CONSTRUCTING κ-LIKE MODELS OF ARITHMETIC

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Abstract

A model $(M, <, \ldots)$ is κ-like if M has cardinality κ but, for all $a \in M$, the cardinality of $\{x \in M : x < a\}$ is strictly less than κ. In this paper we shall give constructions of κ-like models of arithmetic satisfying an arbitrarily large finite part of PA but not PA itself, for various singular cardinals κ. The main results are: (1) for each countable nonstandard $M \models \Pi_2 - \mathrm{Th(PA)}$ with arbitrarily large initial segments satisfying PA and each uncountable κ of cofinality ω there is a cofinal extension K of M which is κ-like; also hierarchical variants of this result for $\Pi_n - \mathrm{Th(PA)}$; and (2) for every $n \geqslant 1$, every singular κ and every $M \models B\Sigma_n + \exp + \neg I\Sigma_n$ there is a κ-like model K elementarily equivalent to M.

1. *Background*

1.1. *Introduction*

A model $(M, <, \ldots)$ is κ-like if and only if $\operatorname{card} M = \kappa$ but for all $a \in M$, the condition

$$\operatorname{card}\{x \in M : x < a\} < \kappa$$

holds. In this paper, we shall construct κ-like models of arithmetic in the cases when κ is singular. All models here will be nonstandard models of the form

$$(M, <, 0, 1, +, \cdot, \ldots)$$

and will satisfy the theory $I\Delta_0 + \exp$ of induction on Δ_0 formulas, together with an axiom stating that $x \mapsto 2^x$ is everywhere defined.

Models of arithmetic which are κ-like were first constructed by MacDowell and Specker by constructing elementary end-extensions of models of PA; there are κ-like models of every complete extension of PA for any uncountable κ. Conversely, if κ is regular then (modulo our base theory $I\Delta_0$) it is easily seen that any κ-like model must satisfy PA. The situation for singular κ is not so clear, however, and it is perhaps surprising that so little work has been done in this area.

My interest in this subject arose initially from a question that T. Slaman drew my attention to, essentially the following.

QUESTION 1.1. Are there κ-like models of arithmetic for singular κ that do not satisfy PA?

This question originally arose from the reverse mathematicians' analysis of Ramsey's theorem for pairs. In this paper we shall show that the answer to this question is 'yes'. Indeed for each n there are κ-like models of $\Pi_n - \mathrm{Th(PA)}$, the Π_n consequences of PA, that do not satisfy PA itself.

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We shall in fact give two results that answer Slaman's question. In the first result we shall construct κ-like cofinal extensions of a countable model M. In the second result we shall simply construct κ-like models elementarily equivalent to a given model M, but the work involved in proving this second result is more substantial because the hypotheses we shall require on the theory of the model M are considerably weaker. Unfortunately, neither result gives any sort of elegant axiomatization of the theory of κ-like models even for singular strong limits κ (that is, for singular κ such that $\lambda < \kappa$ implies $2^\lambda < \kappa$), and this problem is still open.

Note that for κ a strong limit each κ-like model must satisfy exponentiation. To see this, observe that in a model of arithmetic, if $a < b$ and no $x \leqslant b$ satisfies $x = 2^a$, then each $x < b$ gives rise to a unique subset of $\{y : y < a\}$, that is the set of y for which the yth digit in the binary expansion of x is 1. It follows that

$$\operatorname{card}\{x : x < b\} \leqslant 2^{\operatorname{card}\{y : y < a\}}.$$

The main method we shall use is the one of indiscernibles, and is essentially the same as that used by Keisler (1968) in his general model-theoretic characterization of the theory of κ-models for all models in similarity type $(<, \ldots)$.

It follows from results here that the theory of κ-like models is rather weak in the sense that for each finite set T of consequences of PA, PA is strictly stronger than the theory of κ-like models of T. In fact, we have the stronger statement that for T equal to any of $I\Sigma_n$, or $B\Sigma_{n+1} + \exp$ (where $n \geqslant 1$), the theory of κ-like models of T is a conservative extension of T for a large class of sentences which includes all Π_1 sentences (and in particular all the usual consistency statements such as Con T). This is perhaps surprising since it is rather easy to show that the simplest of the first-order combinatorial properties of κ-like models, the first-order scheme expressing that there is no definable $1-1$ map from the model to a proper initial segment, is not finitely axiomatizable, nor is it a consequence of the set of true Π_n sentences for any fixed n. This is discussed further elsewhere (Kaye [**5**]).

1.2. *Notation and previously known results*

To be more formal now, $\mathscr{L}_A$ will be the usual language $\mathscr{L}_A = \{0, 1, +, \cdot, <\}$ of arithmetic. All models will be considered as $\mathscr{L}_A$-structures, which will be assumed to satisfy at least $I\Delta_0 + \exp$. For each a in such a model, we define $2_0(a) = a$, and $2_{r+1}(a) = 2^{2_r(a)}$.

We shall use the usual subtheories $I\Sigma_n$ and $B\Sigma_m$ of PA (Kaye [**4**, Chapter 10]). These are finitely axiomatized for $n \geqslant 1$ and $m \geqslant 2$. Note that by convention $B\Sigma_m$ contains axioms for $I\Delta_0$ but not exp, and that $B\Sigma_2 \vdash I\Sigma_1 \vdash \exp$ but $B\Sigma_1 \nvdash \exp$. We often be interested in models of $I\Delta_0 + \exp$ that satisfy $B\Sigma_n$ but not $I\Sigma_n$ for some n, and we shall refer to such as 'models of $B\Sigma_n + \exp + \neg I\Sigma_n$' even though exp is redundant for $n \geqslant 2$.

In the metatheory (which is ZFC throughout), cardinals are thought of as initial ordinals. Given a model M and $a \in M$, then $< a$ denotes $\{x \in M : x < a\}$, and $\|a\|$ denotes $\operatorname{card}(< a)$. Also $I_\lambda(M)$ denotes $\{a \in M : \|a\| < \lambda\}$, and for a formula $\phi(x)$ of $\mathscr{L}(M)$ we denote by $\phi(M)$ the set $\{x \in M : \phi(x)\}$. If $S \subseteq M$ we define

$$\inf_M(S) = \{x \in M : \forall s \in S\, x < s\} \quad \text{and} \quad \sup_M(S) = \{x \in M : \exists s \in S\, x \leqslant s\}.$$

For a first-order language $\mathscr{L}$ we denote by $\mathscr{L}_{\mathrm{Sk}}$ a *partial* Skolemization of $\mathscr{L}$ obtained by adding $f_\theta(\bar{x})$ for each $\theta(\bar{x}, y)$ in $\mathscr{L}$, and by $T_{\mathrm{Sk}}(\mathscr{L})$ the $\mathscr{L}_{\mathrm{Sk}}$-theory with axioms $\forall \bar{x}, y(\phi(\bar{x}, y) \to \phi(\bar{x}, f_\phi(\bar{x})))$ for all ϕ in $\mathscr{L}$.

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Keisler's theorem on κ-like models is the following.

THEOREM 1.2 (Keisler [6]). *For any first-order theory T in signature $(<,\ldots)$, if T has a κ-like model for some singular strong limit κ, then T has a λ-like model for all singular λ.*

This theorem is proved via two lemmas, both concerned with a particular special kind of indiscernible sequence.

LEMMA 1.3 [6]. *Suppose that K is an $\mathscr{L}^+$-structure with a linear order $<$, where $\mathscr{L}^+$ is an expansion of $\mathscr{L}$, and K has the property that the reduct of any $\mathscr{L}^+$-substructure of K to $\mathscr{L}$ is elementary in K as $\mathscr{L}$-structures (for example, it might be that $\mathscr{L}^+ = \mathscr{L}_{\mathrm{Sk}}$ and $K \models T_{\mathrm{Sk}}(\mathscr{L})$), and suppose that K has elements c_{ij} for $i < \lambda = \mathrm{cf}(\kappa) < \kappa$ and $j < \kappa_i < \kappa$ where $\kappa = \sum_{i<\lambda} \kappa_i$ satisfying*

(a) *$c_{ij} < c_{kl}$ for all $i < k$ or $i = k$ and $j < l$,*

(b) *$\tau(c_{i_1 j_1}, \ldots, c_{i_n j_n}) < c_{ij}$ all $i_1, j_1, \ldots, i_n, j_n, i, j$ with $i_1, \ldots, i_n < i$,*

for all $\mathscr{L}^+$-terms τ in the arguments shown, and

(c) *$\tau(c_{i_1 j_1}, \ldots, c_{i_n j_n}) < c_{ij} \to \tau(c_{i_1 j_1}, \ldots, c_{i_n j_n}) = \tau(c_{i_1 l_1}, \ldots, c_{i_n l_n})$*

for all $\bar{\imath}, \bar{\jmath}$ with $i < i_1, \ldots, i_n$ and $\langle c_{i_1 j_1}, \ldots, c_{i_n j_n}\rangle$, $\langle c_{i_1 l_1}, \ldots, c_{i_n l_n}\rangle$ both increasing, and all $\mathscr{L}^+$-terms τ which here may contain elements c_{kl} for $k \leqslant i$. Then the $\mathscr{L}^+$-substructure of K generated by the c_{ij} is a κ-like substructure of K elementary for $\mathscr{L}$.

Strictly, Keisler worked with the full Skolemized language, but the lemma is true by the same proof, and it will turn out that the criterion we get from it is easier to verify.

The other lemma of Keisler's uses partition properties to get models M of the form required for Lemma 1.3.

LEMMA 1.4 [6]. *Let M be a $\mathscr{L}$-structure linearly ordered by some $<$ in $\mathscr{L}$. Suppose that M is μ-like, where μ is a strong limit. Then the set of sentences* (a), (b) *and* (c) *in Lemma* 1.3 *in new constants c_{ij} (for $i < \lambda$, $j < \kappa_i$, λ and κ_i all being infinite cardinals) are consistent with* $\mathrm{Th}(M)$.

Theorem 1.2 follows by applying Lemma 1.4 to the Skolemization of a μ-like model M, and applying the Compactness Theorem and Lemma 1.3 to get a κ-like model elementary equivalent to M.

Both our constructions go by verifying Keisler's conditions directly. For the first, we shall use methods due to Kirby and Paris [7] on initial segments. In particular we will need their method for proving the following result (here written using my notation defined above, which differs slightly from that of Kirby and Paris).

THEOREM 1.5 [7, Theorem 7(d, e)]. *Let M be a countable model of* PA, *and let $I \subseteq_{\mathrm{e}} M$ be strong. Then, for any infinite cardinal λ, there is an extension $K \succ M$ such that $I = \sup_K(I)$, $I_{\lambda^+}(K) = \inf_K(M \setminus I)$, and* $\mathrm{card}\,(I_{\lambda^+}(K)) = \lambda^+$.

The construction is a beautiful one using indiscernibles. (All that the reader needs to know about 'strong' initial segments here is that there are arbitrarily large strong cuts in any countable model of PA.)

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1.3. *Some Ramsey theory*

Given a set S or $\{x_1, \dots, x_n\}$, we denote by $[S]^k$ or $[x_1, \dots, x_n]^k$ respectively, the set of unordered k-tuples from S or $\{x_1, \dots, x_n\}$. This notation will be used even when k is nonstandard. In all cases of interest here, the underlying set will have a natural linear order on it, so, as usual, we may identify unordered k-tuples with increasing k-tuples.

We shall also use the usual partition relations in the notation of Erdös and Rado. By $\kappa \to (\lambda)^r_\mu$ we mean that for all partitions of $[\kappa]^r$ into μ pieces there is $X \in [\kappa]^\lambda$ such that $[X]^r$ lies entirely in one piece. When this is true, $R^r_\mu(\lambda)$ denotes the least κ for which $\kappa \to (\lambda)^r_\mu$. The *finite Ramsey theorem* says that $R^r_\mu(\lambda)$ is defined and is finite for all $r, \mu, \lambda \in \mathbb{N}$. In fact, this theorem is provable in $I\Delta_0 + \exp$ without difficulty.

We shall actually need some fairly crude upper bounds for the Ramsey numbers $R^n_a(\lambda)$ in the case when n is standard and a, λ are nonstandard elements of a model of arithmetic. Clearly $R^1_a(\lambda) \leqslant a\lambda$. The argument given by Paris and Mills [**8**] (essentially the same argument as given by Erdös and Rado) gives: if $b+1 \geqslant R^{n-1}_a(\lambda)$ and $k \geqslant b \cdot a^{(b+1)^n}$ then $k \geqslant R^n_a(\lambda)$ so, in particular,

$$R^n_a(\lambda) \leqslant 2^{2^{R^{n-1}_a(\lambda)}} \quad \text{for } \lambda \geqslant n^2,$$

hence $R^n_a(\lambda) \leqslant 2_{2n}(a\lambda)$. All of this is formalizable in $I\Delta_0 + \exp$, indeed it is formalized in $I\Delta_0$ for standard n and arbitrary a, λ provided that $2_{2n+1}(a\lambda)$ exists.

We shall also need certain *polarized partition relations*. The partition relation $(\kappa_1, \dots, \kappa_s) \to (\lambda_1, \dots, \lambda_s)^r_\mu$ means that for every partition of $[\kappa_1]^r \times \cdots \times [\kappa_s]^r$ into μ parts, there exist sets $X_i \in [\kappa_i]^{\lambda_i}$ such that $[X_1]^r \times \cdots \times [X_s]^r$ lies entirely in one part. (The notation is Keisler's, and is not the same as that used by Erdös and Rado.) The following is easy and formalizable in $I\Delta_0 + \exp$.

THEOREM 1.6 (Erdös and Rado [**1**]). *For finite cardinals κ_i, λ_i, if*

$$(\kappa_1, \dots, \kappa_s) \longrightarrow (\lambda_1, \dots, \lambda_s)^r_\mu \quad \text{and} \quad \kappa_{s+1} \longrightarrow (\lambda_{s+1})^r_{\mu'},$$

where $\mu' \geqslant \mu^{\kappa_1 \cdot \kappa_2 \cdots \kappa_s}$, then $(\kappa_1, \dots, \kappa_s, \kappa_{s+1}) \to (\lambda_1, \dots, \lambda_s, \lambda_{s+1})^r_\mu$.

COROLLARY 1.7. *For standard s, n and $\lambda_1 \geqslant n^2$, if $k_t > 2_{2n+1}(\lambda_t)$ and $\lambda_{t+1} > \lambda_1^{(k_t)^t}$ for all $1 \leqslant t \leqslant s$, then $(k_1, \dots, k_s) \to (\lambda_1, \dots, \lambda_s)^n_{\lambda_1}$.*

As a special case, for $\lambda \geqslant n^2$ and $k = 2_{2s(n+1)}(\lambda)$, we have

$$(\overbrace{k, \dots, k}^{s}) \longrightarrow (\overbrace{\lambda, \dots, \lambda}^{s})^n_y.$$

Proof. We have $R^n_{\lambda_1}(\lambda_1) \leqslant 2_{2n+1}(\lambda_1) \leqslant k_1$, so $k_1 \to (\lambda_1)^n_{\lambda_1}$. Suppose inductively that $(k_1, \dots, k_t) \to (\lambda_1, \dots, \lambda_t)^n_{\lambda_1}$, then put $\mu = \lambda_1^{k_1 \cdots k_t}$. We have

$$R^n_\mu(\lambda_{t+1}) \leqslant 2_{2n}(\lambda_{t+1} \cdot \mu) = 2_{2n}(\lambda_{t+1} \cdot \lambda_1^{k_1 \cdots k_t}) \leqslant 2_{2n+1}(\lambda_{t+1}),$$

hence $(k_1, \dots, k_t, k_{t+1}) \to (\lambda_1, \dots, \lambda_t, \lambda_{t+1})^n_{\lambda_1}$.

For the special case, note that putting $\lambda_1 = \lambda, k_t = 2_{2n+1}(\lambda_t)$, and $\lambda_{t+1} = \lambda^{(k_t)^s}$ we get $k_s \leqslant 2_{2s(n+1)}(\lambda)$, hence $(k_1, \dots, k_s) \to (\lambda_1, \dots, \lambda_s)^n_\lambda$ and in particular

$$(\overbrace{k, \dots, k}^{s}) \longrightarrow (\overbrace{\lambda, \dots, \lambda}^{s})^n_\lambda$$

as required.

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2. *New results*

2.1. *Constructing cofinal κ-like extensions*

In our first theorem, we shall construct κ-like *cofinal* extensions of a given model M. Here, the assumptions on M are that it is countable, satisfies $\Pi_2 - \mathrm{Th}(\mathrm{PA})$, the Π_2-consequences of PA, and has arbitrarily large initial segments satisfying PA.

We first characterize this situation using indicator theory. There is a Σ_1 formula $Y(x, y) = z$ of the language of arithmetic such that $I\Delta_0$ proves that

$$\forall x \forall y \exists ! z Y(x, y) = z,$$
$$\forall x \forall y \forall x' \forall y' (x \leqslant x' \leqslant y' \leqslant y \rightarrow Y(x, y) \geqslant Y(x', y')),$$
$$\forall x \forall y (Y(x, y) > 0 \rightarrow x < y),$$

and, for all $n \in \mathbb{N}$, PA proves that $\forall x \exists y (Y(x, y) \geqslant n)$. Also, if in a nonstandard model M of $I\Delta_0$ there are a, b with $Y(a, b) > \mathbb{N}$ then there is $I \subseteq_{\mathrm{e}} \mathrm{M}$ with $a \in I < b$ and $I \vDash \mathrm{PA}$. Thus the models M we are considering here are countable models M of $I\Delta_0$ such that for all $a \in M$ there is $b \in M$ with $Y(a, b) > \mathbb{N}$, since any such M has arbitrarily large initial segments satisfying PA, and hence is a union of a chain of models of PA.

THEOREM 2.1. *Let $M \vDash \Pi_2 - \mathrm{Th}(\mathrm{PA})$ be countable and have arbitrarily large initial segments satisfying* PA, *and let κ be an uncountable cardinal of cofinality ω. Then there is a κ-like cofinal extension $K \vDash \Pi_2 - \mathrm{Th}(\mathrm{PA}) + B\Sigma_1$ of M.*

Proof. Suppose that $\kappa = \sum_{i<\omega} \kappa_i$, where each κ_i is uncountable and $\kappa_i < \kappa_{i+1} < \kappa$.

By a standard model-theoretic construction [**4**], there is a countable extension $N_0 \supseteq M$ elementary for Σ_1 formulas with N_0 satisfying PA; we let $M_0 = \sup_{N_0}(M)$, the initial segment of N_0 in which M is cofinal. By compactness, we may of course assume that M is not cofinal in N_0, that is, $M_0 \neq N_0$.

Since both M and M_0 are Δ_0-elementary in N_0, the extension $M \subseteq_{\mathrm{cf}} M_0$ is also elementary for Δ_0 formulas. So if $a, b, c \in M \vDash Y(a, b) = c$ then $M_0 \vDash Y(a, b) = c$ since $Y(x, y) = z$ is Σ_1. It follows from indicator theory and the fact that M is cofinal in M_0 that M_0 also has arbitrarily large initial segments satisfying PA.

We shall make use of the following facts about strong initial segments $I \subseteq_{\mathrm{e}} N$. Such a cut I is *strong* if and only if $I \rightarrow (I)^n_2$ for all $n \in \mathbb{N}$, that is, for all $n \in \mathbb{N}$ and all N-finite sets $A, B \subseteq [N]^n$ with $A \cup B \supseteq [I]^n$, there is an N-finite C such that $C \cap I$ is unbounded in I and $[C]^n \subseteq A$ or $[C]^n \subseteq B$. Also, it turns out that the indicator Y already introduced is an indicator for strong cuts I in models of PA, that is, if $N \vDash \mathrm{PA}$ and $a, b \in N$ with $Y(a, b) > \mathbb{N}$, then there is a strong $I \subseteq_{\mathrm{e}} N$ with $a \in I < b$. This last fact comes from indicator theory (see, for example, [**4**]) and Gaifman's construction of definable types and conservative end-extensions of models of PA (Gaifman [**2**]).

Using the countability of M_0, let $(a_i : i \in \mathbb{N})$ be an ascending sequence of elements of M_0, cofinal in M_0 and such that $Y(a_i, a_{i+1}) > \mathbb{N}$ for all i.

Inductively, we suppose that we have a countable $N_i \succ N_0$, and that $M_i \subseteq_{\mathrm{e}} N_i$ is the initial segment of N_i in which M is cofinal. (So in particular $N_i \vDash \mathrm{PA}$.) By indicator theory, there is $I \subseteq_{\mathrm{e}} M_i$ such that $a_i \in I < a_{i+1}$ and I is strong in N_i. In the same way as in [**7**] we construct a type over N_i. Enumerate all formulas $\phi_j(\bar{x}, \bar{c})$ in the language

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with constants c naming each element of N_i. Suppose that X_j is N_i-finite (that is, is definable in N_i and bounded) and $I \cap X_j \subseteq_{\mathrm{cf}} I$, and consider $A = \{\bar{x} \in [X_j]^n : \phi_j(\bar{x}, \bar{c})\}$ and $B = \{\bar{x} \in [X_j]^n : \neg\, \phi_j(\bar{x}, \bar{c})\}$. By strength there is $X_{j+1} \subseteq X_j$ satisfying the same conditions as X_j such that $[X_{j+1}]^n \subseteq A$ or $[X_{j+1}]^n \subseteq B$. This process is continued, and at the end we have our type $p(x) = \{x \in X_j : j \in \mathbb{N}\}$. By compactness, there is $N \succ N_i$ containing $u_0 < u_1 < u_2 < \cdots < u_j < \cdots$, each u_j (for $j \in \mathbb{N}$) satisfying p. We take N_{i+1} to be the elementary substructure of N generated by N_i and the elements of $U = \{u_j : j \in \mathbb{N}\}$. So N_{i+1} is an elementary extension of N_i.

Now, note that if $\bar{u} \in [U]^n$ and $\bar{c} \in N_i$, and $t(\bar{u}, \bar{c}) < a \in I$ then each $\bar{x} \in [X_j]^n$ satisfies $t(\bar{x}, \bar{c}) < a$ (since $t(\bar{x}, \bar{c}) < a$ is some formula $\phi_{j-1}(\bar{x}, \bar{c})$). As there are I-cofinally many such $\bar{x} \in [X_k]^n$ for each k, then for each k at least two $\bar{x} \neq \bar{y} \in [X_k \cap I]^n$ must satisfy $t(\bar{x}, \bar{c}) = t(\bar{y}, \bar{c})$. But since $t(\bar{x}, \bar{c}) = t(\bar{y}, \bar{c})$ is also some formula $\phi_{k-1}(\bar{x}\bar{y}, \bar{c})$, it follows that for some $k \in \mathbb{N}$, $t(\bar{x}, \bar{c}) = t(\bar{y}, \bar{c})$ for all $\bar{x}, \bar{y} \in [X_k]^n$, and from this it follows that in N_{i+1} we have $t(\bar{u}, \bar{c}) = b$ for some $b \in I$. In other words, $I \subseteq_{\mathrm{e}} N_{i+1}$. Note too that for all $b > I$ in N_i and all $u \in U$ we have that $u < b$, since no $X_j \subseteq \{x : \neg\, x < b\}$.

Continuing in this way, we obtain a chain $N_0 \prec \cdots \prec N_i \prec \cdots$. In the union $N_\omega = \bigcup_i N_i$ we have elements c_{ij} (where c_{ij} is the u_j constructed when N_i was extended to N_{i+1}).

We regard N_0 and N_ω as structures for the language $\mathscr{L}_{\mathrm{A}}$ expanded by adding constants for each $c \in N_0$. It is easy to see that in N_ω, with the choice of the c_{ij} just made, conditions (a) and (c) in Lemma 1.3 are satisfied, where here, of course, the subscripts i, j in c_{ij} range over all finite ordinals. Condition (b) may not be satisfied, however, but we do of course have $a_i < c_{ij} < a_{i+1}$, which will suffice. This is because a_i is in the strong initial segment I used in constructing N_{i+1} from N_i, for all i.

Now we use compactness to build a model of $\mathrm{Th}(N_0, c)_{c \in N_0}$. Consider new constants d_{ij} (with $i < \omega$, $j < \kappa_i$) and consider the theory $\mathrm{Th}(N_0, c)_{c \in N_0}$ together with all sentences

(A) $d_{ij} < d_{kl}$ for all $i < k$ or $i = k$ and $j < l$,

(B) $a_i < d_{ij} < a_{i+1}$ for all i, j, and

(C) $\tau(d_{i_1 j_1}, \ldots, d_{i_n j_n}) < d_{ij} \to \tau(d_{i_1 j_1}, \ldots, d_{i_n j_n}) = \tau(d_{i_1 l_1}, \ldots, d_{i_n l_n})$ for all $\bar{\imath}, \bar{\jmath}, \bar{l}$ with $i < i_1, \ldots, i_n$ and $\langle d_{i_1 j_1}, \ldots, d_{i_n j_n} \rangle$, $\langle d_{i_1 l_1}, \ldots, d_{i_n l_n} \rangle$ both increasing, and all $\mathscr{L}_{\mathrm{A}}(N_0)$-terms τ which here may contain elements d_{kl} for $k \leqslant i$.

This is obviously consistent, because any finite part of it is realised in N_ω by the c_{ij}. So there is a model N of this theory, generated by the d_{ij} and the $c \in N_0$. Clearly $N_0 \prec N$. Let K be $\sup_N \{a_i : i \in \mathbb{N}\}$, so $M \subseteq_{\mathrm{cf}} K$. It is easy to check that K is κ-like, and is a model of $\Pi_2 - \mathrm{Th}(\mathrm{PA})$. Also K is a model of $B\Sigma_1$ since M is not cofinal in N_0, and hence K is a proper initial segment of N.

The proof just given immediately generalizes to give the following theorem.

THEOREM 2.2. *Let $n \geqslant 0$ and let $M \models \Pi_{n+2} - \mathrm{Th}(\mathrm{PA})$ be countable and have arbitrarily large Σ_n-elementary initial segments satisfying* PA, *and let κ be an uncountable cardinal of cofinality ω. Then there is a κ-like cofinal extension $K \models \Pi_{n+2} - \mathrm{Th}(\mathrm{PA}) + B\Sigma_{n+1}$ of M.*

The only change required is a modification of the notion of 'indicator' to give Σ_n-elementary initial segments. This can be done by closing under suitable Σ_n Skolem functions in an obvious way, and the complexity of the formula $Y(x, y) = z$ rises from Σ_1 to Σ_{n+1}. Everything else is essentially the same.

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Notice that the cofinal extensions just constructed are *not* in general elementary. Indeed we might have started with a model of $\Pi_2 - \mathrm{Th}(\mathrm{PA}) + \neg B\Sigma_1$.

On the other hand, if we start with a model M of $\Pi_{n+2} - \mathrm{Th}(\mathrm{PA}) + B\Sigma_{n+1} + \neg I\Sigma_{n+1}$ in Theorem 2.2, the model K that we get is a cofinal extension satisfying $B\Sigma_{n+1}$. By standard results [**4**, Section 7.2], if it were the case that $K \models B\Sigma_{n+2}$ then we would have $M \prec_{\Sigma_{n+2}} K$, so that K satisfies all Σ_{n+3} sentences true in M. But the negation of an axiom of $I\Sigma_{n+1}$ is Σ_{n+3}, and $B\Sigma_{n+2} \vdash I\Sigma_{n+1}$, so $K \not\models B\Sigma_{n+2}$. Thus Theorem 2.2 gives an answer to Question 1.1. Presumably the model K as constructed here is a model of $\neg I\Sigma_{n+1}$ (see also the result in the next section for a reason why this is expected), but I was unable to obtain this extra information about K.

It would be of interest to extend Theorem 2.1 or Theorem 2.2 to other cardinalities κ. A natural question to ask is the following.

QUESTION 2.3. Suppose that M is nonstandard and has an ascending sequence a_i (for $i < \lambda$) with $Y(a_i, a_{i+1}) > \mathbb{N}$ (equivalently, there is $I \subseteq_e M$ with $I \models \mathrm{PA}$ and $a_i \in I < a_{i+1}$) for all i, and κ is an uncountable cardinal of cofinality λ, is there a cofinal extension K of M which is κ-like and satisfies $\Pi_2 - \mathrm{Th}(\mathrm{PA})$?

Note that the main problem in modifying the proof given above lies at the point where we enumerate all formulas with parameters from N_i in order to construct the type p which we use to extend N_i to N_{i+1}.

2.2. *Models of* $B\Sigma_n + \exp + \neg I\Sigma_n$

The main theorem for this section is the following.

THEOREM 2.4. *Let κ be singular, let $n \geqslant 1$ and suppose that*

$$M \models B\Sigma_n + \exp + \neg I\Sigma_n.$$

Then there is a κ-like model $K \models \mathrm{Th}(M)$.

Proof. Without loss of generality, we may start with a countable recursively saturated model M of $B\Sigma_n + \exp + \neg I\Sigma_n$, where $n \geqslant 1$. We shall use Keisler's method to find a κ-like model elementary equivalent to M. Note that, by saturation, for each $a \in M$ we have $M \neq \sup\{2_n(a) : n \in \mathbb{N}\}$. We shall use this simple observation several times in the course of the proof.

Initially, we follow an idea due to Kossak, used also to construct elementary cofinal extensions (Kaye [**3**]). By standard constructions [**3**, pp. 955–956] there is an end-extension K of M, $a \in M$, and arbitrarily small $f \in K \setminus M$ such that f codes a function $< a \to K$ with $M = \sup_K(f''(< a) \cap M)$. As already mentioned, by saturation there is a nonstandard v so that $2_v(a) \in M$, and since $M \models \exp$ and f may be taken as small as we like we may assume that $2_4(f) \in K$.

Now choose $k \in K$ greater than f with $2^f < k$ and $2_3(k) \in K$. To simplify notation, we identify k with its set $< k$ of predecessors and regard it as a structure

$$(k, 0, 1, \oplus, \otimes, <),$$

where $\oplus$ and $\otimes$ are the ternary relations for $+$ and $\cdot$. Seen in this way, k is a recursively saturated model of PT ('Peano Top')—Peano arithmetic (with full

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induction) but having a greatest element. Note that k is recursively saturated because of the usual truth definition for Δ_0 formulas, which applies here even though K may not satisfy exp, since $2_3(k) \in K$. Let us denote the language of k by $\mathscr{L}_k$. Where convenient, we shall work in k together with a partial inductive satisfaction class Sat for k (see [**4**, Section 15.1]) which is defined for all formulas θ of $\mathscr{L}_k$ with Gödel-number less than some small nonstandard $\mu \in k$.

Now regard some $d \in M$ as a set of pairs $\langle \theta, e \rangle$ where θ is a formula of $\mathscr{L}_k$ with free variables $\bar{x}$, y, z, w and each $e < b \in M \setminus \mathbb{N}$. Note that, since $b > \mathbb{N}$, all such d are less than $2^{2^b} \in M$. Given d, and using the truth definition for k, define

$$f_d(\bar{a}) = \begin{cases} \text{the least } y \in k \text{ such that } k \vDash \theta(\bar{a}, y, f, e) \Leftrightarrow \langle \theta, e \rangle \in d & \text{if such exist,} \\ 0 & \text{otherwise.} \end{cases}$$

The f_d may not be functions on M, however.

We need a lemma.

LEMMA 2.5. *If* $\bar{c}, \bar{c}' \in M$ *satisfy* $k \vDash \forall x < 2_n(a)\,(\phi(x, \bar{c}, f) \leftrightarrow \phi(x, \bar{c}', f))$ *for all formulas* ϕ *of* $\mathscr{L}_k$ *and all* $n \in \mathbb{N}$, *then* $\bar{c}$ *and* $\bar{c}'$ *satisfy the same formulas in* M.

Proof. By a standard construction (see, for example, [**3**]), under the hypotheses given there is an automorphism α of k fixing $\{f\} \cup < a$ pointwise and sending $\bar{c}$ to $\bar{c}'$. But M is setwise fixed by α as $M = \sup_k(f''(< a) \cap M)$, so $\alpha \restriction M$ is an automorphism of M.

For the rest of the proof of the theorem, we freely use the truth definition in k. To start with, by saturation of M there is a nonstandard v in M so that $2_v(a) \in M$, and by taking a smaller v if necessary (that is, by taking $v < \mu$) we may assume that the truth definition for k works for all formulas $\phi < v$.

Take finitely many formulas $\theta_i(\bar{x}, y)$ (for $i = 1, \ldots, n$) in the language of M. We shall show that, for any finite set of the conditions in Lemma 1.3 for finitely many terms τ_j built from Skolem terms for the θ_i, there are elements $c_{i,j}$ of M satisfying those conditions. Hence, by compactness, the theory of M is consistent with all of those conditions taken together, and so Th(M) has a κ-like model for all singular cardinals κ. The construction is an inductive one in stages $i = 1, \ldots, N$; at stage i we shall find infinite but M-finite sets C^i_j (for $1 \leqslant j \leqslant i$). The choice of the c_{ij} will be made at the end so that $c_{ij} \in C^N_j$, and the c_{ij} are increasing, as required by Lemma 1.3(a).

For the first step of the induction, we reason as follows. Since $M \vDash \exp$ and the growth of functions required for the finite Ramsey theorem is at most a finite tower of exponentials, there is a_1 greater than $2_v(a)$ in M (here, we might take $a_1 = R^{\alpha}_{2_{v+2}(a)}(2_v(a)) < 2_{2\alpha+v+3}(a)$ for some small nonstandard $\alpha \in M$) and a M-finite set $C^1_1 \subseteq < a_1$ such that card $C^1_1 \geqslant 2_v(a)$ and C^1_1 is indiscernible in k over parameters in $\{f\} \cup < 2_v(a)$.

Now for each $\theta = \theta_i$ for some i, we have to define (in M) the Skolem terms f_θ for all $\bar{x} \in C^1_1$ and all other $\bar{x}$ necessary so that each of the finitely many terms τ_j is defined on the whole of C^1_1. Consider one such θ, and suppose that for some $\bar{c} \in [C^1_1]^{<\omega}$ there is $y \in M$ with $M \vDash \theta(\bar{c}, y)$. In (k, Sat) let

$$d = \{\langle \phi, e \rangle : e < 2_{v-2}(a), \phi < v, k \vDash \phi(\bar{c}, y, e, f)\}.$$

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Certainly, $d < 2_v(a)$ and y satisfies all the formulas in d, so $f_d(\bar{c}) = y'$ exists. By indiscernibility, $f_d(\bar{c}')$ exists for all $\bar{c}' \in [C_1^1]^{<\omega}$, and by the lemma

$$M \vDash \theta(\bar{c}, y) \Leftrightarrow M \vDash \theta(\bar{c}, f_d(\bar{c})) \Leftrightarrow M \vDash \theta(\bar{c}', f_d(\bar{c}')).$$

We define $f_\theta = f_d$ on all $\bar{c} \in [C_1^1]^{<\omega}$. If there is no $y \in M$ and no $\bar{c} \in [C_1^1]^{<\omega}$ with $M \vDash \theta(\bar{c}, y)$, then $f_\theta(\bar{c}) = 0$ for all such $\bar{c}$. Similarly, we define terms like $f_{\theta_1}(\bar{c}, f_{\theta_2}(\bar{c}))$ for $\bar{c} \in C_1^1$. Since $f_{\theta_2}(\bar{c}) = f_d(\bar{c})$ for some d, the same argument as above applies. Thus we can define all $\tau_j(\bar{c})$ in essentially finitely many steps.

Before we can continue the construction, we have to ensure that Lemma 1.3(b) will hold. For the terms τ_j that are being considered, define b_1 to be the maximum of a_1 and

$$\max\{\tau_j(\bar{c}) : \bar{c} \in [C_1^1]^{<\omega}, \text{ all } j\}. \tag{$*$}$$

The point here is that $b_1 \in M$. To see this, notice that the maximum value in $(*)$ is the maximum of finitely many $f_d(\bar{c})$, where $\bar{c}$ ranges over a k-finite set, so this maximum m exists in k. We can obviously ignore the cases when we defined $f_\theta(\bar{c}) = 0$, so by construction we can assume that each $f_d(\bar{c})$ is in M; so each $f_d(\bar{c}) < b$ for all $b > M$ in k and hence $m \in M$. (Note also that we could have argued for each f_d separately as follows: each $f_d(\bar{c})$ is in M, so for some $e < a$, we have $f_d(\bar{c}) < f(e) \in M$. By indiscernibility this is then true for all $\bar{c} \in [C_1^1]^{<\omega}$.) Hence $a_1 \leqslant b_1 \in M$.

Now, we give the inductive step of the construction. Suppose we are given $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_i \leqslant b_i$, all in M, and $C_j^i \subseteq < a_i$ for $1 \leqslant j \leqslant i$ each of which is M-finite, but actually infinite. In fact, we shall assume inductively that each C_j^i has cardinality at least $2_\beta(a)$ for some nonstandard number $\beta \in M$. We also suppose that all of the finitely many Skolem terms under consideration applied to all $\bar{c} \in [\bigcup_{j \leqslant i} C_j^i]^{<\omega}$ give values at most b_i. We shall then define a_{i+1} and an auxiliary set $C_{i+1}^i \subseteq < a_{i+1}$, and then, having done this, we shall choose $C_j^{i+1} \subseteq C_j^i$ for each j, so that there is b_{i+1} as required to make the inductive construction continue.

To define a_{i+1}, take nonstandard $v_i < v$ such that $2_{v_i}(b_i)$ exists in M. Then a_{i+1} is defined to be equal to some Ramsey number in parameter $2_{v_i}(b_i)$ (more precisely, we may take $a_{i+1} = R_\mu^\alpha(\mu) < 2_{2\alpha + v_i + 3}(b_i)$, where $\mu = 2_{v_i+2}(b_i)$ and α is small and nonstandard, so that there is $C_{i+1}^i \subseteq < a_{i+1}$ indiscernible in k for formulas with parameters from $\{f\} \cup < 2_{v_i}(b_i)$ and $\operatorname{card} C_{i+1}^i = 2_{v_i}(b_i)$. Note that $2_v(a) < 2_{v_i}(b_i)$ and so for any $d < 2_v(a)$ and $\bar{x} < 2_{v_i}(b_i)$, we have $\bar{c} \in [C_{i+1}^i]^{<\omega}$; if it happens that $f_d(\bar{c}, \bar{x}) < 2_{v_i}(b_i)$, then by indiscernibility this holds for all $\bar{c} \in [C_{i+1}^i]^{<\omega}$. Moreover, if $\bar{c}, \bar{c}' \in [C_{i+1}^i]^{<\omega}$ are such that $f_d(\bar{c}, \bar{x}) \neq f_d(\bar{c}', \bar{x})$ then this again holds for all $\bar{c}, \bar{c}' \in [C_{i+1}^i]^{<\omega}$. But there are more than $2_{v_i}(b_i)$ such $\bar{c}$, so we have a contradiction. Hence for all $d < 2_v(a)$, $\bar{x} < 2_{v_i}(b_i)$, and $\bar{c} \in [C_{i+1}^i]^{<\omega}$, we have

$$f_d(\bar{c}, \bar{x}) < 2_{v_i}(b_i) \Rightarrow f_d(\cdot, \bar{x}) \text{ is constant on } [C_{i+1}^i]^{<\omega}. \tag{$\dagger$}$$

The next step is to bound all the Skolem terms by some b_{i+1}. We apparently cannot do this directly: a further Ramsey argument is required. Here we shall find subsets $C_j^{i+1} \subseteq C_j^i$ for each j, using a polarized partition relation. Let r be a large standard number at least equal to the maximum number of arguments in any of the terms τ_j. (Recall that there are actually finitely many such τ_j.) By Corollary 1.7 and the fact that each C_j^i has cardinality at least $2_\beta(a)$ for some nonstandard β, there is a small nonstandard γ such that there are $C_j^{i+1} \subseteq C_j^i$ of cardinality at least $2_\gamma(a)$ (for $\beta_j > \mathbb{N}$) such that for all formulas ϕ of k with parameters from $\{f\} \cup < 2_\gamma(a)$ and all $\bar{c}_j, \bar{c}'_j \in C_j^{i+1}$ (for $j = 1, \ldots, i+1$) we have

$$k \vDash \phi(\bar{c}_1, \ldots, \bar{c}_j, \ldots, \bar{c}_{i+1}) \leftrightarrow \phi(\bar{c}'_1, \ldots, \bar{c}'_j, \ldots, \bar{c}'_{i+1}).$$

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Now the argument above defining Skolem functions as certain of the f_d goes through exactly as before, and their maximum value b_{i+1} can be defined just as before.

The inductive construction is carried out $N \in \mathbb{N}$ times, where N is sufficiently large to cover all the cases in the finite set of axioms from Lemma 1.3 we are considering. The conditions in Lemma 1.3 are easily seen to be satisfied for any increasing tuples $c_{i0}, \ldots, c_{in}$ from C_i^N: part (a) was built into the construction (when we found the b_i) and parts (b) and (c) follow rapidly from the construction, the definition of the b_i and from (†).

That concludes the proof of the theorem.

Comparing our two results, it is natural to ask if they can be combined in some way. In particular, is it always true that a countable sufficiently saturated model M of $B\Sigma_n + \exp + \neg I\Sigma_n$ has κ-like cofinal elementary extensions for some uncountable κ? However, it is not clear what the right notion for 'sufficiently saturated' should be here. Note that in the special case of Theorem 2.1 no further saturation conditions were required, but of course the extension was not elementary; also, all countable models of $B\Sigma_n + \exp + \neg I\Sigma_n$ have proper elementary cofinal extensions [**3**], but here the extensions are only countable. The following question is just a sample question in this direction.

QUESTION 2.6. Given a countable recursively saturated $M \models B\Sigma_n + \exp + \neg I\Sigma_n$, and given $\kappa > \omega$ with cofinality ω, is there a κ-like cofinal elementary extension of M?

Note that the argument in the proof of Theorem 2.4 will not work here, since there we could deal only with actually finitely many θ_i at a time.

References

1. P. ERDŐS and R. RADO, 'A partition calculus for cardinal numbers', *Bull. Amer. Math. Soc.* 62 (1956) 427–489.
2. H. GAIFMAN, 'Models and types of Peano's arithmetic', *Ann. Math. Logic* 9 (1976) 223–306.
3. R. KAYE, 'Model-theoretic properties characterizing Peano arithmetic', *J. Symbolic Logic* 56 (1991) 949–963.
4. R. KAYE, *Models of Peano arithmetic*, Oxford Logic Guides 15 (Oxford University Press, 1991).
5. R. KAYE, 'The theory of κ-like models of arithmetic', *Notre Dame J. Formal Logic* 36 (1995) 547–559.
6. H. J. KEISLER, 'Models with orderings', *Logic, methodology and philosophy of science* III (eds. B. van Rootselaar and J. F. Staal; North-Holland, Amsterdam, 1968) 35–62.
7. L. KIRBY and J. PARIS, 'Initial segments of models of Peano's axioms', *Set theory and hierarchy theory* V, *Bierutowice, Poland*, 1976, Lecture Notes in Mathematics 619 (eds. A. H. Lachlan et al.; Springer, Berlin, 1977) 211–226.
8. J. B. PARIS and G. MILLS, 'Closure properties of countable non-standard integers', *Fund. Math.* 103 (1979) 205–215.

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