) implies that if the limit point c_∞ belongs to the main cardioid, then the diameter of the carrots has to tend to 0. Therefore, we can assume in the following that the limit point c_∞ belongs to some limb $L^{H_0}_{p/q}$ of the main cardioid H_0 of $\mathbf{M}$.

(2) Secondly, we apply the Yoccoz parameter puzzle theorem (Theorem 21). There exists a non-decreasing and non-eventually-constant subsequence n_i such that for $i \geq i_0$ the sticks $\Delta_i \subset \overline{\mathcal{P}}_{n_i}(c_\infty)$ (the parameter puzzle piece). Hence, either the diameter tends to 0 or the limiting parameter c_∞ is renormalizable, that is, $c_\infty \in M'$ for some first renormalization copy M' – i.e. a copy which is not contained in any other copy of $\mathbf{M}$, than $\mathbf{M}$ itself – of $\mathbf{M}$ in $L^{H_0}_{p/q}$, of period k, where $q \leq k$.

(3) The key point in the proof of Proposition 8 is that all of the dyadic sticks $\Delta^{\mathbf{M}}_{p/2^n}$ are disjoint from M' because their root points $\Psi(p/2^n)$ are disjoint from M'.
We show that, given $\epsilon > 0$, either the diameter of Δ_i is less than ϵ or we can separate Δ_i from M' by some annulus of definite modulus $\delta = \delta(\epsilon) > 0$. We shall use here the fact that if $\Delta_i \subset V' \subset V \subseteq U$, with U a hyperbolic domain (here $U = \mathbb{C} \smallsetminus M'$) and $\operatorname{mod}(V \smallsetminus \overline{V'}) \geq \delta > 0$, then the hyperbolic diameter of Δ_i in U satisfies $\operatorname{diam}_U(\Delta_i) \leq d(\delta)$. Therefore, if Δ_i tends to M', it follows that the Euclidean diameter has to go to zero.

(4) To construct such annuli and bound their moduli from below, we shall first construct their dynamical counterpart. In the dynamical planes we construct annuli, which are univalent preimages of some annulus admitting a holomorphic motion in a large domain. We then use the observation by Shishikura that holomorphic motions can be used to transfer bounds for (locally) persistent annuli in dynamical space to bounds for corresponding annuli in parameter space (see [**R**]).

(5) We shall follow slightly different paths according to whether M' is a primitive copy or the satellite copy $M_{p/q}$ with root on the cardioid. We start with the primitive case and afterwards indicate the changes, which make the proof in the satellite case.

3.4. *The setup.* As a start, we recall some classical facts from [**DH1**] (see also [**Mi**]) that will be used at different points in the paper.

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3.4.1. *Rays.* The polynomial Q_c is conjugated to z^2 near ∞. The conjugacy ϕ_c which is tangent to the identity at infinity is called the Böttcher coordinate of Q_c at ∞. It has the property that, for $c \notin \mathbf{M}$,

$$\Phi(c) = \phi_c(c).$$

We shall use also Green's functions for $\mathbf{M}$ and $K_c = \{z \mid Q_c^n(z) \text{ is bounded}\}$, i.e., the subharmonic functions $g_{\mathbf{M}}(c) = \log^+ |\phi_c(c)|$ and

$$g_c(z) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |Q_c^n(z)|.$$

Moreover, we shall use the notations $E^{\mathbf{M}}(h)$ and $E^c(h)$ for the *equipotentials* for $g_{\mathbf{M}}$ and g_c of level $h \geq 0$. Similarly, we shall use the notation $F^{\mathbf{M}}(h)$, respectively $O^{\mathbf{M}}(h)$ and $F^c(h)$ respectively $O^c(h)$, for the *closed, respectively open, filled equipotentials* of level or height h and

$$F^{\mathbf{M}}(h) = \{c \mid g_{\mathbf{M}}(c) \leq h\}, \quad O^{\mathbf{M}}(h) = F^{\mathbf{M}}(h) \setminus E^{\mathbf{M}}(h),$$
$$F^c(h) = \{z \mid g_c(z) \leq h\}, \quad O^c(h) = F^c(h) \setminus E^c(h).$$

The *external ray* of argument θ for $\mathbf{M}$ or K_c is the field line of $g_{\mathbf{M}}$ or g_c, which is asymptotic to the half-line $\exp(t + i2\pi\theta)$ at ∞. By the *parameter ray* of angle θ we mean the external ray for $\mathbf{M}$, which is simply the image $R_\theta^{\mathbf{M}} = \Psi(]1, \infty[e^{2i\pi\theta})$. The *dynamical ray* can also be defined as $R_\theta^c = \phi_c^{-1}(]1, \infty[e^{2i\pi\theta})$, when the Julia set is connected, or else this curve might be broken on the critical point and its iterated preimages.

3.4.2. *Combinatorial rotation number.* A periodic point of minimal period $k > 0$ has combinatorial rotation number p/q, $p \wedge q = 1$, if it is the common landing point of a q-cycle of external rays $R_{\theta_0}^c, \ldots, R_{\theta_{q-1}}^c$ for Q_c^k with $0 < \theta_0 < \theta_1 < \cdots < \theta_{q-1} < 0$ and for which,

$$\text{for all } j,\ 0 < j < q : 2^k\theta_j = \theta_{(j+p) \bmod q} \mod 1.$$

Note that this is a condition on the pure arguments. We say that these arguments form a p/q-cycle under the kth iterate of the angle-doubling map. Consequently the q-cycle of rays $R_{\theta_0}^c, \ldots, R_{\theta_{q-1}}^c$ is called a p/q-cycle of external rays for Q_c^k, independent of whether or not it co-lands. For $k = 1$, there is a unique p/q-cycle of rays for every rotation number p/q. The landing point of a fixed ray under Q_c^k can be considered as having rotation number $0/1$ for Q^k. The combinatorial rotation number of a periodic point z_0 is unique, when it exists, and is the same for all points in the orbit of z_0. The snail lemma asserts that if z_0 is the common landing point of a p/q-cycle of external rays for Q_c^k, then z_0 is a k-periodic point, which is either repelling or parabolic with multiplier $\exp(i2\pi p/q)$, and the Douady landing theorem asserts that any such point has a combinatorial rotation number when the Julia set is connected (see also [**P**] for a more complete discussion).

3.4.3. *Copies of* $\mathbf{M}$.

Definition 10. We say that $f : U' \to U$ is quadratic-like if f is a proper holomorphic map of degree 2 between disks U', U such that $\overline{U'} \subset U$. The associated filled Julia set is $K_f = \cap f^{-n}(U)$. A map g is said to be l-renormalizable if g^l admits a restriction which is quadratic-like with a connected filled Julia set.

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Definition 11. Two quadratic-like maps f and g are said to be *hybrid equivalent* if there is a quasi-conformal conjugacy ϕ between f and g defined on neighborhoods of their respective filled Julia sets and such that $\overline{\partial}\phi = 0$ almost everywhere on $K(f)$.

THEOREM 12. *Every quadratic-like map f is hybrid equivalent to a polynomial of the same degree. When $K(f)$ is connected, this polynomial is unique up to affine conjugacy.*

Definition 13. A copy $M' \subsetneq \mathbf{M}$ of the Mandelbrot set $\mathbf{M}$ is the image of $\mathbf{M}$ by a homeomorphism h which preserves the dynamics in the following sense. For $c' = h(c)$, there exists a minimal k and a renormalization, i.e., a quadratic-like restriction, of the iterate $Q_{c'}^k$ to an open disk containing 0, such that this restriction is hybrid equivalent to Q_c. The common minimal k above is called the period of the copy.

3.4.4. *Wakes of H and their combinatorial rotation number.* Any quadratic polynomial has (counting multiplicity) two fixed points. The dynamical ray R_0^c of argument 0 lands (except when $c \in R_0^{\mathbf{M}}$) on one of the fixed points denoted β_c and referred to as the β-fixed point of Q_c. The other fixed point is denoted α_c and is referred to as the α-fixed point. We denote by H_0 the open disk bounded by the main cardioid in the Mandelbrot set. It corresponds to the set of parameters c such that α_c is attracting. The disk H_0 is called the period 1 or central hyperbolic component of $\mathbf{M}$. The multiplier $\lambda_\alpha(c)$ of α_c defines a homeomorphism of $\overline{H}_0$ onto $\overline{\mathbb{D}}$ that is holomorphic in the interior. Each $c = \lambda_\alpha^{-1}(e^{2i\pi p/q})$ with $p \wedge q = 1$ is the common landing point of two external parameter rays, $R^{\mathbf{M}}_{\eta^{p/q}_\pm}$, whose arguments are the nearest neighbors in the unique p/q-cycle under angle doubling. The *p/q-wake of H_0* is the region $W^{H_0}_{p/q}$ bounded by these two rays and disjoint from H_0. The part of $\mathbf{M}$ in $W^{H_0}_{p/q}$ is usually called the *p/q-limb*: $L_{p/q} = L^{H_0}_{p/q} = \mathbf{M} \cap W^{H_0}_{p/q}$.

The p/q wake and limb are characterized dynamically as the set of parameters c for which the fixed point α_c has combinatorial rotation number p/q. According to the Douady landing theorem, any $c \in \mathbf{M}$ for which both fixed points are repelling belongs to some p/q-limb $L^{H_0}_{p/q}$.

In general, a hyperbolic component H of $\mathbf{M}$ is a connected component of the interior of $\mathbf{M}$ such that Q_c has an attracting periodic point α'_c. Its period k is common for all $c \in H$ and is denoted the period of H. The multiplier of α'_c defines a biholomorphic map $\lambda_H : H \longrightarrow \mathbb{D}$, which extends as a homeomorphism of the closures according to the Douady–Hubbard multiplier theorem. The root c_0^H of H is the parameter $\lambda_H^{-1}(1)$, which has a k-periodic cycle of multiplier 1 (possibly of lower prime period k' dividing k). When $H \neq H_0$, i.e., when $k > 1$, then c_0^H is the common landing point of two external parameter rays, say of arguments $\theta_\pm$. Let W^H denote the open sector bounded by the pair of co-landing rays $R^{\mathbf{M}}_{\theta_\pm}$ and containing H. The sector W^H is called the *founding wake* of H. For $c \in W^H$, the pair of external rays $R^c_{\theta_\pm}$ co-land on a repelling periodic point (see Theorem B.1, for example). The corresponding (dynamical plane) sector is denoted by $W^{H,c}$. The periodic point α'_c depends holomorphically on $c \in W^H$ and is attracting precisely when $c \in H$. For each reduced rational p/q, the *p/q-wake $W^H_{p/q}$ of H* is the set of parameters $c \in W^H \backslash \overline{H}_0$ for which the repelling k-periodic point α'_c has rotation number p/q. Then, as for H_0, there is a pair of parameter rays co-landing at $c^H_{p/q} = \lambda_H^{-1}(e^{2i\pi p/q})$ and bounding $W^H_{p/q}$.

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Their arguments are the nearest neighbors in the common kq-cycle of arguments. The *p/q-limb* of H is the intersection $L^H_{p/q} = \mathbf{M} \cap W^H_{p/q}$. If we define $L^H_{0/1} = \mathbf{M} \backslash \overline{W^H}$, then

$$\mathbf{M} = \overline{H} \cup \bigcup_{p/q} L^H_{p/q}.$$

The hyperbolic component H of $\mathbf{M}$ can be viewed as the principal hyperbolic component of a copy $M' \subset W^H$ of $\mathbf{M}$. The copy M' is characterized as those $c \in W^H$ for which Q_c^k has a quadratic-like restriction with connected filled-in Julia set, α-fixed point α'_c and β-fixed point the common landing point of the rays $R^c_{\theta_\pm}$. The period k of the copy is the period of H, namely the period of the attracting cycle $\alpha'(c)$ for $c \in H$.

We denote by the *tuning interval* for M' (or equivalently for H), the interval $I(M') = I(H) = [\theta_-, \theta_+]$, where $\theta_\pm$ are the root arguments of H. The component H and the copy M' are called satellites of some other hyperbolic component H' if the founding wake W^H is the p/q-wake $W^{H'}_{p/q}$ for this other hyperbolic component, and H, M' is called a primitive component or copy, if no such other hyperbolic component exists.

3.4.5. *The dyadic wakes of M'.* (See Figure 4.) Let M' be a copy of $\mathbf{M}$ in $\mathbf{M}$ with tuning interval I and period k, as above. Let $\widehat{\theta}_+ < \widehat{\theta}_-$ be the points in I such that each of the subintervals $I_0 = [\theta_-, \widehat{\theta}_+]$ and $I_1 = [\widehat{\theta}_-, \theta_+]$ map diffeomorphically onto I under σ^k, where $\sigma(\theta) = 2\theta \bmod 1$. Let $I^{M'}$ denote the corresponding σ^k-invariant Cantor set and let $\kappa = \kappa_{M'} : I^{M'} \longrightarrow \Sigma_2$ denote the conjugacy of $\sigma^k : I^{M'} \longrightarrow I^{M'}$ to the shift on $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$ with $\kappa(\theta_-) = \overline{0}$ and $\kappa(\theta_+) = \overline{1}$. Moreover, let $\Pi_2 : \Sigma_2 \longrightarrow \mathbb{T}$ denote the projection $\Pi_2(\underline{\epsilon}) = \sum_i \epsilon_i/2^i$, which semi-conjugates the shift to angle doubling.

Then the pair of rays with arguments $\widehat{\theta}_\pm$ co-land at the principal tip $c'_{1/2} = \chi^{-1}_{M'}(\Psi(1/2))$ of M'. The sector $W'_{1/2}$ bounded by these rays and disjoint from M' is called the *principal wake* of M'.

More generally, for p odd with binary expansion $p = \epsilon_1 \dots \epsilon_n$, $\epsilon_n \neq 0$, the dyadic number $p/2^n$ has two binary expansions (preimages under Π_2) $0.\epsilon_1 \dots \epsilon_n\overline{0}$ and $0.\epsilon_1 \dots \epsilon_{n-1}0\overline{1}$. According to the Douady tuning algorithm, the two points

$$\theta^-_{p/2^n} = \kappa^{-1}(\epsilon_1 \dots \epsilon_n\overline{0}) \quad \text{and} \quad \theta^+_{p/2^n} = \kappa^{-1}(\epsilon_1 \dots \epsilon_{n-1}0\overline{1})$$

are the endpoints of a complementary interval of $I^{M'}$. The corresponding external rays of $\mathbf{M}$ co-land at the relatively dyadic tip $c'_{p/2^n} = \chi^{-1}_{M'}(\Psi(p/2^n))$ of M'. The *$p/2^n$-wake* $W^{M'}_{p/2^n}$ is the sector bounded by these rays and disjoint from M'. For any parameter c in *the relatively dyadic carrot* $\Delta^{M'}_{p/2^n} := \mathbf{M} \cap W^{M'}_{p/2^n}$, the corresponding dynamical rays co-land on a point, which is preperiodic to the relative β-fixed point (see also Theorem B.1).

Denote by W'_0 the sector bounded by the rays of arguments $\theta_\pm$ and not containing M'. Note that for each $p/2^n$, the root $c'_{p/2^n}$ of the corresponding wake is the only point of intersection between M' and (the closure of) the wake. Moreover, any two wakes are disjoint, since rays do not cross.

4. *Yoccoz–Levin inequality*

In order to prove Proposition 8, we need the following strong version, due to Levin, of the Yoccoz parameter inequality. For the version of Proposition 8 leading to a proof of

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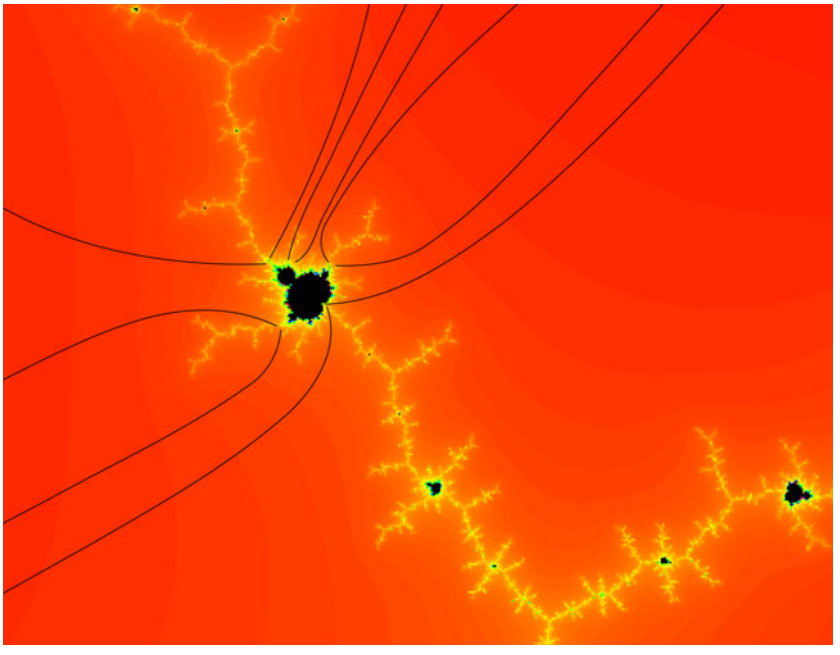


FIGURE 4. Dyadic wakes of M' copy of $\mathbf{M}$.

Theorem 5, the simpler Yoccoz (rather than Levin–Yoccoz) parameter space inequality suffices.

4.1. *Dynamical inequality.*

THEOREM 14. (The Yoccoz–Levin dynamical inequality) *Let H be any hyperbolic component of $\mathbf{M}$ of period k. Let p/q be any non-zero reduced rational and let $W^H_{p/q}$ denote the relative p/q-wake of H, bounded by parameter rays with arguments $0 < \eta_- < \eta_+ < 1$. For $c \in W^H_{p/q}$, let $\lambda = \lambda(c)$ denote the multiplier of the repelling k-periodic point α'_c defining the wake, i.e., which is the common landing point of the kq-periodic rays $R^c_{\eta_\pm}$. Then λ admits a logarithm Λ such that*

$$\left|\Lambda - \frac{p}{q}2\pi i\right| \le \frac{2k \log 2 \cos\varphi}{q} \frac{\pi}{\omega(c)},$$

where $\varphi \in]-\pi/2, \pi/2[$ is the argument of $\Lambda - 2\pi i \frac{p}{q}$ and $\omega(c)$ is π if $c \in \mathbf{M}$ and otherwise equals the angle of vision of the interval $i2\pi[\eta_-, \eta_+]$ from $\operatorname{Log}\Phi(c) \in \{z = x + iy \mid 0 < y < 2\pi\}$.

Proof. Levin proved the fixed point case $k = 1$ in [**L**, Theorem 5.1]; the general case is similar. For completeness, we give a proof in the Appendix. □

We shall use this theorem when c belongs to $\mathbf{M} \cup \Delta^{\mathbf{M}}$, so we need to estimate the quantity $\omega(c)$ for $c \in \Delta^{\mathbf{M}}$ (for $\Delta^{\mathbf{M}}$ see Definition 1). For this, we shall estimate the largest potential of $\Delta^{\mathbf{M}} \cap W^H_{p/q}$. This in turn leads us to estimate the lowest denominator 2^n of dyadics $p/2^n \in [\eta_-, \eta_+]$.

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PROPOSITION 15. *Let H be a period-k hyperbolic component of* $\mathbf{M}$ *with tuning interval* $I(H) = [\theta_-, \theta_+]$ *and let* $p'/2^{m'} \in I(H)$, $1 \le m' < k$ *be the dyadic with the smallest denominator. For any irreducible rational* p/q, *let* $0 < \eta_- < \eta_+ < 1$ *be the arguments of the co-landing parameter rays bounding* $W^H_{p/q}$ *and let* $p/2^m \in [\eta_-, \eta_+]$ *be the dyadic with the smallest denominator. Then*

$$2^{-kq} \le \eta_+ - \eta_- \quad \text{and} \quad m = m' + k(q-2).$$

Proof. As $\eta_- < \eta_+$ are periodic of exact period kq, we have $\eta_+ - \eta_- \ge 1/(2^{kq} - 1) > 2^{-kq}$. For the second inequality, let M' denote the copy of $\mathbf{M}$ with H as the central hyperbolic component. Let $\theta_- < \theta_+ \in I^{M'}$ denote the arguments of the parameter rays co-landing at the root of M'. Let $I = [\theta_-, \theta_+] \supset I_0, I_1$, $I^{M'} \subset I_0 \cup I_1$ and $\kappa : I^{M'} \longrightarrow \Sigma_2$ be as above and write π for the binary projection of Σ_2 onto $\mathbb{T}$ and set $\widehat{\kappa} = \pi \circ \kappa$. Then $\tau_\pm = \widehat{\kappa}(\eta_\pm)$ are the arguments of the parameter rays co-landing at the root of the wake $W^{H_0}_{p/q}$. It is well known that the intervals $\sigma^j([\tau_-, \tau_+])$, $0 \le j < q$ are interiorly disjoint and injective images. Moreover, $0 \in \sigma^{(q-1)}([\tau_-, \tau_+])$ and thus $1/2 \in \sigma^{(q-2)}([\tau_-, \tau_+])$. Consequently, $\sigma^{k(q-2)}$ maps $[\eta_-, \eta_+]$ injectively into I. Moreover, $I \supset \sigma^{k(q-2)}([\eta_-, \eta_+]) \supset (I \setminus (I_0 \cup I_1))$. Let $p'/2^{m'} \in I$ be the dyadic with smallest denominator; then $p'/2^{m'} \in I \setminus (I_0 \cup I_1)$ and $1 \le m' \le k$. Thus, $m = m' + k(q-2)$. □

COROLLARY 16. *For any* $c \in W^H_{p/q} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})$, *the angle of vision* $\omega(c)$ *of* $i2\pi[\eta_-, \eta_+]$ *from* $\mathrm{Log}(\Phi(c))$ *is bounded from below by*

$$\arctan(2\pi 2^{m'-2k}).$$

Proof. The angle is bounded from below by the angle obtained when c belongs to one of the two bounding rays of $W^H_{p/q}$ and on the largest potential of $\Delta^{\mathbf{M}}$. This largest potential is obtained when $g_{\mathbf{M}}(c) = 1/2^m$ with $p/2^m$ the angle of the parameter ray entering the wake with smallest denominator, hence $\log|\Phi(c)| \le 1/2^m$:

$$\begin{aligned}\arctan(2\pi(\eta_+ - \eta_-)/\log|\Phi(c)|) &\ge \arctan(2\pi 2^{-kq}/2^{-(m'+k(q-2))}) \\ &= \arctan(2\pi 2^{m'-2k}).\end{aligned}$$ □

4.2. *Parameter inequality.*

THEOREM 17. (The Yoccoz–Levin parameter inequality) *For any hyperbolic component* H *of* $\mathbf{M}$, *there exists a constant* $C = C_H > 0$ *such that, for any relative* p/q*-wake* $W^H_{p/q}$,

$$\mathrm{diam}(W^H_{p/q} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})) \le \frac{C}{q}.$$

Proof. For $h = 0$, i.e., for the limbs $\mathbf{M} \cap W^H_{p/q}$, this is essentially proved by Hubbard in [**Hu**]. Note that for primitive hyperbolic components, he erroneously states the inequality with $C/\sqrt{q}$ instead of C/q. (The bounds actually give C/q^2 asymptotically when p/q tend to 0 or 1 and he reverts this to $C/\sqrt{q}$.) For the extension, we use the Levin–Yoccoz dynamical inequality above instead of the Yoccoz inequality. By Corollary 16, the angle $\omega(c)$ for $c \in W^H_{p/q} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})$ is bounded from below by

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the angle $\omega_H = \arctan(2\pi 2^{m'-2k})$. The argument is then identical to the argument in Hubbard's paper [**Hu**], except using the Levin–Yoccoz dynamical inequality with the fixed value ω_H. Thus, asymptotically for q large, we can take

$$C_H = \frac{\pi}{\omega_H} C_H^{\text{Yoccoz}},$$

where C_H^{Yoccoz} is the corresponding asymptotic value for the Yoccoz parameter inequality. □

COROLLARY 18. *We can assume that the sequence Δ_k of carrots belongs to $W_{p/q}^{H_0}$ for some fixed p/q.*

Proof. Let us first apply the Yoccoz–Levin parameter inequality. This gives a constant $C = C_{H_0} > 0$ such that, for all r/s,

$$\operatorname{diam}(W_{r/s}^{H_0} \cap (\mathbf{M} \cup \Delta^{\mathbf{M}})) \le \frac{C}{s}. \tag{1}$$

The sequence $\{\Delta_k\}$ of carrots is included in a sequence of wakes $W_{p_k/q_k}^{H_0}$ and the roots belong to the corresponding limbs $L_{p_k/q_k}^{H_0}$. If q_k tends to ∞, then the diameter of Δ_k tends to 0 by (1). On the other hand, if the sequence $\{p_k/q_k\}$ contains a bounded subsequence, then it contains a constant subsequence $p_{k_m}/q_{k_m} = p/q$. Hence, the limit point $c_\infty \in L_{p/q}^{H_0}$. For any Q, the limb $L_{p/q}^{H_0}$ is strongly separated from any of the limbs $L_{p'/q'}^{H_0}$, $p/q \neq p'/q'$ with $q' \le Q$. Hence, either $p_k/q_k = p/q$ for sufficiently large k or for the remaining elements in the sequence q_k tend to infinity, and thus $\operatorname{diam}(\Delta_k)$ tends to zero for these k. In both cases, we can suppose $p_k/q_k = p/q$ for some $p/q \in]0, 1[$. □

5. *Yoccoz puzzles and holomorphic motions*

5.1. *Definition of Yoccoz puzzles.* Recall from §3.4.4 that, for an irreducible rational p/q and $c \in W_{p/q}^{H_0}$, the α-fixed point α_c is the common landing point of the unique p/q-cycle of external rays. Let $\{\theta_0, \ldots, \theta_{q-1}\}$ denote the corresponding p/q-cycle of arguments.

Definition 19. The p/q-'model' graphs are

$$\mathcal{G}_0 = \{z \mid |z| = e\} \cup \bigcup_{0 \le j \le q-1} [1, e]e^{2i\pi\theta_j}, \quad \mathcal{G}_n = Q_0^{-n}(\mathcal{G}_0).$$

For $c \in L_{p/q}^{H_0}$, the p/q-Yoccoz graphs are $\mathcal{Y}_0^c = \overline{\phi_c^{-1}(\mathcal{G}_0)}$, $\mathcal{Y}_n^c = \overline{\phi_c^{-1}(\mathcal{G}_n)} = Q_c^{-n}(\mathcal{Y}_0^c)$.

The puzzle pieces of level n are the closures of the bounded connected components of $\mathbb{C}\backslash\mathcal{Y}_n^c$, where $\mathcal{Y}_n^c$ is the level-n Yoccoz graph.

Definition 20. The p/q-Yoccoz parameter graphs are $\mathcal{Y}_0 = \overline{\Psi(\mathcal{G}_0)} \cap \overline{W_{p/q}^{H_0}}$ and $\mathcal{Y}_n = \overline{\Psi(\mathcal{G}_n)} \cap \overline{W_{p/q}^{H_0}}$. The parameter puzzle pieces of level n are the closures of the bounded connected components of $\mathbb{C}\backslash\mathcal{Y}_n$, where $\mathcal{Y}_n$ is the level-n Yoccoz parameter graph.

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The level-n parameter puzzle piece(s) containing c is the closed subset of $\mathbb{C}$ bounded by the closure of $\Psi \circ \phi_c(\partial P_n^c)$, where P_n^c denotes the (or possibly any of the) q level-n puzzle piece(s) in the dynamical plane containing c. (If c, c' belong to the same level-n parameter puzzle piece $\mathcal{P}_n$, then $\phi_{c'}^{-1} \circ \phi_c$ induces a homeomorphism between the Yoccoz graphs for c and c', at least up to and including level n.)

5.2. *Yoccoz theorem and its application.*

THEOREM 21. (Yoccoz) *For any p/q and any $c \in L_{p/q}^{H_0}$, there are two possibilities: either Q_c is not renormalizable and $\cap \mathcal{P}_n = \{c\}$ for any nest $(\mathcal{P}_n)$ of parameter puzzle pieces containing c, or c is at least once renormalizable, say with first renormalization period k, and there is a first level n such that, for the dynamical puzzle pieces $Int(P_n^c) =: U$ and $Int(P_{n+k}^c) =: U'$, the map $Q_c^k : U' \longrightarrow U$ is quadratic-like with connected filled-in Julia set (fattening U and U' if $k = q$.)*

For a proof, see [**Hu**].

We provide a rough proof of Theorem 4 using the puzzles, the reader will easily supply the details.

THEOREM. (Douady–Hubbard, Yoccoz) *For any copy M' of $\mathbf{M}$ in $\mathbf{M}$,*

$$\mathbf{M} = M' \cup \bigcup_{n \geq 0} \bigcup_{p/2^n \in \mathbb{Q}} \Delta_{p/2^n}^{M'}.$$

Note that M' is denoted $\mathbf{M_0}$ in Theorem 4.

Proof. The copy M' of $\mathbf{M}$ belongs to the limb $L_{p'/q'}^{H_0}$ of the central hyperbolic component H_0 of $\mathbf{M}$, for some $p', q' \in \mathbb{N}$ with $(p', q') = 1$. Let $c \in M'$ be the center of M', i.e., $Q_c^k(c) = c$, where k is the period of M'. Let P_n, $n \in \mathbb{N}$ be the level n (p'/q')-Yoccoz puzzle piece containing the critical value c of Q_c, and let $\mathcal{P}_n$ denote (p'/q')-parameter Yoccoz puzzle piece containing the parameter c. Then, for each n, the map $\Psi \circ \phi_c(z)$ restricted to $\partial P_n \setminus J_c$ extends to a homeomorphism of ∂P_n onto $\partial \mathcal{P}_n$, preserving argument and potential. Also, for each n, the closed puzzle piece P_n contains the ends of the external rays with arguments in $I^{M'}$ from potential 2^{-n} (and down to the end). Hence, the same holds for the corresponding parameter rays and $\mathcal{P}_n$. It follows that any other level-n parameter puzzle piece, as well as $\mathbf{M} \setminus L_{p'/q'}^{H_0}$, is contained in one of the relatively dyadic wakes $W'_{p/2^m}$ of M'. The theorem then follows from the Yoccoz parameter puzzle theorem for renormalizable parameters, which states that

$$M' = \bigcap_{n \geq 0} \overline{\mathcal{P}}_n. \qquad \square$$

COROLLARY 22. *Either the diameter of the sequence Δ_k tends to 0 or c_∞ is in a strict copy M' of $\mathbf{M}$ in $\mathbf{M}$ (i.e., Q_{c_∞} is renormalizable).*

Proof. For any p/q, the corresponding rotation orbit $0 < \theta_0 < \cdots < \theta_{q-1}$ is disjoint from the set of dyadic arguments. Thus, for any p/q, the graph defining the associated p/q puzzle for $L_{p/q}^{H_0}$ is disjoint from $\Delta^{\mathbf{M}}$. Therefore, there exists an increasing sequence n_k

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such that, for $k \geq k_0$, the carrots are in the nest of puzzle pieces: $\Delta_k \subset \overline{\mathcal{P}}_{n_k}(c_\infty)$ (the parameter puzzle piece containing c_∞).

Hence, by the Yoccoz theorem 21, either the diameter tends to 0 or the limiting parameter c_∞ is renormalizable, that is, $c_\infty \in M'$ for some period k first renormalization copy M' of $\mathbf{M}$ in $L^{H_0}_{p/q}$, where $q \leq k$. □

For the rest of the proof, it is enough now to consider the parameters c_∞ in a k-periodic, $k \geq q > 1$ first renormalization copy $\mathbf{M}' \subsetneq L^{H_0}_{p/q}$ of $\mathbf{M}$.

Remark 23. The key point in the proof of Proposition 8 is that all the dyadic carrots $\Delta^{\mathbf{M}}_{p/2^n}$ are disjoint from M' because their root points $\Psi(p/2^n)$ are disjoint from M'. This in turn is because the tuning interval $I(M')$ is σ^k-invariant and does not contain the fixed point $0 = 1$ of σ.

6. *Dynamical estimates*

By the above, we can assume the limit point $c_\infty \in M' \subset L^{H_0}_{p/q}$, where M' is a first renormalization copy of period $k \geq q$. We first consider the case where M' is a primitive first renormalization copy and postpone the q-periodic satellite case until later. Let c_b be the center of the principal hyperbolic component H' of period k of M'. We first define for this parameter c_b, a slicing of some neighborhood of the Julia set that allows us to wrap the carrots $\Delta^{c_b}_{p/2^n} \subset W^{H',c_b}$ in annuli (for the terminology see the founding wake on p. 2032). We prove a lower bound on the moduli of these annuli. Then we deform this picture by a holomorphic motion on some open set $W^{H'} \cap O^{\mathbf{M}}(1)$ (which contains M', except for its root). This will allow us to transfer the dynamical annuli to parameter annuli around the carrots $\Delta^{\mathbf{M}}_{p/2^n} \subset W^{H'}$, keeping a lower bound on the moduli. Finally we treat in a similar manner those carrots which approach the root c_0 of M' from the outside of $W^{H'}$. In the following section, we shall indicate the changes to the above argument that are needed in order to treat the satellite case. Throughout, we shall make use of the reduction of the carrots to sticks, in order to avoid considerations of a carrot intersecting other rays than its dyadic root ray.

6.1. *Slicing of some part of the Julia set.*

(See Figure 5.) Since $M' \subset L^{H_0}_{p/q}$, one can define a Yoccoz puzzle as in Definition 19. Then Theorem 21 says that, for $c \in M'$, there is a first level n such that the puzzle piece P^c_n containing the critical value c defines a quadratic-like map as $Q^k_c : Int(P^c_{n+k}) \to Int(P^c_n)$.

In this subsection, we let $c = c_b$ be the center of the principal hyperbolic component H' of M'. We need some notations to describe an extended puzzle structure for Q_c. Clearly, this structure is common, at least for all $c \in M'$.

- Let $\eta_- < \eta_+$ denote the (rational) arguments of the co-landing pair of external rays for Q_c, which are on the boundary of $P = P^c_{n+k}$ and which separate c from 0. Then, by Douady and Hubbard (see [**DH1**]), the parameter rays $R^{\mathbf{M}}_{\eta_\pm}$ co-land at some Misiurewicz point in $\mathbf{M}$.
- Denote by γ^c_0 the set $\overline{R^c_{\eta_-} \cup R^c_{\eta_+}} \cap F^c(1)$ and let U^c_0 denote the connected component of $O^c(1) \setminus \gamma^c_0$ containing c.

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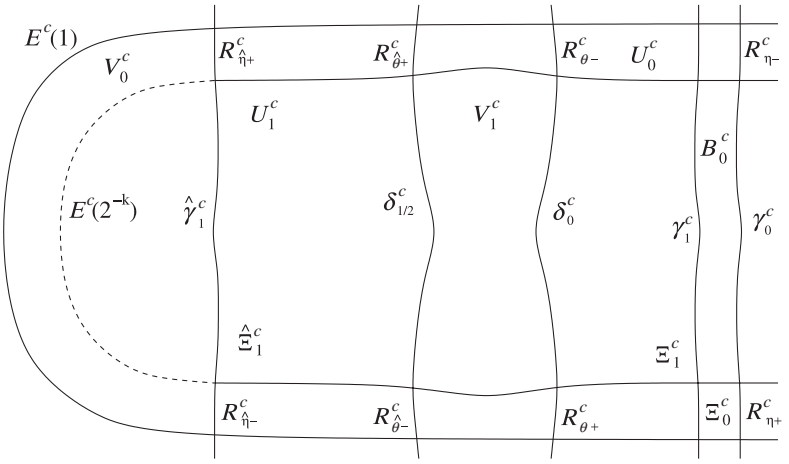


FIGURE 5. The disks U_0^c, U_1^c, and $\widehat\Xi_1^c$. The set $U_0^c \subset O^c(1)$ is the disk which lies at the left of γ_0^c. The set $U_1^c \subset O^c(2^{-k})$ is the disk which is bounded at the right and left by γ_1^c and $\widehat\gamma_1{}^c$, respectively. The arc δ_0^c separates the two subdisks V_0^c and Ξ_0^c of U_0^c, V_0^c is at the left and Ξ_0^c at the right of δ_0^c. The subsets $\widehat\Xi_1^c$, V_1^c and Ξ_1^c of U_1^c are at the left of $\delta^c_{1/2}$, between $\delta^c_{1/2}$ and δ_0, and at the right of δ_0.

- Let U_1^c denote the connected component of $Q_c^{-k}(U_0^c)$ contained in U_0^c and containing c. It is a disk and $f_c := Q_c^k : U_1^c \to U_0^c$ is quadratic-like, with a critical point ω_c satisfying $f_c(\omega_c) = c$.

Next, let us introduce some notations related to the copy M'. Let $0 < \theta_- < \theta_+ < 1$ denote the root arguments of M', i.e., the arguments of the parameter rays co-landing on the root c_0' of M' and bounding its founding wake $W^{H'}$. Recall that for any $c \in W^{H'}$, the dynamical rays $R^c_{\theta_\pm}$ co-land on a repelling k-periodic point denoted β_c'. Moreover, $\theta_\pm$ bounds the tuning interval $I = I(M')$ and the restrictions $\sigma^k : [\theta_-, \widehat\theta_+], [\widehat\theta_-, \theta_+] \longrightarrow I$ are diffeomorphisms, where $\theta_- < \widehat\theta_+ < \widehat\theta_- < \theta_+$.

The angles $\eta_- < \theta_- < \widehat\theta_+ < \widehat\theta_- < \theta_+ < \eta_+$ will remain fixed throughout §§6 and 7.

- Let δ_0^c denote the subarc

$$\delta_0^c = (R^c_{\theta_-} \cup \{\beta_c'\} \cup R^c_{\theta_-}) \cap F^c(1) = \overline{R^c_{\theta_-} \cup R^c_{\theta_+}} \cap F^c(1).$$

- Similarly, let $\delta^c_{1/2}$ denote the subarc $\delta^c_{1/2} = \overline{R^c_{\widehat\theta_-} \cup R^c_{\widehat\theta_+}} \cap F^c(1)$.
- Let V_0^c be the connected component of $U_0^c \smallsetminus \delta_0^c$ containing c. Let Ξ_0^c denote the other connected component. Define recursively disks

$$V_n^c = f_c^{-1}(V_{n-1}^c), \quad \Xi_n^c \cup \widehat\Xi_n^c = f_c^{-n}(\Xi_0^c),$$

where $\Xi_n^c \subset \Xi_{n-1}^c \subset \Xi_0^c$ and $\widehat\Xi_n^c$ is the twin of Ξ_n^c so that the disks $\widehat\Xi_n^c$ also form a nested sequence (see also Figure 6). The restriction $f_c : V_{n+1}^c \longrightarrow V_n^c$ is a branched covering of degree two, whereas the maps $f_c : \Xi_{n+1}^c \longrightarrow \Xi_n^c$ and $f_c : \widehat\Xi_{n+1}^c \longrightarrow \Xi_n^c$ are isomorphisms.
- Let γ_1^c denote the extension to potential level 1 of $Q_c^{-k}(\gamma_0^c) \cap \partial\Xi_0^c$ and let $B_0^c \subset \Xi_0^c$ denote the quadrilateral bounded by γ_0^c, γ_1^c and subarcs of $E^c(1)$. Define recursively the univalently iterated preimages $B_{n+1}^c = f_c^{-1}(B_n^c) \cap \Xi_n^c$.

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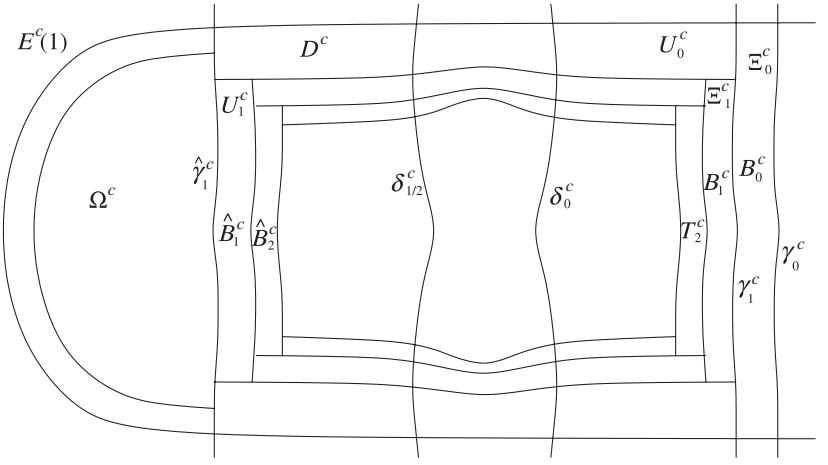


FIGURE 6. The decomposition of the disk D^c, which is bounded by the equipotential $E^c(1)$ and the arc $\delta_{1/2}^c$. The disk Ξ_0^c is the subset of U_0^c to the right of δ_0^c. The disk $\widehat{\Xi}_1^c$ (not labeled) is the subset of U_1^c to the left of $\delta_{1/2}^c$. The disk Ω^c is to the left of $\widehat{\gamma}_1{}^c$ and inside the equipotential $E^c(3/4)$ (not labeled).

- For each $n \geq 1$, let $\widehat{B}_n^c \subset \widehat{\Xi}_n^c$ denote the 'twin' of B_n^c, i.e., the connected component of $f_c^{-1}(B_{n-1}^c) \cap \widehat{\Xi}_n^c$.
- Let $\widehat{\gamma}_1{}^c = Q_c^{-k}(\gamma_0^c) \cap \partial\widehat{\Xi}_1^c$ extend to equipotential level $3/4$ and let Ω^c denote the open disk bounded by $\widehat{\gamma}_1{}^c$ and the subarc of $E^c(3/4) \cap U_0^c$ connecting the endpoints of $\widehat{\gamma}_1{}^c$.
- Let the disk $D^c \subset V_0^c$ denote the connected component of $O^c(1) \setminus \delta_{1/2}^c$ which contains the sets $\widehat{\Xi}_n^c$ as well as Ω^c.

LEMMA 24. *For $c = c_b$, each of the sets Ω^c and $\widehat{B}_n^c$, $n \geq 1$ is relatively compact in D^c. Moreover, there exists $m = m(c_b) > 0$ such that*

$$\operatorname{mod}(D^c \setminus \Omega^c) \geq m, \quad \textit{and} \quad \textit{for all } n \geq 1 \quad \operatorname{mod}(D^c \setminus \overline{\widehat{B}_n^c}) \geq m,$$

so that

$$\textit{for all } n \geq 1 \quad \operatorname{mod}(\Xi_0^c \setminus \overline{B_n^c}) \geq m.$$

Proof. Compactness is by construction. The restriction $f_c : \widehat{\Xi}_1^c \longrightarrow \Xi_0^c$ is biholomorphic so that, for all $n \geq 1$,

$$\begin{aligned} \operatorname{mod}(D^c \setminus \overline{\widehat{B}_{n+1}^c}) &\geq \operatorname{mod}(\widehat{\Xi}_1^c \setminus \overline{\widehat{B}_{n+1}^c}) = \operatorname{mod}(\Xi_0^c \setminus \overline{B_n^c}) \\ &\geq \operatorname{mod}(\Xi_{n-1}^c \setminus \overline{B_n^c}) = \operatorname{mod}(\Xi_0^c \setminus \overline{B_1^c}). \end{aligned}$$

Thus, we may define

$$m(c) = \min\{\operatorname{mod}(D^c \setminus \Omega^c), \operatorname{mod}(D^c \setminus \overline{\widehat{B}_1^c}), \operatorname{mod}(\Xi_0^c \setminus \overline{B_1^c})\}. \qquad \square$$

6.2. *Holomorphic motion of graphs.* A holomorphic motion of a set $X \subset \mathbb{C}$ parameterized by $\mathbb{D}$ is a map $H(\lambda, z)$ from $\mathbb{D} \times X$ to $\mathbb{C}$ that is holomorphic in $\lambda \in \mathbb{D}$ injective in $z \in \mathbb{C}$ and the identity for $\lambda_0 = 0$.

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Douady and Hubbard proved in [**DH1**] that the closure $\overline{R}$ of an external ray R_t^{cb} admits a holomorphic motion when t is eventually periodic and when the orbit of $\overline{R}$ does not meet either a critical point or a parabolic point.

We shall use the following theorem on **M**, presumably due to Douady, Hubbard and Lavaurs. This result is at least folklore, but since we did not find a precise reference in the literature, we have, for the sake of completeness, provided a proof of it in the Appendix.

THEOREM 25. *Let $0 < t_- < t_+ < 1$ be rationals for which the parameter rays $R^{\mathbf{M}}_{t_\pm}$ co-land at some point $c_0 \in \mathbf{M}$, and let $W^{\mathbf{M}}_{t_\pm}$ denote the parameter sector bounded by $R^{\mathbf{M}}_{t_-} \cup \{c_0\} \cup R^{\mathbf{M}}_{t_+}$ and not containing the main cardioid H_0. Then the following hold.*

- *t_-, t_+ have the same preperiod $l \geq 0$ and period $k > 0$, $\sigma^{k+l}(t_\pm) = \sigma^l(t_\pm)$ and $\sigma^{j+i}(t_\pm) \neq \sigma^j(t_\pm)$, for $j < l$ or $i < k$.*
- *The forward orbits of $t_\pm$ under σ do not enter the interval $]t_-, t_+[$.*
- *For any $c \in W^{\mathbf{M}}_{t_\pm}$, the pair of dynamical rays $R^c_{t_\pm}$ co-land at some repelling (pre)periodic point $z(c)$ with $Q_c^{l+k'}(z(c)) = Q_c^l(z(c))$, where k' divides k.*
- *The set $R^c_{t_-} \cup \{z(c)\} \cup R^c_{t_+}$ moves holomorphically with $c \in W^{\mathbf{M}}_{t_\pm}$ and bounds a sector $W^c_{t_\pm}$ containing c and the rays of arguments $t \in]t_-, t_+[$, but not 0.*

Recall that $\theta_\pm$ are the root arguments of M' and that $\eta_\pm$ with $0 < \eta_- < \theta_- < \theta_+ < \eta_+ < 1$ are the arguments of the co-landing pair of parameter rays such that $M' \subset W^{\mathbf{M}}_{\eta_\pm}$ and such that, for $c = c_b \in M'$, the closure of the corresponding co-landing dynamical rays $R^c_{\eta_\pm}$ intersect the boundary of the Yoccoz puzzle piece $P = P_{n+k}$ in an arc γ_0^c, where n is minimal with the property that the restriction $Q_c^k : P_{n+k} \longrightarrow P_n$ is quadratic-like.

Let γ_1^c denote the connected component of $Q_c^{-k}(\gamma_0^c)$ on the boundary of U_1^c and separating 0 and c in the disk $O^c(2^{-k})$, and let $\widehat{\gamma}_1{}^c \subset \partial U_1^c$ denote its non-separating twin. Denote by $\eta'_\pm$, $\eta_- < \eta'_- < \eta'_+ < \eta_+$ the arguments of the rays defining γ_1^c and let $\widehat{\eta}\,'_\pm$, $\eta'_- < \widehat{\eta}\,'_+ < \widehat{\eta}\,'_- < \eta'_+$ denote those for the twin. Then $\eta_- < \eta'_- < \theta_- < \widehat{\theta}_+ < \widehat{\eta}\,'_+ < \widehat{\eta}\,'_- < \widehat{\theta}_- < \theta_+ < \eta'_+ < \eta_+$, $\sigma^k(\eta'_\pm) = \sigma^k(\widehat{\eta}\,'_\pm) = \eta_\pm$ and U_1^c is the middle connected component of $O^c \setminus (\gamma_1^c \cup \widehat{\gamma}_1{}^c)$.

Definition 26.

- Denote by $\gamma_0^{\mathbf{M}}$ and $\delta_0^{\mathbf{M}}$ the arcs
$$\gamma_0^{\mathbf{M}} = \overline{R^{\mathbf{M}}_{\eta_-} \cup R^{\mathbf{M}}_{\eta_+}} \cap F^{\mathbf{M}}(1), \quad \delta_0^{\mathbf{M}} = \overline{R^{\mathbf{M}}_{\theta_-} \cup R^{\mathbf{M}}_{\theta_+}} \cap F^{\mathbf{M}}(1).$$
- Denote by $\Lambda_0 \supset \Lambda_1$ the parameter disks
$$\Lambda_0 = W^{H'} \cap O^{\mathbf{M}}(2), \quad \Lambda_1 = W^{H'} \cap O^{\mathbf{M}}(1).$$
- Denote by $\Lambda_0^P \supset \Lambda_1^P$ the larger parameter disks which are the connected components of $O^{\mathbf{M}}(2) \setminus \overline{R^{\mathbf{M}}_{\eta_-} \cup R^{\mathbf{M}}_{\eta_+}}$ and $O^{\mathbf{M}}(1) \setminus \overline{R^{\mathbf{M}}_{\eta_-} \cup R^{\mathbf{M}}_{\eta_+}}$, respectively, containing Λ_1.
- We extend the definition of the arc γ_0^c and the disk U_0^c to all $c \in \Lambda_0^P$, that is, for such c,
$$\gamma_0^c = \overline{R^c_{\eta_-} \cup R^c_{\eta_+}} \cap F^c(1),$$
and U_0^c denotes the connected component of $O^c(1) \setminus \gamma_0^c$ containing c.
- Let $\widehat{U}_0^c$ denote the connected component of $O^c(2) \setminus \overline{R^c_{\eta_-} \cup R^c_{\eta_+}}$ containing U_0^c.

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PROPOSITION 27. *Fix any base point in Λ_0^P, such as c_b. Then the arc γ_0^c and the set $\overline{\mathbb{C}}\setminus O^c(1)$ move holomorphically with $c \in \Lambda_0^P$ under the motion given by the Böttcher coordinate ϕ_c. In particular, ∂U_0^c moves holomorphically with $c \in \Lambda_0^P$.*

Proof. By definition, γ_0^c is the set $\overline{R_{\eta_-}^c \cup R_{\eta_+}^c} \cap F^c(1)$ and U_0^c denotes the component of $O(1)\setminus\gamma_0^c$ not containing 0. It follows from Theorem B.1 that the dynamical rays $R_{\eta_\pm}^c$ co-land for every $c \in \Lambda_0^P$ and admit a holomorphic motion. Moreover, by stability, they separate 0 from c. Finally, $\overline{\mathbb{C}}\setminus O^c(1) = \psi_c(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}(e))$ moves holomorphically with c because the critical point is not in the set. □

COROLLARY 28. *The Böttcher coordinate ϕ_c also defines a holomorphic motion over Λ_0^P of the rays $R_{\eta'_-}^c$, $R_{\eta'_+}^c$, $R_{\widehat{\eta}'_-}^c$, $R_{\widehat{\eta}'_+}^c$. In particular, they pairwise co-land and this defines a holomorphic motion of γ_1^c and $\widehat{\gamma}_1^c$.*

Proof. The forward orbits of the rays $R_{\eta'_-}$, $R_{\eta'_+}$, $R_{\widehat{\eta}'_-}$, $R_{\widehat{\eta}'_+}$ never enter the sector $W_{\eta_\pm}^c$ containing the critical value and thus they also move holomorphically with $c \in W_{\eta_\pm}^{\mathbf{M}}$. Since they pairwise co-land for $c = c_b$, they pairwise co-land for all $c \in W_{\eta_\pm}^{\mathbf{M}}$, and finally their tips up to and including potential 2^{-k} are dense in γ_1^c and $\widehat{\gamma}_1^c$. □

Notice that Λ_0^P is precisely the set of parameters for which $c \in \widehat{U}_0^c$ and in fact $c \in \partial\widehat{U}_0^c$ if and only if $c \in \partial\Lambda_0^P$. For $c \in \Lambda_0^P$, let $\widehat{U}_1^c$ denote the connected component of $Q_c^{-k}(\widehat{U}_0^c) \cap \widehat{U}_0^c$ which intersects the pairs of rays $R_{\theta_\pm}$ and $R_{\widehat{\theta}_\pm}$, or, equivalently, let $\widehat{U}_1^c$ denote the middle connected component of

$$O^c(2^{1-k})\setminus(\overline{R_{\eta'_-} \cup R_{\eta'_+}} \cup \overline{R_{\widehat{\eta}'_-} \cup R_{\widehat{\eta}'_+}}).$$

Define

$$U_1^c = \widehat{U}_1^c \cap O^c(2^{-k}).$$

Note that, depending on the position of $c \in \Lambda_0^P$, this set may be connected or not. We leave the proof of the following easy lemma to the reader.

LEMMA 29. *The restriction*

$$f_c := Q_c^k : \widehat{U}_1^c \longrightarrow \widehat{U}_0^c \tag{2}$$

is quadratic-like with critical value c and critical point ω_c depending holomorphically on $c \in W_{\eta_\pm}^{\mathbf{M}}$.
For $c \in \Lambda_0^P\setminus\Lambda_1^P$, U_1^c is the union of two disks and the restriction of f_c to U_1^c is non-ramified.
For $c \in \Lambda_1^P$, U_1^c is a disk whose boundary ∂U_1^c moves holomorphically with $c \in \Lambda_1^P$ and the restriction $f_c : U_1^c \longrightarrow U_0^c$ is quadratic-like. The filled Julia set K_c' is connected if and only if $c \in M'$.

LEMMA 30.

$$E^c(1) \cup \bigcup_{j=0}^{k-1} Q_c^j(R_{\theta_-}^c \cup \{\beta_c'\} \cup R_{\theta_+}^c) \cup Q_c^j(R_{\widehat{\theta}_-}^c \cup \{c o \beta_c'\} \cup R_{\widehat{\theta}_+}^c)$$

moves holomorphically with $c \in \Lambda_0$.

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Proof. Here we use Theorem B.1. For any $c \in \Lambda_0$, the dynamical rays $R^c_{\theta_\pm}$ co-land at the repelling k-periodic point β'_c and the rays $R^c_{\widehat{\theta}_\pm}$ co-land at the f_c-preimage $c\alpha\beta'_c$ of β'_c, all of which move holomorphically with $c \in \Lambda_0$. Moreover, the set

$$\bigcup_{j=0}^{k-1} Q_c^j(R^c_{\theta_-} \cup \{\beta'_c\} \cup R^c_{\theta_+}) \cup Q_c^j(R^c_{\widehat{\theta}_-} \cup \{c\alpha\beta'_c\} \cup R^c_{\widehat{\theta}_+})$$

does not enter the sector $W^c_{\theta_\pm}$, which contains the critical value. Hence,

$$E^c(1) \cup \bigcup_{j=0}^{k-1} Q_c^j(R^c_{\theta_-} \cup \{\beta'_c\} \cup R^c_{\theta_+}) \cup Q_c^j(R^c_{\widehat{\theta}_-} \cup \{c\alpha\beta'_c\} \cup R^c_{\widehat{\theta}_+})$$

moves holomorphically with $c \in \Lambda_0$. □

COROLLARY 31. *In particular,* $\delta_0^c := \overline{R_{\theta_-} \cup R_{\theta_+}} \cap F^c(1)$ *and* $\delta^c_{1/2} := \overline{R_{\widehat{\theta}_-} \cup R_{\widehat{\theta}_+}} \cap F^c(1)$ *move holomorphically with* $c \in \Lambda_0$.

For $c \in \Lambda_1$, the set ∂U_1^c does admit a holomorphic motion. For $c \in \Lambda_0 \setminus \Lambda_1$, it does not. However, we have the following lemma.

LEMMA 32. *For* $c \in \Lambda_0 \setminus \Lambda_1$, *consider* $\widetilde{V}^c$, *the subdisk of* U_0^c *bounded by the arcs* δ_0^c, $\delta^c_{1/2}$ *and containing the* f_c *critical point* $\omega(c)$. *Then* $\partial U_1^c \setminus \widetilde{V}^c$ *moves holomorphically with* $c \in \Lambda_0$.

Proof. Indeed, the only place where the boundary of ∂U_1^c does not admit a holomorphic motion is when the parameter c crosses $\partial\Lambda_1 \cap \Lambda_0$. At such parameters, the equipotential of ∂U_0^c becomes a figure of eight with crossing contained in $\widetilde{V}^c$. □

COROLLARY 33. *Let* V_0^c *be the connected component of* $U_0^c \setminus \delta_0^c$ *containing* $\delta^c_{1/2}$. *Then* ∂V_0^c *admits a holomorphic motion over* Λ_0. *Moreover, for* $c \in \Lambda_1$, *the preimage* $V_1^c = f_c^{-1}(V_0^c)$ *is a disk whose boundary* ∂V_1^c *admits a holomorphic motion over* Λ_1.
For $c \in \Lambda_0 \setminus \Lambda_1$, *the preimage* V_1^c *is the union of two disks.*
Finally, for any $c \in \Lambda_0$, $f_c : V_1^c \to V_0^c$ *is a degree-two (possibly) ramified covering.*

Proof. By the lemma, the arcs δ_0^c, $\delta^c_{1/2}$ move holomorphically over Λ_0. Moreover, $\partial V_0^c \setminus \delta_0^c \subset U_0^c$, so this part also moves holomorphically with c over Λ_0 and its preimage moves holomorphically with $c \in \Lambda_1$. □

Note that for $c \in \Lambda_0 \backslash \Lambda_1$, the preimage V_1^c is the union of two disks contained in $\widetilde{V}^c$.

LEMMA 34. *Using the quadratic-like restrictions* f_c *of* Q_c^k *for* $c \in \Lambda_0$, *we can extend the definitions of* V_n^c, Ξ_n^c, B_n^c, $\widehat{B}_n^c$ *and* $\widehat{\Xi}_n^c$, *for* $c = c_b$, *to any* $c \in \Lambda_0$.

Proof. They are simply the pull-back by f_c of sets like U_0^c and V_0^c that are well defined in Λ_0 and whose boundary admits a holomorphic motion. □

We recall here the definitions for $c \in \Lambda_0$ that extend the ones given for c_b.

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- Define recursively disks

$$V_n^c = f_c^{-1}(V_{n-1}^c), \quad \Xi_n^c \cup \widehat{\Xi}_n^c = f_c^{-n}(\Xi_0^c),$$

where $\Xi_n^c \subset \Xi_{n-1}^c \subset \Xi_0^c$ and $\widehat{\Xi}_n^c$ is the twin of Ξ_n^c so that the disks $\widehat{\Xi}_n^c$ also form a nested sequence (see also Figure 6). The restriction $f_c : V_{n+1}^c \longrightarrow V_n^c$ is a branched covering of degree two, whereas the maps $f_c : \Xi_{n+1}^c \longrightarrow \Xi_n^c$ and $f_c : \widehat{\Xi}_{n+1}^c \longrightarrow \Xi_n^c$ are isomorphisms.
- Let γ_1^c denote the extension to potential level 1 of $Q_c^{-k}(\gamma_0^c) \cap \partial\Xi_0^c$ and let $B_0^c \subset \Xi_0^c$ denote the quadrilateral bounded by γ_0^c, γ_1^c and subarcs of $E^c(1)$. Define recursively the univalently iterated preimages $B_{n+1}^c = f_c^{-1}(B_n^c) \cap \Xi_n^c$.
- For each $n \geq 1$, let $\widehat{B}_n^c \subset \widehat{\Xi}_n^c$ denote the 'twin' of B_n^c, i.e., the connected component of $f_c^{-1}(B_{n-1}^c) \cap \widehat{\Xi}_n^c$.
- Extend $\widehat{\gamma}_1^c = Q_c^{-k}(\gamma_0^c) \cap \partial U_1^c$ to equipotential level 3/4 and let Ω^c denote the connected component of $O^c(3/4) \setminus \widehat{\gamma}_1^c$ disjoint from U_1^c.
- Let the disk $D^c \subset V_0^c$ denote the connected component of $O^c(1) \setminus \delta_{1/2}^c$ which contains the sets $\widehat{\Xi}_n^c$ as well as Ω^c.

Then the lemma given below follows.

LEMMA 35. *The graph*

$$G^c = \partial D^c \cup \partial\Omega^c \cup \bigcup_{n=1}^{\infty}(\partial\widehat{\Xi}_n^c \cup \partial\widehat{B}_n^c) \cup \bigcup_{n=0}^{\infty}(\partial\Xi_n^c \cup \partial B_n^c) \cup \partial U_0^c$$

moves holomorphically with $c \in \Lambda_0$.

Proof. Note that $Q_c^n(G^c)$ does not meet the critical point 0 for any $n \geq 0$ for $c \in \Lambda_0$: when the parameter $c \in \Lambda_0$, the critical value c does not belong to $\overline{\Xi_0^c}$. Hence, the Böttcher coordinate is defined and depends holomorphically on c, on the dense subset $(\partial\Xi_0^c \cup \partial B_0^c) \setminus K_c$ of $(\partial\Xi_0^c \cup \partial B_0^c)$, so that the latter moves holomorphically with c. Secondly, f_c depends holomorphically on c and its critical value c again still does not belong to $\overline{\Xi_0^c}$. Hence, the iterated univalent preimages of $(\partial\Xi_0^c \cup \partial B_0^c)$ inside Ξ_0^c and $\overline{\widehat{\Xi}_1^c}$ also depend holomorphically on c. This takes care of

$$\bigcup_{n=1}^{\infty}(\partial\widehat{\Xi}_n^c \cup \partial\widehat{B}_n^c) \cup \bigcup_{n=0}^{\infty}(\partial\Xi_n^c \cup \partial B_n^c).$$

Finally, ∂D^c moves holomorphically with c because $\delta_{1/2}^c$ does, and $\partial\Omega^c$ moves holomorphically with c because $\widehat{\gamma}_1^c$ does. □

COROLLARY 36. *For* $c \in \Lambda_0$, *there exists* $m = m(c) > 0$ *such that for all* $n \geq 1$

$$\operatorname{mod}(D^c \setminus \Omega^c) \geq m, \quad \operatorname{mod}(D^c \setminus \overline{\widehat{B}_n^c}) \geq m \quad \textit{and} \quad \operatorname{mod}(\Xi_0^c \setminus \overline{B_n^c}) \geq m.$$

Proof. The proof follows from Lemma 24 and the fact that the boundaries of these sets admit a holomorphic motion. □

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6.3. *Dynamical sticks.* We define the relatively dyadic wakes $W_0^c = \mathbb{C} \setminus \overline{W_{\theta_\pm}^c}$ and $W_{1/2}^c = W_{\widehat{\theta}_\pm}^c$ as the open set bounded by the closure of the rays $R_{\widehat{\theta}_\pm}^c$ and disjoint from W_0^c. For any $c \in \Lambda_0$, the filled-in Julia set $K_c' \subset K_c$, defined above, coincides with the points in the filled-in Julia set of K_c, whose orbits never enter the relatively dyadic wakes W_0^c and $W_{1/2}^c$ (see (2)).

The key point in the proof of Proposition 8 is that if a carrot intersects Λ_0, then it is entirely contained in Λ_0 and its dynamical counterpart in the dynamical planes of Q_c is either contained in $W_{1/2}^c$ or has a univalent forward image which is. In order to prove the theorem, we shall wrap the dynamical counterpart of each dyadic carrot inside the relatively dyadic wake $W_{1/2}^c$ in an annulus in $W_{1/2}^c$ moving holomorphically with $c \in \Lambda_0$ and of modulus bounded uniformly from below. We do this in the next subsection.

Let $\Delta^c = \{\Delta_{p/2^m}^c \mid p \text{ odd}, m \geq 0\}$ denote the dynamical field of dyadic sticks. They are defined as $\phi_c^{-1}(\Delta_{p/2^m})$, where we use for ϕ_c^{-1} the maximal radial extension. (We can then view the carrots/sticks $\Delta_{p/2^m}^{\mathbf{M}}$ of $\mathbf{M}$ as the set of parameters c for which the critical value c belongs to the corresponding carrot/stick $\Delta_{p/2^m}^c$ of the filled Julia set K_c.)

LEMMA 37. *The graph*

$$G^c = \partial D^c \cup \partial \Omega^c \cup \bigcup_{n=1}^{\infty} (\partial \widehat{\Xi}_n^c \cup \partial \widehat{B}_n^c) \cup \bigcup_{n=0}^{\infty} (\partial \Xi_n^c \cup \partial B_n^c) \cup \partial U_0^c$$

does not intersect any of the dynamical dyadic sticks $\Delta_{p/2^m}^c$.

Proof. All such sticks are at dyadic angles and only Δ_0^c extends further than potential $1/2$. □

COROLLARY 38. *Such a stick is thus wrapped by an annulus of modulus at least* $m(c) > 0$, *contained in* D^c *and thus disjoint from* K_c'.

Proof. No dynamical plane stick $\Delta_{p/2^n}^c$ intersects G^c. Hence, any such dyadic stick in the relative dyadic wake $W_{1/2}^c$ of the filled Julia set K_c' for $f_c : U_1^c \longrightarrow U_0^c$ is contained in one of the sets Ω^c or $\widehat{B}_n^c$ for some $n \geq 1$. The proof then follows from Corollary 36. □

7. *Transfer to the parameter plane*

We shall use an argument to transfer bounds on moduli of dynamical annuli to bounds on moduli of corresponding parameter annuli. This argument was pioneered by Shishikura (see also [**R**]).

We have fixed a base point, namely $c_b \in \Lambda_1 \subset \Lambda_0$, the center of the central hyperbolic component of M'. Note that on $F(2) \supset \Lambda_0$, the Böttcher coordinate at ∞ defines a holomorphic motion of the set $\overline{\mathbb{C}} \setminus F^{c_b}(1)$ extending the Böttcher motion of G^{c_b}. By Slodkowski's extension theorem, there exists a global holomorphic motion $H : \Lambda_0 \times \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ over Λ_0 with base point c_b and extending the Böttcher motion of the graph G^{c_b} union $\overline{\mathbb{C}} \setminus F^{c_b}(1)$; in particular, we obtain holomorphic motions of $\overline{U_0^{c_b}} \supset \overline{D^{c_b}}$. As usual, for $c \in \Lambda_0$, write $H_c(\cdot) = H(c, \cdot)$, then each map $H_c : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is a quasi-conformal homeomorphism with a dilatation bounded uniformly above by $\log \mathrm{d}_\Lambda(c, c_b)$, where $\mathrm{d}_\Lambda(\cdot, \cdot)$ denotes the hyperbolic distance in Λ_0.

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As in the dynamical plane, let

$$\delta_0^{M'} = \overline{R_{\theta_-}^{\mathbf{M}} \cup R_{\theta_-}^{\mathbf{M}}} \cap F^{\mathbf{M}}(1), \quad \delta_{1/2}^{M'} = \overline{R_{\widehat{\theta}_-}^{\mathbf{M}} \cup R_{\widehat{\theta}_-}^{\mathbf{M}}} \cap F^{\mathbf{M}}(1)$$

denote the parameter rays of the same angle as δ_0^c, $\delta_{1/2}^c$ for $c \in \Lambda_0$. Define similarly to D^c the parameter disk $D^{M'} \subset \Lambda_1 \setminus M'$ as the connected component of $O^{\mathbf{M}}(1) \setminus \delta_{1/2}^{M'}$ that is disjoint from M'. Then $D^{M'}$ is relatively compact in Λ_0. Let $\chi : \Lambda_0 \longrightarrow O(2) \cap W_{\theta_\pm}^{c_b}$ be the map $\chi(c) = H_c^{-1}(c)$. Then χ is locally quasi-regular with dilatation $K(c)$ bounded by $\log \mathrm{d}_\Lambda(c, c_b)$ (see [**R**], for example). (Douady and Hubbard use the term topologically holomorphic in [**DH2**].) By construction, the restriction

$$\chi : \partial D^{M'} \longrightarrow \partial D^{c_b}$$

is a homeomorphism given by $\chi(c) = \psi^{c_b} \circ \Phi^{\mathbf{M}}(c) = \psi^{c_b} \circ \phi_c(c)$ on $\partial D^{M'}$. Hence, the restriction $\chi : \overline{D^{M'}} \longrightarrow \overline{D^{c_b}}$ is a quasi-conformal homeomorphism with dilatation bounded by $K = K_{1/2} = \max\{\log \mathrm{d}_\Lambda(c, c_b) \mid c \in \overline{D^{M'}}\}$.

Define $\Omega^{M'} = \chi^{-1}(\Omega^{c_b}) \subset D^{M'}$ and $\widehat{B}_n^{M'} = \chi^{-1}(\widehat{B}_n^{M'}) \subset D^{M'}$, $n \geq 1$. Let $\Delta_q = \Delta_q^{\mathbf{M}}$, $q = p/2^n$ be any dyadic stick contained in the relatively dyadic 1/2-wake $W_{1/2}^{M'}$ of M'. Then it is contained in one of the disks $\Omega^{M'}$ or $\widehat{B}_n^{M'}$ and is thus wrapped in an annulus with a modulus bounded from below by $K_{1/2} \cdot m(c_b)$, according to Lemma 24.

Rename $D^c =: D_{1/2}^c$, $D^{M'} =: D_{1/2}^{M'}$ and $\chi =: \chi_{1/2}$.

We have thus proved the following proposition.

PROPOSITION 39. *Any dyadic stick $\Delta_{p/2^n}$ in the relative 1/2-wake $W_{1/2}^{M'}$ of M' is wrapped in an annulus contained in the disk $D_{1/2}^{M'}$ and of modulus bounded uniformly from below by $m(c_b)/K_{1/2}$, where $K_{1/2} = \max\{\log d_\Lambda(c, c_b) | c \in D_{1/2}^{M'}\}$. Moreover, $m(c_b)$ is some constant depending only on the center $c_b \in M'$.*

Moreover, $D_{1/2}^{M'}$ is disjoint from M' and the annuli are q-c images of corresponding annuli in the dynamical plane of Q_{c_b}.

We shall prove by induction the similar statements for any other relatively dyadic wake $W_{r/2^s}^{M'}$ of M'. The only difference is that the bounds on the dilatation of the q-c homeomorphisms $\chi_{r/2^s}$, and hence on the moduli of annuli in the $W_{r/2^s}^{M'}$- wake, depend on $r/2^s$. Worse, the bounds degenerate when $r/2^s \to 0$ or 1. As a remedy for this, we shall apply the Levin–Yoccoz parameter inequality once more. The details follow.

Recall that V_0^c is the connected component of $U_0^c \setminus \delta_0^c$ containing the critical point ω_c and $V_n^c = f_c^{-n}(V_0^c)$. We shall also need the extension $\widetilde{V}_1^c = V_0^c \setminus \overline{D^c}$ of V_1^c and its iterated preimages $\widetilde{V}_n^c = f_c^{-(n-1)}(\widetilde{V}_1^c)$. Define parameter disks Λ_s, $s > 1$ by

$$\Lambda_s = \{c \mid c \in \widetilde{V}_{s-1}^c\}.$$

Evidently, $\Lambda_s \supset \Lambda_{s+1}$. Note that the condition $c \in \widetilde{V}_s^c$ is equivalent to $f_c^s(\omega_c) \in \widetilde{V}_1^c$. Rename $G^c =: G_1^c$ and define recursively $G_{s+1}^c = f_c^{-1}(G_s^c) \cup G_1^c$ for $s \geq 1$. Then, as noted above, G_1^c moves holomorphically in $\Lambda_0 \supset \Lambda_1$ and we shall prove as part of the induction on $s \geq 2$ that, for $c \in \Lambda_s$, the critical value $c \notin G_{s-1}^c$ so G_s^c moves holomorphically over Λ_s.

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For $s = 2$ notice that, by the above, $c \in \overline{D^c_{1/2}}$ if and only if $c \in \overline{D^{M'}_{1/2}}$. Thus, $c \notin G^c_1$ for any $c \in \Lambda_2$, so G^c_2 moves holomorphically with $c \in \Lambda_2$. For $c \in \Lambda_2$, let $D^c_{j/2^2}$ for $r = 1, 3$ denote the connected components of $f_c^{-1}(D^c_{1/2})$ containing the $r/2^2$ dyadic decorations and define $D^{M'}_{r/2^2} \subset \Lambda_2$ as the parameter disks bounded by the corresponding parameter ray segments and equipotential level.

Rename the previous holomorphic motion H as H_1 and let H_2 denote the restriction of H_1 to $\Lambda_2 \times (\overline{\mathbb{C}} \setminus \widetilde{V}_1^{c_b})$. Extend H_2 to a motion including $\widetilde{V}_1^{c_b} \setminus V_1^{c_b}$ using the Böttcher motion and extend H_2 further to $\Lambda_2 \times f_{c_b}^{-1}(\overline{D^{c_b}_{1/2}})$ by $f_c^{-1} \circ H_1(c, f_{c_b}(z))$, where the inverse branches are taken so as to map $D^{c_b}_{r/2^2}$ quasi-conformally onto $D^c_{r/2^2}$. Finally, use Slodkowski's extension theorem to extend this holomorphic motion to a holomorphic motion of $\overline{\mathbb{C}}$ over the disk Λ_2 (i.e., extend the motion by a motion of $V_2^{c_b}$). By the same argument as above, the map $\chi_2 : \Lambda_2 \longrightarrow \widetilde{V}_1^{c_b}$ given by $\chi_2(c) = (H_2)_c^{-1}(c)$ is a locally quasi-regular map. Again by construction, $\chi_2 : \partial D^{M'}_{r/2^2} \longrightarrow \partial D^{c_b}_{r/2^2}$ are homeomorphisms so that the restrictions $\chi_2 : D^{M'}_{r/2^2} \longrightarrow \partial D^{c_b}_{r/2^2}$ are quasi-conformal. However, on the sets $D^{M'}_{r/2^2}$, the holomorphic motion H_2 is a conjugacy between the holomorphic maps f_{c_b} and f_c. Hence, the dilatation of $(H_2)_c$ at z equals that of $(H_1)_c$ at $f_{c_b}(z)$. Thus, similarly to the first case, the dilatation of χ_2 on $D^{M'}_{r/2^2}$ is bounded by $K_{r/2^s} = \max\{\log \mathrm{d}_{\Lambda_0}(c, c_b) \mid c \in \overline{D^{M'}_{r/2^2}}\}$. Arguing as in the first case corresponding to $s = 1$ completes the case $s = 2$. The inductive step is completely similar and is left to the reader. We have thus proved the following proposition.

PROPOSITION 40. *Any dyadic stick $\Delta_{p/2^n}$ in the relative $r/2^s$-wake $W^{M'}_{r/2^s}$ of M' is wrapped in an annulus contained in the disk $D^{M'}_{r/2^s}$ and of modulus bounded uniformly from below by $m(c_b)/K_{r/2^s}$, where $K_{r/2^s} = \max\{\log d_\Lambda(c, c_b) | c \in D^{M'}_{r/2^s}\}$. In addition, $m(c_b)$ is some constant depending only on the center $c_b \in M'$.*

Moreover, $D^{M'}_{r/2^s}$ is disjoint from M' and the annuli are q-c images of corresponding annuli in the relative $W^{c_b}_{1/2}$-wake in the dynamical plane of Q_{c_b}.

To complete the proof that the diameters of the dyadic sticks $\Delta_k \subset W^{H'}$ converge to zero, we shall apply the Levin–Yoccoz parameter inequality once more. Let H' denote the central hyperbolic component of M'. Then, for k sufficiently large, the root c_k of the dyadic stick Δ_k belongs to the p'_k/q'_k limb $L^{H'}_{p'_k/q'_k}$ of H'. Applying the Yoccoz–Levin parameter inequality Theorem 17 to H', we find that the diameter of $L^{H'}_{p'/q'}$ union its attached dyadic sticks is bounded uniformly above by C/q' for some constant $C = C_{H'}$. Arguing as in the beginning of the first reduction, we can assume that $p'_k/q'_k = p'/q'$ for all large k. The set $W^{H'}_{p'/q'} \cap \Lambda_1$ is relatively compact in Λ_0, so

$$\sup\{\log \mathrm{d}_\Lambda(c, c_b) \mid c \in W^{H'}_{p'/q'} \cap \Lambda_1\} = K = K^{H'}_{p'/q'} < \infty.$$

Hence, the dyadic sticks Δ_k either have diameters which *a priori* tend to zero or such sticks are separated from M' by an annulus in $\Lambda_0 \setminus M'$ of modulus at least $m(c_b)/K$. In the last case, their diameters are forced to converge to 0 *a posteriori* because the roots $c_k \in \Delta_k$ converge to $c_\infty \in M'$.

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This completes the proof that if $\Lambda_0 \ni c_k \to c_\infty \in M'$, then the diameters of Δ_k converge to zero.

Recall that $\Lambda^P = \Lambda_0^P$ is the parameter subdisk of $O^{\mathbf{M}}(2) \setminus \overline{R_{\eta_-}^{\mathbf{M}} \cup R_{\eta_+}^{\mathbf{M}}}$ containing Λ_0. We have taken care of the case $\Delta_k \subset \Lambda_0$. For the case $\Delta_k \subset \Lambda_P \setminus \Lambda_0$, we necessarily have $c_k \to c'$, where c' denotes the root of M'. To prove that the diameters of Δ_k also converge to zero in this case, let

$$\Lambda_n^P = \{c \in \Lambda_P \mid c \in U_n^c\},$$

where $U_n^c = f_c^{-n}(U_0^c)$ (it may or may not be connected), for $n > 1$.

For any $c \in \Lambda_0$, the sets $\partial \Xi_n^c$, $n \geq 0$ move holomorphically with c. Define $A_n^c = \Xi_n^c \setminus \Xi_{n+1}^c$; then A_n^c are quadrilaterals with a-sides, the boundary arcs $\partial A_n^c \cap R_{\theta_-}^c$ and $\partial A_n^c \cap R_{\theta_+}^c$. Moreover, ∂A_n^c even move holomorphically with $c \in (\Lambda_0 \cup \Lambda_n^P)$. Let

$$A_n^{M'} = \{c \in \Lambda_n^P \mid c \in A_n^c\}$$

denote the corresponding parameter quadrilaterals. Then the root c' of M' belong to Λ_n^P, for all n. Choose by Slodkowski's extension theorem a holomorphic motion

$$H^0 : \Lambda_0^P \times A_0^{c'} \longrightarrow \mathbb{C}$$

over Λ_0^P with base point c' of the quadrilateral $A_0^{c'}$ extending the Böttcher motion of its boundary.

For $c \in \Lambda_n^P$, the restriction $f_c^n : A_n^c \longrightarrow A_0^c$ is biholomorphic. Hence, we may lift the motion H^0 to a holomorphic motion

$$H^n : \Lambda_n^P \times A_n^{c'} \longrightarrow \mathbb{C}.$$

As with the annuli above, define quasi-conformal homeomorphisms

$$\rho_n : A_n^{M'} \longrightarrow A_n^{c'}$$

by $\rho_n(c) = \left(H_c^n\right)^{-1}(c)$. Then, as above, the distortion of these homeomorphisms is bounded by the distortion of the q-c homeomorphisms $H_c^0(\cdot)$, $c \in A_n^{M'}$. The distortion is thus bounded by

$$K = \sup\{\log \mathrm{d}_{\Lambda_0^P}(c, c') \mid c \in A_n^{M'}\},$$

which is uniformly bounded, because $A_n^{M'} \subset \Lambda_1^P \subset\subset \Lambda_0^P$. Thus, all the quadrilaterals $A_n^{M'}$ have moduli bounded uniformly from below by $\mathrm{mod}\,(A_0^{c'})/K$. Moreover, the a-sides of these quadrilaterals are all contained in the two rays $R_{\theta_-}^{M'}$ and $R_{\theta_+}^{M'}$ co-landing at c'. By the Grötzsch inequality for quadrilaterals, the Euclidean diameters of $A_n^{M'}$ tend to zero and the closures converge to c'. By construction, no dyadic stick intersects the boundary of any of the $A_n^{M'}$. Thus, in this case also, the diameter of Δ_k converges to 0 when $k \to \infty$. This completes the proof in the case when c_∞ belongs to a primitive first renormalization copy.

8. *The satellite case*

In the complementary satellite case, $M' = M_{p/q}$ with the central hyperbolic component $H_{p/q}$ attached at the internal argument $\exp(i2\pi p/q)$ from the central hyperbolic

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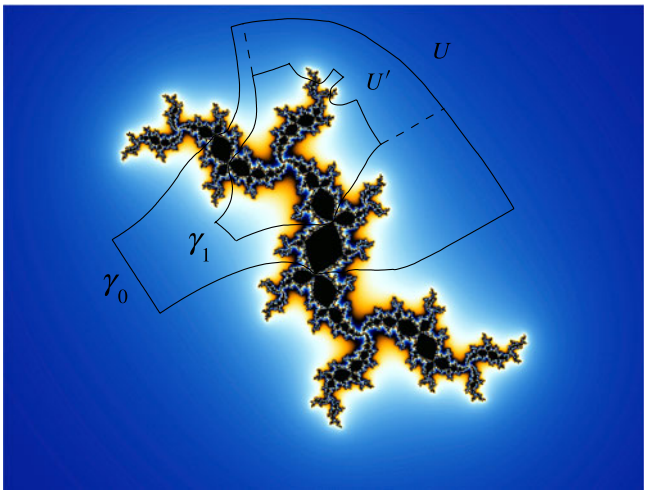


FIGURE 7. The disks U_0^c, U_1^c, B_0^c and Ξ_0^c.

component H_0 of $\mathbf{M}$. As above, let $\theta_- < \theta_+$ be the arguments of the parameter rays co-landing at the root and bounding the wake $W_{p/q}^{H_0}$. Recall that $c_\infty \in M_{p/q}$ is the limiting parameter of the roots of dyadic sticks and that these dyadic sticks are eventually contained in $W_{p/q}^{H_0}$.

We apply the Yoccoz–Levin parameter inequality Theorem 17 as we have done twice above. This reduces the problem to the case where c_∞ belongs to a relative p'/q'-limb $L_{p'/q'}^{H_{p/q}}$ of $H_{p/q}$ for some $p'/q' \neq 0/1$ and the dyadic sticks Δ_k are subsets of the corresponding wake $W_{p'/q'}^{H_{p/q}}$ for large k. Denote by $\tau_- < \tau_+$ the arguments of the parameter rays bounding $W_{p'/q'}^{H_{p/q}}$ and define $\Lambda = \Lambda_0 = W_{p'/q'}^{H_{p/q}} \cap F^{\mathbf{M}}(2)$. Let $I^{M'} = I^{M_{p/q}}$ denote the Cantor set of arguments of parameter rays accumulating M', as given by the Douady tuning algorithm. Then $\tau_\pm \in I^{M'}$ and each has a unique preimage $\widehat{\tau}_\pm = \sigma^{-q}(\tau_\pm) \cap I^{M'}$ different from itself. For $c \in \Lambda$, let U_0^c denote the disk containing the fixed point α^c of Q_c and bounded by the segments $\overline{(R^c_{\sigma^i(\widehat{\tau}_-)} \cup R^c_{\sigma^i(\widehat{\tau}_+)}) \cap F^c(1)}$ for $0 < i < q$ union the connecting subarcs of $E^c(1)$. Denote by ι^c the open subarc of $\partial U_0^c \cap E^c(1)$ intersecting the rays $R^c_{\theta_\pm}$ and let $\gamma_0^c = \partial U_0^c \setminus \iota^c$. As in the primitive case, write $\delta_0^c = \overline{(R^c_{\theta_-} \cup R^c_{\theta_+}) \cap F^c(1)}$ for $c \in \Lambda$.

Then the whole setup is similar to the primitive case. We can thus define Ω^c, Ξ_n^c, $B^c{}_n$, $\widehat{\Xi}_{n+1}^c$, $\widehat{B}_{n+1}^c$ for $n \geq 0$ and G^c, all of which move holomorphically with $c \in \Lambda$. There are, however, two differences: the first is that the center of $H_{p/q}$ does not belong to Λ. The arguments we used in the primitive case are insensitive to a change of base point c_b to another point in the interior of M'. We shall thus take as the base point $c_b \in \Lambda$, the center of the central hyperbolic component $H_{p'/q'}^{H_{p/q}} \subset W_{p'/q'}^{H_{p/q}}$. The second difference is

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that the Yoccoz–Levin parameter inequality is applied to the sublimbs of the hyperbolic component $H^{H_{p/q}}_{p'/q'}$. We leave the details to the reader.

This completes the satellite case and thus completes the proof of Proposition 8.

9. *Proof of Theorem 5*

The proof of Theorem 5 is completely analogous to the proof above of Theorem 2. Let $\mathbf{M_0}$ be any period k renormalization copy of $\mathbf{M}$ inside a limb $L_{p/q} \subset \mathbf{M}$, $(p, q) = 1$. In the arguments above, replace the carrot field $\Delta = \Delta^{\mathbf{M}}$ of $\mathbf{M}$ by the relative carrot field $\Delta^{\mathbf{M_0}}$ of $\mathbf{M_0}$ (for definitions, see Definition 3). As any decoration $\Delta^{\mathbf{M_0}}_{r/2^n}$ is contained in the relatively dyadic wake $W^{\mathbf{M_0}}_{r/2^n}$ bounded by the $r/2^n$ pair of co-landing relatively dyadic rays of $\mathbf{M_0}$, and as external rays do not cross, the above arguments apply with the rays replaced by the corresponding rays relative to $\mathbf{M}_0$.

We shall give in more detail the case when $\mathbf{M_0}$ is a first renormalization copy and indicate the induction step to higher renormalizations. Write $M' := \mathbf{M_0}$ to simplify notation and let $I^{M'}$ denote the corresponding σ^k-invariant Cantor set as defined in §3.4.5. The map $\Pi_2 \circ \kappa_{M'} : I^{M'} \longrightarrow \mathbb{T}$ is $2:1$ above any of the dyadic arguments $r/2^n$, where it is injective above any other argument, because $\kappa_{M'} : I^{M'} \longrightarrow \Sigma_2$ is a homeomorphism. For any dyadic $r/2^n$, the two rays of arguments in $\kappa^{-1}_{M'}(\Pi_2^{-1}(r/2^n))$ co-land at the dyadic tip $\gamma_0(r/2^n)$ and thus define a sector $W'_{r/2^n}$ disjoint from M'. We call the set of rays with arguments $\zeta' \in I^{M'}$, additionally marked by $\zeta = \Pi_2 \circ \kappa_{M'}(\zeta')$, the relative rays of M' because $R^{\mathbf{M}}_{\zeta'}$ plays the same role relative to M' as does the ray $R^{\mathbf{M}}_{\zeta}$ relative to $\mathbf{M}$. The only exception is that for dyadics $r/2^n$, it is the whole sector $W'_{r/2^n}$ which relative to M' plays the role that for $\mathbf{M}$ is played by the ray $R^{\mathbf{M}}_{r/2^n}$.

The relatively dyadic carrot $\Delta^{M'}_{r/2^n}$ is the closure of $\mathbf{M} \cap W'_{r/2^n}$, i.e., this intersection union its root $\gamma_0(r/2^n)$. We need to apply the relative Yoccoz puzzle theorem and Yoccoz inequality. The $1:1$ preimage under $\Pi_2 \circ \kappa_{M'}$ of the unique p'/q'-cycle $(p', q') = 1$ for σ is the relative p'/q'-cycle for σ_2^k in $I^{M'}$. For c in the p'/q'-limb $L'_{p'/q'} = \chi^{-1}_{M'}(L_{p'/q'})$, we may refine the p/q-Yoccoz puzzle (coming from $M' \subset L_{p/q}$) with the relative p'/q'-puzzle (see also [**Hu**]). This leads to the Yoccoz puzzle theorem for M'. In the general case where M' is a level $m > 1$ renormalization copy, we would be refining not just the p/q-Yoccoz puzzle but the iterated refinements by the relative Yoccoz puzzles of renormalization levels $m' < m$. Note, however, that $L'_{p'/q'} = L^{H'}_{p'/q'} \cap M'$, where $H' = \chi^{-1}_{M'}(H_0)$ is the central hyperbolic component of M' and $L^{H'}_{p'/q'}$ was defined in §3.4.4 and is used in the relative Yoccoz inequality for H'.

Use the Yoccoz parameter inequality and the iterated Yoccoz parameter puzzle theorem relative to M' to prove that $\operatorname{diam}(\Delta'_k) \to 0$ for any sequence of decorations $(\Delta'_k)_k$ with roots c_k converging to a relatively non-renormalizable parameter $c_\infty \in M'$. Secondly, consider the case $c_\infty \in M''$, where $M'' \subset M'$ is a first – relative to M' – renormalization copy of M, belonging to some p'/q'-limb of the central hyperbolic component H' of M'. Use again the p'/q' Yoccoz puzzle relative to M' to define the parameter disk Λ containing M'' similarly as we defined Λ for M' above, and define also Λ^P and γ_n analogously, i.e., with the aid of the p'/q'-puzzle piece P and rays relative to M' given by Theorem 21

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for M'. Similarly, let δ_n be defined in terms of the pairs of co-landing rays which are dyadic relative to M''. From here, the proof proceeds analogously.

A. Appendix. *Levin's escape locus PLY-inequality*

In this appendix, we supply for completeness proofs of the two theorems we refer to, but for which we have not been able to find either adequate or complete proofs in the literature.

THEOREM A.1. (The Levin–Yoccoz dynamical inequality) *Let H be any hyperbolic component of $\mathbf{M}$ of period k. Let p/q be any non-zero reduced rational and let $W^H_{p/q}$ denote the relative p/q-wake of H, bounded by parameter rays with arguments $0 < \eta_- < \eta_+ < 1$. For any $c \in W^H_{p/q}$, let λ denote the multiplier of the repelling k-periodic common landing point α' of the kq-periodic rays $R^c_{\eta_\pm}$. Then α' has combinatorial rotation number p/q and λ has a logarithm Λ such that*

$$|\Lambda - p/q 2\pi i| \leq \frac{2k \log 2\cos\theta}{q} \frac{\pi}{\omega(c)},$$

where $\theta \in]-\pi/2, \pi/2[$ is the argument of $\Lambda - p/q 2\pi i$ and $\omega(c)$ is the angle of vision of the interval $i2\pi[\eta_-, \eta_+]$ from $\operatorname{Log} \phi_c(c) \in \{z = x + iy \mid 0 < y < 2\pi\}$.

Proof. Levin proved the fixed point case $k = 1$ in [**L**, Theorem 5.1]; the general case is similar. For completeness, we sketch a proof here. Let us first recall the proof of the Yoccoz inequality (or the Pommerenke–Levin–Yoccoz inequality); full details can be found in [**P**]. Let T denote the quotient torus $T = D^*/Q_c^k$, where $D^* = \{z \mid 0 < |z - \alpha'| < r\}$ and $r > 0$ is chosen so small that Q_c^k is univalent on $D = D^* \cup \{\alpha'\}$ and $D \subset\subset Q_c^k(D)$. Let $\Pi : D^* \longrightarrow T$ denote the natural projection. The two rays $R^c_{\eta_\pm}$ belong to the same orbit and define combinatorial rotation number p/q for α'. Let $\gamma = \Pi(D^* \cap R^c_{\eta_-}) = \Pi(D^* \cap R^c_{\eta_+})$. Then γ is a Jordan curve and thus the pair (T, γ) has a conformal modulus which satisfies a Grötzsch inequality.

Let $w_\pm = \exp(i2\pi\eta_\pm)$. Then $Q_0^k(w_-) = w_+$ and $Q_c^{kq}(w_\pm) = w_\pm$. Similarly to T above, let $\widehat{T}$ denote the quotient torus $\widehat{T} = \widehat{D}^*/Q_c^k$, where $\widehat{D}^*$ is a small punctured disk centered at, say, w_- and let $\widehat{\Pi} : \widehat{D}^* \longrightarrow \widehat{T}$ denote the natural projection. Then $\widehat{\Pi}(\widehat{D}^* \cap \mathbb{S}^1)$ are two disjoint Jordan curves in $\widehat{T}$, with complements two disjoint, symmetric and straight annuli A_i and A_o. Moreover, $\widehat{\gamma} = \widehat{\Pi}(\widehat{D}^* \cap R_{\eta_-})$ is the Jordan equator of A_o and

$$\operatorname{mod}(A_i) + \operatorname{mod}(A_o) = 2\operatorname{mod}(A_o) = \operatorname{mod}(\widehat{T}, \widehat{\gamma}).$$

If $c \in \mathbf{M}$, so that K_c is connected, the Böttcher coordinate at ∞ induces an isomorphism between A_o and $\Pi(S)$, where S is the connected component of $D^* \cap B^c(\infty)$ containing the end of $R^c_{\eta_-}$. Hence, the Grötzsch inequality for (T, γ) implies that

$$\operatorname{mod}(A_o) \leq \operatorname{mod}(T, \gamma). \tag{A.1}$$

Writing out the values of these two numbers explicitly yields the Yoccoz dynamical inequality: the torus T is isomorphic to $\mathbb{C}^*/\lambda z$ via the linearizer for Q_c^k at α', or, equivalently, to $\mathbb{C}/(\mathbb{Z}\Lambda + \mathbb{Z}i2\pi)$ via the log-linearizer. Let $\Pi_u : \mathbb{C} \longrightarrow T$ denote the universal covering corresponding to the last isomorphism. Then the Jordan curve $\gamma = \Pi(D^* \cap R^c_{\eta_-})$ lifts under Π_u to an arc Γ, which is invariant under the translation

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$z \mapsto z + L$, where $L = q\Lambda - pi2\pi$ for some appropriate logarithm Λ of λ. A simple computation shows that

$$\mathrm{mod}\,(T, \gamma) = \frac{2\pi\,\cos\theta}{q|L|},$$

where θ is the angle between the vector L and the positive real axis. A similar computation shows that

$$2\mathrm{mod}\,(A_o) = \mathrm{mod}\,(\widehat{T}, \widehat{\gamma}) = \frac{2\pi}{kq\,\log 2}.$$

Hence, (A.1) is equivalent to

$$|\Lambda - \frac{p}{q} i2\pi| \leq \frac{2k\,\log 2\,\cos\theta}{q}, \tag{A.2}$$

which is the Yoccoz inequality.

If $c \notin \mathbf{M}$, let $0 \leq \theta < 1$ denote the argument of c, i.e., $c \in R^c_\theta$. Then the Böttcher coordinate ϕ_c at infinity does not extend to a biholomorphic map between $B^c(\infty)$ and $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, but it almost does: it extends to a biholomorphic map of $\overline{\mathbb{C}} \setminus F^c(h)$ onto $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}(\mathrm{e}^h)$, where $h = g_c(c)/2$. Let ψ_c denote the inverse of this extension; then ψ_c extends continuously to $C(0, \mathrm{e}^h)$, but this extension is not injective because $0 = \psi_c(\exp(h + 2\pi i\theta/2)) = \psi_c(\exp(h + i2\pi(\theta + 1)/2))$. Let $C = g_c(c) + i2\pi\theta$, $N_0 = [\mathrm{e}^{i2\pi\theta}, \phi_c(c)]$ and $N_n = Q_0^{-n}(N_0)$. Define $N^0_\theta = \cup_{n\geq 0} N_n$ and $N^1_\theta = \cup_{n\geq 0} N_1$. Then $Q_0(N^1_\theta) = N^0_\theta$ and ψ_c is easily seen to extend by iterated lifting to a univalent map from $\overline{\mathbb{C}}_\theta := \overline{\mathbb{C}} \setminus (\overline{\mathbb{D}} \cup N^1_\theta)$ into $B^c(\infty)$. The map Q_0 lifts under $\exp(z)$ to the map $z \mapsto 2z$ on $\mathbb{C}$, that is, exp is a simultaneous linearizer for all the repelling periodic points of Q_0. The corresponding lifted sets $\widetilde{N}^j_\theta = \log(N^j_\theta)$, $j = 0, 1$, are invariant under translation by $i2\pi$ and $2\widetilde{N}^1_\theta = 2\widetilde{N}^0_\theta$. Thus, if $w = \exp(i2\pi\tau) \in \mathbb{S}^1$ is periodic and if $0 \leq \tau < 1$ does not belong to the orbit of θ, then $\mathbb{C}_\theta$ contains a definite sector around the horizontal $\widetilde{R}_\tau = \{t + i2\pi\tau \mid t > 0\}$, which projects to R^0_τ under exp: let $\tau_l < \tau < \tau_r$ be the arguments closest to θ of points in the orbit of w. Then the sectors $\widetilde{S}_l$ between $\widetilde{R}_{\tau_l} = \{t + i2\pi\tau_l \mid t > 0\}$ and the oblique line through $i2\pi\tau_l$ in the direction $v_l = C - i2\pi\tau_l$, and $\widetilde{S}_r$ between $\widetilde{R}_{\tau_r} = \{t + i2\pi\tau_r \mid t > 0\}$ and the oblique line through $i2\pi\tau_r$ in the direction $v_r = C - i2\pi\tau_r$, are contained in $\mathbb{C}_\theta$: if not, some line segment L with $\exp(L) \in N_n$ for some $n \geq 1$ intersects, say, $\widetilde{S}_l$. But then $2^n L$ intersects the sector $2^n\widetilde{S}_l$ with top point $2^n\tau_l$, and is also congruent modulo $i2\pi$ to $L_0 = [i2\pi\theta, C]$ with $\exp(L_0) = N_0$. Since $2^n\tau_l$ is an argument for a point in the orbit of w, this contradicts that τ_l is the closest such argument for points in the orbit of w.

Consequently, the sector $\widetilde{S}$ around $\widetilde{R}_\tau$ bounded by the two lines through $i2\pi\tau$ and of directions v_l and v_r is contained in $\mathbb{C}_\theta$.

In the case at hand, $c \in W^H_{p/q}$ implies that $\eta < \theta < \eta_+$ and, for $\eta = \eta_-$, we have $\eta_- = \eta_l$, $\eta_+ = \eta_r$. Let ω_l and ω_r denote the angle of inclination of the vectors $C - i2\pi\tau_l$ and $C - i2\pi\tau_r$, respectively. Then the opening angle ω of $\widetilde{S}$ equals $\omega_r - \omega_l$ and the sector $\widetilde{S}$ projects to a straight subannulus A^θ_o of A_o with

$$\mathrm{mod}\,(A^\theta_o) = \frac{\omega}{\pi}\mathrm{mod}\,(A_o).$$

Arguing as for the proof of the Yoccoz inequality, we obtain

$$\mathrm{mod}\,(A^\theta_o) = \frac{\omega}{\pi}\mathrm{mod}\,(A_o) \leq \mathrm{mod}\,(T, \gamma).$$

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Properly rewritten as with (A.2) above, this is the Levin–Yoccoz inequality, except for the interpretation of ω. This interpretation is, however, an elementary exercise in planar geometry and is left to the reader. By continuity, the inequality even holds on $\partial W^H_{p/q}$, where either ω_l or ω_r, but not both, is zero. □

B. Appendix. *The Lavaurs theorem revisited*

THEOREM B.1. *Let $0 < t_- < t_+ < 1$ be rationals for which the parameter rays $R^{\mathbf{M}}_{t_\pm}$ co-land at some point $c_0 \in \mathbf{M}$, and let $W_{t_\pm}$ denote the parameter sector bounded by $R^{\mathbf{M}}_{t_-} \cup \{c_0\} \cup R^{\mathbf{M}}_{t_+}$ and not containing the main cardioid H_0. Then the following hold.*

- *t_-, t_+ have the same preperiod $l \geq 0$ and period $k > 0$: $\sigma^{k+l}(t_\pm) = \sigma^l(t_\pm)$ and $\sigma^{j+i}(t_\pm) \neq \sigma^j(t_\pm)$ for $j < l$ or $i < k$.*
- *The forward orbits of $t_\pm$ under σ do not enter the interval $]t_-, t_+[$.*
- *For any $c \in W_{t_\pm}$, the pair of dynamical rays $R^c_{t_\pm}$ co-land at some repelling (pre)periodic point $z(c)$ with $Q_c^{l+k'}(z(c)) = Q_c^l(z(c))$, where k' divides k.*
- *The set $R^c_{t_-} \cup \{z(c)\} \cup R^c_{t_+}$ moves holomorphically with $c \in W_{t_\pm}$ and bounds a sector $W^c_{t_\pm}$ containing c and the rays of arguments $t \in]t_-, t_+[$, but not 0.*

Proof. This theorem is often attributed to Pierre Lavaurs, Douady and Hubbard. We supply a proof here for completeness. Douady and Hubbard showed that any irreducible rational $t = r/s$ can be written uniquely as $r/s = m/2^l(2^k - 1)$, where $l \geq 0$ is the preperiod and $k > 0$ is the (eventual) period. They proved that if J_c is connected, then the ray R^c_t lands on a point $z(c)$ with $Q_c^l(z(c))$ parabolic or repelling with exact period k', $k = qk'$ and combinatorial rotation number p/q for some p. If $l > 0$, then $Q_c^{l-1}(z(c))$ is strictly preperiodic, and if $l = 0$ and $k > 1$, then $z(c)$ admits at least two external arguments. If $z(c)$ admits more than one external argument, say t, t', then these have the same preperiod l and exact period k. If $q > 1$, then $z(c)$ admits precisely q external arguments, all of which are preperiodic to points in the same orbit.

They proved, moreover, in [**DH1**] that the parameter ray $R^{\mathbf{M}}_t$ lands on a parameter $c_0 \in \mathbf{M}$. More precisely, if $l > 0$ then $z(c_0) = c_0$ and c_0 admits the same external arguments in the parameter plane and in the dynamical plane of Q_{c_0}. If $l = 0$, then $z(c_0)$ is parabolic with multiplier $\exp(i2\pi p/q)$, the critical value c_0 is in the immediate basin of $z(c_0)$ and the ray $R^{c_0}_t$ is adjacent to c_0, i.e., is not separated from c_0 by any other ray landing on $z(c_0)$. If, in addition, $k > 1$, so that $z(c_0)$ admits at least two external arguments, then the parameter c_0 admits precisely two external arguments $0 < t_- < t_+ < 1$ (one of which is t by hypothesis). Moreover, the rays $R^{c_0}_{t_\pm}$ co-land on $z(c_0)$ and separate the critical value c_0, together with the rays of arguments $t' \in]t_-, t_+[$, from the critical point 0.

Let $0 < t_- < t_+ < 1$, c_0 and $W_{t_\pm}$ be as in the statement of the theorem. By the above, the first bullet holds and thus we need only show the last three bullets. As above, we shall treat separately the preperiodic case $l > 0$ and the periodic case $l = 0$. Suppose first that $l > 0$ and let Θ denote the set of external arguments of $c_0 = z(c_0)$. The set

$$R^c = \bigcup_{\theta \in \Theta} \overline{R^c_\theta}$$

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moves holomorphically with c in $\mathbb{C}\setminus\widehat{R}^{\mathbf{M}}$, where

$$\widehat{R}^{\mathbf{M}} = \bigcup_{\theta\in\Theta}\bigcup_{j=1}^{k+l-1}\overline{R^{\mathbf{M}}_{\sigma^j(\theta)}}.$$

This is because the Böttcher coordinate ϕ_c depends holomorphically on c and thus R^c moves holomorphically with c as long as c does not belong to the strict forward orbit of R^c. Let $0<\theta_-<\theta_+<1$ denote, respectively, the smallest and the largest element of Θ. Then $W_{t\pm}\subseteq W_{\theta_\pm}$ and, for the last two bullets, it suffices to prove that $W_{\theta_\pm}\cap\widehat{R}^{\mathbf{M}}=\emptyset$. Moreover, this in turn follows once we have proved the following stronger form of the second bullet:

$$\left(\bigcup_{\theta\in\Theta}\bigcup_{j=1}^{k+l-1}\sigma^j(\theta)\right)\cap[\theta_-,\theta_+]=\emptyset. \tag{B.1}$$

To this end, let us consider the Hubbard tree T^{c_0} for Q_{c_0}. In this strictly preperiodic case, T^{c_0} is the minimal connected subset of $K_c=J_c$ containing the orbit

$$\mathcal{O}^{c_0}(0)=\bigcup_{j=0}^{k+l}Q^j_{c_0}(0).$$

As the orbit $\mathcal{O}^{c_0}(0)$ is forward invariant, so is T^{c_0}. Moreover, any extremal point of T^{c_0} belongs to $\mathcal{O}^{c_0}(0)$ by minimality. As $Q^j_{c_0}$ is a local homeomorphism on a neighborhood of the critical value $c_0=Q_{c_0}(0)$ for all j, the point c_0 is necessarily an extremal point of T^{c_0} because otherwise T^{c_0} would have at most one extremal point, the critical point 0. This implies (B.1) and completes the preperiodic case $l>0$. Notice that in the preperiodic case the connected component $W^{\mathbf{M}}_{c_0}$ of $\mathbb{C}\setminus\widehat{R}^{\mathbf{M}}$ containing $W_{\theta_\pm}$ is in fact a neighborhood of the closure of $W_{\theta_\pm}$. Hence, bullets 3 and 4 remain valid for $c\in W^{\mathbf{M}}_{c_0}$.

The periodic case $l=0$ is similar and yet slightly different. The common landing point c_0 of the two parameter rays $R^{\mathbf{M}}_{t_\pm}$ is the root of a hyperbolic component $H\neq H_0$. Let us rename c_0 as c_1 and use c_0 to denote the center of H. As above, the dynamical rays $R^c_{t_\pm}$ move holomorphically on $\mathbb{C}\setminus\widehat{R}^{\mathbf{M}}$, where

$$\widehat{R}^{\mathbf{M}}=\bigcup_{j=0}^{k-1}\overline{R^{\mathbf{M}}_{\sigma^j(t_-)}\cup R^{\mathbf{M}}_{\sigma^j(t_+)}}.$$

To prove the theorem, it suffices to prove that $R^{c_0}_{t_-}$ and $R^{c_0}_{t_+}$ co-land at a repelling periodic k'-period point $z(c_0)$ with combinatorial rotation number p/q and that

$$\left(\bigcup_{j=0}^{k-1}\sigma^j(t_-)\cup\sigma^j(t_+)\right)\cap\,]t_-,t_+\,[=\emptyset. \tag{B.2}$$

Haïssinsky proved, using trans quasi-conformal surgery, that there exists a homeomorphism $\Xi:\overline{\mathbb{C}}\longrightarrow\overline{\mathbb{C}}$ coinciding with $\psi^{c_1}\circ\phi^{c_0}$, on the basins of infinity and conjugating dynamics, except on a topological disk in the immediate basin $\Lambda(0)$ for the super-attracting k-periodic point 0 for Q_{c_0}. We can even choose Ξ so that $\Xi(0)=0$ and $\Xi(c_0)=c_1$. Let $\chi:\Lambda(c_0)\longrightarrow\mathbb{D}$ denote the Böttcher coordinate on $\Lambda(c_0)=Q_{c_0}(\Lambda(0))$, i.e., $\chi\circ Q^k_{c_0}=(\chi)^2$. Then

$$\Xi^{-1}(R^{c_1}_{t_-}\cup\{z(c_1)\}\cup R^{c_1}_{t_+})=R^{c_0}_{t_-}\cup\{z(c_0)\}\cup R^{c_0}_{t_+},$$

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where

$$z(c_0) = \Xi^{-1}(z(c_1)) = \lim_{w \nearrow 1} \chi(w)$$

is a repelling, k'-periodic point on the boundary of $\Lambda(c_0)$ with combinatorial rotation number p/q, at which the rays $R^{c_0}_{t_\pm}$ land. Moreover, $R^{c_0}_{t_\pm}$ are adjacent to c_0 because $R^{c_1}_{t_\pm}$ are adjacent to c_1, and the sector $W^{c_0}_{t_\pm} = \Xi^{-1}(W^{c_1}_{t_\pm})$ contains c_0 and the rays of arguments $t' \in]t_-, t_+[$, but does not contain 0. To complete the proof we argue, as above, that c_0 is an extremal point in the Hubbard tree T_{c_0} for Q_{c_0}. It follows that $T_{c_0} \cap \Lambda(c_0) = \chi^{-1}([0, 1[)$. Hence (B.2), and thus the theorem follows. □

References

[DH1] A. Douady and J. H. Hubbard. *Etude dynamique des polynômes complexes*. Publications mathématiques d'Orsay, 1984.

[DH2] A. Douady and J. H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Éc. Norm. Supér.* **t.18** (1985), 287–343.

[Du] D. Dudko. The Decoration Theorem for Mandelbrot and Multibrot sets, manuscript. arXiv:1004.0633v1.

[Ha] P. Haïssinsky. Chirurgie parabolique. *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), 195–198.

[Hu] J. H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J. C. Yoccoz. *Topological Methods in Modern Mathematics*. Eds. Goldberg and Phillips. Publish or Perish, Houston, TX, 1993, pp. 467–511.

[L] G. M. Levin. Disconnected Julia set and rotation sets. *Ann. Sci. Éc. Norm. Supér.* **t.29** (1996), 1–22.

[Mi] J. W. Milnor. Periodic orbits, externals rays and the Mandelbrot set: an expository account. Géométrie complexe et systèmes dynamiques (Orsay, 1995). *Astérisque* (261) (2000), 277–333 xiii.

[McM] C. T. McMullen. The Mandelbrot set is universal. *The Mandelbrot Set, Theme and Variations (London Mathematical Society Lecture Note Series, 274)*. Ed. Tan Lei. Cambridge University Press, Cambridge, 2000.

[P] C. L. Petersen. On the Pommerenke–Levin–Yoccoz inequality. *Ergod. Th. & Dynam. Syst.* **13** (1993), 785–806.

[PR1] C. L. Petersen and P. Roesch. Parabolic tools. *J. Difference Equ. Appl.* **16**(05-06) (2010), 715–738.

[PR2] C. L. Petersen and P. Roesch. *The Yoccoz Combinatorial Analytic Invariant (Fields Institute Communications, 53)*. American Mathematical Society, 2008.

[PR3] C. L. Petersen and P. Roesch. The Parabolic Mandelbrot set, manuscript.

[PR4] C. L. Petersen and P. Roesch. Carrots in Mandelbrot-like families, manuscript in preparation.

[Po] C. Pommerenke. *Boundary Behaviour of Conformal Maps*. Springer, 1992.

[R] P. Roesch. Holomorphic motions and puzzles. *The Mandelbrot Set, Theme and Variations (London Mathematical Society Lecture Note Series, 274)*. Ed. Tan Lei. Cambridge University Press, 2000, pp. 117–131.

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