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Abstract

We provide a complete classification of when the homeomorphism group of a stable surface, Σ, has the automatic continuity property: Any homomorphism from $\mathrm{Homeo}(\Sigma)$ to a separable group is necessarily continuous. This result descends to a classification of when the mapping class group of Σ has the automatic continuity property. Towards this classification, we provide a general framework for proving automatic continuity for groups of homeomorphisms. Applying this framework, we also show that the homeomorphism group of any stable, second-countable Stone space has the automatic continuity property. Under the presence of stability, this answers two questions of Mann.

1. Introduction

Given a topological group, a natural question is: How does the algebra of the group determine the topology of the group? Surprisingly, the answer to this question can be 'almost entirely'. A topological group, G, has the *automatic continuity property* if every group homomorphism from G to a separable group H is continuous. Recent examples of groups with this property are the homeomorphism groups of the following spaces: Cantor set, $\mathbb{R}$ [RS07], compact manifolds [Ros08, Man16], compact manifolds with a Cantor set and a finite set removed [Man24], and some infinite-type surfaces [Vla24]. There is also a classification theorem [Dic24] for pure mapping class groups of infinite-type surfaces (allowing for noncompact boundary components). See [Ros09, RS24] for surveys of the question, together with some historical context.

In this paper, we prove a classification theorem for the homeomorphism and mapping class groups of a large class of infinite-type surfaces. For the collection of stable surfaces (see Definitions 3.3 and 4.3), this gives a complete answer to two questions of Mann ([AIM, Question 4.8] and [Man24, Question 2.4]). All our surfaces Σ will be connected, orientable and without boundary.

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Theorem A. *Let Σ be a connected, stable, orientable surface without boundary. The homeomorphism group and mapping class group of Σ have automatic continuity if and only if every end of Σ is telescoping. That is, every end is one of the following:*

(i) *an isolated puncture;*

(ii) *of Cantor type; or*

(iii) *is not isolated in the space of ends accumulated by genus and is a successor with all predecessors of Cantor type.*

The stability assumption on Σ rules out various pathological phenomena. In particular, stability ensures that $\mathrm{Homeo}(\Sigma)$-orbits of ends are locally closed and hence each orbit is either locally a Cantor set or consists of isolated points (Lemma 4.7), Σ has only finitely many equivalence classes of maximal ends (Proposition 4.9), and allows us to prove an alternate characterization of 'telescoping' (Lemma 4.16). We make use of each of these in order to obtain a complete classification. In § 4.10, we offer an unstable surface for which we do not know whether its mapping class group satisfies automatic continuity.

More formally, we say that an end, x, of the surface Σ is *stable* if there exists a neighborhood U of x such that any further neighborhood $U' \subset U$ of x is homeomorphic to U. We then say that a surface is stable if every end is stable. Not only does this definition allow us to avoid various pathologies, but it also defines a rich and interesting class of surfaces. In particular, this class is uncountable, containing all surfaces with countable end space and either finite genus or having every end accumulated by genus, as well as finite-genus surfaces with Cantor sets of ends (i.e. those arising from complex dynamics [Cal09, CC22]), among many others. The class of stable surfaces also fits into the coarse-geometry classification results in [MR23] and is a natural class of surfaces for which to define infinite-diameter, hyperbolic, curve graph analogs á la the omnipresent arc graph [FGM21] or the grand arc graph [BNV23]. In particular, stability allows one to run versions of the 'Anderson trick' [And58] in order to decompose certain group elements into products of commutators. We direct the reader to [Vla24, § 5.5] for an excellent exposition on Anderson's method with a focus on surfaces.

Here we make ample use of the language and perspective provided by [MR23], particularly with respect to the partial order on the space of ends of a surface. Condition (iii) in Theorem A makes use of this partial order. This condition may be interpreted as saying that the end in question has an immediate predecessor and so does not exhibit 'limit' behavior (a useful analogy is the distinction between a successor ordinal and a limit ordinal) and the $\mathrm{Homeo}(\Sigma)$-orbit of this predecessor is locally homeomorphic to a Cantor set.

The techniques used to prove the positive direction are extensions of those in [RS07, Ros08, Man16, Man24]. In particular, the 'telescoping' condition can be thought of as requiring each end to have a neighborhood that exhibits similar behavior to a punctured disk. This condition allows one to decompose any neighborhood of an end into infinitely many homeomorphic 'tubes' such that one can shift the collection of tubes either towards or away from the end. We also use an Eilenberg–Mazur swindle [Bas63, Maz59] in order to deal with the issue of locally writing certain homeomorphisms as commutators. This type of argument does not appear in the previous proofs of automatic continuity. For the negative direction, we build on tools developed in [Dom22]. All of the discontinuous maps we exhibit factor through $\mathrm{Map}(\Sigma)$ and are constructed in [Dom22] via actions on Gromov hyperbolic metric spaces.

Remark 1.1. We note that some of our techniques used to prove the failure of automatic continuity do not require the surface to be stable. In particular, if Σ (not necessarily stable) has an end that (1) has a countable $\mathrm{Homeo}(\Sigma)$-orbit and (2) has either a countable predecessor or

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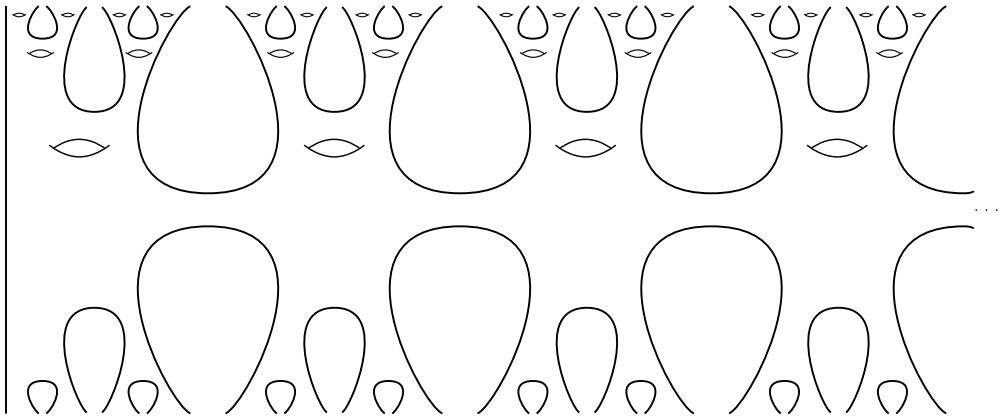


Figure 1. An example of a surface for which Theorem A applies in the positive direction.

is isolated in the space of ends accumulated by genus, then $\mathrm{Homeo}(\Sigma)$ and $\mathrm{Map}(\Sigma)$ will fail to have automatic continuity. This follows by directly applying the techniques of § 4.8. Similarly, if Σ has an end with a countable orbit that is also the accumulation point of a sequence of pairwise incomparable ends, then $\mathrm{Homeo}(\Sigma)$ and $\mathrm{Map}(\Sigma)$ fail to have automatic continuity.

Notably, all of the discontinuous maps that we make use of have targets that contain isomorphic copies of $\mathbb{Q}$. This is perhaps not a coincidence, and further evidence for the following question of Conner (the question was originally stated for countable codomains; we extend it to separable codomains).

Question 1.2 ([CC19, Question 5.2] and [CPV21, Conner's Conjecture]). If H is a torsion-free separable group that does not contain an isomorphic copy of $\mathbb{Q}$, must every homomorphism from a completely metrizable group to H be continuous?

Example 1.3. Here, we construct a few surfaces for which our theorems apply that were not covered by any previous results.

(i) Consider the surface, M, with end space two Cantor sets $C_1 \cup C_2$ so that $C_1 \cap C_2$ is exactly a single point. Furthermore, each end in C_1 is accumulated by genus. See Figure 1. Every end of this surface is telescoping and so $\mathrm{Homeo}(M)$ and $\mathrm{Map}(M)$ have the automatic continuity property. The end space $C_1 \cap C_2$ is the prototypical example of a telescoping end.

(ii) Let L be the Loch Ness monster surface, i.e. the surface with infinite genus and a single end. If N is the connect sum of L with any orientable, boundaryless surface, then $\mathrm{Homeo}(N)$ and $\mathrm{Map}(N)$ fail to have the automatic continuity property. Here, L can also be replaced by any surface whose end space has a countable orbit under the action of the homeomorphism group.

The general techniques for proving automatic continuity developed below also apply to other classes of groups. We apply them to homeomorphism groups of end spaces, i.e. second-countable Stone spaces (closed subsets of a Cantor set). Alternatively, by Stone duality [Sto36], these groups can also be thought of as automorphism groups of countable Boolean algebras.

Theorem B. *Let X be a second-countable Stone space. If X is stable, then* $\mathrm{Homeo}(X)$ *has automatic continuity.*

This answers [Man24, Question 2.5] under the stability hypothesis. In particular, the stability condition can be thought of as a type of 'local homogeneity'. Here, stability is defined as in

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[MR23] (see Definition 3.3 and Proposition 3.2). We also prove an alternate characterization (Proposition 3.2) of stability that allows us to view each point in X as satisfying a type of 'telescoping' condition.

By a classical result of Mazurkiewicz and Sierpiński [MS20], every countable Stone space is homeomorphic to a countable ordinal of the form $\omega^\alpha \cdot n + 1$, where α is a countable ordinal and $n \in \mathbb{N}$. Notably, all countable ordinals are stable and so, as a corollary, we have the following.

Corollary 1.4. *Let $\omega^\alpha \cdot n + 1$ be a compact, countable ordinal equipped with the order topology. The group* Homeo$(\omega^\alpha \cdot n + 1)$ *has the automatic continuity property.*

Concurrent with this paper, Hernández Hernández, Hrušák, Rosendal, and Valdez have also obtained the previous work. In fact, when $n = 1$ they prove that Homeo$(\omega^\alpha + 1)$ has ample generics; a stronger condition that implies automatic continuity. They can then leverage this ample-generics property to show that Homeo$(\omega^\alpha \cdot n + 1)$ has automatic continuity for arbitrary n.

Theorem A and Corollary 1.4 offer some insight as to how the cases of surfaces and end spaces differ. We have that Homeo$(\omega^\alpha \cdot n + 1)$ always has automatic continuity, but if $\Sigma_{\alpha,n}$ is a surface with such an end space, then Homeo$(\Sigma_{\alpha,n})$ fails to have automatic continuity. Upon reflection, this is perhaps not surprising, since the techniques used to prove this failure of automatic continuity are directly coming from the topology of the surface and the mapping class group. In particular, the hyperbolic spaces used in [Dom22] are built out of curve graphs using techniques from [MM99, BBF15].

The only examples known to us of Stone spaces whose homeomorphism groups have discontinuous homomorphisms come from constructions in [Ros09, Man24]. For these examples, one has infinitely many finite orbits of cardinality larger than one. Notably, the condition on stability does not allow this to happen. This suggests the following question towards a classification for general second-countable Stone spaces.

Question 1.5. Does there exist a second-countable Stone space X so that Homeo(X) does not surject onto an infinite product of finite groups and Homeo(X) fails to have automatic continuity?

Returning to our original question, 'How does the algebra of the group determine the topology of the group?', an immediate application of the automatic continuity property is to say that the topology on the group is essentially unique.

Corollary 1.6. *Let Σ be a stable, orientable surface without boundary, all of whose ends are telescoping. The groups* Homeo(Σ) *and* Map(Σ) *both have unique Polish group topologies.*

Corollary 1.7. *Let X be a stable, second-countable Stone space. The group* Homeo(X) *has a unique Polish group topology.*

This follows from some standard arguments in descriptive set theory, e.g. see [Vla24, § 2.5].

Curiously, our arguments show that for the topological groups we consider, automatic continuity is equivalent to another property, a priori unrelated to automatic continuity. We say that a topological group G is *locally perfect* if, for every neighborhood of the identity U, there exists a smaller neighborhood $V \subset U$ so that every element of V can be written as a product of commutators of elements in U. We say that a topological group G is *locally uniformly perfect* if there is some N so that, for every neighborhood of the identity U, there exists a smaller neighborhood $V \subset U$ so that every element of V can be written as a product of N commutators of elements in U.

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Corollary 1.8. *Let G be one of: the homeomorphism group of a stable, second-countable Stone space; the homeomorphism group of a stable surface; or the mapping class group of a stable surface. The following are equivalent:*

(i) *G satisfies automatic continuity;*
(ii) *G is locally perfect;*
(iii) *G is locally uniformly perfect.*

Of course, in the case of the homeomorphism group of a stable, second-countable Stone space, (i)–(iii) always hold, and in the other two cases they hold if and only if all ends are telescoping.

In the case of surfaces, the proofs follow directly from the arguments in § 4.3, § 4.4 and § 4.8.

1.1 The five-step program

We first provide a general outline for how to prove the automatic continuity property for a Polish group of homeomorphisms. We will follow this general outline in § 3 and § 4 for Stone spaces and surfaces, respectively. We remark that this general outline is inspired by the arguments found in [RS07, Ros08, Man16, Man24]. We expect this general framework to be applicable to other spaces and Polish groups of transformations. In particular, our arguments should be applicable for subgroups of homeomorphisms provided they contain 'enough' maps of an appropriate type.

For all our proofs of automatic continuity, we will actually prove that the Steinhaus property holds and apply a result of Rosendal and Solecki [RS07].

Definition 1.9. A topological group G has the *Steinhaus property* if there exists an $n \in \mathbb{N}$ such that, whenever W is a symmetric subset of G such that countably many translates of W cover G, W^n contains an open neighborhood of the identity in G.

Proposition 1.10 [RS07, Proposition 2]. *If a topological group G has the Steinhaus property, then it also has the automatic continuity property.*

We remark that we are not aware of any Polish groups, let alone homeomorphism groups of surfaces, that are known to have the automatic continuity property but are not Steinhaus. Moreover, every proof of the automatic continuity property that we are aware of passes through the Steinhaus property.

Throughout this section, we give vague definitions that will each be modified to fit the subsequent sections. All of our proofs require some sort of *locally telescoping* condition. This condition can be thought of as a type of strong local homogeneity of the space. In particular, it will allow us to decompose a neighborhood, Ω, of a point into an uncountable collection of *bricks*. These bricks will all be pairwise homeomorphic via homeomorphisms supported only on Ω. Additionally, each brick and its complement must contain uncountably many bricks. This may give some insight into why we require ends to be of Cantor type in Theorem A.

Example 1.11. Here we give three examples of bricks. The first is in a Stone space and the second two are in surfaces.

(i) Let $X = \{1/n | n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ equipped with subspace topology. A brick of X is then a countably infinite subset of points (not including 0) so that its complement is also countably infinite. Such a subset is sometimes referred to as a *moiety* in the literature. Here, $\mathrm{Homeo}(X)$ is isomorphic to the symmetric group on $\mathbb{N}$.
(ii) Let Σ_1 be a punctured sphere. Then a brick about the puncture, p, of Σ_1 is a locally finite collection of disjoint annuli $\{A_i\}$ such that each A_i separates p from A_j for all $j < i$.

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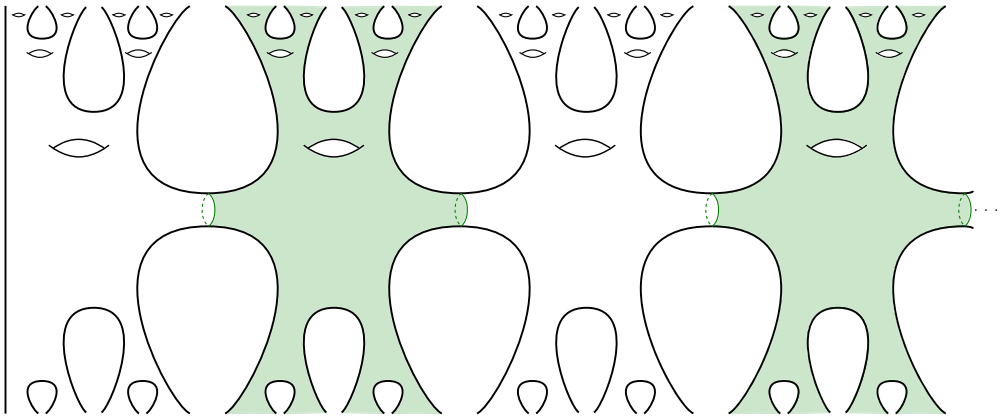


Figure 2 (colour online). An example of a brick (shaded) comprised of drawtubes in a surface.

(iii) Let M be the surface described in Example 1.3 above and let $e = C_1 \cap C_2$. Then we say a drawtube is a subsurface A so that
- A is bounded by two curves, γ_1 and γ_2, so that γ_2 is contained in the complementary component of γ_1 witnessing the end e, and
- the end space of A intersects both C_1 and C_2 non-trivially.

We then define a brick to be a countably infinite, locally finite, disjoint union of drawtubes. See Figure 2.

Now, let $G = \mathrm{Homeo}(X)$ for some compact space X and W be a symmetric set in G so that countably many translates of W cover G. First, we will always apply a standard Baire category argument (Lemma 2.3) to find an open neighborhood of the identity $U \subset G$ so that W^2 is dense in it. This U will determine some finite partition of X into $\Omega_1 \sqcup \Omega_2 \sqcup \cdots \sqcup \Omega_m$ so that any $g \in G$ preserving each of the Ω_i belongs to U. Next, we use compactness to further partition each Ω_i into telescoping neighborhoods. This reduces the problem of proving Steinhaus to the following claim. We let $G(\Omega)$ denote the subgroup of G consisting of homeomorphisms supported on $\Omega \subset X$.

Claim 1. *Let Ω be telescoping with respect to y. There exists an $N \geq 0$ so that if $W \subset G$ is symmetric, with countable many translates covering $G(\Omega)$, and W^2 dense in $G(\Omega)$, then $G(\Omega) \subset W^N$.*

Proving this claim will proceed through several standard steps.

Step 1: Fragmentation. First, we will fragment any $g \in G(\Omega)$ into two maps, $g = g_1 g_2$, so that each g_i fixes a point $y_i \in \Omega$ and so that Ω is telescoping with respect to y_i. Next, we fragment again to write such a map as a product of two maps supported on bricks. The final outcome is that, given any $g \in G$, we can write $g = g_1 g_2 g_3 g_4$ with each g_i supported on a brick in Ω. This reduces the problem to finding some N so that any map supported on a brick is in W^N.

Step 2: Finding commutators. Next, we want to realize any map supported on a brick as a uniformly finite product of commutators, each of which is supported on a single brick. This step will always proceed by using a version of the Anderson trick [And58]. How exactly it is implemented will vary depending on our situation and the exact form of the 'telescoping condition' above. This will depend on what types of shift maps exist in G. For example, in the case of a zero-dimensional space, we have a well-defined shift map supported on a single brick that we can use to write any map as a single commutator. However, in the surface case we will

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only have shift maps supported in each connected component of a brick. In this case we will have to fragment again using an Eilenberg–Mazur swindle [Bas63, Maz59], taking advantage of the existence of certain infinite products of maps.

Step 3: Diagonal argument. We now use the above in order to find one specific 'good' brick $\mathcal{B}$ in Ω so that any g supported on $\mathcal{B}$ is in W^N. This part of the argument will be fairly standard throughout. In fact, the arguments in this step are taken almost directly from [Man16, Man24, Ros08, RS07].

Step 4: Pigeonhole. Next, we run a pigeonhole argument to find elements of W^N that move any brick into our preferred brick $\mathcal{B}$. This will make use of some part of the 'telescoping condition' that guarantees that we can map any brick onto any other brick. This will again proceed in a fairly standard fashion in each case, following [Man24].

Step 5: Wrapping up. Finally, we put all of the above pieces together to prove Claim 1. Then, applying a Baire category argument, Lemma 2.3, we upgrade this to obtain Steinhaus, and thus automatic continuity, for $G(\Omega)$ whenever Ω is a telescoping neighborhood.

2. Background on automatic continuity

Definition 2.1. A topological group G has the *automatic continuity property (AC)* if every homomorphism from G to a separable group is continuous.

Using an induction construction we can see that if a closed countable-index subgroup fails to have AC, then so does the larger group. See also the introduction of [Ros09] for a more direct proof of this.

Lemma 2.2. *Let G be a topological group, $H < G$ a countable index subgroup, and $f : H \to A$ a discontinuous homomorphism to a countable discrete group. Then there is a separable topological group B (isomorphic to $S_\omega = \mathrm{Homeo}(\omega + 1)$) and a discontinuous homomorphism $F : G \to B$.*

Proof. Take $X = A$ equipped with the discrete topology. The subgroup H acts via left multiplication on X via f, i.e. $h \cdot x = f(h)x$. We construct a larger set Y on which G acts, extending the action of H on X. This will not be a literal extension, but it will be close enough.

Start with the action of G on $G \times X$ by $g \cdot (g', x) = (gg', x)$. Define the following equivalence relation on $G \times X$:

$$(g, x) \sim (gh, h^{-1}x) \quad \text{for all } h \in H.$$

Let Y be the quotient set. Note that

- G acts on Y via $g \cdot [g', x] = [gg', x]$, and
- if $\{g_i\}$ is a set coset representative so that $G = \bigsqcup_{i\in\mathbb{N}} g_i H$, then

$$Y = \bigsqcup_{i\in\mathbb{N}} \{[g_i, x] | x \in X\}.$$

Moreover, we can canonically identify $\{[1, x] | x \in X\}$ with X via the map $[1, x] \mapsto x$ and under this identification we have $\{[g_i, x] | x \in X\} = g_i X$ so that $Y = \bigsqcup_{i\in\mathbb{N}} g_i X$.

We now have a homomorphism $F : G \to \mathrm{Aut}(Y) =: B$, the group of bijections of Y. Note first that Y is countable. Indeed, $Y = \bigsqcup_{i\in\mathbb{N}} g_i X$, a countable union of countable sets. Thus, the group of bijections $\mathrm{Aut}(Y)$ with the permutation topology is isomorphic to S_ω. It remains to show that F is discontinuous. It suffices to argue that the restriction to H is discontinuous. The restricted action preserves $X \subset Y$, so it suffices to argue that $H \to \mathrm{Aut}(X)$ is discontinuous.

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The image of this homomorphism is contained in the group of left translations of $X = A$, which can be identified with A. In fact, the image of this homomorphism can be exactly identified with the image of the original $f : H \to A$. Thus, we see that the homomorphism $H \to \mathrm{Aut}(X)$ is discontinuous by assumption. □

The starting point of all our arguments will be the following standard application of the Baire category theorem.

Lemma 2.3. *Let G be a Polish group. If W is a symmetric subset of G such that countably many translates of W cover G, then the closure of W^2 contains a neighborhood of the identity in G.*

Proof. Write $G = \bigcup g_i W$. By Baire category, some $\overline{g_i W}$ has non-empty interior. Choose some $g_i w \in g_i W$ that belongs to this interior. Then the closure of $w^{-1} g_i^{-1} \cdot g_i W = w^{-1} W \subseteq W^2$ contains a neighborhood of the identity. □

3. End spaces

Let X be a compact totally disconnected metrizable space (i.e. a second-countable Stone space). In this section, we investigate whether the group $\mathrm{Homeo}(X)$ has the automatic continuity property. More generally, we fix a collection $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ of closed subsets of X and consider the group $\mathrm{Homeo}(X, \{X_\alpha\}_{\alpha \in \mathcal{A}})$ of homeomorphisms that preserve each X_α (setwise). We think of each $\alpha \in \mathcal{A}$ as a color, so elements of $\mathrm{Homeo}(X, \{X_\alpha\}_{\alpha \in \mathcal{A}})$ are color-preserving homeomorphisms of X. To simplify notation, we omit the colors and talk about $\mathrm{Homeo}(X)$, but coloring is understood. That is, when we say $\phi : U \to V$ is a homeomorphism from $U \subset X$ to $V \subset X$, we always assume that, for every $\alpha \in \mathcal{A}$,

$$\phi(U \cap X_\alpha) = V \cap X_\alpha.$$

The coloring is motivated by § 4, where we need to distinguish between planar and non-planar ends of a surface. In the first reading, the reader may think about the case $\mathcal{A} = \emptyset$. The goal of this section is to prove the following theorem. The definition of stability is below, in Definition 3.3.

Theorem 3.1. *Let X be a second-countable Stone space. If X is stable (relative to the given colors), then $\mathrm{Homeo}(X)$ has the Steinhaus property and therefore the automatic continuity property.*

By a neighborhood U in X, we always mean a clopen neighborhood, that is, a set that is both open and closed. We say a family of subsets $Y_k \subset X$ descends to x and write $Y_k \searrow x$ if, for every sequence $x_k \in Y_k$, we have $x_k \to x$.

Proposition 3.2. *Let X be a second-countable Stone space, x a point in X, and U a neighborhood of x. Then the following are equivalent.*

(i) *For every neighborhood $U' \subset U$ of x there is a homeomorphism $\Phi : U \to U'$ fixing x.*

(ii) *For every neighborhood $U' \subset U$ there is a neighborhood $U'' \subset U'$ of x and a homeomorphism $\Phi : U \to U''$ fixing x.*

(iii) *There is a decomposition*

$$U - \{x\} = \bigsqcup_{k=1}^{\infty} Y_k \tag{3.1}$$

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where Y_k are clopen, disjoint, $Y_k \searrow x$ and, for every $k \geq 1$, the set Y_{k+1} contains a homeomorphic copy of Y_k. Furthermore, for every sequence of indices k_n,

$$U - \{x\} \cong \bigsqcup_{n=1}^{\infty} Y_{k_n}. \tag{3.2}$$

Proof. The implication (i) $\Longrightarrow$ (ii) is immediate. We start by proving (ii) $\Longrightarrow$ (iii). Let

$$U = U_0 \supset U_1 \supset U_2 \supset \cdots$$

be a nested sequence of clopen neighborhoods of x such that $U_i \searrow x$ and there are homeomorphisms

$$\Psi_i : U_i \to U_{i+1}, \quad i = 0, 1, 2, \ldots$$

fixing x. This is possible by starting from a nested sequence that descends to x and making them smaller to make sure they are all homeomorphic to U. We construct disjoint clopen sets V_k, Y_k and an index sequence p_k inductively as follows. Set $p_0 = 0$, $p_1 = 1$ and set

$$Y_1 = V_1 = U_0 - U_1.$$

Assume now that the index $p_k > 0$ and sets $V_k, Y_k \subset U - \{x\}$ are given. Let

$$\Phi_k : U_{p_{k-1}} \to U_{p_k} \quad \text{be the composition} \quad \Phi_k := \Psi_{(p_k-1)} \circ \cdots \circ \Psi_{(p_{k-1}+1)} \circ \Psi_{p_{k-1}}.$$

Then $\Phi_k(Y_k)$ is a closed subset of U_{p_k} disjoint from x. Hence, we can choose an index $p_{k+1} > p_k$ large enough such that

$$\Phi_k(Y_k) \cap U_{p_{k+1}} = \emptyset.$$

Then define

$$Y_{k+1} = U_{p_k} - U_{p_{k+1}} \quad \text{and} \quad V_{k+1} = Y_{k+1} - \Phi_k(Y_k).$$

The sets Y_k are clopen and disjoint. Since $Y_{k+1} \subset U_{p_k}$ and $U_i \searrow x$, we also have $Y_k \searrow x$. Also,

$$U - \{x\} = \bigsqcup_{k=1}^{\infty} (U_{p_k} - U_{p_{k+1}}) = \bigsqcup_{k=1}^{\infty} Y_k. \tag{3.3}$$

To see the last two assertions of (iii), note that Y_{k+1} is homeomorphic to $Y_k \sqcup V_{k+1}$. By induction, this implies

$$Y_k \cong \bigsqcup_{j=1}^{k} V_j.$$

Now, let a sequence of indices k_n be given and let $A = \bigsqcup_{n=1}^{\infty} Y_{k_n}$. Since a copy of V_j appears in Y_k for every $k \geq j$, we can break this further and write

$$A = \bigsqcup_{n=1}^{\infty} \bigsqcup_{j=1}^{k_n} V_{n,j},$$

where $V_{n,j}$ is a homeomorphic copy of V_j in Y_{k_n}. Changing the order of unions, we have

$$A = \bigsqcup_{j=1}^{\infty} \bigsqcup_{n=1}^{\infty} V_{n,j}.$$

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Similarly, using (3.3), we have

$$U - \{x\} = \bigsqcup_{k=1}^{\infty} Y_k = \bigsqcup_{k=1}^{\infty} \bigsqcup_{j=1}^{k} V_{k,j} = \bigsqcup_{j=1}^{\infty} \bigsqcup_{k=1}^{\infty} V_{k,j},$$

where $V_{k,j}$ is a homeomorphic copy of V_j. We can now construct a homeomorphism from A to $U - \{x\}$ by sending $V_{k,j}$ homeomorphically to $V_{n,j}$. This map is continuous since, for every j,

$$V_{n,j} \searrow x \quad \text{and} \quad V_{k,j} \searrow x \quad \text{as } n \to \infty.$$

This finishes the proof of (ii) $\Longrightarrow$ (iii).

To see (iii) $\Longrightarrow$ (ii), we observe that, for every $U' \subset U$, there is $N > 0$ such that $Y_k \subset U'$ for $k \geq N$. Otherwise, there is a point $x_k \in Y_k - U$. But x_n does not limit to x, which contradicts the assumption that $Y_n \searrow x$. Hence,

$$U \cong \bigsqcup_{k=N}^{\infty} Y_k \subset U'.$$

It remains to show (ii) $\Longrightarrow$ (i). Let $U' \subset U$ be given. Let $U = U_1 \supset U_2 \supset \cdots$ and $\Psi_i : U_i \to U_{i+1}$ be as before (which exist after assuming (ii)). We construct disjoint clopen sets W_k and an index sequence q_k as follows. Set $q_1 = 1$ and set $W_1 = U - U'$. Assuming $q_k > 0$ and $W_k \subset U - \{x\}$ are given, let

$$\Phi'_k : U_{q_{k-1}} \to U_{q_k} \quad \text{be the composition} \quad \Phi'_k := \Psi_{(q_k - 1)} \circ \cdots \circ \Psi_{(q_{k-1}+1)} \circ \Psi_{q_{k-1}}$$

and choose an index $q_{k+1} > q_k$ large enough such that

$$\Phi'_k(W_k) \cap U_{q_{k+1}} = \emptyset.$$

Then define $W_{k+1} = \Phi_k(W_k)$. The sets W_k are all homeomorphic, disjoint and descend to x. We can now build a homeomorphism from $U \to U'$ by sending W_k to W_{k+1} and fixing everything else. This map is continuous since $W_k \searrow x$. This finishes the proof. □

Definition 3.3 [MR23, Definition 4.14]. A neighborhood U of a point $x \in X$ is *stable* if it satisfies the three equivalent conditions in Proposition 3.2. We say that a point in X is *stable* if it has a stable neighborhood. We say the space X is *stable* if every point of X is stable.

Property (iii) of stability will play the role of a type of 'telescoping' property for second-countable Stone spaces.

Remark 3.4. In [MR23], property (ii) from above was originally given as the definition of stability. A version of property (i) also shows up in [MR23, Lemma 4.17]. Similarly, there is a discussion of stability and properties (i) and (ii) in [Vla24, § 4]. Additionally, in [Vla24, Lemma 5.4.5], Vlamis proves (ii) $\Longrightarrow$ (iii) in the case that the maximal ends contained in the neighborhood U are of Cantor type.

Before proceeding with the proof of Theorem 3.1, we give some examples.

Example 3.5. Here are two examples of stable Stone spaces.

(1) The Cantor set is stable and thus its homeomorphism group has AC. Our theorem thus recovers the result of [RS07].

(2) Every countable compact Stone space is stable and hence has automatic continuity. This follows from the classification theorem that realizes compact Stone spaces as countable ordinals [MS20].

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Remark 3.6. Adding a coloring to the space may change whether the space is stable. For example, let $X = C \times D$, where C is the Cantor set and $D = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{n\geq 1}\}$. Then this space is stable and hence $\mathrm{Homeo}(X)$ has the automatic continuity property. However, if we now color each $C \times \{d\}$ with its own color for $d \in D$, the space is no longer stable and we do not know whether the color-preserving homeomorphism group has automatic continuity. This is an end-space analog of the surface example given in § 4.10.

Unless otherwise stated, throughout the following we assume that X is a second-countable Stone space, $x \in X$, and $\Omega \subset X$ is a stable neighborhood of x.

We have a decomposition

$$\Omega \setminus \{x\} = \bigsqcup_{k=1}^{\infty} Y_k \tag{3.4}$$

as given by Proposition 3.2.

Denote

$$Y[n] := \bigsqcup_{i=1}^{n} Y_i \quad \text{and more generally} \quad Y[n,m] = \bigsqcup_{i=n}^{m} Y_i.$$

Proposition 3.7. *For every $1 \leq n < m$ and every embedding $\phi : Y[n] \to Y[m]$ onto a clopen subset, there is some $p > m$ and a homeomorphism $h : Y[p] \to Y[p]$ that agrees with ϕ on $Y[n]$.*

The proof is based on the following lemma.

Lemma 3.8. *Let A, B be topological spaces and let $\phi : A \to B$ be an embedding onto a clopen subset of B. Then there is a homeomorphism $\tilde{\phi}$ of $A \sqcup B \to B \sqcup A$ that restricts on A to ϕ.*

Proof. Since $\phi(A)$ is clopen in B, we have $B \cong \phi(A) \sqcup C$ for some clopen subset C of B. We define

$$\tilde{\phi} : A \sqcup B \to B \sqcup A \cong \phi(A) \sqcup C \sqcup A$$

to be ϕ on A and a homeomorphism from B to $C \sqcup A$. □

Proof of Proposition 3.7. Take $A = Y[n]$, $B = Y[m]$. Then $Y[2m]$ contains a clopen copy of $B \sqcup A$. Now apply the lemma and extend the resulting homeomorphism by the identity to all of $Y[2m]$. □

In this setting, we say that a *brick*, $\mathcal{B}$, is an infinite union of the Y_k so that the complement in $X \setminus \{x\}$ is also an infinite union of Y_j. A *subbrick*, $\mathcal{C} \subset \mathcal{B}$, of a brick $\mathcal{B}$ is a brick $\mathcal{C}$ such that $\mathcal{B} \setminus \mathcal{C}$ is also a brick. Note that the furthermore statement of property (iii) from Proposition 3.2 implies that all bricks are homeomorphic. We will need the following proposition on the existence of shift maps on bricks.

Lemma 3.9. *If Ω is a stable neighborhood with respect to x, the Y_j are as in (3.1) and $\mathcal{B} \subset \Omega$ is a brick, then there is a homeomorphism $\sigma : \Omega \to \Omega$ fixing x such that the collection of $\sigma^i(\mathcal{B})$ are pairwise disjoint bricks for $i \in \mathbb{Z}$. Furthermore, the σ-orbit of every point in $\Omega - \{x\}$ accumulates to x.*

Proof. Using a bijection between $\mathbb{N}$ and $\mathbb{Z}^2$, write

$$\Omega \smallsetminus \{x\} = \bigsqcup_{\kappa \in \mathbb{Z}^2} Y_\kappa \quad \text{so that} \quad \mathcal{B} = \bigsqcup_{\kappa \in \mathbb{Z} \times \{0\}} Y_\kappa.$$

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Note that, for every $m \in Z$, $\bigsqcup_{\kappa\in\mathbb{Z}\times\{m\}} Y_\kappa$ is a brick. Since all bricks are homeomorphic, there exists a homeomorphism $\sigma : \Omega \to \Omega$ so that σ sends

$$\bigsqcup_{\kappa\in\mathbb{Z}\times\{m\}} Y_\kappa \quad \text{homeomorphically to} \quad \bigsqcup_{\kappa\in\mathbb{Z}\times\{m+1\}} Y_\kappa$$

for all $m \in \mathbb{Z}$. □

We now start the five-step program with the assumption that Ω is a stable neighborhood of the point $x \in \Omega$.

Standing Assumptions. We first set up some notation and standing assumptions. Unless explicitly stated, these will hold until the end of § 3.

(1) Denote by $G(\Omega)$ the subgroup of $G = \mathrm{Homeo}(X)$ consisting of homeomorphisms supported on Ω; thus $G(\Omega)$ can be naturally identified with $\mathrm{Homeo}(\Omega)$.
(2) Fix the decomposition $\Omega \setminus \{x\} = \bigsqcup_{k=1}^{\infty} Y_k$ from (3.1).
(3) Assume that $W < G$ is a symmetric set so that countably many translates cover G and W^2 is dense in $G(\Omega)$ (i.e. $\overline{W^2} \supset G(\Omega)$).
(4) For any brick, $\mathcal{B}$, we write $G(\mathcal{B})$ for the closed subgroup of homeomorphisms of Ω or X supported on $\mathcal{B} \cup \{x\}$.

3.1 Step 1: Fragmentation

We first fragment an element of $G(\Omega)$ into two maps that have fixed points in the orbit of x.

Lemma 3.10. *Any $h \in G(\Omega)$ can be written as $h = h_1 h_2$ so that each h_i belongs to $G(\Omega)$ and fixes some point in the orbit $G(\Omega) \cdot x$.*

Proof. Suppose $h(x) \neq x$ and assume $h^{-1}(x) \in Y_k$ for some k. By Lemma 3.9, the orbit of x accumulates on x. Hence, there exists $j \neq k$ so that Y_j intersects $G(\Omega) \cdot x$. Note that $h(Y_j)$ is disjoint from a neighborhood of x, so it is contained as a clopen subset of some finite union $Y_{k_1} \cup \cdots \cup Y_{k_r}$. Using Proposition 3.7, we can find a homeomorphism h_1 of X supported on only finitely many Y_s and agreeing with h on Y_j (to apply Proposition 3.7, first extend $h|_{Y_j}$ to $\bigsqcup_{i\leq j} Y_i$ by sending $\bigsqcup_{i<j} Y_i$ arbitrarily to a clopen set disjoint from $h(Y_j)$). Thus, $h = h_1 h_2$, where h_1 fixes x and $h_2 = h_1^{-1} h$ is the identity on Y_j and so fixes a point in $G \cdot x$. □

Note that Ω is a stable neighborhood with respect to any point in the $G(\Omega)$-orbit of x. Thus, the previous lemma allows us to reduce our problem to only considering maps in $\mathrm{Homeo}(\Omega, x)$. Next we fragment again, this time into bricks.

Lemma 3.11. *Any $f \in \mathrm{Homeo}(\Omega, x)$ can be written as $f = gh$ so that both $g, h \in \mathrm{Homeo}(\Omega, x)$ are supported on the closure of a brick.*

Proof. Choose indices n_k and m_k, where $m_k \in [n_{k-1} + 1, n_k - 1]$ inductively as follows. Let $n_0 = 0$ and $m_1 = 1$. Then, assuming $n_1, \ldots, n_{k-1}$ and $m_1, \ldots, m_k$ are given, choose $n_k > m_k$ large enough so that

$$f^{\pm 1}(Y_{m_k}) \subset \bigsqcup_{i=n_{k-1}+1}^{n_k - 1} Y_i$$

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and then choose m_{k+1} large enough so that

$$f^{\pm 1}(Y_{m_{k+1}}) \cap \Big(\bigsqcup_{i=1}^{n_k} Y_i \Big) = \emptyset.$$

Define

$$\mathcal{A}_m = \bigsqcup_{k=1}^{\infty} Y_{m_k} \quad \text{and} \quad \mathcal{A}_n = \bigsqcup_{k=1}^{\infty} Y_{n_k}.$$

Let $\mathcal{A}'_m$ and $\mathcal{A}'_n$ be complementary bricks to $\mathcal{A}_m$ and $\mathcal{A}_n$, respectively. Note that $f(\mathcal{A}_m)$ is disjoint from $\mathcal{A}_n$. Therefore, there is a homeomorphism $g : X \to X$ with support in $\mathcal{A}'_n$ such that $f|_{\mathcal{A}_m} = g|_{\mathcal{A}_m}$. In particular, $h = g^{-1}f$ preserves the brick $\mathcal{A}_m$. But g preserves the brick $\mathcal{A}_n$ and $f = gh$. This finishes the proof. □

Thus, combining these two lemmas, we see that any map $h \in G(\Omega)$ can be fragmented as $h = h_1h_2h_3h_4$, where each h_i is in $G(\Omega)$ and is supported on a brick (although not necessarily centered around the same point). Our new goal is to show that any map supported on a brick satisfies a Steinhaus condition.

3.2 Step 2: Finding commutators

Lemma 3.12. *Let $\mathcal{A}$ be a brick around x and $\mathcal{B}$ a subbrick of $\mathcal{A}$. For every $h \in G(\mathcal{B})$, there exists some $u, v \in G(\mathcal{A})$ so that $h = [u, v]$.*

Proof. We apply Lemma 3.9 to find a homeomorphism $v \in \mathrm{Homeo}(\Omega, x)$ supported on $\mathcal{A}$ so that $v^i(\mathcal{B})$ are pairwise disjoint for $i \in \mathbb{Z}$. Note that we can apply this lemma since any brick is itself homeomorphic to $\Omega \setminus \{x\}$. Now, let u be the map that is exactly a copy of h on each $v^i(\mathcal{B})$ for $i \geq 0$, i.e. $u = \prod_{i=0}^{\infty} vhv^{-1}$. Then we have $h = [u, v]$, as desired. □

3.3 Step 3: Diagonalization

Next we will find a 'good' subbrick. Namely, any map supported on this subbrick will be in W^8. First, we see that we can find a subbrick on which we can approximate maps. A similar step is carried out in [RS07, Ros08, Man16, Man24].

Lemma 3.13. *Let $\mathcal{A}$ be a brick and $\mathcal{A}_1, \mathcal{A}_2, \ldots$ pairwise disjoint subbricks. Then*

(i) *there is some $i \geq 1$ such that any homeomorphism of $\mathcal{A}_i$ extends to a homeomorphism of X supported on $\mathcal{A}$ that belongs to g_iW; and*

(ii) *moreover, every homeomorphism of $\mathcal{A}_i$ extends to a homeomorphism of X supported on $\mathcal{A}$ that belongs to W^2.*

Proof. We prove (i) by contradiction. Suppose, for every i, there is a homeomorphism h_i of $\mathcal{A}_i$ that does not extend to a homeomorphism of X supported on $\mathcal{A}$ that belongs to g_iW. Define h to be a homeomorphism of $\bigcup_i \mathcal{A}_i$ that agrees with h_i on $\mathcal{A}_i$. Extend h by the identity on $X \setminus \mathcal{A}$. Then $h \in g_iW$ for some i, contradicting the choice of h_i. For (ii), let $\tilde{h} \in G(\mathcal{A})$ be a homeomorphism in g_iW extending the given homeomorphism h of $\mathcal{A}_i$, and let $\tilde{f} \in g_iW \cap G(\mathcal{A})$ extend the identity on $\mathcal{A}_i$. Then $\tilde{f}^{-1}\tilde{h} \in W^2$ extends h. □

Next we upgrade this approximation.

Lemma 3.14. *Let $\mathcal{A}$ be a brick. Then there exists a subbrick $\mathcal{Z}$ of $\mathcal{A}$ so that $G(\mathcal{Z}) \subset W^8$.*

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Proof. Apply Lemma 3.13 to $\mathcal{A}$ and label the resulting subbrick $\mathcal{B}$. Repeat this process to $\mathcal{B}$ and label the subbrick $\mathcal{C}$. Let $\mathcal{Z}$ be any subbrick of $\mathcal{C}$. Now apply Lemma 3.12 to $\mathcal{Z}$ as a subbrick of $\mathcal{C}$. Therefore, for $h \in G(\mathcal{Z})$, there exists some $u, v \in G(\mathcal{C})$ so that $h = [u, v]$.

The support of u is contained in $\mathcal{C}$, and so we can apply Lemma 3.13 to find a $\bar{u}$ such that $\bar{u} \in W^2$, $\bar{u}$ is supported on $\mathcal{B}$, and $\bar{u}|_{\mathcal{C}} = u$. Similarly, the support of v is contained in $\mathcal{C} \subset \mathcal{B}$. Lemma 3.13 implies that there exists $\bar{v}$ such that $\bar{v} \in W^2$, $\bar{v}$ is supported on $\mathcal{A}$, and $\bar{v}|_{\mathcal{B}} = v$. In particular, $\bar{v}$ is the identity in $\mathcal{B} - \mathcal{C}$.

Since $\bar{u}$ is the identity in $X - \mathcal{B}$ and $\bar{v}$ is the identity in $\mathcal{B} - \mathcal{C}$, we have $[\bar{u}, \bar{v}]$ is the identity in $X - \mathcal{C}$. Thus, we have $[\bar{u}, \bar{v}] = [u, v] = h$ and hence $h \in W^8$. □

3.4 Step 4: Pigeonhole

In the previous step we found a 'good' subbrick. Next we will see that we can conjugate any map supported on any given brick by maps in W in order to have support on this good brick (up to a finite amount of error). This will allow us to prove that any map supported on a brick is in W^{24}.

Lemma 3.15. *Every $h \in \mathrm{Homeo}(\Omega, x)$ supported on a brick is in W^{24}.*

Proof. Suppose h is supported on a brick $\mathcal{A} = \bigsqcup_i A_i$. Let $\mathcal{Z}$ be as in Lemma 3.14 applied to the complementary brick to $\mathcal{A}$, so that $\mathcal{Z}$ and $\mathcal{A}$ are disjoint. Write $\mathcal{Z} = \bigsqcup_{i=1}^{\infty} \mathcal{Z}_i$, where each $\mathcal{Z}_i$ is itself a brick. Fix uncountably many infinite subsets $\Lambda_\alpha \subset \mathbb{N}$ that pairwise intersect in finite sets, and set $\mathcal{Z}_\alpha = \bigsqcup_{i \in \Lambda_\alpha} \mathcal{Z}_i$. Fix a homeomorphism $f_\alpha : X \to X$ that fixes x and interchanges $\mathcal{Z}_\alpha$ and its complement in $X \smallsetminus \{x\}$. By the pigeonhole principle, there exist $\alpha \neq \beta$ so that $f_\alpha, f_\beta \in g_i W$ for some i, and in particular $f_\beta^{-1} f_\alpha \in W^2$. We claim that $F := (f_\beta^{-1} f_\alpha) h (f_\beta^{-1} f_\alpha)^{-1}$ is supported on $\mathcal{Z}$ plus a finite number of the A_i.

To prove the claim, first note that the support of F is contained in $\bigcup_{i=1}^{\infty} f_\beta^{-1} f_\alpha(A_i)$. Recall that $\mathcal{A}$ is disjoint from $\mathcal{Z}$ and hence from $\mathcal{Z}_\alpha$, so $f_\alpha(A_i)$ is contained in $\mathcal{Z}_\alpha$ for all i. Furthermore, if i is sufficiently large, then $f_\alpha(A_i)$ will be disjoint from $\mathcal{Z}_\beta$, since $\mathcal{Z}_\alpha \cap \mathcal{Z}_\beta$ is disjoint from a neighborhood of x. It follows that, for all sufficiently large i, $f_\beta^{-1} f_\alpha(A_i) \subset Z_\beta \subset Z$, which proves the claim.

Thus, we can write $F = F_1 F_2$, where F_1 is supported on finitely many of the Y_k, $\bigsqcup_{k=1}^{n} Y_k$, and $F_2 \in G(\mathcal{Z})$. By Lemma 3.14 we have that $F_2 \in W^8$. It only remains to check that F_1 is in W^{12}. By the property (iii) of Proposition 3.2, there is a homeomorphism $g \in G(\Omega)$ that maps $\bigsqcup_{i=1}^{n} Y_i$ into $\mathcal{Z}$. Since W^2 is dense in $G(\Omega)$, we take $\tilde{g} \in W^2$ with this same property (this $\tilde{g}$ may not fix x and may not belong to $G(\Omega)$). Thus, we have that $g F_1 g^{-1} \in G(\mathcal{Z}) \subset W^8$ and therefore $F_1 \in W^{12}$.

Therefore, we have $F = (f_\beta^{-1} f_\alpha) h (f_\beta^{-1} f_\alpha)^{-1} \in W^{20}$ and so $h \in W^{24}$. □

3.5 Step 5: Wrapping up

We will first verify a version of the Steinhaus condition for stable neighborhoods.

Theorem 3.16. *Let X be a second-countable Stone space, Ω a stable neighborhood of $x \in X$, and $G = \mathrm{Homeo}(X)$. Suppose W is a symmetric set in G such that*

- *G is covered by countably many sets $g_i W$, $g_i \in G$, and*
- *W^2 is dense in $G(\Omega)$.*

Then $G(\Omega) \subseteq W^{96}$.

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Proof. Let $h \in G$. We first apply Lemmas 3.10 and 3.11 in order to write $h = h_1h_2h_3h_4$, where each $h_i \in G(\Omega)$ fixes a point $x_i \in G(\Omega) \cdot x$ so that Ω is telescoping with respect to x_i and the support of h_i is contained in a brick around x_i. Next, we apply Lemma 3.15 to see that each h_i is contained in W^{24}. We conclude that $h \in W^{96}$. □

Finally, we can upgrade this to the case that X is stable. We first need a short lemma on stable spaces.

Lemma 3.17. *Suppose X is stable. Then X can be written as a finite disjoint union of spaces, each of which is a stable neighborhood of one of its points.*

Proof. Using compactness, cover X by finitely many clopen sets $T_1, \dots, T_N$ so that each T_i is a stable neighborhood of a point $x_i \in T_i$, with all x_i distinct from each other. The sets T_i may not be disjoint. We perform the following operation to make them disjoint. Suppose $T_1 \cap T_2 \neq \emptyset$. If $x_1 \notin T_2$, then replace T_1 with $T_1 \smallsetminus T_2$; this space is also stable with respect to x_1, as stability passes to sub-neighborhoods. Similarly, if $x_2 \notin T_1$, we can replace T_2 by $T_2 \smallsetminus T_1$. Now suppose x_1, x_2 are both in $T_1 \cap T_2$. Write $T_1 \cap T_2 = A_1 \sqcup A_2$, where A_i are clopen and $x_i \in A_i$. Then replace T_1 with $T_1 \smallsetminus A_2$ and T_2 with $T_2 \smallsetminus A_1$. Again, both are stable neighborhoods of the same points. Thus, in all cases we replaced T_1, T_2 with two disjoint stable neighborhoods of the same points and with the same union as T_1 and T_2. Continuing in this way produces the desired partition. □

We will need a slight extension of Lemmas 3.13–3.15 and Theorem 3.16.

Suppose we have a finite collection of distinct points $x_i \in X$, $i = 1, \dots, N$, and pairwise disjoint clopen subsets Ω_i, $i = 1, \dots, N$, so that Ω_i is a stable neighborhood of x_i for all i. We will abbreviate this by $\vec{\Omega}$ and $\vec{x} \in \vec{\Omega}$. A *multibrick* $\vec{\mathcal{B}}$ is a collection of bricks $\mathcal{B}_i$ in each Ω_i centered at x_i. The group $\mathrm{Homeo}(\vec{\Omega})$ is the group of homeomorphisms of $\sqcup\Omega_i$ preserving each Ω_i, and $\mathrm{Homeo}(\vec{\Omega}, \vec{x})$ is the subgroup that in addition preserves each x_i.

Proposition 3.18. *Lemmas 3.13–3.15 and Theorem 3.16 hold when $G(\Omega)$, $\mathrm{Homeo}(\Omega, x)$ and bricks are replaced with $\mathrm{Homeo}(\vec{\Omega})$, $\mathrm{Homeo}(\vec{\Omega}, \vec{x})$ and multibricks.*

Proof. Proofs remain valid, *mutatis mutandis*. □

We are finally ready to prove the main theorem of this section.

Proof of Theorem 3.1. We verify the Steinhaus property. Let W be a symmetric set in $G = \mathrm{Homeo}(X)$ so that G is the union of countably many translates of W. By Lemma 2.3, there is an open neighborhood $\mathcal{U}$ of the identity in G so that W^2 is dense in it. There is a finite clopen partition $A_1 \sqcup \cdots \sqcup A_m$ of X so that any $g \in G$ that preserves each A_i belongs to $\mathcal{U}$. Now, applying Lemma 3.17 to each A_i produces a finer clopen partition $\Omega_1 \sqcup \cdots \sqcup \Omega_N$ of X, where each partition element is a stable neighborhood. This finer partition determines an open set $\mathcal{U}' \subset \mathcal{U}$. Every $g \in \mathcal{U}'$ thus restricts on each subset in the partition to a map on a stable neighborhood. We can apply the Proposition 3.18 version of Theorem 3.16 to see that $g \in W^{96}$. □

4. Surfaces

The goal of this section is to prove the classification theorem for stable surfaces as stated in the introduction (recalled here).

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Theorem A. *Let Σ be a stable, orientable surface without boundary. The homeomorphism group and mapping class group of Σ have automatic continuity if and only if every end of Σ is telescoping. That is, every end is one of the following:*

(i) *an isolated puncture;*
(ii) *of Cantor type;*
(iii) *not isolated in the space of ends accumulated by genus and is a successor with all predecessors of Cantor type.*

4.1 Background

We recall some terminology and structural lemmas on end spaces of surfaces introduced in [MR23]. For a surface Σ we denote by $E(\Sigma)$ the space of ends of Σ. This space is defined as

$$E(\Sigma) = \varprojlim_{K} \pi_0(\Sigma \setminus K),$$

where the inverse limit is taken over all connected, compact subsurfaces of Σ. We denote by

$$E_g(\Sigma) \subset E(\Sigma)$$

the subset of $E(\Sigma)$ consisting of non-planar ends. The end space is equipped with the inverse limit topology. This makes it into a second-countable Stone space and $E_g(\Sigma)$ into a closed subset. When the underlying surface is fixed, we will often shorten the pair $(E(\Sigma), E_g(\Sigma))$ to just $E(\Sigma)$.

Every homeomorphism of Σ induces a homeomorphism of $E(\Sigma)$ that preserves $E_g(\Sigma)$. Hence, when we talk about a homeomorphism between subsets of x, we always assume that our maps preserve the set $E_g(\Sigma)$. That is, we think of points in $E_g(\Sigma)$ as being colored following the convention of § 3. Furthermore, this pair $(E(\Sigma), E_g(\Sigma))$ effectively classifies infinite-type surfaces.

Theorem 4.1 [Ker23, Ric63]. *Let Σ be an orientable surface with finitely many boundary components. Then Σ is determined up to homeomorphism by the triple $(g, b, (E(\Sigma), E_g(\Sigma)))$, where $g \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the genus of Σ, $b \in \mathbb{Z}_{\geq 0}$ is the number of boundary components of Σ, and the pair $(E(\Sigma), E_g(\Sigma))$ is considered up to homeomorphism. Furthermore, the quotient map* $\mathrm{Homeo}(\Sigma) \to \mathrm{Homeo}(E(\Sigma), E_g(\Sigma))$ *is a continuous surjection.*

Given a clopen subset $U \subseteq (E(\Sigma), E_g(\Sigma))$, we will let Σ_U denote a connected subsurface of Σ, closed in Σ, with either infinite or zero genus so that $\partial\Sigma_U$ is a single simple closed curve and the space of ends of Σ_U, $E(\Sigma_U)$, is U. Given a neighborhood U of an end $x \in E$, we will say that Σ_U is a neighborhood of x in the surface Σ. Furthermore, if we have a pair $U \subset V$, we will assume that Σ_U and Σ_V are chosen so that $\Sigma_U \subset \Sigma_V$. Note that Σ_U has infinite genus if and only if $U \cap E_g(\Sigma) \neq \emptyset$.

Lemma 4.2. *Given subsurfaces Σ_U and Σ_V, we have Σ_U is homeomorphic to Σ_V if and only if U is homeomorphic to V.*

Proof. A homeomorphism between Σ_U and Σ_V induces a homeomorphism between their end spaces that preserves the sets of planar and non-planar ends. Hence, if Σ_U if homeomorphic to Σ_V, then U is homeomorphic to V.

In the other direction, as in § 3, any homeomorphism $\phi : U \to V$ is assumed to send the set $U \cap E_g(\Sigma)$ to $V \cap E_g(\Sigma)$. In particular, Σ_U has infinite genus (or genus zero) if and only if Σ_V does. Since they both have one boundary component, the same genus, and homeomorphic end spaces, they are homeomorphic by the classification of surfaces. □

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Definition 4.3 [MR23, Definition 4.14]. We say a surface Σ is *stable* if $E(\Sigma)$ is stable as in Definition 3.3. If U is a stable neighborhood of x in $E(\Sigma)$, then Σ_U is a *stable* neighborhood of x in Σ.

Definition 4.4 [MR23, Definition 4.1]. Let $\preceq$ be the preorder on $E(\Sigma)$ defined by $y \preceq x$ if, for every neighborhood U of x, there exists a neighborhood V of y and a homeomorphism $f \in \mathrm{Homeo}(S)$ so that $f(V) \subset U$. Write $x \sim y$ if $y \preceq x$ and $x \preceq y$. Recall that, by [MR24, Theorem 1.2], if $x \sim y$ then there is a homeomorphism $\phi : \Sigma \to \Sigma$ such that $\phi(x) = y$. Denote by $E(x)$ the equivalence class of x.

Definition 4.5. We say $y \prec x$ is $y \preceq x$ and $y \not\sim x$. If $y \not\preceq x$ and $x \not\preceq y$, we say x and y are *incomparable*. An end, x, is a *successor* if there exists finitely many incomparable ends $y_1, \ldots, y_n$, with $n \neq 0$, so that each $y_i \prec x$ and if $z \prec x$, then $z \preceq y_i$ for some i. Each y_i is a *predecessor* of x.

Abusing notation, we denote the partial order on the set of equivalence classes induced by $\preceq$ again by $\preceq$.

Proposition 4.6 [MR23, Proposition 4.7]. *The partial order $\preceq$ has maximal elements. Furthermore, for any maximal element x, $E(x)$ is either a finite set of points or a Cantor set.*

We say that an end y is of *Cantor type* if there exists some neighborhood U of y such that $E(y) \cap U$ is homeomorphic to a Cantor set. Note that $E(y)$ may not be a Cantor set if y is not maximal. For example, it is possible that only a finite number of points in $\overline{E}(y)$ are non-planar. However, note the following.

Lemma 4.7. *If Σ is stable, then, for every end y of Σ, the set $E(y)$ is locally closed. In particular, an end y is of Cantor type if and only if $E(y)$ is uncountable.*

Proof. Let V be a stable neighborhood of y. For every $z \in V$, $E(z)$ intersects every stable neighborhood of y, hence $z \preceq y$. This implies that either $z \sim y$ and $z \in E(y)$ or $y \not\preceq z$ and $z \notin \overline{E(y)}$. That is,

$$\overline{E(y)} \cap V = E(y) \cap V$$

and hence $E(y) \cap V$ is closed.

To see the second assertion, it is immediate from the definition that if y is of Cantor type, then $E(y)$ is uncountable. In the other direction, since $E(y)$ is uncountable and $E(\Sigma)$ is compact, there exists at least one point in $E(y)$ that is an accumulation point of $E(y)$. Moreover, because all stable neighborhoods of points in $E(y)$ are homeomorphic to one another, every point in $E(y) \cap V$ is an accumulation point of $E(y)$. This implies that $E(y) \cap V$ is a perfect set. Given that $E(\Sigma)$ is totally disconnected, $E(y) \cap V$ forms a Cantor set, which means that y is of Cantor type. □

Note that, for a maximal end x, saying x is of Cantor type is the same as saying $E(x)$ is a Cantor set. This is because, for every point $y \in \overline{E(x)}$, we have $x \preceq y$ and, since x is maximal, $y \preceq x$.

Remark 4.8. Here we describe an example to demonstrate that the assumption of stability is needed for Lemma 4.7. We will use the toolkit developed in [MR24, § 2]. Namely, we use the fact that there are surfaces D_n with one boundary component whose end space is a single point z_n such that these points are not *comparable*. That is, if $n \neq m$, the D_n surface does not include a neighborhood of z_m in D_m as a subsurface. We refer to the surfaces D_n as disks to emphasize that they have one boundary component.

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Armed with these, we construct our example as follows. Starting with a sphere minus a Cantor set, we picture S^2 as a one-point compactification of $\mathbb{R}^2$ and the Cantor set C as the standard 1/3-Cantor set in $\mathbb{R}^2$. That is, C is obtained from $[0,1]$ by removing 2^n open intervals of size $1/3^n$. We refer to these as the *missing intervals*.

We now *decorate* the Cantor set as follows. For every missing interval of size $1/3^n$, place $\mathbb{Z}$ copies of disks D_n. That is, remove a $\mathbb{Z}$-family of disks from $\mathbb{R}^2$ that accumulate to the end points of the missing interval and replace them with homeomorphic copies of D_n. Do this for every $n \geq 1$ and denote the resulting surface Σ.

Now, for every n, Σ has exactly 2^n ends that are accumulation points of $E(z_n)$. We denote this set by B_n. The union $B = \bigcup_n B_n$ is a countable and dense subset of the Cantor set C. We claim that all other points in the Cantor set are of the same type. That is, for $y \in C - B$, we have $E(y) = C - B$. To see this, we note that all intervals of type

$$\left[\frac{k}{3^n}, \frac{3k+1}{3^{n+1}}\right]$$

that are not contained in a missing interval are homeomorphic to each other by a horizontal translation and they are homeomorphic to intervals of type

$$\left[\frac{3k-1}{3^{n+1}}, \frac{k}{3^n}\right]$$

by a flip and a horizontal translation. We leave the details to the reader.

Now, for every $y \in C - B$, $E(y) = C - B$ is uncountable but it is not locally closed. Note that, for every n, we can choose a small enough neighborhood of y that does not intersect $E(z_n)$. Hence, the homeomorphism type of a small neighborhood of y does not stabilize and so Σ is not stable.

Proposition 4.9. *If Σ is a stable surface, then Σ has only finitely many maximal equivalence classes of ends.*

This proposition arises from the following. The version we have written here is not exactly the same as was written in [MR23], but follows directly from their statements.

Lemma 4.10 [MR23, Proposition 4.8 and Remark 4.15]. *If U is a stable neighborhood of x, then U has only a single maximal equivalence class of ends. Furthermore, x is a maximal end of U and is either the unique maximal end of U or is of Cantor type.*

Proof of Proposition 4.9. Let Σ be a stable surface. Using compactness of the end space of Σ, we can cover $E(\Sigma)$ by a finite collection of stable neighborhoods $U_1, \ldots, U_n$. Now, by the previous lemma, each U_i has only a single type of maximal end. Furthermore, each maximal end in $E(\Sigma)$ must appear as a maximal end of some U_i. Therefore, we see that n is an upper bound on the number of maximal equivalence classes of ends. □

Next, we recall the following.

Lemma 4.11 [MR23, Lemma 4.18]. *Let $x, y \in E(\Sigma)$ and assume x has a stable neighborhood V_x and that x is an accumulation point of $E(y)$. Then, for any sufficiently small clopen neighborhood U_y of y, $U_y \cup V_x$ is homeomorphic to V_x.*

We restate this lemma slightly.

Corollary 4.12. *Let Σ be a stable surface and $V \subset E(\Sigma)$ be a clopen set such that, for every $y \in (E \setminus V)$, $E(y)$ has an accumulation point in V. Then V is homeomorphic to $E(\Sigma)$.*

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Proof. For every $y \in E \setminus V$, let $x \in V$ be an accumulation point of $E(y)$. Let $V_x \subset V$ and $U_y \subset E \setminus V$ be as in Lemma 4.11. Then

$$V \cup U_y = (V \setminus V_x) \cup (V_x \cup U_y) \simeq (V \setminus V_x) \cup V_x \simeq V.$$

Since $E \setminus V$ is compact, it can be covered with finitely many such sets $U_{y_1}, \ldots, U_{y_k}$. Making these sets smaller, we can assume they are disjoint. Then, from the above argument, we have

$$E = (V \sqcup U_{y_1}) \sqcup U_{y_2} \sqcup \cdots \sqcup U_{y_k} \simeq (V \sqcup U_{y_2}) \sqcup U_{y_3} \sqcup \cdots \sqcup U_{y_k} \simeq \cdots \simeq (V \sqcup U_{y_k}) \simeq V.$$

This finishes the proof. □

4.2 Telescoping and drawtubes

From now on, we will always be working under the assumption that Σ is a stable surface.

Definition 4.13. We say that an end x of Σ is *telescoping* if it is one of the following:

(i) an isolated puncture;
(ii) of Cantor type;
(iii) not isolated in $E_g(\Sigma)$ and is a successor with all predecessors of Cantor type.

Definition 4.14. For an end x of Σ, a *tube around* x is a subsurface $A \subset \Sigma$ with $\partial A = \alpha_1 \sqcup \alpha_2$ such that each α_i is a single simple closed curve, α_1 is the boundary of Σ_{U_1} and α_2 is the boundary of Σ_{U_2} where $U_2 \subset U_1$ are nested stable neighborhoods of x.

Let U be a stable neighborhood of x, let A be a tube in the interior of Σ_U around x, and let A' be the component of $\Sigma_U - A$ that does not contain x. We say A is a *drawtube*[1] for the neighborhood U of x if both A and A' contain all types of ends in U other than possibly x itself. If x is an isolated puncture, then Σ_U is a punctured disk and every tube in Σ_U around x is a drawtube.

Now we check some basic properties of stable neighborhoods and drawtubes relative to a telescoping end, as well as prove an alternate characterization of telescoping.

Lemma 4.15. *Let x be a telescoping end of Σ and let U be a stable neighborhood of x. Then any two drawtubes in Σ_U around x are homeomorphic via a homeomorphism of Σ_U that fixes x.*

Proof. Let A and B be two drawtubes in Σ_U. Let A' be the component of $\Sigma_U - A$ that does not contain x and A'' be the component that does contain x. Let B' and B'' be similarly defined.

Let $V \subset U$ be a small neighborhood of x that is disjoint from $E(A)$ and $E(B)$. The end space $E(A')$ is a subset of the end space of $E(U - V)$. Every $y \in E(U - V)$ is either a predecessor of x or accumulates to a predecessor of x. Either way, $E(y)$ has an accumulation point in $E(A')$. Corollary 4.12 implies that $E(A')$ is homeomorphic to $E(U - V)$. Also, since x is a stable end, Σ_V is homeomorphic to $A \cup A''$. Therefore, there is a homeomorphism ϕ_A of Σ_U fixing x that sends $A \cup A''$ to Σ_V and A' to $\Sigma_U - \Sigma_V$. Similarly, there is a homeomorphism ϕ_B of Σ_U fixing x that sends $B \cup B''$ to Σ_V and B' to $\Sigma_U - \Sigma_V$. Then $\phi_B^{-1}\phi_A$ sends A' to B'.

Hence, we can assume $A' = B'$. Now, applying the same argument, we can find a homeomorphism ψ_A of $A \cup A''$ fixing x that sends A'' to Σ_V and a homeomorphism ψ_B of $B \cup B''$ fixing x that sends B'' to Σ_V. Then $\phi_B^{-1}\phi_A$ sends A to B and we are done. □

[1]The name drawtube comes from the tubes of a telescope that can slide within each other, either for assistance in focusing or in order to make the telescope more portable. Just as one would a retractable telescope, we will be sliding our drawtubes back and forth.

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Lemma 4.16. *Let x be an end and U a stable neighborhood of x. Then x is a telescoping end of Σ if and only if there exists a subdivision*

$$\Sigma_U = \bigcup_{i=0}^{\infty} Y_i$$

of Σ_U into tubes Y_i around x so that

(i) *$Y_i \cap Y_j = \emptyset$ if $|i-j| > 1$,*

(ii) *Y_i and Y_{i+1} intersect only along one of their boundary components, and*

(iii) *for $0 \le i \le j$, we have $\bigcup_{k=i}^{j} Y_k$ is homeomorphic to Y_0.*

In particular, each $E(Y_i)$ contains all types of ends in U other than possibly x itself, and for $i \ge 1$, Y_i is a drawtube.

Proof. Assume x is a telescoping end and let U be a stable neighborhood of x. Let

$$U = U_0 \supset U_1 \supset U_2 \supset \cdots$$

be a family of nested subsets of U that descends to x, $U_k \searrow x$. We immediately note that if x is an isolated puncture, then we can take the collection $\{Y_i\}$ to be nested tubes that descend to x. From now on, we assume that x is not an isolated puncture. First we show that, for k_1 large enough, $U - U_{k_1}$ contains all types of ends in U other than possibly x itself.

There are two cases to consider. If x is of Cantor type then, for every end $y \in U$, $E(y) \cap U$ accumulates to x and hence it accumulates to every point in $E(x) \cap U$. So, we only need to take k_1 large enough so that $(U - U_{k_1}) \cap E(x)$ is not empty and this ensures that $(U - U_{k_1})$ contains all types of ends. If x is not of Cantor type, then it has finitely many predecessors $y_1, \ldots, y_n$ and they are all of Cantor type. Then, for every end $z \in U$, $E(z)$ accumulates to some y_i and hence to all points in $E(y_i) \cap U$. Hence, if k_1 is large enough so that $(U - U_k) \cap E(y_i)$ is not empty for $i = 1, \ldots, n$, then $(U - U_k)$ contains all ends except possibly x itself.

We proceed in this way and choose indices $k_1, k_2, \ldots$ such that $U_{k_i} - U_{k_{i+1}}$ contains all types of ends in U other than possibly x itself. Let Σ_U be a stable neighborhood of x in Σ and denote the boundary of Σ_U by α_0. Choose a sequence of curves α_i in Σ_U that exit the end x such that the component of $\Sigma_U - \alpha_i$ that contains x has the end space U_{k_i}. Let Y_i be the tube bounded by α_i and α_{i+1}. Then $\Sigma_U = \bigcup_{i=0}^{\infty} Y_i$ and the first two assumptions of the lemma hold by construction.

To see that the final assumption holds, it is sufficient to show that $Y_0 \cup Y_1$ is homeomorphic to both Y_0 and Y_1. If Σ_U has infinite genus, then x is a non-planar end. Since x is not isolated in E_G, some ends in some Y_i (and hence every Y_i) have to be non-planar. Therefore, every Y_i has infinite genus. Otherwise, Σ_U has genus zero and hence every Y_i has genus zero. Hence, $Y_0 \cup Y_1$, Y_0 and Y_1 have the same genus. Also, they all have two boundary components. We only need to check that their end spaces are homeomorphic. But this follows from Corollary 4.12 applied to $E(Y_0) \subset E(Y_0 \cup Y_1)$ or to $E(Y_1) \subset E(Y_0 \cup Y_1)$. This finishes the proof of the one direction.

To see the other direction, let Σ_U be a stable neighborhood of x in Σ and let $\Sigma_U = \bigcup_{i=0}^{\infty} Y_i$ be a decomposition of Σ_U into tubes as described in the statement. We show that x is a telescoping end by checking the conditions of Definition 4.13.

If x is the only non-planar end in Σ_U, then there is a tube Y_k that has a finite non-zero genus. The Y_0 also has finite non-zero genus and hence Y_0 cannot be homeomorphic to $Y_0 \cup \cdots \cup Y_k$, which contradicts assumption (iii). Therefore, either x is a planar end, or it is not isolated in $E_G(\Sigma)$.

If x is an isolated puncture or is of Cantor type, then we are done by definition. Otherwise, we need to show that x is a successor and the predecessors are of Cantor type. By Lemma 4.9,

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$E(Y_0)$ has finitely many different maximal types, say $y_1, \dots, y_n$. Then every $z \in U$, $E(z) \cap E(Y_0)$ is non-empty and hence $E(z)$ accumulates to some y_i. This, by Definition 4.5, implies that x is a successor.

The assumption (iii) in particular implies that $E(Y_0)$ is homeomorphic to $E(Y_k)$ for every k, and in particular $E(y_i) \cap E(Y_k)$ is non-zero for every k. If $E(y_i) \cap E(Y_0)$ is finite, then $E(y_0) \cap (Y_0 \cup Y_1)$ is larger than $E(y_0) \cap Y_0$ and hence Y_0 cannot be homeomorphic to $Y_0 \cup Y_1$, which contradicts assumption (iii). If $E(y_i) \cap E(Y_0)$ is infinite then $E(y_i)$ has an accumulation point in $E(Y_0)$. But y_i is maximal in $E(Y_0)$, hence the accumulation point has to be in $E(y_i)$. Therefore, every point in $E(y_i) \cap E(Y_0)$ is an accumulation point of $E(y_i)$, and $E(y_i)$ is uncountable. Lemma 4.7 implies that y_i is of Cantor type. This holds for every $i = 1, \dots, n$, therefore x is a telescoping end. □

We call a decomposition as in this previous lemma a *telescoping decomposition* of the neighborhood Σ_U. We also give yet another characterization of telescoping in Lemma 4.35. We will not make use of this alternate characterization until § 4.9 and so do not include it here, but the curious reader may want to jump ahead.

Remark 4.17. The definition of a telescoping surface has appeared previously in the literature [MR23, Vla25]. The definition we give here can be thought of as a local version of these previous notions. In particular, we are referring to a telescoping *end* as opposed to a telescoping *surface*. As such, our definition only requires the decomposition, indexed by $\mathbb{N}$, of some stable neighborhood into homeomorphic pieces. Meanwhile, for a *surface* to be telescoping it must have a decomposition of the entire surface, indexed by $\mathbb{Z}$, and with respect to two fixed maximal ends. Since the automatic continuity property is closed under taking countable index sub- or super-groups, we must consider this local version of telescoping (see § 4.9). On the other hand, [MR23, Vla25] is concerned with the coarsely bounded and strong distortion properties, respectively. Both of these are *not* closed under taking countable index sub- or super-groups and hence one does not need to consider local telescoping properties.

We have now also seen that the definition of telescoping, Definition 4.13, fits into the same framework as Proposition 3.2 and Definition 3.3 from the end-space case. Thus, we can use this to define a brick in the surface setting.

Definition 4.18. A *brick* is a subsurface $\mathcal{B}$ of Σ_U that is a disjoint union of drawtubes so that all complementary components contain all types of ends in U except possibly x.

For example, fixing a tubular decomposition $\Sigma_U = \bigcup_{i=0} Y_i$, the union $\cup Y_{n_i}$ is a brick for any infinite and co-infinite subset $\{n_i\} \subset \{1, 2, \dots\}$ (it is important not to include Y_0 since Y_0 is not a drawtube). From Lemma 4.15 we see that for any two bricks there is a homeomorphism of Σ_U taking one to the other. In particular, any brick can be enlarged by inserting drawtubes into the complementary components, including the complementary component that contains $\partial\Sigma_U$, and therefore we could choose a homeomorphism between two bricks to fix a neighborhood of $\partial\Sigma_U$ that contains a drawtube.

Standing Assumptions. As in the end-space case, we now set up some notation and standing assumptions. Unless explicitly stated, these will hold until the end of § 4.7.

(1) If A is a tube or a finite-type subsurface of Σ, we will use $G(A)$ to denote the subgroup of $G = \mathrm{Homeo}(\Sigma)$ consisting of maps supported on the interior of A. If U is a stable neighborhood of a telescoping end x, then $G(\Sigma_U)$ is the set of homeomorphisms supported on some $\Sigma_{U'}$ so that $U \smallsetminus U'$ contains all types of ends in U except possibly x.

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(2) Fix a telescoping end x and $\Sigma_U = \bigcup_{i=0}^{\infty} Y_i$ a subdivision as in Lemma 4.16.
(3) Assume that $W \subset G$ is a symmetric set so that G is covered by countably many sets $g_i W$ with $g_i \in G$ and W^2 is dense in $G(\Sigma_U)$ (i.e. $\overline{W^2} \supset G(\Sigma_U)$).
(4) For any brick, $\mathcal{B}$, we write $G(\mathcal{B})$ for the closed subgroup of homeomorphisms of Σ_U or Σ supported on $\mathcal{B}$.

4.3 Step 1: Fragmentation

Lemma 4.19. *Any $h \in G(\Sigma_U)$ can be written as $h = h_1 h_2$ so that each h_i belongs to $G(\Sigma_U)$ and fixes some point in $G(\Sigma_U) \cdot x$.*

Proof. Since $h \in G(\Sigma_U)$, by definition there is a smaller stable neighborhood $\Sigma_{U'}$ so that h is supported in $\Sigma_{U'}$ and $U \smallsetminus U'$ contains all types of ends as U except possibly x. We can modify the subdivision Y_i so that $\Sigma_{U'} = \bigcup_{i=1}^{\infty} Y_i$ (e.g. choose a homeomorphism $\Sigma_U \to \Sigma_{U'}$ fixing x and let the new Y_i be the image of the (old) Y_{i-1} and set $Y_0 = U \setminus U'$). Suppose $h(x) \neq x$ and assume $h^{-1}(x) \in Y_i$ for some $i > 0$. Choose some $j \neq i$, $j > 0$ and note that $h(Y_j)$ is disjoint from a neighborhood of x, so it is contained in some finite union $Y_{k_1} \cup \cdots \cup Y_{k_r}$. Using the classification of surfaces we can find a homeomorphism $h_1 \in G(\Sigma_U)$ supported on only finitely many of the tubes including Y_0 and agreeing with h on Y_j. Thus, $h = h_1 h_2$, where h_1 fixes x and $h_2 = h_1^{-1} h$ is the identity on Y_0 and Y_j. Finally, we note that since Y_i and Y_j are both drawtubes, they are homeomorphic, and so Y_j contains an end in $G(\Sigma_U) \cdot x$ and h_2 fixes this end. □

Again, note that if U is a stable neighborhood of an end x, then U is also a stable neighborhood of $g \cdot x$ for all $g \in G(\Sigma_U)$. Keeping this in mind, the statements that follow will apply when one replaces the telescoping end x with an end of the form $g \cdot x$ for $g \in G(\Sigma_U)$. We will slightly abuse notation and not keep track of this distinction.

Lemma 4.20. *Any $h \in G(\Sigma_U)$ that fixes x can be written as $h = h_1 h_2$ so that each h_i is supported on a brick.*

Proof. Given h, choose two sequences (n_i) and (m_i) so that $1 = n_1 < m_1 < n_2 < m_2 < \cdots$, $h(Y_1 \cup Y_2 \cup \cdots \cup Y_{n_i})$ is disjoint from Y_{m_i} and $h(Y_{n_{i+1}})$ is disjoint from Y_{m_i}. Then define h_1 to agree with h on each Y_{n_i} and to be the identity on Y_{m_i} for all i. This can be done by the classification of surfaces. Then h_1 and $h_2 = h_1^{-1} h$ are both supported on bricks of Σ_U. □

Just as in the end-space case, these two lemmas allow us to write any $h \in G(\Sigma_U)$ as $h = h_1 h_2 h_3 h_4$ with each $h_i \in G(\Sigma_U)$ supported on a brick limiting to a fixed point in the orbit of x.

4.4 Step 2: Finding commutators

For surfaces, this step will be drastically different than in the end-space case. For end spaces, since we were dealing with totally disconnected spaces, we could use maps that shift an entire brick and restrict to the identity elsewhere. Now, the topology of the surface obstructs using such a shift. Instead, we will fragment our map again using an Eilenberg–Mazur swindle [Bas63, Maz59] into two maps, still supported on bricks, that are themselves commutators.

Definition 4.21. Let A be a drawtube in Σ_U. We say that $f \in G(A)$ has *alternating support* if $\mathrm{supp}(f) = A_1 \sqcup A_2$, with each A_i a drawtube and with all three complementary components of $A_1 \sqcup A_2$ in A drawtubes, and so that there exists $h \in G(A)$ such that $h(A_1) = A_2$ and $f|_{A_2} = h(f|_{A_1})^{-1} h^{-1}$. If $\mathcal{A}$ is a brick, then we say that $f \in G(\mathcal{A})$ has *alternating support* if $f|_A$ has alternating support for each tube component $A \in \mathcal{A}$.

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Lemma 4.22. *If $f \in G(A)$ has alternating support, then f can be written as a single commutator of maps in $G(A)$. Similarly, if $f \in G(\mathcal{A})$, for $\mathcal{A}$ a brick, has alternating support, then f can be written as a single commutator of maps in $G(\mathcal{A})$.*

Proof. Let h be as in the definition of alternating support. Let f_1 denote the homeomorphism that agrees with f on A_1 and is the identity outside of A_1. Then we can write

$$f = f_1 h (f_1)^{-1} h^{-1} = [f_1, h].$$

If $\mathcal{A}$ is a brick, then we can simply apply this to each tube of $\mathcal{A}$ and collect terms since the tubes are all pairwise disjoint □

Lemma 4.23 (Eilenberg–Mazur swindle). *Let $f \in G(\mathcal{A})$ for $\mathcal{A}$ a brick of Σ_U. There exist two bricks, $\mathcal{A}_1$ and $\mathcal{A}_2$ in Σ_U, and $h_i \in G(\mathcal{A}_i)$ so that h_i has alternating support for $i = 1, 2$ and $f = h_1 h_2$.*

Proof. The brick $\mathcal{A}$ is the disjoint union of drawtubes ordered linearly, which we write as

$$A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots,$$

with A_{i+1} separating A_i from x for $i > 0$. Denote by $f_j = f|A_j$ the restriction of f to A_j. We will also use f_j to denote the extension by the identity to the rest of Σ_U. Thus, we can write f as the infinite composition $f_1 \circ f_2 \circ \cdots$. We will denote by B_0 the closure of the complementary component of $\mathcal{A}$ in Σ_U that contains the boundary component, and we will denote by B_i the drawtube that shares a boundary component with both A_i and A_{i+1} for $i > 0$. Thus,

$$\Sigma_U = B_0 \cup A_1 \cup B_1 \cup A_2 \cup B_2 \cup \cdots,$$

respecting the linear order.

Next, we subdivide each B_i, $i > 0$, into $4i + 1$ drawtubes, writing in linear order

$$B_i = C_{i,1} \cup C_{i,2} \cup \cdots \cup C_{i,2i} \cup D_i \cup E_{i,1} \cup E_{i,2} \cup \cdots \cup E_{i,2i}.$$

We will also fix a homeomorphism between B_0 and every drawtube A we named so far, so that the boundary curve of Σ_U is mapped to the boundary component of A not separated from x by the other boundary component of A. In particular, we have preferred homeomorphisms between any two tubes above by composing the homeomorphisms with B_0.

Now define a new homeomorphism $h_1 : \Sigma_U \to \Sigma_U$ as follows:

- $h_1|A_i = f_i$;
- $h_1|B_0 = \mathrm{id}$;
- $h_1|D_i = \mathrm{id}$ for $i > 0$;
- $h_1|E_{i,2j-1} = \phi_{i,2j-1} f_j \phi_{i,2j-1}^{-1}$ for $j = 1, \ldots, i$, where $\phi_{i,2j-1} : A_j \to E_{i,2j-1}$ is the preferred homeomorphism;
- $h_1|C_{i,2j} = \psi_{i,2j} f_j^{-1} \psi_{i,2j}^{-1}$ for $j = 1, \ldots, i$, where $\psi_{i,2j} : A_j \to E_{i,2j}$ is the preferred homeomorphism;
- $h_1|C_{i,2j-1} = \mathrm{id}$, $h_1|E_{i,2j} = \mathrm{id}$, for $j = 1, \ldots, i$.

We also define $h_2 : \Sigma_U \to \Sigma_U$ in the same way, except that $h_2|A_i = \mathrm{id}$ for $i > 0$. Thus, $f = h_1 h_2^{-1}$ and it remains to argue that h_1, h_2 have alternating support on suitable bricks.

We start with h_2. We consider the supporting brick $B_1 \sqcup B_2 \sqcup \cdots$ and show that $h_2|B_i$ has alternating support for every i. Indeed, write B_i as a subdivision into five drawtubes:

$$B_i = C_{i,1} \cup (C_{i,2} \cup C_{i,3} \cup \cdots \cup C_{i,2i}) \cup D_i \cup (E_{i,1} \cup E_{i,2} \cup \cdots \cup E_{i,2i-1}) \cup E_{i,2i}.$$

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Then $h_2|B_i$ is the identity on the first, third and fifth of these subtubes. Let g_i be a homeomorphism of B_i that restricts to the preferred homeomorphisms $C_{i,2j} \to E_{i,2j-1}$, $j = 1, 2, \dots, i$. This homeomorphism exists by the classification of surfaces, as in the proof of Lemma 4.15, and it conjugates h_2 restricted to the second tube to the inverse of the restriction on the fourth tube, as required.

Now consider h_1. It will be convenient to subdivide each D_i into three drawtubes as $D_i = D_i^1 \cup D_i^2 \cup D_i^3$. We consider the supporting brick, which is the complement of $\bigcup_{i=1}^{\infty} D_i^2$ and a subtube of B_0 containing $\partial\Sigma_U$. Thus, a tube of this brick has the form

$$D_i^3 \cup (E_{i,1} \cup E_{i,2} \cup \dots \cup E_{i,2i} \cup A_{i+1}) \cup C_{i+1,1} \cup (C_{i+1,2} \cup \dots \cup C_{i+1,2i+2}) \cup D_{i+1}^1$$

and is written as the union of five tubes (the very first tube in the brick is slightly different and its consideration is left to the reader). Note that h_1 is the identity on the first, third and fifth tubes. As in the case of h_2, we can construct a homeomorphism k_i of this tube that restricts to preferred homeomorphisms $E_{i,2j-1} \to C_{i+1,2j}$ for $j = 1, 2, \dots, i$, and $A_{i+1} \to C_{i+1,2i+2}$. Then k_i will conjugate the restriction of h_1 to the second tube to the inverse of its restriction on the fourth tube.

The following schematic summarizes the construction. The given map f can be represented as

$$\mathbf{1}\quad \mathbf{2}\quad \mathbf{3}\quad \mathbf{4}\quad \cdots,$$

where we only keep indices from various f_i. The map h_1 can then be represented as

$$\mathbf{1}\overline{1}\,1\mathbf{2}\overline{1\,2}\,12\mathbf{3}\overline{1\,2\,3}\,123\mathbf{4}\overline{1\,2\,3\,4}\cdots,$$

where italicized numbers refer to the restriction to subdivision tubes, and number i means that the map is conjugate to f_i, while the bar denotes the inverse. Between any two drawtubes with consecutive numbers, there is a drawtube on which the map is the identity, giving a buffer. Each block of italicized numbers between two bold numbers defines a map on a single drawtube with alternating supports:

$$\mathbf{1}\,\overbrace{\overline{1}\,1}\,\mathbf{2}\,\overbrace{\overline{1\,2}\,12}\,\mathbf{3}\,\overbrace{\overline{1\,2\,3}\,123}\,\mathbf{4}\,\overbrace{\overline{1\,2\,3\,4}\,1234}\cdots.$$

Therefore, h_2 has alternating supports.

But so does h_1:

$$\underbrace{\mathbf{1}\overline{1}}\,\underbrace{1\mathbf{2}\overline{1\,2}}\,\underbrace{12\mathbf{3}\overline{1\,2\,3}}\,\underbrace{123\mathbf{4}\overline{1\,2\,3\,4}}\cdots.$$

Here we get alternating supports corresponding to the groups of numbers indicated after adding a drawtube on each side of the group (and using the definition of a brick that guarantees that there is a drawtube to the left of the first group). □

The culmination of this step is to combine these two lemmas in order to write a map supported on a brick as a product of two commutators, each of which are also supported on the brick.

Lemma 4.24. *Any $h \in G(\Sigma_U)$ supported on a brick can be written as $h = h_1 h_2$ so that each $h_i \in G(\Sigma_U)$ is supported on a brick $\mathcal{A}_i$ in Σ_U and can be written as a single commutator in $G(\mathcal{A}_i)$.*

4.5 Step 3: Diagonalization

Again, we next want to find a 'good' brick. The first lemma is identical to the end-space case.

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Lemma 4.25. *Let $\mathcal{A}$ be a brick and $\mathcal{A}_1, \mathcal{A}_2, \ldots$ pairwise disjoint subbricks. Then*

(i) *there is some $i \geq 1$ such that any homeomorphism of $\mathcal{A}_i$ extends to a homeomorphism of Σ supported on $\mathcal{A}$ that belongs to $g_i W$, and*

(ii) *moreover, every homeomorphism of $\mathcal{A}_i$ extends to a homeomorphism of Σ supported on $\mathcal{A}$ that belongs to W^2.*

Proof. This is exactly the same proof as the proof of Lemma 3.13. □

Next we upgrade this approximation. Notably, here we will only be doing it for maps with alternating support.

Lemma 4.26. *Let $\mathcal{A}$ be a brick. Then there exists a subbrick $\mathcal{Z}$ of $\mathcal{A}$ so that any homeomorphism with alternating support on $\mathcal{Z}$ is in W^8.*

Proof. Apply the previous lemma to $\mathcal{A}$ (and an arbitrary pairwise disjoint collection of subbricks) and label the resulting subbrick $\mathcal{B}$. Repeat this process for $\mathcal{B}$ and label the subbrick $\mathcal{Z}$. Let $f \in G(\mathcal{Z})$ have alternating support. Then, by Lemma 4.22, $f = [u, v]$ for $u, v \in G(\mathcal{Z})$. The support of u is contained in $\mathcal{Z}$ and so we can apply the previous lemma to find $\bar{u}$ such that $\bar{u} \in W^2$, $\bar{u}$ is supported on $\mathcal{B}$, and $\bar{u}|_{\mathcal{Z}} = u$. Similarly, the support of v is contained in $\mathcal{B}$ and so we can find $\bar{v}$ such that $\bar{v} \in W^2$, $\bar{v}$ is supported on $\mathcal{A}$, and $\bar{v}|_{\mathcal{B}} = v$. Thus, we have $h = [\bar{u}, \bar{v}] \in W^8$. □

4.6 Step 4: Pigeonhole

Next we see that, except for a finite error, we can push any brick into a specific subbrick.

Lemma 4.27. *For any brick $\mathcal{A}$ and subbrick $\mathcal{B}$, there exists $a_1, a_2 \in W^2$ so that for all but finitely many tubes A in $\mathcal{A}$, either $a_1 A \in \mathcal{B}$ or $a_2 A \in \mathcal{B}$.*

Proof. Write $\mathcal{B} = \bigsqcup_{i=1}^{\infty} B_i$, with each B_i a drawtube. Let Λ_α, $\alpha \in \mathbb{R}$, be an uncountable collection of infinite subsets of $\mathbb{N} \setminus \{1\}$ so that if $\alpha \neq \beta$, then $\Lambda_\alpha \cap \Lambda_\beta$ is finite. Write $\mathcal{B}_\alpha$ for the subbrick of $\mathcal{B}$ consisting of tubes indexed by Λ_α. Then, for every α, choose a homeomorphism $f_\alpha \in G(\Sigma_U)$ with the following property. Say $\Lambda_\alpha = \{n_1, n_2, n_3, \ldots\}$ with $1 < n_1 < n_2 < n_3 < \cdots$. Then f_α sends B_{n_i} homeomorphically onto the drawtube cobounded by B_{n_i} and $B_{n_{i+1}}$, and this component onto $B_{n_{i+1}}$. Note that these maps exist by Lemma 4.15.

Since the collection $\{\Lambda_\alpha\}_{\alpha \in \mathbb{R}}$ is uncountable, there exist some pair $\alpha \neq \beta$ so that f_α and f_β are in the same left translate of W. See Figure 3 for a schematic of the arrangement of these tubes and the maps f_α and f_β. Thus, we have that $f_\alpha^{-1} f_\beta$ and $f_\beta^{-1} f_\alpha$ are both in W^2. Set $a_1 = f_\beta^{-1} f_\alpha$ and $a_2 = f_\alpha^{-1} f_\beta$. Let $\mathcal{A}_0 = \mathcal{B}_\alpha \cap \mathcal{B}_\beta$, note that this set is finite, let $\mathcal{A}_1 = \mathcal{A} \setminus \mathcal{B}_\alpha$ and $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{B}_\beta$. Note that $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$. Now, given any tube A in $\mathcal{A}$ that is not in the finite exceptional set $\mathcal{A}_0$, we have that either $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$. If $A \in \mathcal{A}_1$, then $a_1 A \in \mathcal{B}_\beta \subset \mathcal{B}$ and if $A \in \mathcal{A}_2$, then $a_2 A \in \mathcal{B}_\alpha \subset \mathcal{B}$. □

Finally, we can put all of these pieces together to see that any map supported on a brick is in W^{72}.

Lemma 4.28. *Every $h \in G(\Sigma_U)$ supported on a brick is in W^{72}.*

Proof. First apply Lemma 4.23 to h to find $h_1, h_2 \in G(\Sigma_U)$ with support on bricks $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Abusing notation, we replace h with one of the h_i and will show that such a map is in W^{36}.

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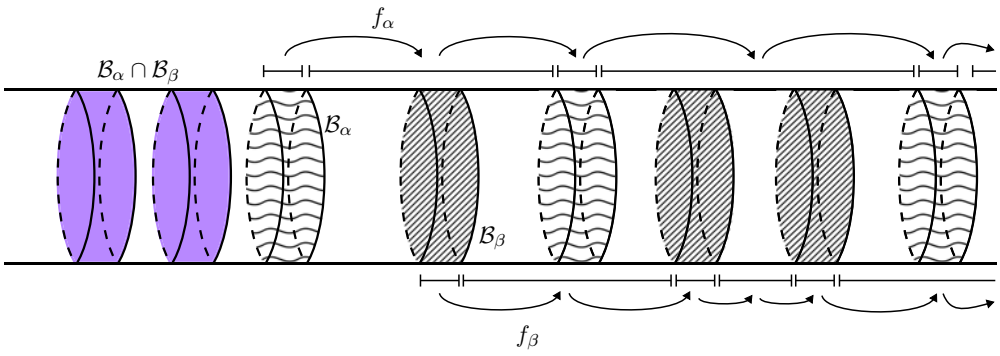


Figure 3 (colour online). A schematic of the pigeonhole argument. The initial two tubes represent the finitely many tubes in $\mathcal{B}_\alpha \cap \mathcal{B}_\beta$. The wavy-lined and straight-lined tubes are the tubes of $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$, respectively.

Apply Lemma 4.26 to $\mathcal{A}$ to find $\mathcal{Z}$ so that any map in $G(\mathcal{Z})$ with alternating support is in W^8. Next, apply Lemma 4.27 to $\mathcal{Z}$ as a subbrick of $\mathcal{A}$ to obtain maps $a_1, a_2 \in W^2$. Write $\mathcal{A} = F \sqcup X \sqcup Y$, where F is the finite collection of tubes for which Lemma 4.27 fails, X is the set of tubes for which $a_1 A \in \mathcal{Z}$ for all $A \in X$, and Y is the complement of these two sets. We can write $h = h_F h_X h_Y$ for the restrictions of h to each of these sets. Note that $a_1 h_X a_1^{-1}$ and $a_2 h_Y a_2^{-1}$ are two maps with alternating support in $\mathcal{Z}$ and therefore $h_X, h_Y \in W^{12}$.

It remains to deal with h_F. Let $g \in G(\Sigma_U)$ be a map that sends the set F of tubes into $\mathcal{Z}$. By assumption, W^2 is dense in $G(\Sigma_{U'})$ and so we may approximate g by $g' \in W^2$ (which may not belong to $G(\Sigma_U)$ or even be an identity outside of Σ_U). Thus, we see that $g' h_F g'^{-1}$ is again a map with alternating support in $\mathcal{Z}$ and hence $h_F \in W^{12}$. We conclude that $h \in W^{36}$ and thus our original map is contained in W^{72}. □

4.7 Step 5: Wrapping up

We again first verify a version of the Steinhaus condition for a stable neighborhood of a telescoping end.

Theorem 4.29. *Let Σ_U be a stable neighborhood of a telescoping end x. Suppose W is a symmetric set in $G = \mathrm{Homeo}(\Sigma)$ such that*

- *$\mathrm{Homeo}(\Sigma)$ is covered by countably many sets $g_i W$ with $g_i \in \mathrm{Homeo}(\Sigma)$, and*
- *W^2 is dense in $G(\Sigma_U)$.*

Then $G(\Sigma_U) \subset W^{288}$.

Proof. Let $h \in G(\Sigma_U)$. We first apply Lemmas 4.19 and 4.20 in order to write $h = h_1 h_2 h_3 h_4$, where each $h_i \in G(\Sigma_U)$ fixes an end $x_i \in G(\Sigma_U) \cdot x$ so that Σ_U is stable with respect to x_i and the support of h_i is contained in a brick around x_i. Next, we apply Lemma 4.28 to see that each h_i is contained in W^{72}. We conclude that $h \in W^{288}$. □

Remark 4.30. Let $D \subset \Sigma$ be a closed disk. By considering a fixed point near the boundary as a marked point, we can also obtain a version of the previous theorem for $G(D)$. That is, under the same assumptions, we have that $G(D) \subset W^{288}$.

As in the end-space case, we technically need a slight extension of all the results in §§ 4.4–4.6 and the above theorem. See Proposition 3.18 and the discussion immediately preceding it. In

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other words, we can run the previous proofs simultaneously on any finite collection of pairwise disjoint stable neighborhoods of telescoping ends.

Proposition 4.31. *Suppose $\{U_i\}$ is a finite collection of pairwise disjoint stable neighborhoods of telescoping ends x_i, and suppose Σ_{U_i} is a pairwise disjoint collection of corresponding subsurfaces. Let $W \subset \mathrm{Homeo}(\Sigma)$ be a symmetric set, so that countably many translates cover $\mathrm{Homeo}(\Sigma)$, and so that W^2 is dense in each $G(\Sigma_{U_i})$. Then, any homeomorphism h of Σ which is an identity outside of $\bigcup_i \Sigma_{U_i}$ with restrictions to each Σ_{U_i} belonging to $G(\Sigma_{U_i})$ is in W^{288}.*

We are finally ready to prove one implication in Theorem A.

Theorem 4.32. *Let Σ be a stable surface. If every end of Σ is telescoping, then AC holds for $\mathrm{Homeo}(\Sigma)$.*

Proof. We will verify the Steinhaus property. Let W be a symmetric subset of $\mathrm{Homeo}(\Sigma)$ so that countably many translates cover the whole group. We will show that W^{4896} contains a neighborhood of the identity. We can apply Lemma 2.3 to see that there exists some neighborhood of the identity $\mathcal{U} \subset \mathrm{Homeo}(\Sigma)$ so that W^2 is dense in $\mathcal{U}$. The plan is to cover the end space by surfaces of the form Σ_U, given $h \in \mathrm{Homeo}(\Sigma)$ close to the identity, represented as a composition of maps supported on the Σ_U (or compact subsurfaces), and apply Proposition 4.31. The difficulty is that homeomorphisms in Σ_U require an extra drawtube contained in Σ_U on which the map is the identity, and the realizations of these drawtubes have to be pairwise disjoint, for otherwise we cannot work simultaneously on all of them.

Step 1. Here we construct a collection of pairwise disjoint Σ_{U_i} that cover the end space. We temporarily work in the Fruedenthal compactification of Σ, which we denote by $\hat{\Sigma}$. Note that $\mathrm{Homeo}(\Sigma) = \mathrm{Homeo}(\hat{\Sigma}, E(\Sigma), E_g(\Sigma))$, the group of homeomorphisms that fix the ends of Σ, setwise. Fix a metric on $\hat{\Sigma}$. There will be $\epsilon > 0$ so that if a homeomorphism moves points $< \epsilon$ it belongs to $\mathcal{U}$. Using Lemma 3.17, we can cover $E(\Sigma)$ by a finite collection U_i of pairwise disjoint clopen sets so that each U_i is a stable neighborhood of one of its points x_i. Furthermore, we can construct pairwise disjoint subsurfaces Σ_{U_i} with one boundary component γ_i with end space U_i. We can also choose these sets and subsurfaces so that they have diameter $< \epsilon$. We now decompose the given homeomorphism h, assumed close to the identity, into three homeomorphisms, $h = h_1h_2h_3$, so that each is close to the identity (i.e. for every neighborhood $\mathcal{V}$ of the identity, there is a neighborhood $\mathcal{V}'$ so that if $h \in \mathcal{V}'$, then $h_1, h_2, h_3 \in \mathcal{V}$). The simple closed curves γ_i are moved only slightly by h, so we can choose h_3 to be supported on pairwise disjoint annular neighborhoods of γ_i and so that hh_3^{-1} fixes each γ_i pointwise. Then hh_3^{-1} is the composition h_1h_2, where h_1 is a homeomorphism of the compact surface bounded by the γ_i and it is close to the identity, and h_2 is supported on the union of Σ_{U_i}. We can then use the Edwards–Kirby fragmentation [EK71] and Remark 4.30 (performed simultaneously) to see that both $h_1, h_3 \in W^{864}$ (here we need $864 = 3 \times 288$ because the fragmentation produces three families of pairwise disjoint disks). See also [Man16, Ros08]. We are thus reduced to working with h_2 and we rename it h.

Step 2. Write $\Sigma_{U_i} = \bigcup_{j=0}^{\infty} Y_j^i$ as a tubular decomposition and let the boundary of the first tube Y_0^i be $\gamma_i \sqcup \delta_i$. Exactly as in Step 1, we can write $h = h_1h_2h_3$, where h_3 is supported in a neighborhood of the δ_i, h_1 is supported on the tubes Y_0^i, and h_2 is supported on the stable neighborhoods $\bigcup_{j=1}^{\infty} Y_j^i$. In particular, $h_3 \in W^{864}$ as before. Note that $h_2|\Sigma_{U_i}$ belongs to $G(\Sigma_{U_i})$, since we arranged that it is the identity on the first tube Y_0^i. Thus, $h_2 \in W^{288}$ by Proposition 4.31. It remains to deal with h_1, which we rename h.

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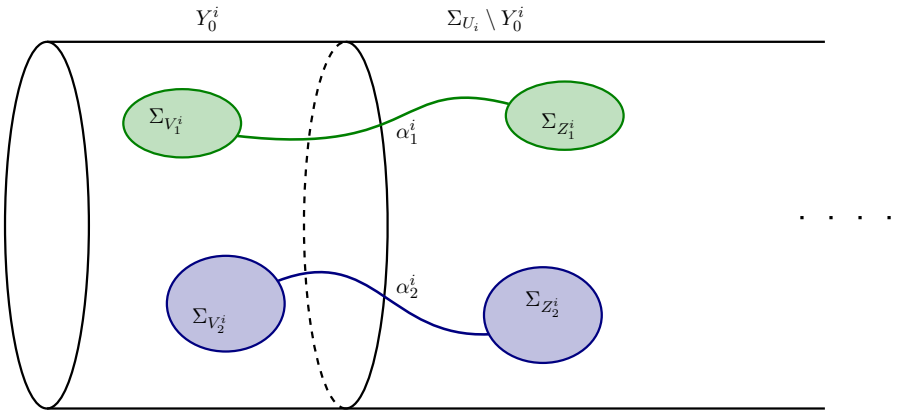


FIGURE 4 (colour online). An example of how to disjointly extend stable neighborhoods in Step 3.

Step 3. We now assume h is supported on $\cup Y_0^i$. First, repeat Step 1 for each tube Y_0^i and write $h = h_1 h_2 h_3$ with $h_1, h_3 \in W^{864}$ and with h_2 supported on a finite collection of pairwise disjoint stable neighborhoods $\Sigma_{V_j^i} \subset Y_0^i$, and in fact we may assume that h_2 is the identity in a neighborhood of each boundary $\partial\Sigma_{V_j^i}$, which we call γ_j^i. Again, rename $h = h_2$. Choose pairwise disjoint $Z_j^i \subset U_i$ near x_i (and so away from the ends of Y_0^i) homeomorphic to V_j^i. Let $\Sigma_{Z_j^i}$ be the corresponding surfaces and arrange that they are pairwise disjoint, disjoint from all the tubes in the support of h, and $\Sigma_{Z_j^i} \subset (\Sigma_{U_i} \setminus Y_0^i)$. Choose pairwise disjoint arcs α_j^i connecting γ_j^i with $\partial\Sigma_{Z_j^i}$, intersecting the two curves only at the endpoints, and disjoint from all other such curves. Also, ensure that $\alpha_j^i \subset \Sigma_{U_i}$. Use α_j^i to boundary-connect sum the surface $\Sigma_{V_j^i}$ cut off by γ_j^i with $\Sigma_{Z_j^i}$. See Figure 4 for a schematic of this arrangement. The new surface is also a stable neighborhood of the same end as $\Sigma_{V_j^i}$, but we now added 'luft', i.e. a tube where h is the identity, and now we can apply Proposition 4.31 to conclude that $h \in W^{288}$. Putting everything together, we get the power $17 \times 288 = 4896$. □

4.8 Failure of automatic continuity

In this section, we provide tools in order to generate discontinuous homomorphisms. All of our discontinuous homomorphisms will factor through the mapping class group $\mathrm{Map}(\Sigma) = \mathrm{Homeo}(\Sigma)/\mathrm{Homeo}_0(\Sigma)$ and appear as an application of the following theorem. We note that this theorem provides a general tool for building discontinuous homomorphisms for closed subgroups of $\mathrm{Map}(\Sigma)$. It is not stated explicitly in this form in [Dom22], but follows from the arguments in §§ 7, 8, and 10. First, we need to recall the definition of a nondisplaceable subsurface and a nondisplaceable sequence.

DEFINITION 4.33. Let $K \subset \Sigma$ be a connected subsurface and $G < \mathrm{Map}(\Sigma)$ a subgroup. We say that K is *G-nondisplaceable* if K and $g(K)$ have an essential intersection for all $g \in G$. A *G-nondisplaceable sequence* is a sequence $\{K_i\}_{i=1}^{\infty}$ so that

(i) each K_i is homeomorphic to a fixed finite-type surface K of sufficient complexity to carry a pseudo-Anosov homeomorphism,
(ii) $K_i \cap K_j = \emptyset$ for all $i \neq j$,
(iii) the collection $\{K_i\}$ eventually has trivial intersection with every finite-type subsurface of S, and
(iv) each K_i is G-nondisplaceable.

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Theorem 4.34 [Dom22, §§ 7, 8, and 10]. *If G is a closed subgroup of $\mathrm{Map}(\Sigma)$ and has a G-nondisplaceable sequence $\{K_i\}_{i=1}^{\infty}$ such that G contains $\mathrm{Map}(K_i)$ for all i, then G fails to have automatic continuity. In particular, there exists a discontinuous homomorphism $G \to \mathbb{Q}$.*

Proof. Let $\mathcal{K} = \{K_i\}_{i=1}^{\infty}$ be a G-nondisplaceable sequence so that each K_i is homeomorphic to a fixed finite-type surface K. Let $f \in \mathrm{Map}(K)$ be a pseudo-Anosov mapping class and write $f_i \in G$ for the map defined to be f on K_i and the identity elsewhere. Then, for $A = (a_i)_{i=1}^{\infty}$ any unbounded sequence of natural numbers, we form the mapping class

$$f_{\mathcal{K},A} := \prod_{i=1}^{\infty} f_i^{a_i} \in G.$$

Now we apply [Dom22, Theorem 7.1] to see that $f_{\mathcal{K},A}$ cannot be written as a product of commutators in G. We remark that [Dom22, Theorem 7.1] is stated specifically in the case that G is the closure of compactly supported mapping classes. However, the steps of the proof only require one to have a G-nondisplaceable sequence. In particular, the proof relies on constructing a sequence of Bestvina–Bromberg–Fujiwara projection complexes [BBF15] and the projection axioms are verified in [Dom22, Lemma 3.8] for any collection of pairwise overlapping finite-type subsurfaces in an infinite-type surface. The proof then proceeds by using actions on these projection complexes to define quasimorphisms that coarsely count the exponents a_i. Once again, this step only relies on the fact that we began with a pseudo-Anosov defined on a finite-type subsurface by appealing to [BF02, Proposition 11] and [BBF16, Proposition 2.9] (see also [Dom22, Lemmas 7.4–7.6]).

Next we run the exact same argument as in [Dom22, § 8] to see that any such element $f_{\mathcal{K},A}$, with $A = (i!)_{i=1}^{\infty}$, generates a copy of $\mathbb{Q}$ in $H_1(G;\mathbb{Z})$. By factoring through the abelianization, we obtain a homomorphism $G \to \mathbb{Q}$. Finally, we note that any finite sub-product of $f_{\mathcal{K},A}$ is trivial in $H_1(G;\mathbb{Z})$ (see [Dom22, § 8.1.2] for technicalities in the low-genus case) and hence this map must necessarily be discontinuous. In fact, one can use this to generate $2^{2^{\aleph_0}}$ many discontinuous homomorphisms (see [Dom22, § 10]). □

4.9 Proof of Proposition A

We need one final lemma that gives an alternate topological characterization of telescoping.

Lemma 4.35. *Let Σ be a stable surface. If an end $x \in E(\Sigma)$ is not telescoping, then either*

(i) *x is isolated in $E_g(\Sigma)$,*
(ii) *x has a predecessor y with $E(y)$ countable, or*
(iii) *there exists a family of nested stable neighborhoods $\{U_n\}$ descending to x such that each tube $U_n \setminus U_{n+1}$ contains an end z_n with $E(z_n) \cap (U_0 \smallsetminus U_n) = \emptyset$.*

Proof. Let $x \in E(\Sigma)$ be an end that is neither telescoping nor isolated in $E_g(\Sigma)$. In particular, x is not isolated in $E_g(\Sigma)$ and $E(x)$ is a discrete set (by Lemma 4.7). Let U be a stable neighborhood of x and $U_k \searrow x$ a family of nested subsets of U that descends to x. For each k, let M_k denote the set of equivalence classes in $E(\Sigma)$ that both intersect $U_k \setminus U_{k+1}$ and are maximal in $U_k \setminus U_{k+1}$. Note that each of $U_k \setminus U_{k+1}$ is a stable surface and so M_k is finite for each k. We now have two cases to consider.

Case 1. Suppose $\bigcup_{k=0}^{\infty} M_k$ is finite. In other words, there exists some k_0 such that every equivalence class of end in $U \setminus \{x\}$ intersects $U_0 \setminus U_{k_0}$. This implies that x has finitely many predecessors

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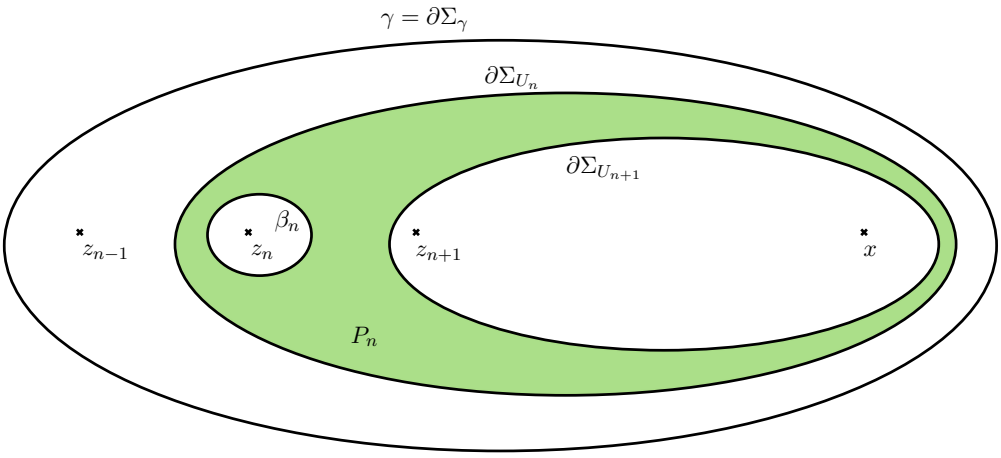


Figure 5 (colour online). Building a nondisplaceable pair of pants, P_n, in Case (iii).

and hence is a successor. Therefore, since we assumed that x is *not* telescoping, one of these predecessors, y, must not be of Cantor type. Thus, by Lemma 4.7, we have that $E(y)$ is countable.

Case 2. Suppose $\bigcup_{k=0}^{\infty} M_k$ is infinite. Thus, since each M_k is finite, we must have that the tubes $U_k \setminus U_{k+1}$ are seeing new equivalence classes of ends as $k \to \infty$. Therefore, after passing to a subsequence of the U_k, we can find a family $\{U_n\}$ satisfying (iii). □

We are now ready to prove our main classification theorem for surfaces.

Proof of Proposition A. The positive direction is exactly the statement of Theorem 4.32. Now we assume that Σ has an end $x \in E(\Sigma)$ that is not telescoping. Our discontinuous map will factor through the mapping class group, $\mathrm{Map}(\Sigma)$.

Since $E(x)$ is not of Cantor type, $E(x)$ is countable by Lemma 4.7 and x is isolated in $E(x)$. Thus, we can find a simple separating curve γ so that γ separates x from all other ends in $E(x)$. Let Σ_γ be the subsurface containing x with boundary γ. We may assume, without loss of generality, that $E(\Sigma_\gamma)$ is a stable neighborhood of x. Note that the stabilizer, $\mathrm{Stab}(\gamma)$, is a closed countable index subgroup of $\mathrm{Map}(\Sigma)$. Thus, by Lemma 2.2, it suffices to build a discontinuous homomorphism from $\mathrm{Stab}(\gamma)$ to a countable discrete group. Our new goal is to find a $\mathrm{Stab}(\gamma)$-nondisplaceable sequence in order to apply Theorem 4.34. The proof breaks down into the three cases from Lemma 4.35.

Case (i). Suppose that $x \in E_g(\Sigma)$ is isolated. We claim that any simple curve α in Σ_γ that separates x from γ is $\mathrm{Stab}(\gamma)$-nondisplaceable. Indeed, α cuts Σ_γ into two components, one with finite genus and one with infinite genus. If $f \in \mathrm{Stab}(\gamma)$ mapped α completely into either component, it would change the genus of the finite-genus piece. As f fixes γ, this cannot happen. Thus, we conclude that $f(\alpha) \cap \alpha \neq \emptyset$. Finally, to build a $\mathrm{Stab}(\gamma)$-nondisplaceable sequence, we can take sufficiently spaced out sequences of such curves $\{\alpha_i\}$ that converge to x and for each α_i take a genus-two subsurface K_i that contains it and is disjoint from all other such subsurfaces.

Case (ii). Suppose that x has a countable predecessor, y. Write $E'(y) = E(y) \cap \Sigma_\gamma$ and note that $E'(y)$ is still countably infinite. Now, this case follows exactly as in Case 1, except points in $E'(y)$ play the role of genus.

Case (iii). Let $\{U_n\}$ be a family of nested stable neighborhoods descending to x such that each tube $U_n \setminus U_{n+1}$ contains an end z_n with $E(z_n) \cap (U_0 \setminus U_n) = \emptyset$. We may assume, without loss of

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generality, that $U_0 = E(\Sigma_\gamma)$. We will build a nondisplaceable pair of pants for each fixed n. Let P_n be the pair of pants made up of the curves $\partial\Sigma_{U_n}$, $\partial\Sigma_{U_{n+1}}$, and a separating curve, β_n, that cuts off all ends of $U_n \setminus U_{n+1}$. See Figure 5. We claim that these P_n are $\mathrm{Stab}(\gamma)$-nondisplaceable. If $f \in \mathrm{Stab}(\gamma)$, then $f(P_n)$ cannot land in the component cut off by $\partial\Sigma_{U_n}$ since $E(\Sigma_\gamma) \setminus U_n$ contains no points of $E(z_n)$. Similarly, if $f(P_n)$ landed in the component cut off by $\partial\Sigma_{U_{n+1}}$, then $f(E(\Sigma_\gamma) \setminus U_n)$ would intersect $E(z_n)$. Finally, $f(P_n)$ cannot land in the component cut off by β_n since P_n must separate γ and x. As above, we can expand these pairs of pants to subsurfaces of sufficiently high complexity and pass to a disjoint subsequence to obtain a $\mathrm{Stab}(\gamma)$-nondisplaceable sequence.

We can now apply Theorem 4.34 to the $\mathrm{Stab}(\gamma)$-nondisplaceable sequences in each of these cases in order to see that $\mathrm{Stab}(\gamma)$ fails to have automatic continuity. Finally, we use Lemma 2.2 to conclude that $\mathrm{Map}(\Sigma)$ and hence $\mathrm{Homeo}(\Sigma)$ fails to have automatic continuity. □

4.10 Unknown example

In this final section we construct an unstable surface Σ where our techniques fail to decide automatic continuity. The end space of Σ will contain countably many incomparable Cantor sets of ends that themselves converge onto a Cantor set. Building these Cantor sets while avoiding ends that violate Theorem A (or the examples of Remark 1.1) will take some care.

We construct the end space of Σ inductively. For colored second-countable Stone spaces $U_1, \ldots, U_k$, let $C(U_1, \ldots, U_k)$ be the Cantor set with a copy of U_i added into every 'missing interval' of the Cantor set. Similarly, let $C^g(U_1, \ldots, U_k)$ be the same except that points in the Cantor set are non-planar. When $k = 0$ we use the notation $C(\emptyset)$ and $C^g(\emptyset)$. Let p be an isolated point. We define our *level-one* Cantor sets at follows:

$$L_{0,1} = C^g(\emptyset), \quad L_{1,1} = C(\emptyset), \quad L_{2,1} = C^g(p), \quad \text{and} \quad L_{3,1} = C(p).$$

To reiterate, $L_{0,1}$ is a non-planar Cantor set, $L_{1,1}$ is a planar Cantor set, $L_{2,1}$ is a Cantor set with each point accumulated by a sequence of punctures, and $L_{3,1}$ is a non-planar Cantor set where each point is accumulated by a sequence of punctures. Note that the maximal ends in these spaces are not comparable. Also, every point in these spaces satisfies the assumptions of Theorem A.

Now we can use all non-empty subsets of $\{L_{1,1}, L_{2,1}, L_{3,1}\}$ to make level-two sets. Thus, we obtain four level-two sets:

$$L_{0,2} = C(L_{1,1}, L_{2,1}), \quad L_{1,2} = C(L_{2,1}, L_{3,1}), \quad L_{2,2} = C(L_{1,1}, L_{3,1}),$$

and

$$L_{3,2} = C(L_{1,1}, L_{2,1}, L_{3,1}).$$

Note that $L_{0,1}$ is incomparable to all level-two sets. Then we reserve $L_{0,2}$ and again use non-empty subsets of $\{L_{1,2}, L_{2,2}, L_{3,2}\}$ to construct level-three sets. We recursively continue this process to construct sets at all levels. That is, for all $n \geq 1$,

$$L_{0,n+1} = C(L_{1,n}, L_{2,n}), \quad L_{1,n+1} = C(L_{2,n}, L_{3,n}), \quad L_{2,n+1} = C(L_{1,n}, L_{3,n}),$$

and

$$L_{3,n+1} = C(L_{1,n}, L_{2,n}, L_{3,n}).$$

Each time, we reserve $L_{0,n}$ so the maximal points in $L_{0,n}$ are not comparable to any point in $L_{i,m}$ for $m \geq n$. Thus, we obtain the countable collection $L_{0,1}, L_{0,2}, L_{0,3}, \ldots$ of incomparable Cantor sets.

Finally, for each $n \in \mathbb{N}$, we let Λ_n be the surface with end space $L_{0,n}$ and a single boundary component and infinite genus where the ends labeled non-planar are accumulated by genus.

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We then construct Σ by beginning with the surface obtained by thickening a rooted binary tree, removing a disk from each pair of pants, and, along the missing disks at level n, gluing a copy of Λ_n. Each end of Σ is either an isolated puncture or of Cantor type. Furthermore, Σ has countably many maximal types of ends, corresponding to each of the levels $L_{0,n}$, and one additional maximal Cantor type of end, L_∞, which is accumulated by the $L_{0,n}$ as $n \to \infty$. The ends of L_∞ are non-telescoping and so we cannot run our argument for automatic continuity. However, we are also not able to find a nondisplaceable sequence of subsurfaces in order to apply Theorem 4.34 to build a discontinuous map. As such, it is still open whether $\mathrm{Homeo}(\Sigma)$ or $\mathrm{Map}(\Sigma)$ has automatic continuity.

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Conflicts of interest

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