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RESEARCH ARTICLE

Integer-valued polynomials satisfying growth constraints

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Abstract

We consider polynomials which take integer values on the integers (IVPs), and satisfy an additional growth condition on the natural numbers. Elkies and Speyer, answering a question by Dimitrov, showed that there is a critical exponential growth threshold, such that there are infinitely many IVPs with growth above the threshold and finitely many IVPs below that threshold (of arbitrary degree). In this paper, we give more refined estimates for the number of IVPs satisfying such exponential growth constraints. In addition, we consider a similar problem on the integers, where not necessarily symmetric growth conditions are imposed. Notably, the critical threshold is determined by the logarithmic capacity of an explicit domain.

1. Introduction

Ruzsa's conjecture asks whether a function $f : \mathbb{N} \to \mathbb{Z}$ which satisfies

$$f(m) - f(n) \equiv 0 \pmod{m-n} \quad \text{for all } n, m \in \mathbb{N}$$

and $|f(n)| < e^n$ for all $n \in \mathbb{N}$ is a polynomial (see [17]). Perhaps being motivated by this conjecture, Dimitrov [7] asked if there are infinitely many *integer-valued polynomials* P of arbitrary degree that satisfy

$$|P(n)| \le A^n \text{ for all } n \in \mathbb{N} = \{0, 1, 2, \dots\}, \tag{1.1}$$

where $A > 1$ is fixed. Recall that an integer-valued polynomial (IVP) P is an algebraic polynomial taking integer values on the natural numbers $\mathbb{N}$ (or equivalently on the integers $\mathbb{Z}$). For more on IVPs see the book [5] and also the survey by Cahen and Chabert [4].

Elkies and Speyer [7] proved that the threshold lies at the *golden ratio*, that is, if $A < \varphi = \frac{1}{2}(1+\sqrt{5})$ there are only finitely many such polynomials (of all degrees), while if $A > \varphi$ there are infinitely many of them. To the best of our knowledge the answer for $A = \varphi$ is still not known.

Our first theorem provides a small (asymptotic) improvement to the existence result.

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Theorem 1.1. *For any $\varepsilon > 0$, there are infinitely many integer-valued polynomials P that satisfy the inequalities*

$$|P(n)| \le \left(\sqrt{2\varphi} + \varepsilon\right)\varphi^n, \quad \text{for all } n \in \mathbb{N}.$$

Remark. We are not aware of any explicit example of an infinite family of IVPs satisfying the above inequalities (even after replacing the constant $\sqrt{2\varphi}$ by any larger absolute constant).

It is also possible to give ℓ^2-type results.

Theorem 1.2. *There are no (nontrivial) IVPs P satisfying*

$$\sum_{n\ge 0} |P(n)|^2 \varphi^{-2n} < \varphi.$$

For any $\varepsilon > 0$, there are infinitely many IVPs P that satisfy

$$\sum_{n\ge 0} \tfrac{1}{n+1} |P(n)|^2 \varphi^{-2n} \le \frac{2}{\pi} + \varepsilon.$$

Remark. Notice that for the constant IVP, $P \equiv 1$, it holds

$$\sum_{n\ge 0} \varphi^{-2n} = \varphi,$$

and moreover, taking $P(n) = \frac{1}{2}n(n-3)$, one can check that

$$\sum_{n\ge 0} \frac{1}{n+1} |P(n)|^2 \varphi^{-2n} = 4\varphi^2 \log(\varphi) - \frac{15}{2\varphi} \approxeq 0.40 < \frac{2}{\pi}.$$

This bound can be improved, but we do not know the optimal bound (if one exists).

It is known [5, Chap. 1] that the sequence of binomial polynomials $\{\binom{x}{k}\}_{k\in\mathbb{N}}$ forms a basis over $\mathbb{Z}$ for IVPs. That is, an IVP P of degree at most d can be written uniquely in the form

$$P(x) = P_{\mathbf{c}}(x) = \sum_{k=0}^{d} c_k \binom{x}{k}, \quad \text{with} \quad \mathbf{c} = (c_0, \dots, c_d) \in \mathbb{Z}^{d+1}.$$

Hence, counting IVPs P that satisfy the inequalities (1.1) is equivalent to counting integer lattice points inside the *convex* body

$$\mathcal{C}_A(d) = \left\{ \mathbf{c} = (c_0, \dots, c_d) \in \mathbb{R}^{d+1} : \sup_{n\in\mathbb{N}} |P_{\mathbf{c}}(n) A^{-n}| \le 1 \right\}.$$

It is well known that in order to bound the number of $\mathbb{Z}^{d+1}$ lattice points inside $\mathcal{C}_A(d)$ from below, it is sufficient to estimate $\mathrm{Vol}(\mathcal{C}_A(d))$. On the other hand, to obtain an upper bound for the number of lattice points one has to use the special structure of this convex set.

Remark. Note that since a polynomial of degree d is determined by its values at (say) $\{0, \dots, d\}$, there are finitely many polynomials of *fixed* degree d that satisfy any of the inequalities in the theorems above.

2. The results

We are interested in estimating the number of IVPs of degree at most d satisfying growth conditions on the integers such as (1.1) as the degree d tends to infinity.

We begin by introducing some notation. Let $A, B > 1$ be fixed constants. We put $\mathbf{c} = \mathbf{c}(d) = (c_0, \dots, c_d)$, and define the convex body

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$$\mathcal{C}^{\infty}_{A,B}(d;t) = \Big\{ \mathbf{c} \in \mathbb{R}^{d+1} : \sup_{n\in\mathbb{N}} |P_{\mathbf{c}}(n)A^{-n}| \le t, \ \sup_{n\in\mathbb{N}^+} |P_{\mathbf{c}}(-n)B^{-n}| \le t \Big\},$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^+ = \{1, 2, 3, \dots\}$. Notice that the $\mathbb{Z}^{d+1}$ lattice points in $\mathcal{C}^{\infty}_{A,B}(d;t)$ correspond to IVPs P of degree at most d satisfying the bounds

$$|P(n)| \le tA^n, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad |P(-n)| \le tB^n, \quad \forall n \in \mathbb{N}^+.$$

Similarly, one may consider the ℓ^2 bounds

$$\sum_{n\in\mathbb{N}} |P(n)|^2 A^{-2n} \le t^2, \quad \text{and} \quad \sum_{n\in\mathbb{N}^+} |P(-n)|^2 B^{-2n} \le t^2,$$

corresponding to $\mathbb{Z}^{d+1}$ lattice points inside the set

$$\mathcal{C}^{2}_{A,B}(d;t) = \Big\{ \mathbf{c} \in \mathbb{R}^{d+1} : \sum_{n\in\mathbb{N}} |P_{\mathbf{c}}(n)|^2 A^{-2n} \le t^2, \ \sum_{n\in\mathbb{N}^+} |P_{\mathbf{c}}(-n)|^2 B^{-2n} \le t^2 \Big\}.$$

It turns out that the volume of the convex bodies above is closely related to the *logarithmic capacity* of certain domains in the complex plane. The logarithmic capacity of a compact set $E \subset \mathbb{C}$ is given by (see [13, Chap. 5])

$$\mathrm{Cap}(E) := \exp\Big(-\inf_{\nu} \int_{E\times E} \log \frac{1}{|z-w|} \, \mathrm{d}\nu(z) \, \mathrm{d}\nu(w) \Big),$$

where the infimum is taken over *probability* measures ν supported on the set E.

Put

$$\mathbb{D}_1 = \mathbb{D}\Big(\frac{1}{A^2-1}, \frac{A}{A^2-1} \Big), \ \mathbb{D}_2 = \mathbb{D}\Big(\frac{B^2}{1-B^2}, \frac{B}{B^2-1} \Big),$$

where $\mathbb{D}(z,r)$ is the disk centered at z and of radius r. Notice that if $A, B > 1$, then $\mathbb{D}_1 \cap \mathbb{D}_2 = \varnothing$. See Figure 1.

It is known that the capacity of a disk is equal to its radius (e.g., [13, Cor. 5.2.2]), for example, $\mathrm{Cap}(\mathbb{D}_1) = \frac{A}{A^2-1}$. Note that the solution to the equation $\frac{A}{A^2-1} = 1$ with $A > 1$ is the golden ratio φ. This value of A is the *critical threshold* for the volume of the set of polynomial coefficients $\mathbf{c}$, for which the polynomial $P_{\mathbf{c}}$ satisfies the one-sided bound (1.1).

For the two-sided bounds, the following result gives the volume of the set of polynomial coefficients, and the asymptotic number of IVPs (depending on the degree) above the critical threshold.

Theorem 2.1. *Let $A, B > 1, t > 1$ be fixed constants, $\mathbb{D}_1, \mathbb{D}_2$ as above, and*

$$\gamma_{A,B} := \mathrm{Cap}(\mathbb{D}_1 \cup \mathbb{D}_2).$$

Then, as $d \to \infty$,

$$\log \mathrm{Vol}\Big(\mathcal{C}^{2}_{A,B}(d;t) \Big) = -\tfrac{1}{2} \log \gamma_{A,B} \cdot d^2 - \tfrac{1}{2} d \log d + O(d \log t), \quad \textit{and}$$
$$\log \mathrm{Vol}\Big(\mathcal{C}^{\infty}_{A,B}(d;t) \Big) = -\tfrac{1}{2} \log \gamma_{A,B} \cdot d^2 + O(d \log t).$$

Moreover, when $\gamma_{A,B} \in (0,1)$ we have

$$\log \#\big\{ \mathcal{C}^{2}_{A,B}(d;t) \cap \mathbb{Z}^{d+1} \big\} = \log \mathrm{Vol}\Big(\mathcal{C}^{2}_{A,B}(d;t) \Big) + O(d \log t).$$

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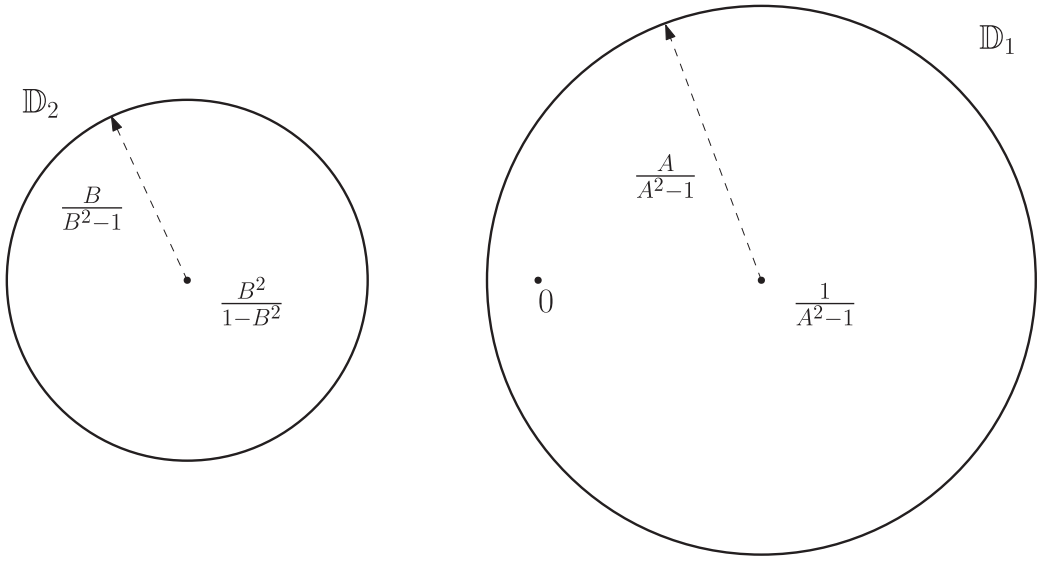


Figure 1. *The disks $\mathbb{D}_1$ and $\mathbb{D}_2$.*

Remark. By the monotonicity of logarithmic capacity, we see that the region where $\gamma_{A,B} \in (0,1)$ is contained in the set $A, B \geq \varphi$.

Remark. Using the Green function for an annulus (e.g., [6, V.15.7]) one can derive an expression for the capacity of the union of two disjoint disks in terms of theta functions. More precisely, one can show that

$$\gamma_{A,B} = \tfrac{1}{2} \exp\left(\frac{\log^2 A + \log^2 B}{2\log(AB)}\right) \frac{\theta_1'(0,\alpha)}{\sqrt{|\theta_1(i\log A,\alpha)\theta_1(i\log B,\alpha)|}},$$

where $\alpha = (AB)^{-1}$, θ_1 is a theta function (see, e.g., [1, 16.27 f.]). Here we used a modification of a formula due to Ransford, see [12, Example 4.7].

See Figure 2 for an illustration of the level sets of $\gamma_{A,B}$ using Mathematica. The dashed line corresponds to the critical level set $\gamma_{A,B} = 1$.

At the critical threshold, we can prove the analogs of Theorems 1.1 and 1.2, up to the calculation of the explicit constants (which depend on A, B).

Theorem 2.2. *Let $A, B > 1$ be fixed constants as in Theorem 2.1 that satisfy $\gamma_{A,B} = 1$. There is a constant $C_1 = C_1(A,B) > 0$ so that there are infinitely many IVPs P satisfying*

$$|P_{\boldsymbol{c}}(n)| \leq C_1 A^n, \text{ for all } n \in \mathbb{N} \text{ and } |P_{\boldsymbol{c}}(-n)| \leq C_1 B^n, \text{ for all } n \in \mathbb{N}^+.$$

Theorem 2.3. *Let $A, B > 1$ be fixed constants as in Theorem 2.1 that satisfy $\gamma_{A,B} = 1$. There is a constant $C_2 = C_2(A,B) > 0$ so that there are no IVPs P that satisfy*

$$\sum_{n\in\mathbb{N}} |P_{\boldsymbol{c}}(n)|^2 A^{-2n} \leq C_2, \text{ and } \sum_{n\in\mathbb{N}^+} |P_{\boldsymbol{c}}(-n)|^2 B^{-2n} \leq C_2.$$

Moreover, there is a constant $C_3 = C_3(A,B) > 0$ such that there are infinitely many IVPs P satisfying

$$\sum_{n\in\mathbb{N}} \frac{|P_{\boldsymbol{c}}(n)|^2}{n+1} A^{-2n} \leq C_3, \text{ and } \sum_{n\in\mathbb{N}^+} \frac{|P_{\boldsymbol{c}}(-n)|^2}{n+1} B^{-2n} \leq C_3.$$

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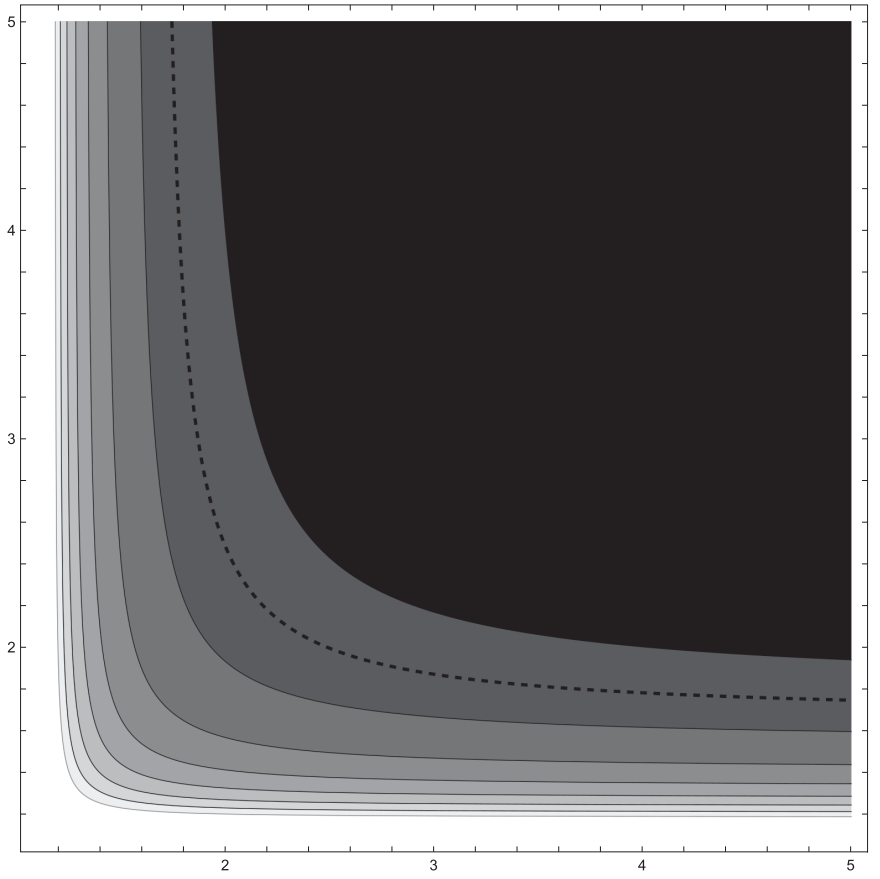


Figure 2. *Level sets of* $\gamma_{A,B}$.

Notation

We will use the following notation for the disk and the circle. $\mathbb{D}(a,r) = \{z \in \mathbb{C}\colon |z-a| < r\}$, $\mathbb{T}(a,r) = \partial\mathbb{D}(a,r)$. We also write $\mathbb{D} = \mathbb{D}(0,1)$, $\mathbb{T} = \partial\mathbb{D}$ for the unit disk and unit circle.

We write $\mathrm{d}|z|$ to denote integration with respect to arc-length measure, and $\mathrm{d}A(z)$ denotes integration with respect to area measure (both are not normalized). That is, if Γ is a (rectifiable) curve, and G is a domain, then

$$\int_\Gamma 1\,\mathrm{d}|z| = \mathrm{length}(\Gamma), \qquad \int_G 1\,\mathrm{d}A(z) = \mathrm{area}(G).$$

3. Outline of the proofs of Theorems 1.1 and 1.2

Here we give a summary of the ideas behind the proofs of the theorems with a growth condition on the natural numbers. Using the binomial polynomial basis

$$\binom{z}{k} = \frac{z(z-1)\dots(z-k+1)}{k!}, \quad k \in \mathbb{N},$$

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we let

$$P(z) = P_{\mathbf{c}}(z) = \sum_{k=0}^{d} c_k \binom{z}{k}$$

be a polynomial of degree d. The polynomial $P_{\mathbf{c}}$ is integer-valued if and only if $\mathbf{c} = (c_0, \dots, c_d) \in \mathbb{Z}^{d+1}$. The key observation is that the *generating function*

$$g_P(z) := \sum_{n\in\mathbb{N}} P(n) z^n = \frac{1}{1-z} \sum_{k=0}^{d} c_k \left(\frac{z}{1-z}\right)^k, \qquad \forall |z| < 1,$$

can be utilized to obtain information on $\mathbf{c}$ given the sequence $\{P(n)\}_{n\in\mathbb{N}}$ and vice versa.

3.1. The ℓ^∞ constraint

Let $t > 0$. We first consider the (easier) problem of the asymptotic volume of the set

$$\left\{ \mathbf{c} \in \mathbb{R}^{d+1} : \sup_{n\in\mathbb{N}} |P_{\mathbf{c}}(n) A^{-n}| \le t \right\}.$$

Using theorems by Ball and Vaaler, and some approximations (see Section 4 for the details), we will arrive at the following result

$$\frac{t^{d+1}}{\sqrt{\det \mathcal{M}}} \le \mathrm{Vol}\left(\left\{ \mathbf{c} \in \mathbb{R}^{d+1} : \sup_{n\in\mathbb{N}} |P_{\mathbf{c}}(n) r^n| \le \tfrac{1}{2} t \right\} \right) \le \frac{C_1^{d+1} t^{d+1}}{\sqrt{\det \mathcal{M}}},$$

where $C_1 > 0$ is an absolute constant, and

$$\mathcal{M} = \mathcal{M}(d; r) = \left[\sum_{n=0}^{\infty} \binom{n}{j} \binom{n}{k} r^{2n} \right]_{j,k=0}^{d}.$$

Observe that by the orthogonality of the monomials

$$\begin{aligned} \sum_{n=0}^{\infty} P_1(n) P_2(n) r^{2n} &= \frac{1}{2\pi r} \int_{r\mathbb{T}} \left[\sum_{m=0}^{\infty} P_1(m) w^m \right] \left[\sum_{n=0}^{\infty} P_2(n) \bar{w}^n \right] \mathrm{d}|w| \\ &= \frac{1}{2\pi r} \int_{r\mathbb{T}} g_{P_1}(w) \overline{g_{P_2}(w)} \, \mathrm{d}|w|, \end{aligned}$$

where the integrals are with respect to arc-length.

By a change of variables, using the fractional linear transformation $w = z/(1-z)$, we conclude that

$$\mathcal{M} = \left[\frac{1}{2\pi r} \int_{\mathbb{T}_1} w^j \bar{w}^k \, \mathrm{d}|w| \right]_{j,k=0}^{d}$$

is a *moment matrix*, where

$$\mathbb{D}_1 = \mathbb{D}\left(\frac{r^2}{1-r^2}, \frac{r}{1-r^2} \right), \quad \mathbb{T}_1 = \partial \mathbb{D}_1 = \mathbb{T}\left(\frac{r^2}{1-r^2}, \frac{r}{1-r^2} \right).$$

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It is known that (see, e.g., Appendix A)

$$\det \mathcal{M} = \prod_{k=0}^{d} \|q_k\|^2_{L^2(\mu)},$$

where q_k is the *monic* orthogonal polynomial (of degree k) with respect to μ – the arc-length measure on $\mathbb{T}_1$ (weighted by $\frac{1}{2\pi r}$). These polynomials and their norms are given explicitly by

$$q_k(w) = \big(w - \tfrac{r^2}{1-r^2}\big)^k, \qquad \|q_k\|^2_{L^2(\mu)} = \frac{1}{r}\Big(\frac{r}{1-r^2}\Big)^{2k+1},$$

hence

$$\det \mathcal{M} = \frac{1}{r^{d+1}}\Big(\frac{r}{1-r^2}\Big)^{(d+1)^2}.$$

The critical threshold is achieved when $\frac{r}{1-r^2} = 1$, that is $r = \varphi^{-1}$. Theorem 1.1 which concerns lattice points is obtained using a related theorem of Vaaler.

3.2. *The ℓ^2 constraint*

Here we have, for $r < 1$,

$$\frac{1}{2\pi r}\int_{r\mathbb{T}} |g_P(z)|^2 \mathrm{d}|z| = \sum_{n=0}^{\infty} |P(n)|^2 r^{2n} = \|a_n\|^2_{\ell^2},$$

and

$$\frac{1}{\pi r^2}\int_{r\mathbb{D}} |g_P(z)|^2 \mathrm{d}A(z) = \sum_{n=0}^{\infty} \frac{|P(n)|^2}{n+1} r^{2n} = \Big\| \frac{a_n}{\sqrt{n+1}} \Big\|^2_{\ell^2},$$

where we write $a_n = a_n(r) = P(n)r^n$, and the second integral is with respect to planar Lebesgue measure.

Again making the change of variables $w = z/(1-z)$, we find that

$$\|a_n\|^2_{\ell^2} = \frac{1}{2\pi r}\int_{\mathbb{T}_1} |h_{\mathbf{c}}(w)|^2 \mathrm{d}|w|, \qquad \Big\| \frac{a_n}{\sqrt{n+1}} \Big\|^2_{\ell^2} = \frac{1}{\pi r^2}\int_{\mathbb{D}_1} |h_{\mathbf{c}}(w)|^2 \frac{\mathrm{d}A(w)}{|1+w|^2},$$

where $\mathbb{D}_1$ and $\mathbb{T}_1$ are as above, and

$$h_{\mathbf{c}}(w) = \sum_{k=0}^{d} c_k w^k.$$

Let q_k be the (Szegő) monic orthogonal polynomial with respect to the measure μ as above, and t_k be the (Bergman) monic orthogonal polynomial, with respect to weighted area measure

$$\mathrm{d}\nu(w) = \frac{\mathrm{d}A(w)}{\pi r^2 |1+w|^2} \quad \text{on} \quad \mathbb{D}_1.$$

We are not aware of any simple formula for the t_ks (see, e.g., [10, Remark 8.1]). However, by results of Korovkin (see Section 5.1) we have, with $r = \varphi^{-1}$, and some $q \in (0,1)$,

$$\|t_k\|^2_{L^2(\nu)} = \frac{1}{k+1} + O(q^k).$$

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We obtain the correspondences

$$\sum_{n=0}^{\infty} |P_{\mathbf{c}}(n)|^2 \varphi^{-2n} \le t^2 \iff \sum_{k=0}^{d} |\alpha_k(\mathbf{c})|^2 \|q_k\|^2_{L^2(\mu)} = \varphi \sum_{k=0}^{d} |\alpha_k(\mathbf{c})|^2 \le t^2,$$

where $h_{\mathbf{c}} = \sum_{k=0}^{d} \alpha_k(\mathbf{c}) q_k$, and

$$\sum_{n=0}^{\infty} \frac{|P(n)|^2}{n+1} \varphi^{-2n} \le t^2 \iff \sum_{k=0}^{d} |\beta_k(\mathbf{c})|^2 \|t_k\|^2_{L^2(\nu)} \le t^2,$$

where now $h_{\mathbf{c}} = \sum_{k=0}^{d} \beta_k(\mathbf{c}) t_k$. The proof of the first part of Theorem 1.2 follows by observing that since q_k are monic polynomials, if $h_{\mathbf{c}}$ has integer coefficients, then the first, in *decreasing* order of k, nonzero coefficient $\alpha_k(\mathbf{c})$ is the same integer. The second part of the theorem is deduced from a classical theorem of van der Corput on the existence of lattice points inside symmetric convex bodies with a large volume.

4. Preliminaries

4.1. Generating functions

To an algebraic polynomial P we associate the generating function

$$g(z) = g_P(z) = \sum_{n=0}^{\infty} P(n) z^n, \tag{4.1}$$

which converges for $|z| < 1$. We also denote by g_k the generating function of the binomial polynomial $\binom{\cdot}{k}$. We leave the proof of the following claim as an exercise to the reader.

Claim 4.1. *For $|z| < 1$ and $k \ge 0$, we have*

$$g_k(z) = \frac{1}{1-z} \left(\frac{z}{1-z} \right)^k.$$

The above claim provides an analytic continuation of g_P to a meromorphic function.

Claim 4.2. *For any polynomial P, the function g_P is analytic in $\mathbb{C} \setminus \{1\}$ with $g_P(\infty) = 0$, and in addition*

$$g_P(z) = -\sum_{n=1}^{\infty} \frac{P(-n)}{z^n} \quad \text{for all } |z| > 1. \tag{4.2}$$

Remark. For a more general claim see [11, Lecture 10, Remark 1].

Proof. Any polynomial P of degree d can be written in a unique way as

$$P(x) = \sum_{k=0}^{d} c_k \binom{x}{k},$$

hence it is sufficient to prove the claim for binomials. By Claim 4.1, for $k \in \mathbb{N}$ we have that

$$g_k(z) = \sum_{n \in \mathbb{N}} \binom{n}{k} z^n = \frac{z^k}{(1-z)^{k+1}}, \qquad |z| < 1,$$

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where the right hand side is analytic in $z \in \mathbb{C} \setminus \{1\}$ and vanishes at infinity. Let $\varepsilon \in (0,1)$. Using the vanishing of g_k at infinity, and Cauchy's integral formula, we have

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|w|=\varepsilon} \frac{w^k}{(1-w)^{k+1}}\frac{\mathrm{d}w}{w^{n+1}} = -\frac{1}{2\pi i}\int_{|w-1|=\varepsilon} \frac{w^k}{(1-w)^{k+1}}\frac{\mathrm{d}w}{w^{n+1}}.$$

We will use $w^z = \exp(z\log w)$, where the principal branch of log, cut along the ray $(-\infty, 0]$, is chosen. Observe that with this choice,

$$Q(z) := -\frac{1}{2\pi i}\int_{|w-1|=\varepsilon} \frac{w^k}{(1-w)^{k+1}}\frac{\mathrm{d}w}{w^{z+1}},$$

is an entire function with at most polynomial growth at infinity (this can be seen by choosing ε of order $|z|^{-1}$). Hence by Liouville's theorem Q *is* a polynomial. Since the polynomials Q and $\binom{\cdot}{k}$ coincide on the (infinite) set $\mathbb{N}$, we conclude they are equal. In particular, for $n \in \mathbb{N}$,

$$\binom{-n}{k} = -\frac{1}{2\pi i}\int_{|w-1|=\varepsilon} \frac{w^{k+n}}{(1-w)^{k+1}}\frac{\mathrm{d}w}{w}.$$

By making a change of variables $\zeta = 1/w$, so that $\mathrm{d}w/w = -\mathrm{d}\zeta/\zeta$, and shifting the contour we find

$$\binom{-n}{k} = \frac{1}{2\pi i}\int_{|\zeta^{-1}-1|=\varepsilon} \frac{1}{(\zeta-1)^{k+1}}\frac{\mathrm{d}\zeta}{\zeta^n} = \frac{1}{2\pi i}\int_{|\zeta|=\varepsilon}\left[-\frac{1}{(\zeta-1)^{k+1}}\right]\frac{\mathrm{d}\zeta}{\zeta^n}.$$

Therefore, again using Cauchy's formula, for sufficiently small $|\zeta|$,

$$\sum_{n\geq 1}\binom{-n}{k}\zeta^{n-1} = -\frac{1}{(\zeta-1)^{k+1}} \implies \sum_{n\geq 1}\binom{-n}{k}\zeta^{n} = -\frac{\zeta}{(\zeta-1)^{k+1}}.$$

Finally, putting $z = 1/\zeta$ we obtain

$$\sum_{n\geq 1}\binom{-n}{k}z^{-n} = -\frac{z^k}{(1-z)^{k+1}} = -g_k(z), \qquad \text{where } |z| > 1,$$

as claimed. □

The following properties of the generating function are crucial for us.

Corollary 4.3. *For a polynomial $P(z) = \sum_{k=0}^{d} c_k\binom{z}{k}$ of degree d, the function g_P can be analytically continued to $\mathbb{C}\setminus\{1\}$ so that*

$$g_P(z) = \frac{1}{1-z}\sum_{n=0}^{d} c_k\left(\frac{z}{1-z}\right)^k, \quad z \in \mathbb{C}\setminus\{1\}.$$

Moreover, for $r > 0$,

$$\frac{1}{2\pi r}\int_{r\mathbb{T}} |g_P(w)|^2 \mathrm{d}|w| = \begin{cases} \sum_{n=0}^{\infty} |P(n)|^2 r^{2n}, & r < 1; \\ \sum_{n=1}^{\infty} |P(-n)|^2 r^{-2n}, & r > 1, \end{cases}$$

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and

$$\frac{1}{\pi r^2}\int_{r\mathbb{D}} |g_P(w)|^2 \mathrm{d}A(w) = \sum_{n=0}^{\infty} \frac{|P(n)|^2}{n+1} r^{2n}, \quad \textit{when } r<1,$$
$$\frac{r^2}{\pi}\int_{(r\mathbb{D})^c} |g_P(w)|^2 \frac{\mathrm{d}A(w)}{|w|^4} = \sum_{n=1}^{\infty} \frac{|P(-n)|^2}{n+1} r^{-2n}, \quad \textit{when } r>1.$$

Remark. Under the conformal map $z \mapsto \frac{1}{z}$ the area measure $\mathrm{d}A(z)$ transforms to $\frac{\mathrm{d}A(z)}{|z|^4}$.

Proof. The first part follows from the previous claims. The second part follows by expanding $|g_P(w)|^2$ as in (4.1) and (4.2) in the variables $w, \overline{w}$ and observing that

$$\frac{1}{2\pi r}\int_{r\mathbb{T}} w^j \overline{w}^k \mathrm{d}|w| = r^{2j}\delta_{j,k}, \quad \text{when } j,k \in \mathbb{Z}.$$

Similarly, the formulas with respect to area measure follow from

$$\frac{1}{\pi}\int_{r\mathbb{D}} w^j \overline{w}^k \mathrm{d}A(w) = \frac{r^{2(j+1)}}{j+1}\delta_{j,k}, \quad \text{when } r<1 \text{ and } j,k \in \mathbb{N},$$

and

$$\frac{1}{\pi}\int_{(r\mathbb{D})^c} \frac{1}{w^j}\frac{1}{\overline{w}^k}\frac{\mathrm{d}A(w)}{|w|^4} = \frac{r^{-2(j+1)}}{j+1}\delta_{j,k}, \quad \text{when } r>1 \text{ and } j,k \in \mathbb{N}. \qquad \square$$

The next result follows from the above by polarization. We leave the details to the reader.

Corollary 4.4. *For $j,k \geq 0$ we have,*

$$\sum_{n=0}^{\infty} \binom{n}{j}\binom{n}{k} r^{2n} = \frac{1}{2\pi r}\int_{r\mathbb{T}} g_j(w)\overline{g_k(w)}\mathrm{d}|w|, \quad r \in (0,1),$$

and

$$\sum_{n=1}^{\infty} \binom{-n}{j}\binom{-n}{k} r^{-2n} = \frac{1}{2\pi r}\int_{r\mathbb{T}} g_j(w)\overline{g_k(w)}\mathrm{d}|w|, \qquad r>1.$$

4.2. Tools for the ℓ^∞ case

Let $N,k \in \mathbb{N}$ with $N>k$, and let Σ be an $N\times k$ matrix of rank k. We use the following version of a result by Vaaler [15, Thms. 1 and 2], see also Gluskin-Kashin [8].

Theorem 4.5. *We have that*

$$V_N := \mathrm{Vol}\left(\left\{\boldsymbol{c}=(c_1,\ldots,c_k)\in\mathbb{R}^k : \sup_{n\leq N}\left|\sum_{j=1}^{k} c_j\Sigma_{n,j}\right| \leq \tfrac{1}{2}\right\}\right) \geq \frac{1}{\sqrt{\det \Sigma^t\Sigma}}.$$

Moreover, if $R\in\mathbb{N}$ is such that

$$R^2\cdot \det\Sigma^t\Sigma \leq 1,$$

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then there are at least $2R$ *nontrivial solutions* $\boldsymbol{c} \in \mathbb{Z}^k$ *to the system of inequalities*

$$\left|\sum_{j=1}^{k} c_j \Sigma_{n,j}\right| \le 1 \qquad \text{for all} \quad n \in \{1, \dots, N\}.$$

In the other direction we use a result of Ball [2] (also Gluskin-Kashin [8]).

Theorem 4.6. *We have that*

$$\mathrm{Vol}\left(\left\{\boldsymbol{c} = (c_1, \dots, c_k) \in \mathbb{R}^k : \sup_{n \le N} \left|\sum_{j=1}^{k} c_j \Sigma_{n,j}\right| \le 1\right\}\right) \le \frac{\sqrt{2}^{N-k}}{\sqrt{\det \Sigma^t \Sigma}}.$$

For the convenience of the reader we prove the following standard result in Appendix A.

Theorem 4.7. *Let* μ *be a measure on* $\mathbb{C}$ *with infinite support and finite moments of all orders. Define the n-th moment matrix by*

$$\mathcal{M}(n; \mu) = \left[M_{k,m}\right]_{k,m=0}^{n}, \qquad M_{k,m} = \int_{\mathbb{C}} z^j \bar{z}^k \, \mathrm{d}\mu(z).$$

Then

$$\det \mathcal{M}(n; \mu) = \prod_{j=0}^{n} \left\|Q_j\right\|_{L^2(\mu)}^2,$$

where Q_j *are the orthogonal monic polynomials, with respect to* μ.

Let $\eta \in (0, 1)$. Suppose $\Gamma = \mathbb{T}_1 \cup \mathbb{T}_2$ is the union of two circles whose radii are bounded between η and $\frac{1}{\eta}$, and their interiors are separated by at least η and at most $\frac{1}{\eta}$. Denote the $n \times n$ moment matrix, with respect to arc-length measure on Γ, by $\mathcal{M}_\Gamma(n)$. We will need an effective lower bound for the smallest eigenvalue of $\mathcal{M}_\Gamma(n)$, which is given by

$$\lambda_n^{-1} = \max\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|P(e^{i\theta})\right|^2 \mathrm{d}\theta : \ \deg P = n, \ \int_\Gamma |P(z)|^2 |\mathrm{d}z| = 1\right\}.$$

The following result is obtained by combining [16, Theorem 7.3 and Eq. (10.5)].

Proposition 4.8. *Using the same notation as above, and under the same assumptions, we have*

$$\lambda_n \ge \exp(-C_\Gamma n),$$

where the constant $C_\Gamma > 0$ *depends continuously on the Green function of the exterior of* Γ *(and its derivatives).*

Remark. Under the assumptions above the constant C_Γ can be bounded in terms of η only.

4.2.1. Elementary results

We use the standard bound (that follows from the inequality $n! \ge (n/e)^n$)

$$\binom{n}{k} \le \left(\frac{en}{k}\right)^k, \qquad n > k \ge 1. \tag{4.3}$$

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Claim 4.9. *Let $a > 0$ and $C \geq e$, then we have*

$$\sup_{1 \leq x \leq a/C} \left(\frac{a}{x}\right)^x \leq C^{a/C}.$$

Proof. The function $x \log \frac{a}{x}$ is increasing on $[1, a/C]$. □

Lemma 4.10. *For every $A > 1$ there exists a constant $C_A > 1$ such that if P is a polynomial of degree d satisfying $|P(n)| \leq A^n$ for all $0 \leq n \leq C_A d$, then $|P(n)| \leq A^n$ for all $n \in \mathbb{N}$.*

Proof. The Lagrange interpolation formula states that

$$P(x) = \sum_{j=0}^{d} P(j) L_j(x), \qquad \text{where} \quad L_j(x) = \prod_{\substack{0 \leq m \leq d \\ m \neq j}} \frac{x - m}{j - m}.$$

Notice that for $n \geq d + 1$

$$|L_j(n)| \leq n^{d+1} \prod_{\substack{0 \leq m \leq d \\ m \neq j}} (j - m)^{-1} \leq \frac{n^{d+1}}{[(\lfloor d/2 \rfloor)!]^2} \leq \left(\frac{6n}{d}\right)^{d+1}.$$

Let C_A be a constant depending only on A to be chosen later. Under the assumption of the lemma, we have that $|P(n)| \leq A^n$ for all $0 \leq n \leq C_A d$. In addition, for $n \geq d + 1$

$$|P(n)| \leq \left(\frac{6n}{d}\right)^{d+1} \sum_{j=0}^{d} A^j \leq \frac{1}{A-1} \left(\frac{6An}{d}\right)^{d+1}.$$

If we choose C_A large enough so that

$$\text{(i)} \ \exp(2/C_A) \leq A \quad \text{(ii)} \ 6AC_A \leq A^{C_A/2} \quad \text{(iii)} \ (A-1)^{-1} \leq A^{C_A/2},$$

then we have that

$$\frac{1}{A-1} \left(\frac{6An}{d}\right)^{d+1} \leq A^n, \qquad \text{for all} \quad n \geq C_A d.$$

In particular, we may assume C_A is decreasing with A. □

A similar result holds for $P(-n)$ (e.g., define $Q(z) = B^{-1} P(-z - 1)$ and apply the lemma). Hence, one can prove the following generalization. We leave the details to the reader.

Lemma 4.11. *For every $A, B > 1$ there exists a constant $C_{A,B} > 1$ such that if P is a polynomial of degree d satisfying $|P(n)| \leq A^n$ and $|P(-n)| \leq B^n$ for all $0 \leq n \leq C_{A,B} \cdot d$, then $|P(n)| \leq A^n$ for all $n \in \mathbb{N}$ and $|P(-n)| \leq B^n$ for all $n \in \mathbb{N}$.*

The following lemma will allow us to approximate the value of the determinant of the moment matrix.

Lemma 4.12. *Let $A > 1$, then for $L > C_A$ (where $C_A > e$ is the constant from Lemma 4.10), $n \in \mathbb{N}$ and $k, m \in \{0, \ldots, n-1\}$ we have*

$$\sum_{j > Ln} A^{-2j} \binom{j}{k} \binom{j}{m} \leq \frac{A}{A-1} \exp((2 - L \log A) n).$$

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Proof. We will prove the result for $k, m \geq 1$, the other cases being similar. For $j > Ln$, we have by (4.3) and Claim 4.9,

$$A^{-2j}\binom{j}{k}\binom{j}{m} \leq e^{k+m}A^{-2j}\frac{j^{k+m}}{k^k m^m} \leq e^{2n}A^{-2j}L^{2j/L}.$$

Therefore, if $L > e$ is large enough so that $L^{1/L} < \sqrt{A}$ we get

$$\sum_{j>Ln} A^{-2j}\binom{j}{k}\binom{j}{m} \leq e^{2n}\sum_{j>Ln} A^{-j},$$

and the result follows. □

Similarly one can prove the following extension; we skip the details of the proof (one has to use $\left|\binom{-n}{k}\right| \leq \binom{2n}{k}$ for $n \geq 0$ and $k \in \{0, \dots, n-1\}$).

Lemma 4.13. *Let $A, B > 1$, then for $L > C_{A,B}$ (where $C_{A,B} > e$ is the constant from Lemma 4.11), $n \in \mathbb{N}$ and $k, m \in \{0, \dots, n-1\}$ we have*

$$\sum_{j<-Ln} B^{2j}\binom{j}{k}\binom{j}{m} + \sum_{j>Ln} A^{-2j}\binom{j}{k}\binom{j}{m} \leq C'_{A,B}\exp((4 - L\log\min\{A, B\})n),$$

where $C'_{A,B}$ is some constant depending continuously on A, B.

4.2.2. Bounds for the determinant of the moment matrix

Let $\mathcal{A}$ be an $n \times n$ Hermitian matrix, and denote its eigenvalues by $\lambda_n(\mathcal{A}) \leq \cdots \leq \lambda_1(\mathcal{A})$. We will use the following result (see, e.g., [3, III.2]).

Theorem 4.14 (Weyl's inequalities). *Let $\mathcal{A}, \mathcal{B}$ be $n \times n$ Hermitian matrices.*
For all $k \in \{1, \dots, n\}$ we have

$$\lambda_k(\mathcal{A}) + \lambda_n(\mathcal{B}) \leq \lambda_k(\mathcal{A} + \mathcal{B}) \leq \lambda_k(\mathcal{A}) + \lambda_1(\mathcal{B}).$$

We will use the following notation for the entries of moment matrices:

$$\Xi_1(A, k, m; M) = \sum_{j=0}^{M} A^{-2j}\binom{j}{k}\binom{j}{m},$$

$$\Xi_2(A, B, k, m; M) = \sum_{j=-M}^{-1} B^{2j}\binom{j}{k}\binom{j}{m} + \sum_{j=0}^{M} A^{-2j}\binom{j}{k}\binom{j}{m}.$$

Proposition 4.15. *Let $d \in \mathbb{N}$, $\varepsilon \in (0, 1/2)$, and $A \in (1+\varepsilon, 1/\varepsilon)$. Consider the following (symmetric) matrices*

$$\mathcal{M}_d^{(1)}(A) = [\Xi_1(A, k, m, \infty)]_{k,m=0}^{d},$$
$$\mathcal{M}_d^{(1)}(A; M) = [\Xi_1(A, k, m, M)]_{k,m=0}^{d}.$$

If $L > e$ is sufficiently large (depending only on ε), then

$$\det \mathcal{M}_d^{(1)}(A; \lfloor Ld \rfloor) \geq 2^{-(d+1)} \det \mathcal{M}_d^{(1)}(A).$$

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Remark. Notice that $\mathcal{M}_d^{(1)}(A)$ and $\mathcal{M}_d^{(1)}(A;M)$ are positive-definite matrices. In particular, we may write

$$\mathcal{M}_d^{(1)}(A) = \mathcal{N}_d^t \mathcal{N}_d,$$

where $\mathcal{N}_d$ is the $\infty \times (d+1)$ matrix whose entries are given by $[\mathcal{N}_d]_{k,m} = A^{-k}\binom{k}{m}$. Similarly for $\mathcal{M}_d^{(1)}(A;M) = (\mathcal{N}_d(M))^t \mathcal{N}_d(M)$, where $\mathcal{N}_d(M)$ is the $(M+1)\times(d+1)$ truncation of $\mathcal{M}_d^{(1)}(A)$. By the Cauchy-Binet formula we have that

$$\det \mathcal{M}_d^{(1)}(A;M) = \det\big(\mathcal{N}_d^t(M)\mathcal{N}_d(M)\big) \le \det\big(\mathcal{N}_d^t\mathcal{N}_d\big) = \det \mathcal{M}_d^{(1)}(A).$$

Proof. We let $\mathcal{A} = \mathcal{M}_d^{(1)}(A)$ and define the matrix

$$\mathcal{B} = \left[\sum_{j>\lfloor Ld\rfloor}^{\infty} A^{-2j}\binom{j}{k}\binom{j}{m}\right]_{k,m=0}^{d}.$$

Let $\lambda_{d+1}(\mathcal{A})$ be the smallest eigenvalue of the positive definite matrix $\mathcal{A}$. Observe that by Corollary 4.4 and a change of variables (see Section 6), $\mathcal{A}$ is a moment matrix. Thus, by Proposition 4.8, there is a constant $C_1 = C_1(\varepsilon)$ such that $\lambda_{d+1}(\mathcal{A}) \ge \exp(-C_1 d)$. Using Lemma 4.12, we get a pointwise bound for the entries of $\mathcal{B}$, which in turn bounds its operator norm (up to a factor $d+1$ which can be subsumed into the exponential). Thus, we may choose $L = L(\varepsilon)$ sufficiently large so that for all d large enough $\lambda_1(\mathcal{B}) \le \frac{1}{2}\lambda_{d+1}(\mathcal{A})$.

Hence, Weyl's inequalities give

$$\begin{aligned}\det(\mathcal{M}_d(A;\lfloor Ld\rfloor)) = \det(\mathcal{A}-\mathcal{B}) &= \prod_{j=1}^{d+1}\lambda_j(\mathcal{A}-\mathcal{B}) \\ &\ge \prod_{j=1}^{d+1}[\lambda_j(\mathcal{A}) - \lambda_1(\mathcal{B})] \ge \prod_{j=1}^{d+1}\tfrac{1}{2}\lambda_j(\mathcal{A}) = 2^{-(d+1)}\det(\mathcal{A}).\end{aligned}$$

Note that the constant $\frac{1}{2}$ is arbitrary and can be replaced by (e.g.) $1-\exp(-d)$. □

Using Lemma 4.13, one can prove the following generalization.

Proposition 4.16. *Let $d \in \mathbb{N}$, $\varepsilon \in (0,1/2)$, and $A, B \in (1+\varepsilon, 1/\varepsilon)$. Consider the following (symmetric) matrices*

$$\begin{aligned}\mathcal{M}_d^{(2)}(A,B) &= [\Xi_2(A,B,k,m,\infty)]_{k,m=0}^{d}, \\ \mathcal{M}_d^{(2)}(A,B;M) &= [\Xi_2(A,B,k,m,M)]_{k,m=0}^{d}.\end{aligned}$$

If $L > e$ is sufficiently large (depending only on ε), then

$$\det \mathcal{M}_d^{(2)}(A,B;\lfloor Ld\rfloor) \ge 2^{-(d+1)} \det \mathcal{M}_d^{(2)}(A,B).$$

4.3. *Tools for the ℓ^2 case*

Below we will need an estimate for the number of lattice points inside ellipsoids of a special type.

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Proposition 4.17. *Let $d \in \mathbb{N}$ and $\gamma, t > 0$ be parameters (that may depend on d), and define the ellipsoid*

$$\mathcal{E} = \mathcal{E}(d;\gamma,t) = \left\{ \boldsymbol{x} = (x_0,\dots,x_d) \in \mathbb{R}^{d+1} : \sum_{k=0}^{d} x_k^2 \gamma^{2k} \le t^2 \right\}.$$

Moreover, let Ψ be a lower unipotent matrix, that is, an $n \times n$ lower triangular matrix, with diagonal elements all equal to 1. We denote by $\Psi(\mathcal{E})$ the image of $\mathcal{E}$ under Ψ. Note that this map preserves volume.

For $\gamma \le 1 \le t$ we have

$$-C_1(d+1) \le \log \frac{\#\big(\Psi(\mathcal{E}) \cap \mathbb{Z}^{d+1}\big)}{\mathrm{Vol}(\mathcal{E})} \le C_2(d+1)\log\min\{d+1,(1-\gamma)^{-1}\},$$

where $C_1, C_2 > 0$ are absolute constants.

Remark. Notice that $\mathrm{Vol}(\mathcal{E}) = \gamma^{-d(d+1)/2} t^{d+1} \mathrm{Vol}\big(B_2^{d+1}\big)$.

Proof. The lower bound (with $C_1 = \log 2$) follows from [9, Sec. 7, Thm. 4].

Let $\mathbf{z}_1, \mathbf{z}_2 \in \Psi(\mathcal{E}) \cap \mathbb{Z}^{d+1}$ be two different lattice points. Notice that by the structure of the matrix Ψ, we have that

$$\|\mathbf{x}_1 - \mathbf{x}_2\|_\infty := \|\Psi^{-1}(\mathbf{z}_1) - \Psi^{-1}(\mathbf{z}_2)\|_\infty \ge 1.$$

Now let $\mathbf{x} = (x_0,\dots,x_d) \in \mathcal{E}$ be a preimage of one of the $\mathbb{Z}^{d+1}$ lattice points in $\Psi(\mathcal{E})$. If $\mathbf{y} = (y_0,\dots,y_d) \in \left[-\frac{1}{2},\frac{1}{2}\right]^{d+1}$, then

$$\begin{aligned}\sum_{k=0}^{d}(x_k+y_k)^2\gamma^{2k} &= \sum_{k=0}^{d} x_k^2\gamma^{2k} + 2\sum_{k=0}^{d} x_k y_k \gamma^{2k} + \sum_{k=0}^{d} y_k^2\gamma^{2k} \\ &\le 2\left[t^2 + \tfrac{1}{4}\min\{d+1,(1-\gamma^2)^{-1}\}\right],\end{aligned}$$

since $2|x_k y_k| \le x_k^2 + y_k^2$. Therefore, if we denote by $\mathcal{Q}$ the union of all the cubes with side length 1 centered at preimages of lattice points inside $\mathcal{E}$, we see that

$$\mathcal{Q} \subset C_2(1+t^{-2}\min\{d+1,(1-\gamma)^{-1}\}\mathcal{E} \subset C_2\min\{d+1,(1-\gamma)^{-1}\}\mathcal{E}.$$

Since the cubes forming $\mathcal{Q}$ are of volume 1 and have disjoint interiors, we get the required bound for the number of lattice points. □

The following theorem can be proved explicitly using conformal maps, but it also follows as a very special case of [16, Thm. 9.1]. The fine properties of the normalizing constants can be found in [16, Sec. 6].

Theorem 4.18. *Let $\eta \in (0,1)$. Suppose $\Gamma = \mathbb{T}_1 \cup \mathbb{T}_2$ is the union of two circles, whose interiors are separated by at least η and at most $\frac{1}{\eta}$, and whose radii are bounded between η and $\frac{1}{\eta}$. Denote by $\{q_k\}_{k=0}^{\infty}$ the monic orthogonal polynomial of degree k on Γ with respect to arc-length, and let*

$$\beta_k^2 = \int_\Gamma |q_k|^2 \,\mathrm{d}\sigma,$$

be its normalization constant (squared). The asymptotics of β_k are given by

$$\beta_k = \mathrm{Cap}(\Gamma)^k f_k(\Gamma)(1+o(1)), \qquad k \to \infty, \tag{4.4}$$

where f_k are functions that can be bounded between positive constants depending only on η. The implicit error term can also be bounded in terms of η only.

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We will also require corresponding results for Bergman polynomials (i.e., orthogonal with respect to area measure). The following result closely follows [10, Theorem 4.1]. We give the details in Appendix B.

Theorem 4.19. *Let $\eta \in (0,1)$. Suppose $G = \mathbb{D}_1 \cup \mathbb{D}_2$ is the union of two disks, separated by at least η and at most $\frac{1}{\eta}$, whose radii are bounded between η and $\frac{1}{\eta}$. Let $\{t_k\}_{k=0}^{\infty}$ denote the sequence of monic Bergman orthogonal polynomials associated with G, that is*

$$\int_G t_{k_1}(z)t_{k_2}(z)\,\mathrm{d}A(z) = \gamma_{k_1}^2 \cdot \begin{cases} 1, & k_1 = k_2, \\ 0, & k_1 \neq k_2, \end{cases}$$

where $\mathrm{d}A$ denotes the Lebesgue measure on $\mathbb{C}$. There are constants $0 < C_1(\eta) < C_2(\eta)$ such that for $k \in \mathbb{N}$

$$C_1 \frac{\mathrm{Cap}(\Gamma)^k}{\sqrt{k+1}} \leq \gamma_k \leq C_2 \frac{\mathrm{Cap}(\Gamma)^k}{\sqrt{k+1}}.$$

5. ℓ^2 volume and number of lattice points

In this section we prove Theorem 1.2, part of Theorem 2.1, and Theorem 2.3.

We recall some notation. Put $\mathbf{c} = \mathbf{c}(d) = (c_0, \ldots, c_d) \in \mathbb{R}^{d+1}$ and let

$$P(x) = P_{\mathbf{c}}(x) = \sum_{k=0}^{d} c_k \binom{x}{k},$$

$$g_{\mathbf{c}}(z) = g_P(z) = \frac{1}{1-z} \sum_{n=0}^{d} c_k \left(\frac{z}{1-z}\right)^k, \quad z \in \mathbb{C} \setminus \{1\}.$$

Let $A, B > 1$ be fixed constants, and define the sequences

$$a_n(\mathbf{c}) = P_{\mathbf{c}}(n)A^{-n},\ n \in \mathbb{N}, \quad b_n(\mathbf{c}) = P_{\mathbf{c}}(-n)B^{-n},\ n \in \mathbb{N}^+,$$

and the sets

$$\mathcal{C}^2_{d,A,B}(t) := \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \|a_n(\mathbf{c})\|_{\ell^2(\mathbb{N})} \leq t,\ \|b_n(\mathbf{c})\|_{\ell^2(\mathbb{N}^+)} \leq t\right\},$$

and

$$\mathcal{D}^2_{d,A,B}(t) := \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \left\|\frac{a_n(\mathbf{c})}{\sqrt{n+1}}\right\|_{\ell^2(\mathbb{N})} \leq t,\ \left\|\frac{b_n(\mathbf{c})}{\sqrt{n+1}}\right\|_{\ell^2(\mathbb{N}^+)} \leq t\right\}.$$

5.1. ℓ^2 volume estimates

We now estimate the volume of the sets $\mathcal{C}^2$ and $\mathcal{D}^2$. By Corollary 4.3 we have

$$\mathcal{C}^2_{d,A,B}(t) = \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \frac{A}{2\pi}\int_{A^{-1}\mathbb{T}} |g_{\mathbf{c}}(z)|^2 \mathrm{d}|z| \leq t^2,\ \frac{1}{2\pi B}\int_{B\mathbb{T}} |g_{\mathbf{c}}(z)|^2 \mathrm{d}|z| \leq t^2\right\},$$

and

$$\mathcal{D}^2_{d,A,B}(t) = \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \frac{A^2}{\pi}\int_{A^{-1}\mathbb{D}} |g_{\mathbf{c}}(z)|^2 \mathrm{d}A(z) \leq t^2,\ \frac{B^2}{\pi}\int_{(B\mathbb{D})^c} |g_{\mathbf{c}}(z)|^2 \frac{\mathrm{d}A(z)}{|z|^4} \leq t^2\right\}.$$

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Notice that under a (one-to-one) change of variables $z = \phi(w)$, the arc-length and area integrals transform as follows,

$$\int_{\phi(\Gamma)} f(z)\,\mathrm{d}|z| = \int_{\Gamma} f(\phi(w))|\phi'(w)|\,\mathrm{d}|w|, \qquad \int_{\phi(G)} f(z)\,\mathrm{d}A(z) = \int_{G} f(\phi(w))|\phi'(w)|^2\,\mathrm{d}A(w).$$

We now make the change of variables via the fractional linear transformation,

$$w = \psi(z) = \frac{z}{1-z}, \quad \phi = \psi^{-1}, \quad z = \phi(w) = \frac{w}{1+w}, \quad \phi'(w) = \frac{1}{(1+w)^2},$$

such that

$$h_{\mathbf{c}}(w) := (1+w)^{-1}(g_{\mathbf{c}} \circ \phi)(w) = \sum_{k=0}^{d} c_k w^k,$$

and in particular $h_{\mathbf{c}}$ is a polynomial. Moreover,

$$\psi(A^{-1}\mathbb{T}) = \mathbb{T}\left(\frac{1}{A^2-1}, \frac{A}{A^2-1}\right) =: \mathbb{T}_1, \quad \psi(B\mathbb{T}) = \mathbb{T}\left(\frac{B^2}{1-B^2}, \frac{B}{B^2-1}\right) =: \mathbb{T}_2.$$

Thus,

$$\mathcal{C}^2_{d,A,B}(t) = \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \frac{A}{2\pi}\int_{\mathbb{T}_1} |h_{\mathbf{c}}(w)|^2 \mathrm{d}|w| \le t^2,\ \frac{1}{2\pi B}\int_{\mathbb{T}_2} |h_{\mathbf{c}}(w)|^2 \mathrm{d}|w| \le t^2\right\},$$

and

$$\mathcal{D}^2_{d,A,B}(t) = \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \frac{A^2}{\pi}\int_{\mathbb{D}_1} |h_{\mathbf{c}}(w)|^2 \frac{\mathrm{d}A(w)}{|1+w|^2} \le t^2,\ \frac{B^2}{\pi}\int_{\mathbb{D}_2} |h_{\mathbf{c}}(w)|^2 \frac{\mathrm{d}A(w)}{|w|^4|1+w|^{-2}} \le t^2\right\},$$

where

$$\mathbb{D}_1 := \mathbb{D}\left(\frac{1}{A^2-1}, \frac{A}{A^2-1}\right), \quad \mathbb{D}_2 := \mathbb{D}\left(\frac{B^2}{1-B^2}, \frac{B}{B^2-1}\right).$$

5.1.1. Volume of $\mathcal{C}^2$

Let $\{q_k(w)\}_{k=0}^{\infty}$ be the sequence of monic orthogonal polynomials with respect to arc-length measure on the curve $\Gamma := \mathbb{T}_1 \cup \mathbb{T}_2$ in increasing degree. Then, we can write

$$h_{\mathbf{c}}(w) = \sum_{k=0}^{d} \alpha_k(\mathbf{c}) q_k(w), \quad \text{where} \quad \alpha_d = c_d, \qquad \beta_k^2 := \int_{\Gamma} |q_k(w)|^2 \mathrm{d}|w|.$$

Furthermore,

$$\mathcal{E}(\beta_0, \ldots, \beta_d; t_1) \subset \mathcal{C}^2_{d,A,B}(t) \subset \mathcal{E}(\beta_0, \ldots, \beta_d; t_2), \tag{5.1}$$

where $\mathcal{E}(\beta_0, \ldots, \beta_d; t)$ is the ellipse

$$\mathcal{E}(\beta_0, \ldots, \beta_d; t) = \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \sum_{k=0}^{d} |\alpha_k(\mathbf{c})|^2 \beta_k^2 \le t^2\right\},$$

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and

$$t_1 := \sqrt{2\pi \min\{A^{-1}, B\}}\, t, \qquad t_2 := \sqrt{2\pi (A^{-1} + B)}\, t.$$

Let $\eta \in (0,1)$, and suppose $A, B > 1$ satisfy the conditions of Theorem 4.18. In particular, A and B are bounded away from 1 and ∞ in terms of η. Hence, the constants $\{\beta_k\}_{k\in\mathbb{N}}$ satisfy

$$\beta_k = \mathrm{Cap}(\Gamma)^k g_k(A, B, d),$$

where g_k are functions that are bounded between positive constants depending only on η. Therefore, by (5.1),

$$[C_1(\eta) t_1]^{d+1} \le \frac{\mathrm{Vol}\Big(\mathcal{C}^2_{d,A,B}(t)\Big)}{\mathrm{Vol}\big(B_2^{d+1}\big)\,\mathrm{Cap}(\Gamma)^{-d(d+1)/2}} \le [C_2(\eta) t_2]^{d+1}.$$

This proves the ℓ^2 volume asymptotics in the first part of Theorem 2.1. The second part of the theorem follows by an application of Proposition 4.17, with the ellipsoid $\Psi^{-1}(\mathcal{E}(\beta_0, \dots, \beta_d; t_\ell))$, where Ψ is the change of basis given by the α_ks.

Now, notice that the leading coefficient of $h_{\mathbf{c}}$ is a nonzero integer. Hence, it follows that there is a constant $C_\eta > 0$ such that

$$\int_{\mathbb{T}_1} |h_{\mathbf{c}}(w)|^2 \mathrm{d}|w| \ge C_\eta \quad \text{or} \quad \int_{\mathbb{T}_2} |h_{\mathbf{c}}(w)|^2 \mathrm{d}|w| \ge C_\eta.$$

If we choose t sufficiently small, depending on η, then we conclude that there are no integer lattice points inside $\mathcal{C}^2_{d,A,B}(t)$. This proves the first part of Theorem 2.3.

5.1.2. Volume of $\mathcal{D}^2$

For Bergman orthogonal polynomials notice that first we can bound the weights $|1+w|^{-2}$ on $\mathbb{D}_1$, and $|w|^{-4}|1+w|^2$ on $\mathbb{D}_2$ from above and from below in terms of η. Thus, there are constants $c^{(j)} = c^{(j)}(\eta)$, $j \in \{1, 2\}$, such that

$$\begin{aligned}\mathcal{D}^2_{d,A,B}(t) &\subset \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon c^{(1)} \int_{\mathbb{D}_1} |h_{\mathbf{c}}(w)|^2 \mathrm{d}A(w) \le t^2, \quad c^{(2)} \int_{\mathbb{D}_2} |h_{\mathbf{c}}(w)|^2 \mathrm{d}A(w) \le t^2\right\} \\ &\subset \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \tfrac{1}{2} \min\{c^{(1)}, c^{(2)}\} \left(\int_{\mathbb{D}_1} + \int_{\mathbb{D}_2}\right) |h_{\mathbf{c}}(w)|^2 \mathrm{d}A(w) \le t^2\right\},\end{aligned}$$

and similarly in the other direction with a different constant.

Now, let t_k be the Bergman monic orthogonal polynomials with respect to area measure on $\mathbb{D}_1 \cup \mathbb{D}_2$, and let γ_k be their normalizing constants. By Theorem 4.19 we have

$$C_1(\eta) \frac{\mathrm{Cap}(\Gamma)^{k+1}}{\sqrt{k+1}} \le \gamma_k \le C_2(\eta) \frac{\mathrm{Cap}(\Gamma)^{k+1}}{\sqrt{k+1}},$$

which, together with the previous discussion, gives

$$\Big(\widetilde{C_1}(\eta) t\Big)^{d+1} \le \frac{\mathrm{Vol}\Big(\mathcal{D}^2_{d,A,B}(t)\Big)}{\mathrm{Cap}(\Gamma)^{-d(d+1)/2}} \le \Big(\widetilde{C_2}(\eta) t\Big)^{d+1}.$$

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Here we used the asymptotics of the volume of the Euclidean ball

$$\mathrm{Vol}\Big(B_2^{d+1}\Big) = \left(\frac{2\pi e(1+o(1))}{d+1}\right)^{(d+1)/2}, \quad \text{as} \quad d \to \infty.$$

Hence, choosing t sufficiently large, the volume of $\mathcal{D}^2$ is exponentially large in d (as long as $\mathrm{Cap}(\Gamma) \leq 1$). Applying [9, Sec. 7.2, Thm. 1] proves the second part of Theorem 2.3.

5.2. Refined estimate for the critical growth constraint on the natural numbers

By essentially the same argument as in Section 5.1 we find

$$\begin{aligned}\mathcal{C}^2_{d,A}(t) &= \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \sum_{n\in\mathbb{N}} |P_{\mathbf{c}}(n)A^{-n}|^2 \leq t^2\right\} = \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \frac{A}{2\pi}\int_{\mathbb{T}_1} |h_{\mathbf{c}}(w)|^2 \mathrm{d}|w| \leq t^2\right\} \\ &= \left\{\mathbf{c} \in \mathbb{R}^{d+1} \colon \frac{A}{2\pi}\sum_{k=0}^{d} |\alpha_k(\mathbf{c})|^2\beta_k^2 \leq t^2\right\},\end{aligned}$$

where as before

$$\mathbb{T}_1 = \mathbb{T}\left(\frac{1}{A^2-1}, \frac{A}{A^2-1}\right), \qquad h_{\mathbf{c}}(w) = \sum_{k=0}^{d} c_k w^k.$$

Here the monic orthogonal polynomials with respect to arc-length on $\mathbb{T}_1$ and their normalization constants are given explicitly by

$$p_k(w) = (w - (A^2-1)^{-1})^k, \qquad \beta_k^2 = 2\pi\left(\frac{A}{A^2-1}\right)^{2k+1}.$$

Thus, taking the prefactor $A/(2\pi)$ into account, we have

$$\mathrm{Vol}\Big(\mathcal{C}^2_{d,A}(t)\Big) = \left(\frac{A^2-1}{A}\right)^{(d+1)^2/2} \mathrm{Vol}\Big(B_2^{d+1}\Big)\left(\frac{t}{\sqrt{A}}\right)^{d+1} = \frac{(A^2-1)^{(d+1)^2/2}}{A^{(d+1)(d+2)/2}} \mathrm{Vol}\Big(B_2^{d+1}\Big)t^{d+1}.$$

Evidently if c_k is the largest nonzero coefficient of $h_{\mathbf{c}}$, then $\int_{\mathbb{T}_1} |h_{\mathbf{c}}(w)|^2 \mathrm{d}|w| \geq c_k^2\beta_k^2$. Hence, when $A = \varphi$ and $t < \sqrt{\varphi}$ there are no integer-valued polynomials P satisfying

$$\sum_{n\in\mathbb{N}} |P_{\mathbf{c}}(n)\varphi^{-n}|^2 \leq t^2.$$

This proves the first part of Theorem 1.2. We now turn to the proof of the second part.

We will use the results from [14, Chap. III]. We denote by S.x and S.x.y theorem and equation numbers from this paper.

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Denote

$$\begin{aligned}\mathcal{D}^2_{d,\varphi}(t) &= \left\{\mathbf{c}\in\mathbb{R}^{d+1}\colon \sum_{n\in\mathbb{N}}\frac{1}{n+1}|P_{\mathbf{c}}(n)\varphi^{-n}|^2 \le t^2\right\}\\ &= \left\{\mathbf{c}\in\mathbb{R}^{d+1}\colon \frac{\varphi^2}{\pi}\int_{\mathbb{D}_1}|h_{\mathbf{c}}(w)|^2\frac{\mathrm{d}A(w)}{|1+w|^2}\le t^2\right\}\\ &= \left\{\mathbf{c}\in\mathbb{R}^{d+1}\colon \frac{\varphi^2}{\pi}\sum_{k=0}^{d}|\widetilde{\alpha}_k(\mathbf{c})|^2\gamma_k^2\le t^2\right\},\end{aligned}$$

where now γ_k are the normalizing constants of monic orthogonal polynomials with respect to the weight $\frac{\mathrm{d}\mathbb{A}(w)}{|1+w|^2}$ on $\mathbb{D}_1=\mathbb{D}(\varphi^{-1},1)$, and $\widetilde{\alpha}_k(\mathbf{c})$ are the coefficients after the change of basis.

For fixed $\varepsilon\in(0,1)$, we will find a condition on $t=t(\varepsilon)$ so that the volume of $\mathcal{D}^2_{d,\varphi}(t)$ is at least $(2+\varepsilon)^{d+1}$. Then an application of [9, Sec. 7.2, Thm. 1] proves there are infinitely many integer-valued polynomials P satisfying the growth condition

$$\sum_{n\in\mathbb{N}}\frac{1}{n+1}|P_{\mathbf{c}}(n)\varphi^{-n}|^2\le t^2.$$

The asymptotics of γ_k follow from Theorem S.3.1 (in Suetin's notation it is equal to λ_k^{-1}). We introduce some notation required for the statement of the results.

Let Φ be the conformal map from the exterior of $\mathbb{D}_1$ to the exterior of the unit disk $\mathbb{D}$, satisfying $\Phi(\infty)=\infty$ and $\Phi'(\infty)>0$, that is, the map $\Phi(z)=z-\varphi^{-1}$. We need to introduce an analytic function $D:\mathbb{D}^c\to\mathbb{D}_1^c$ such that

$$|D(w)|^2=\frac{1}{\left|1+\left(w+\varphi^{-1}\right)\right|^2}=\frac{1}{|w+\varphi|^2}\quad\text{on}\quad|w|=1.$$

Explicitly it is given by the following expression

$$D(w)=\exp\left(-\frac{1}{4\pi}\int_0^{2\pi}\log\frac{1}{|e^{i\theta}+\varphi|^2}\frac{e^{i\theta}+w}{e^{i\theta}-w}\,d\theta\right).$$

Using Suetin's notation, let

$$g(z)=\frac{\Phi'(z)}{D(\Phi(z))}=\frac{\Phi'(z)}{D\left(z-\varphi^{-1}\right)}=\frac{1}{D(z-\varphi^{-1})},\quad\text{and}\quad\alpha_0=g(\infty)=\frac{1}{D(\infty)}.$$

Since the function $z+\varphi$ does not vanish in $\mathbb{D}$, the mean value theorem for harmonic functions gives

$$\log D(\infty)=\frac{1}{4\pi}\int_0^{2\pi}\log\frac{1}{|e^{i\theta}+\varphi|^2}\,d\theta=-\frac{1}{2\pi}\int_0^{2\pi}\log|e^{i\theta}+\varphi|\,d\theta=-\log\varphi.$$

Now by Theorem S.3.1 we have

$$\gamma_k^2=\frac{\pi}{\alpha_0^2(k+1)}(1+o(1))=\frac{\pi}{\varphi^2(k+1)}(1+o(1)).$$

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By the asymptotic formula for the volume of the Euclidean ball and Stirling's approximation, we have

$$\mathrm{Vol}\Big(\mathcal{D}^2_{d,\varphi}(t)\Big) = \mathrm{Vol}\Big(B_2^{d+1}\Big)\prod_{k=0}^{d}\gamma_k^{-1}\left(\frac{\sqrt{\pi t}}{\varphi}\right)^{d+1} = (\sqrt{2\pi})^{d+1}t^{d+1}\cdot\exp(o(1)).$$

Finally, we conclude that if $t \geq \sqrt{2/\pi}+\varepsilon$, then the required condition is satisfied.

6. ℓ^∞ volume and lattice points

Let $A, B > 1$ be fixed constants, and recall the sequences

$$a_n(\mathbf{c}) = P_{\mathbf{c}}(n)A^{-n},\ n\in\mathbb{N}, \quad b_n(\mathbf{c}) = P_{\mathbf{c}}(-n)B^{-n},\ n\in\mathbb{N}^+.$$

In this section we consider the sets

$$\mathcal{C}^\infty_{d,A,B}(t) = \big\{\mathbf{c}\in\mathbb{R}^{d+1}\colon \|a_n(\mathbf{c})\|_{\ell^\infty(\mathbb{N})}\leq t,\ \|b_n(\mathbf{c})\|_{\ell^\infty(\mathbb{N}^+)}\leq t\big\},$$

and

$$\mathcal{C}^\infty_{d,A,B}(t;M) = \left\{\mathbf{c}\in\mathbb{R}^{d+1}\colon \max_{n\in\{0,\dots,M\}}|a_n(\mathbf{c})|\leq t,\ \max_{n\in\{1,\dots,M\}}|b_n(\mathbf{c})|\leq t\right\}.$$

By Lemma 4.11 there is a constant $C_{A,B}>1$, such that for $M > C_{A,B}\cdot d$ we have

$$\mathcal{C}^\infty_{d,A,B}(t) = \mathcal{C}^\infty_{d,A,B}(t;M) \quad \text{as sets.}$$

Below we fix $M = \lceil C_{A,B}\cdot d\rceil$.

Let $\Sigma_M = \Sigma_M(d,t;A,B)$ be a $(2M+1)\times(d+1)$ matrix, whose entries are given by

$$\Sigma_{j,k} = \begin{cases} \frac{1}{t}\binom{j}{k}B^j & ,\ j\in\{-M,\dots,-1\};\\ \frac{1}{t}\binom{j}{k}A^{-j}, & j\in\{0,\dots,M\},\end{cases}$$

where $k\in\{0,\dots,d\}$. The semi-infinite matrix $\Sigma_\infty = \Sigma_\infty(d,t;A,B)$ is defined analogously. We have that

$$\mathcal{C}^\infty_{d,A,B}(t;M) = \left\{(c_0,\dots,c_d)\in\mathbb{R}^{d+1}\colon \sup_{|j|\leq M}\left|\sum_{k=0}^{d}c_k\Sigma_{j,k}\right|\leq 1\right\}.$$

Denote by $V_M = V_M(d,t;A,B)$ the volume of the set $\mathcal{C}^\infty_{d,A,B}(t;M)$ in $\mathbb{R}^{d+1}$. By Theorems 4.5 and 4.6 we have

$$\frac{1}{\sqrt{\det\Sigma_M^t\Sigma_M}} \leq V_M \leq \frac{2^{M-d/2}}{\sqrt{\det\Sigma_M^t\Sigma_M}} \leq \frac{(C_{A,B})^d}{\sqrt{\det\Sigma_M^t\Sigma_M}}. \tag{6.1}$$

The Cauchy-Binet theorem gives

$$\det\big(\Sigma_M^t\Sigma_M\big) \geq \det\big(\Sigma_\infty^t\Sigma_\infty\big).$$

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Notice that by Corollary 4.4 and a change of variables

$$\begin{aligned}\left(\Sigma_\infty^t\Sigma_\infty\right)_{j,k} &= \frac{1}{t^2}\left[\sum_{n=-\infty}^{-1}\binom{n}{j}\binom{n}{k}B^{2n}+\sum_{n=0}^{\infty}\binom{n}{j}\binom{n}{k}A^{-2n}\right]\\ &= \frac{1}{t^2}\left[\sum_{n=1}^{\infty}\binom{-n}{j}\binom{-n}{k}B^{-2n}+\sum_{n=0}^{\infty}\binom{n}{j}\binom{n}{k}A^{-2n}\right]\\ &= \frac{1}{t^2}\left[\frac{1}{\pi B}\int_{B\mathbb{T}} g_j(z)\overline{g_k(z)}\mathrm{d}|z|+\frac{A}{\pi}\int_{A^{-1}\mathbb{T}} g_j(z)\overline{g_k(z)}\mathrm{d}|z|\right]\\ &= \frac{1}{t^2}\left[\frac{1}{\pi B}\int_{\mathbb{T}_2} w^j\bar{w}^k\mathrm{d}|w|+\frac{A}{\pi}\int_{\mathbb{T}_1} w^j\bar{w}^k\mathrm{d}|w|\right],\end{aligned}$$

where

$$g_j(z)=\frac{1}{1-z}\left(\frac{z}{1-z}\right)^j,\quad \mathbb{T}_1=\mathbb{T}\left(\frac{1}{A^2-1},\frac{A}{A^2-1}\right),\quad \mathbb{T}_2=\mathbb{T}\left(\frac{B^2}{1-B^2},\frac{B}{B^2-1}\right).$$

By Theorem 4.7 we have that

$$\det\left(\Sigma_\infty^t\Sigma_\infty\right)=\prod_{k=0}^{d}\|q_k\|^2_{L^2(\mu_t)},$$

where q_k are monic orthogonal polynomials with respect to arc-length on $\mathbb{T}_1\cup\mathbb{T}_2$, weighted by A on $\mathbb{T}_1$ and $\frac{1}{B}$ on $\mathbb{T}_2$. The measure μ_t is defined similarly, scaled by $\frac{1}{\pi t^2}$.

Let $\eta\in(0,1)$. Assume $A,B>1$ are such that the condition of Theorem 4.18 is satisfied. Then, we have that

$$\|q_k\|_{L^2(\mu_t)}\asymp_\eta t^{-1}\operatorname{Cap}(\Gamma)^k,\quad \text{where}\quad \Gamma=\mathbb{T}_1\cup\mathbb{T}_2.$$

Notice we can reduce to the case of arc-length (weighted by $\frac{1}{t}$), by bounding the weight of the measure μ_t from below and from above, by factors depending only on η (ignoring the dependence on A,B). Hence,

$$\begin{aligned}\log\det\left(\Sigma_\infty^t\Sigma_\infty\right) &= d(d+1)\log\operatorname{Cap}(\Gamma)-2(d+1)\log t+O_\eta(d)\\ &= d^2\log\operatorname{Cap}(\Gamma)-2d\log t+O_\eta(d).\end{aligned}\tag{6.2}$$

Therefore,

$$\log\operatorname{Vol}\left(\mathcal{C}^\infty_{d,A,B}(t)\right)\geq-\frac{1}{2}d^2\log\operatorname{Cap}(\Gamma)+d\log t+O_\eta(d).$$

By Proposition 4.16 (perhaps after increasing the constant $C_{A,B}$ in the definition of M), and the upper bound in (6.1) we obtain a matching upper bound for $\log\operatorname{Vol}\left(\mathcal{C}^\infty_{d,A,B}(t)\right)$. This proves the ℓ^∞ volume result in the first part of Theorem 2.1. It remains to prove Theorems 1.1 and 2.2.

Fix $A,B>1$ such that $\gamma_{A,B}=\operatorname{Cap}(\Gamma)=1$ (in particular, we may assume $A,B>\frac{3}{2}$). By the second part of Theorem 4.5, if $\det\left(\Sigma_\infty^t\Sigma_\infty\right)$ (which depends on d) tends to 0 as $d\to\infty$, then there are infinitely many IVPs satisfying the condition of Theorem 2.2. Hence by (6.2) for t sufficiently large depending on η the required result is obtained.

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To prove Theorem 1.1 we follow a similar path, now taking the constants into account. Consider the set

$$\mathcal{C}_d^{\infty}(t) = \left\{ \mathbf{c} \in \mathbb{R}^{d+1} \colon \sup_{n\in\mathbb{N}} |P_{\mathbf{c}}(n)\varphi^{-n}| \le t \right\} = \left\{ \mathbf{c} = (c_0, \dots, c_d) \colon \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{d} c_k \Sigma_{n,k} \right| \le \frac{1}{2} \right\},$$

where

$$\Sigma_{n,k} = \Sigma_{n,k}(t) = \frac{1}{2t} \binom{n}{k} \varphi^{-n}.$$

We have that

$$\left(\Sigma_{\infty}^{t} \Sigma_{\infty}\right)_{j,k} = \frac{1}{4t^2} \sum_{n=0}^{\infty} \binom{n}{j} \binom{n}{k} \varphi^{-2n} = \frac{1}{4t^2} \frac{\varphi}{\pi} \int_{\varphi^{-1}\mathbb{T}} g_j(z) \overline{g_k(z)} \mathrm{d}|z| = \frac{1}{4t^2} \frac{\varphi}{\pi} \int_{\mathbb{T}_1} w^j \bar{w}^k \mathrm{d}|w|,$$

where $\mathbb{T}_1 = \mathbb{T}(1/\varphi, 1)$.

Let q_k be the monic orthogonal polynomials with respect to the arc-length measure on $\mathbb{T}_1$, which are given by

$$q_k(w) = \left(w - 1/\varphi\right)^k.$$

Hence, by Theorem 4.7 we have (taking the normalization into account)

$$\det\left(\Sigma_{\infty}^{t} \Sigma_{\infty}\right) = \left(\frac{\varphi}{2t^2}\right)^{d+1}.$$

By Lemma 4.10, there exists some constant $C_0 > 0$, such that we have

$$\mathcal{C}_d^{\infty}(t) = \left\{ \mathbf{c} = (c_0, \dots, c_d) \colon \sup_{n \le C_0 d} \left| \sum_{k=0}^{d} c_k \Sigma_{n,k} \right| \le \frac{1}{2} \right\},$$

for all d. Therefore, by the second part of Theorem 4.5 (and the monotonicity of the determinant), the set $\mathcal{C}_d^{\infty}(2t)$ contains at least $2\left\lfloor \det\left(\Sigma_{\infty}^{t} \Sigma_{\infty}\right)^{-1/2} \right\rfloor$ integer lattice points. Hence for any $\varepsilon > 0$, if we choose $t = \sqrt{\varphi/2} + \varepsilon/2$ there are infinitely many IVPs P satisfying

$$|P(n)| \le \left(\sqrt{2\varphi} + \varepsilon\right) \varphi^n \quad \text{for all } n \ge 0.$$

This concludes the proof of Theorem 1.1.

A. Moment matrix and orthogonal polynomials

Let μ be a measure on $\mathbb{C}$ with infinite support and all moments (so that no nontrivial algebraic polynomial can vanish on the support of μ). Define the n-th moment matrix

$$\mathcal{M}(n; \mu) = \left[M_{k,m}\right]_{k,m=0}^{n}, \qquad M_{k,m} = \int_{\mathbb{C}} z^k \bar{z}^m \, \mathrm{d}\mu(z).$$

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These matrices are positive definite. To see this, note that for $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, if we put $P_n(z) = \sum_{k=0}^n a_k z^k$, then

$$0 < \|P_n\|^2_{L^2(\mu)} = \int_{\mathbb{C}} \left| \sum_{k=0}^n a_k z^k \right|^2 \mathrm{d}\mu(z) = \sum_{k,m=0}^n a_k \overline{a_m} \int_{\mathbb{C}} z^k \bar{z}^m \, \mathrm{d}\mu(z)$$
$$= \sum_{k,m=0}^n a_k \overline{a_m} M_{k,m} = \mathbf{a}^* \mathcal{M}(n;\mu)\mathbf{a}.$$

Now put $D_n = D_n(\mu) = \det \mathcal{M}(n;\mu)$. We can define *monic* orthogonal polynomials with respect to μ by

$$Q_0(z) = 1, \quad Q_n(z) = \frac{1}{D_{n-1}} \det \begin{bmatrix} M_{0,0} & & M_{0,n} \\ \vdots & \ddots & \vdots \\ M_{n-1,0} & & M_{n-1,n} \\ 1 & z \ \dots & z^n \end{bmatrix}.$$

The orthogonality follows, since for $j \in \{0, \dots, n\}$

$$D_{n-1} \int_{\mathbb{C}} z^j \overline{Q_n(z)} \, \mathrm{d}\mu(z) = \det \begin{bmatrix} M_{0,0} & & M_{0,n} \\ \vdots & \ddots & \vdots \\ M_{n-1,0} & & M_{n-1,n} \\ M_{j,0} & M_{j,1} \ \dots & M_{j,n} \end{bmatrix} = \begin{cases} 0, & j \in \{0, \dots, n-1\}; \\ D_n, & j = n. \end{cases}$$

From the above we also find that

$$\|Q_n\|^2_{L^2(\mu)} = \int_{\mathbb{C}} z^n \overline{Q_n(z)} \, \mathrm{d}\mu(z) = \frac{D_n}{D_{n-1}},$$

while $D_0 = M_{0,0} = \mu(\mathbb{C}) = \|Q_0\|^2_{L^2(\mu)}$. We conclude that

$$D_n = \prod_{j=0}^n \|Q_j\|^2_{L^2(\mu)}.$$

B. Bergman polynomials on two disks

Here we prove Theorem 4.19. We use the method of the paper [10] (simplified for a domain which is the union of two disks).

Let $\eta \in (0,1)$, $G = \mathbb{D}_1 \cup \mathbb{D}_2$ be the *disjoint* union of two disks, whose radii are bounded between η and $\frac{1}{\eta}$, and are separated by at least η and at most $\frac{1}{\eta}$. Put $\Gamma = \partial G = \mathbb{T}_1 \cup \mathbb{T}_2$.

We denote by $\{t_n\}_{n=0}^\infty$ the sequence of *monic* Bergman orthogonal polynomials with respect to area measure on G. Write γ_k for the normalizing constant of t_k (here we do not consider the weighted case of Bergman polynomials).

For p a polynomial of degree n we have by the orthogonality of the monomials

$$\int_{r\mathbb{T}} |p(z)|^2 \mathrm{d}|z| \le \frac{2(n+1)}{r^2} \int_{r\mathbb{D}} |p(z)|^2 \mathrm{d}A(z).$$

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Recall that β_k^2 minimizes $\int_\Gamma |p_k|^2 \mathrm{d}|z|$ for monic polynomials, and γ_k^2 minimizes $\int_G |p_k|^2 \mathrm{d}A(z)$. Therefore, we have

$$\gamma_n^2 = \int_{\mathbb{D}_1} |t_n(z)|^2 \mathrm{d}A(z) + \int_{\mathbb{D}_2} |t_n(z)|^2 \mathrm{d}A(z) \geq \frac{f_3(\eta)}{n+1} \int_\Gamma |t_n(z)|^2 \mathrm{d}|z| \geq \frac{f_3(\eta)}{n+1} \beta_n^2,$$

where f_3 is some positive continuous function of $\eta \in (0,1)$.

Using Theorem 4.18, we conclude the lower bound in Theorem 4.19

$$\gamma_n \geq C_1 \cdot \frac{\mathrm{Cap}(\Gamma)^{n+1}}{\sqrt{n+1}}.$$

To prove the upper bound for γ_n we will use the following (very special case of) [16, Thms. 8.3 and 5.6], concerning Chebyshev polynomials.

Theorem B.1. *Let $\Gamma = \partial G = \mathbb{T}_1 \cup \mathbb{T}_2$ be the boundary of the disks, as in Theorem 4.19. There is a constant M depending continuously on the Green function $\mathfrak{g}$ of Γ (w.r.t. ∞) and its derivatives, such that for each n, there is a monic polynomial Q of degree n satisfying*

$$\log |Q(z)| \leq M + n \log \mathrm{Cap}(\Gamma), \qquad \forall z \in \Gamma.$$

Remark. By the assumptions M can be bounded (continuously) in terms of η.

Now we repeat the proof of [10, Lemma 8.2] in a simplified form. Let $\mathfrak{g} = \mathfrak{g}_\Omega$ be the Green function (always w.r.t. ∞) for

$$\Omega := G^c = \mathbb{C} \setminus (\mathbb{D}_1 \cup \mathbb{D}_2).$$

The function $\mathfrak{g}$ can be extended harmonically into the Green function (up to a constant) of a larger domain Ω_τ which corresponds to the level set $\mathfrak{g} = \log \tau$ with $\tau \in (0,1)$. This τ may be chosen to be a continuous function of η. Moreover, fixing some $\rho = \rho_\eta \in (\tau, 1)$ in a continuous way in η, we have for $R \geq \rho$

$$\mathfrak{g}_R(z) := \mathfrak{g}_{\Omega_R}(z) = \mathfrak{g}(z) - \log R, \qquad \Omega_R := \{\mathfrak{g} \geq \log R\}.$$

Thus $\mathrm{Cap}\,(\partial\Omega_R) = R\mathrm{Cap}\,(\partial G) = R\mathrm{Cap}\,(\Gamma)$. By Theorem B.1 applied to $\partial\Omega_\rho$ we have a polynomial Q of degree n such that

$$\log |Q(z)| \leq M + n \log \mathrm{Cap}\big(\partial\Omega_\rho\big), \qquad \forall z \in \partial\Omega_\rho,$$

where M depends continuously on η. By the maximum principle (for the unbounded domain Ω_ρ) this inequality extends to

$$\log |Q(z)| \leq M + n(\log \mathrm{Cap}\big(\partial\Omega_\rho\big) + \mathfrak{g}_\rho(z)), \qquad \forall z \in \Omega_\rho.$$

Since $\mathfrak{g}_R(z) + \log \mathrm{Cap}(\partial\Omega_R) = \log \mathrm{Cap}(\Gamma)$ does not depend on R, we have for $R \geq \rho$

$$\log |Q(z)| \leq M + n \log \mathrm{Cap}(\partial\Omega_R) \implies |Q(z)| \leq M' R^n \,\mathrm{Cap}(\Gamma)^n, \qquad \forall z \in \partial\Omega_R.$$

Now, by the usual maximum principle

$$|Q(z)| \leq M' R^n \,\mathrm{Cap}(\Gamma)^n, \qquad z \in \Omega_R^c,\ \rho \leq R < \infty. \tag{B.1}$$

We proceed to bound the $L^2(G)$-norm of Q as in [10] by splitting G into Ω_ρ^c and the rest. We bound the first part using (B.1). The integral over $G \setminus \Omega_\rho$ is bounded by foliating it by the level sets of $\partial\Omega_R$ using (B.1) and the coarea formula. For the convenience of the reader we provide the details below.

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Write the integral over G as follows

$$\int_G |Q|^2 \mathrm{d}A = \int_{\Omega_\rho^c} |Q|^2 \mathrm{d}A + \int_{G\cap\Omega_\rho} |Q|^2 \mathrm{d}A =: I_1 + I_2.$$

We bound I_1 using (B.1) by

$$I_1 \le \mathrm{Area}(G) \max_{z\in\partial\Omega_\rho} |Q|^2 \le (M')^2 \,\mathrm{Area}(G)\rho^{2n}\,\mathrm{Cap}(\Gamma)^{2n}.$$

In order to bound I_2 we foliate $G\cap\Omega_\rho$ by the level sets $\partial\Omega_R$ of $\mathfrak{g}_\Omega$, and use the coarea formula. Notice that $\nabla\mathfrak{g}_\Omega$ is (uniformly) bounded away from zero on $G\cap\Omega_\rho$. Thus, by (B.1)

$$I_2 = \int_\rho^1 \int_{\partial\Omega_R} \frac{|Q(z)|^2}{|\nabla \exp(\mathfrak{g}_\Omega(z))|}\mathrm{d}|z|\mathrm{d}R \le M'' \int_\rho^1 \mathrm{Length}(\partial\Omega_R)(M')^2 R^{2n}\,\mathrm{Cap}(\Gamma)^{2n}\mathrm{d}R.$$

In order to bound the length of the level sets (uniformly), notice that

$$-2\pi = \int_{\partial\Omega_R} \frac{\partial\mathfrak{g}}{\partial n}\mathrm{d}s = \int_{\partial\Omega_R} \nabla\mathfrak{g}\cdot n\,\mathrm{d}s = -2\pi\int_{\partial\Omega_R} |\nabla\mathfrak{g}|\mathrm{d}s$$

so that

$$2\pi = \int_{\partial\Omega_R} |\nabla\mathfrak{g}|\mathrm{d}s \ge \min_{\partial\Omega_R} |\mathfrak{g}|\cdot\mathrm{Length}(\partial\Omega_R) \ge C_\eta\,\mathrm{Length}(\partial\Omega_R).$$

Finally, this gives the following bound for I_2

$$I_2 \le C_\eta\,\mathrm{Cap}(\Gamma)^{2n}\int_\rho^1 R^{2n}\mathrm{d}R \le C_\eta\cdot\frac{\mathrm{Cap}(\Gamma)^{2n}}{n+1}.$$

Now we conclude by noting that

$$\gamma_n^2 = \int_G |t_n|^2\mathrm{d}A(z) \le \int_G |Q|^2\mathrm{d}A(z) \le C'_\eta\cdot\frac{\mathrm{Cap}(\Gamma)^{2n+2}}{n+1}.$$

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