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SIMULTANEOUS NIVEN NUMBERS IN ARITHMETIC PROGRESSIONS FOR POWER-RELATED BASES

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Abstract

Recently, Harrington *et al.* ['Every arithmetic progression contains infinitely many b-Niven numbers', *Bull. Aust. Math. Soc.* **109**(3) (2024), 409–413] proved that every arithmetic progression contains infinitely many base-b Niven numbers for any fixed $b \geq 2$. We use a sparse repunit construction to treat a structured two-base version of the same problem, showing that every arithmetic progression with common difference relatively prime to b contains infinitely many integers that are simultaneously b-Niven and b^k-Niven (indeed, we can obtain simultaneous b^ℓ-Niven-ness for $\ell = 1, \ldots, k$).

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1. Introduction

A positive integer is called a *Niven number* (or *Harshad number*) in a given base if it is divisible by the sum of its digits in that base. These numbers were introduced in the decimal setting by Kaprekar [8] and have since been studied from a variety of analytic and combinatorial perspectives (see [1–6, 9–11, 13, 15]).

Recently, Harrington *et al.* [7] proved that every arithmetic progression contains infinitely many base-b Niven numbers for any fixed $b \geq 2$. Inspired by their analysis, we investigate a variation of their question for certain *simultaneous* Niven conditions in multiple bases.

We treat a particular structured two-base version of the problem for the pair of bases $\{b, B\}$, where $B = b^k$ is a power of b. For a single base b, a multiplicative-order repunit argument already gives an elementary proof of infinitude in any arithmetic progression whose common difference is relatively prime to the base. We show how to combine that idea with a digit-sum compatibility construction—a version of Witno's 'sparse repunits' [15]—to obtain explicit simultaneous b- and b^k-Niven numbers in every such progression. As a corollary, we show that it is possible to force Niven-ness in arithmetic progressions simultaneously for any (finite) number of powers of b.

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The restriction to power-related bases is not merely a convenience, but reflects a genuine structural simplification. When $B = b^k$, each base-B digit corresponds to a block of k base-b digits, so by arranging for the base-B digits to be small (indeed, in $\{0, 1, \ldots, b-1\}$), we can ensure that passing from base B to base b creates no carries across blocks and, hence, preserves digit sums. This digit-block compatibility provides a concrete mechanism for linking Niven conditions across the two bases. Our approach exploits this linkage together with a multiplicative-order 'repunits with gaps' construction that simultaneously forces the desired residue class modulo m and divisibility by a prescribed digit sum.

1.1. Main result. For positive integers $m \geq 1$ and r with $0 \leq r < m$, we write

$$S_{m,r} := \{n \in \mathbb{Z}_{\geq 1} : n \equiv r \pmod{m}\}$$

for the arithmetic progression $r + t \cdot m$. We denote the sum of the base b digits of n by $\mathsf{s}_b(n)$.

THEOREM 1.1. *Let $b \geq 2$ and $k \geq 1$ be integers, and set $B = b^k$. If $m \geq 1$ is such that $\gcd(m, b) = 1$, then for any r with $0 \leq r < m$, there exist infinitely many integers $n \in S_{m,r}$ such that*

$$\mathsf{s}_b(n) \mid n \quad \text{and} \quad \mathsf{s}_B(n) \mid n;$$

that is, there are infinitely many n in the arithmetic progression $S_{m,r}$ that are simultaneously b- and b^k-Niven. Moreover, for any $s_0 \geq 1$, there exists such an n with $\mathsf{s}_b(n) = \mathsf{s}_B(n) \geq s_0$.

2. Preliminaries

2.1. Digit sums and base expansions. For an integer base $g \geq 2$, every positive integer c has a unique base-g expansion

$$c = \sum_{i=0}^{L} d_i g^i, \quad d_i \in \{0, 1, \ldots, g-1\},\ d_L \neq 0;$$

in this case, the sum of the base-g digits of c is

$$\mathsf{s}_g(c) = \sum_{i=0}^{L} d_i.$$

For convenience, we set $\mathsf{s}_g(0) = 0$. We say that c is *g-Niven* if $\mathsf{s}_g(c) \mid c$.

We also write

$$\operatorname{rad}(c) := \prod_{\substack{p \mid c \\ p \text{ prime}}} p$$

for the *radical* of an integer $c \geq 1$, that is, the product of the distinct prime divisors of c, with the convention that $\operatorname{rad}(1) = 1$.

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2.2. Compatibility between bases b and $B = b^k$. For power-related bases b and $B = b^k$, a base-B digit occupies a block of k base-b digits. The following elementary observation linking those digit blocks is central to our construction.

PROPOSITION 2.1. *Let $b \geq 2$ and $k \geq 1$ be integers, and set $B = b^k$. Suppose that n admits a base-B expansion of the form*

$$n = \sum_{i=0}^{L} d_i B^i \quad \text{with } d_i \in \{0, 1, \ldots, b-1\} \text{ for all } i \text{ and } d_L \neq 0.$$

Then,

$$\mathsf{s}_b(n) = \mathsf{s}_B(n) = \sum_{i=0}^{L} d_i.$$

PROOF. We write $B^i = b^{ki}$, so

$$n = \sum_{i=0}^{L} d_i b^{ki}.$$

Since $0 \leq d_i \leq b-1$, the base-b expansion of $d_i b^{ki}$ consists of a single digit d_i in position ki and 0s elsewhere. The positions ki are distinct, so when summing over i, each base-b position receives at most one nonzero digit and therefore no carries occur. Hence, the base-b digit sum equals $\sum_i d_i$. The base-B digit sum is also $\sum_i d_i$ by definition. □

REMARK 2.2. More generally, if $n = \sum_{i=0}^{L} d_i B^i$ is the base-B expansion with digits d_i such that $0 \leq d_i < B = b^k$, then expanding each coefficient d_i in base b uses at most k digits, occupying the block of positions $ki, ki+1, \ldots, ki+k-1$; these blocks are disjoint for different i, so no carries occur between them. Consequently,

$$\mathsf{s}_b(n) = \sum_{i=0}^{L} \mathsf{s}_b(d_i).$$

Proposition 2.1 is the special case in which $d_i < b$ for all i, so that $\mathsf{s}_b(d_i) = d_i$.

2.3. Multiplicative order. For integers $a \in \mathbb{Z}$ and $N \geq 1$ with $\gcd(a, N) = 1$, the *multiplicative order* of a modulo N is

$$\mathrm{ord}_N(a) := \min\{\omega \in \mathbb{Z}_{\geq 1} : a^\omega \equiv 1 \pmod{N}\};$$

$\mathrm{ord}_N(a)$ exists by finiteness of $(\mathbb{Z}/N\mathbb{Z})^\times$ whenever $N \geq 2$. (When $N = 1$, the congruence $a^\omega \equiv 1 \pmod 1$ holds for all ω and the definition gives $\mathrm{ord}_1(a) = 1$.)

We also use the Chinese remainder theorem (CRT) in the standard form: if $\gcd(M_1, M_2) = 1$, then for any residues $x_1 \pmod{M_1}$ and $x_2 \pmod{M_2}$, there exists a unique residue class $x \pmod{M_1 M_2}$ satisfying both congruences (see, for example, [12, Section 2.3]).

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3. The construction

We now introduce our simultaneous Niven number construction. Throughout, fix integers $b \geq 2$ and $k \geq 1$, set $B = b^k$, and fix an arithmetic progression $S_{m,r}$ with $m \geq 1$, $0 \leq r < m$ and $\gcd(m, b) = 1$.

Step 1: admissible digit sums. We choose integers $s \geq 1$ simultaneously satisfying

$$s \equiv r \pmod{m} \quad \text{and} \quad \gcd(s, b) = 1; \tag{3.1}$$

we call such s *admissible*.

Step 2: the spacing parameter. Since $B = b^k$, we have $\gcd(B, ms) = 1$ if and only if $\gcd(b, ms) = 1$. For each admissible s, the hypothesis that $\gcd(s, b) = 1$ thus implies $\gcd(ms, b) = 1$, given that we have chosen m and b such that $\gcd(m, b) = 1$. Hence, we can define

$$\omega_s := \operatorname{ord}_{ms}(B). \tag{3.2}$$

Step 3: a sparse repunit in base B. Define

$$n_s := \sum_{j=0}^{s-1} B^{j\omega_s}; \tag{3.3}$$

equivalently, since the sum is geometric,

$$n_s = \frac{B^{s\omega_s} - 1}{B^{\omega_s} - 1}.$$

We prove in Proposition 4.1 that $n_s \equiv r \pmod{m}$, $\mathtt{s}_b(n_s) = \mathtt{s}_B(n_s) = s$ and $s \mid n_s$; that is, for any admissible s, the constructed integer n_s is simultaneously b-Niven and B-Niven. We then complete the proof of Theorem 1.1 in Section 5 by showing that there are infinitely many admissible s.

REMARK 3.1. By construction, the integer n_s in (3.3) is the length-s *repunit* in the auxiliary base B^{ω_s}, that is,

$$n_s = 1 + (B^{\omega_s}) + (B^{\omega_s})^2 + \cdots + (B^{\omega_s})^{s-1}.$$

Equivalently, in base B, it is a *sparse repunit*, with 1s separated by gaps of length $\omega_s - 1$. This point of view connects our construction to the study of generalised repunits by Witno [15] (see also [14]). In Witno's notation (which slightly overloads with ours, so we adopt it only temporarily for this remark): for integers $n \geq 1$, $k \geq 1$ and base $b \geq 2$, we set

$$R_{n,b,k} := \sum_{j=0}^{n-1} b^{jk} = \frac{b^{nk} - 1}{b^k - 1},$$

which in base b is a string of n 1s separated by blocks of $k - 1$ 0s; equivalently, $R_{n,b,k} = R_{n,b^k}$ is a generalised repunit in base b^k. Our n_s is precisely of Witno's form with base b replaced by B and gap parameter k replaced by ω_s.

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Witno's Theorem 4.2 gives a multiplicative order criterion for when such sparse repunits are b-Niven. In contrast, we *choose* the spacing $\omega_s = \mathrm{ord}_{ms}(B)$ so that

$$B^{\omega_s} \equiv 1 \pmod{ms},$$

which forces the congruence $n_s \equiv s \pmod{ms}$ and hence, $s \mid n_s$, while simultaneously enforcing the arithmetic progression constraint $n_s \equiv r \pmod{m}$.

REMARK 3.2. We note also that any positive integer ω with $B^{\omega} \equiv 1 \pmod{m}$ and $B^{\omega} \equiv 1 \pmod{s}$ suffices for the congruence computations below; taking $\omega_s = \mathrm{ord}_{ms}(B)$ as in (3.2) is a convenient canonical choice guaranteeing these congruences (indeed, the stronger congruence $B^{\omega_s} \equiv 1 \pmod{ms}$).

4. Verification of the desired properties

PROPOSITION 4.1. *Assume that s satisfies (3.1), and define ω_s and n_s by (3.2) and (3.3), respectively. Then:*

(1) $n_s \equiv r \pmod{m}$, *so* $n_s \in S_{m,r}$;
(2) $\mathtt{s}_B(n_s) = \mathtt{s}_b(n_s) = s$;
(3) $s \mid n_s$.

In particular, n_s is simultaneously b-Niven and B-Niven.

PROOF. As $\omega_s = \mathrm{ord}_{ms}(B)$, by construction, we have $B^{\omega_s} \equiv 1 \pmod{ms}$. Hence, for every $j \geq 0$,

$$B^{j\omega_s} = (B^{\omega_s})^j \equiv 1 \pmod{ms}. \tag{4.1}$$

Expanding (3.3) modulo ms using (4.1) gives

$$n_s = \sum_{j=0}^{s-1} B^{j\omega_s} \equiv \sum_{j=0}^{s-1} 1 = s \pmod{ms}. \tag{4.2}$$

Reducing (4.2) modulo m yields $n_s \equiv s \pmod{m}$ and since $s \equiv r \pmod{m}$, we obtain $n_s \equiv r \pmod{m}$, proving (1). Reducing (4.2) modulo s instead yields

$$n_s \equiv s \equiv 0 \pmod{s},$$

proving (3).

For (2), in the base-B expansion of n_s, the summands are distinct powers of B with coefficient 1, so no carries occur in base B: there are 1s in positions $0, \omega_s, 2\omega_s, \ldots, (s-1)\omega_s$ and 0s in all other positions. It follows that $\mathtt{s}_B(n_s) = s$. Since all base-B digits of n_s lie in $\{0, 1\} \subseteq \{0, 1, \ldots, b-1\}$, Proposition 2.1 then gives $\mathtt{s}_b(n_s) = \mathtt{s}_B(n_s) = s$. □

5. Infinitude of the admissible s

It remains to show that admissible s exist infinitely often and that the resulting n_s are distinct.

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LEMMA 5.1. *For any $b \geq 2$ and $m \geq 1$ with $\gcd(m, b) = 1$, there exist infinitely many integers $s \geq 1$ satisfying (3.1) for any residue class $r \pmod m$. Moreover, for any $s_0 \geq 1$, there exists such an s with $s \geq s_0$.*

PROOF. Setting $q := \operatorname{rad}(b)$, we have $\gcd(m, q) = 1$ because $\gcd(m, b) = 1$. By the CRT, the simultaneous congruences

$$s \equiv r \pmod m \quad \text{and} \quad s \equiv 1 \pmod q$$

have a solution $s \equiv s^* \pmod{mq}$. Let s^* denote the least positive integer in that residue class. Every integer

$$s = s^* + t(mq) \qquad (t \geq 0)$$

then satisfies $s \equiv r \pmod m$ and $s \equiv 1 \pmod q$; hence, for such s, we have $\gcd(s, q) = 1$. Since q has the same prime divisors as b, this implies $\gcd(s, b) = 1$. Taking t arbitrarily large yields infinitely many admissible s; by choosing t large enough, we can ensure that $s \geq s_0$. □

LEMMA 5.2. *If s and s' are distinct admissible integers, then $n_s \neq n_{s'}$.*

PROOF. We prove the contrapositive: if $n_s = n_{s'}$, then *a priori*,

$$\mathtt{s}_B(n_s) = \mathtt{s}_B(n_{s'}). \tag{5.1}$$

However, by Proposition 4.1(2), we have $\mathtt{s}_B(n_s) = s$ and $\mathtt{s}_B(n_{s'}) = s'$; hence, (5.1) implies that $s = s'$. □

PROOF OF THEOREM 1.1. By Lemma 5.1, there exist infinitely many admissible integers s. For each such s, Proposition 4.1 shows that $n_s \in S_{m,r}$ and, moreover, n_s is simultaneously b-Niven and B-Niven. Lemma 5.2 shows the n_s associated to admissible s are all distinct, giving infinitely many integers with the desired properties.

For the strengthening, given $s_0 \geq 1$, we choose an admissible $s \geq s_0$ (which exists by Lemma 5.1). Proposition 4.1(2) then gives $\mathtt{s}_b(n_s) = \mathtt{s}_B(n_s) = s \geq s_0$, as desired. □

6. An illustrative example

Take $b = 2$, $k = 3$, so $B = b^k = 8$. Let $m = 5$ and $r = 3$. We choose $s \equiv 3 \pmod 5$ and $\gcd(s, 2) = 1$; the smallest such choice is $s = 3$. We have

$$8 \equiv 8 \pmod{15}, \quad 8^2 \equiv 64 \equiv 4 \pmod{15}, \quad 8^4 \equiv 16 \equiv 1 \pmod{15},$$

and since $8^2 \not\equiv 1 \pmod{15}$, we have $\omega_s = 4$.

The construction then gives

$$n_s = 1 + 8^4 + 8^8 = 1 + 4096 + 16777216 = 16781313.$$

Now, $n_s \equiv 3 \pmod 5$, so $n_s \in S_{5,3}$. Also, since $8^4 \equiv 1 \pmod 3$, we have

$$n_s \equiv 1 + 1 + 1 \equiv 0 \pmod 3,$$

so $3 \mid n_s$.

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In base 8, we have

$$n_s = 1 \cdot 8^8 + 1 \cdot 8^4 + 1 \cdot 8^0 = (100010001)_8;$$

hence, $\mathtt{s}_8(n_s) = 3$ as expected. In base 2,

$$n_s = 1 \cdot 2^{24} + 1 \cdot 2^{12} + 1 \cdot 2^0 = (1000000000001000000000001)_2,$$

with exactly three 1s (at positions 0, 12 and 24); hence, $\mathtt{s}_2(n_s) = 3$.

Thus, we see that n_s is simultaneously 2-Niven and 8-Niven, and lies in the progression $n \equiv 3 \pmod 5$.

7. Concluding remarks

We have proven that for every pair of power-related bases $\{b, b^k\}$ and every arithmetic progression $S_{m,r}$ with $\gcd(m, b) = 1$, there are infinitely many integers in $S_{m,r}$ that are simultaneously b-Niven and b^k-Niven. The argument is elementary and rests on two simple ideas:

(1) digit-sum compatibility for bases b and b^k when base-b^k digits lie in $\{0, \ldots, b-1\}$;
(2) congruence forcing via multiplicative order to ensure both the prescribed progression and divisibility by the chosen digit sum.

Theorem 1.1 only asks for $\mathtt{s}_b(n) \mid n$ and $\mathtt{s}_B(n) \mid n$, with no relationship required between $\mathtt{s}_b(n)$ and $\mathtt{s}_B(n)$. Yet, our construction yields the stronger property

$$\mathtt{s}_b(n_s) = \mathtt{s}_B(n_s)$$

because the base-B digits are chosen to lie in $\{0, 1\} \subseteq \{0, 1, \ldots, b-1\}$, allowing the digit-sum compatibility obtained in Proposition 2.1. This feature appears to be an artefact of the method, but it is convenient and may be useful in other simultaneous-base questions.

We note also that our proof of Theorem 1.1 is completely explicit: for a chosen admissible s, we compute $\omega_s = \mathrm{ord}_{ms}(B)$ and then construct n_s via (3.3). The resulting integers grow very rapidly (roughly of the order of $B^{(s-1)\omega_s}$), so this is not an efficient enumeration method, but it does yield a concrete infinite family and a constructive existence proof.

7.1. Relation to the order–repunit method in the coprime case. For a fixed base b and modulus coprime to b, using multiplicative order to force congruences is standard. In particular, for a chosen integer s with $\gcd(s, b) = 1$, setting

$$\omega_s^{(b)} := \mathrm{ord}_{ms}(b), \quad n_s^{(b)} := \sum_{j=0}^{s-1} b^{j\omega_s^{(b)}} \tag{7.1}$$

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gives $b^{\omega_s^{(b)}} \equiv 1 \pmod{ms}$, and so $n_s^{(b)} \equiv s \pmod{ms}$ and $\mathsf{s}_b(n_s^{(b)}) = s$. Choosing $s \equiv r \pmod{m}$ then places $n_s^{(b)}$ in a prescribed arithmetic progression. (This is precisely the special case $k = 1$ of Theorem 1.1.)

The present construction is a structured refinement of this idea, using B instead of b in (7.1). Thus, as noted in Remark 3.1, n_s is a length-s repunit in the auxiliary base B^{ω_s}, appearing in base B as a sparse repunit (see [15]) with digits equal to 1. As in the single-base–coprime-case argument, multiplicative order enforces the required congruences, but here, the power-related structure additionally allows the same sparse repunit to have identical digit sums in bases b and B.

7.2. Extension to multiple power-related bases. If ℓ is a positive divisor of k, then the integers n_s constructed in our proof of Theorem 1.1 are automatically b^ℓ-Niven. Indeed, writing $k = \ell t$ gives

$$n_s = \sum_{j=0}^{s-1} b^{kj\omega_s} = \sum_{j=0}^{s-1} (b^\ell)^{tj\omega_s},$$

so the base-b^ℓ expansion of n_s has exactly s digits equal to 1, and thus, $\mathsf{s}_{b^\ell}(n_s) = s$.

In our numerical example in Section 6, we also have $n_s = (1000001000001)_4$, so that n_s is 4-Niven in addition to being 2- and 8-Niven. Here, $4 = 2^2$ and $\ell = 2$ does not correspond to a divisor of $k = 3$, so this additional 4-Niven-ness is an arithmetical coincidence. In general, applying the construction given for $\{b, b^k\}$ does not guarantee b^ℓ-Niven-ness for intermediate exponents ℓ with $\ell \nmid k$. For instance, again take $b = 2$ and $k = 3$ (so $B = 8$), and choose $m = 1$ and $s = 7$. Then, $\omega_s = \mathrm{ord}_7(8) = 1$, so the construction gives

$$n_s = 8^6 + \cdots + 8^2 + 8 + 1 = 299593,$$

which is simultaneously 2- and 8-Niven:

$$299593 = (1001001001001001001)_2 = (1111111)_8,$$

with $\mathsf{s}_2(299593) = \mathsf{s}_8(299593) = 7$ by construction. However, $299593 = (1021021021)_4$ with $\mathsf{s}_4(299593) = 10 \nmid 299593$, so n_s is not 4-Niven.

Nevertheless, the divisibility observation above implies that an extension of our construction makes it possible to pick up any finite collection of power-related bases $\{b, b^2, \ldots, b^k\}$.

COROLLARY 7.1. *Let $b \geq 2$ and $k \geq 1$ be integers, and put $K = \mathrm{lcm}(1, 2, \ldots, k)$. If $m \geq 1$ is such that $\gcd(m, b) = 1$, then for any r with $0 \leq r < m$, there exist infinitely many integers $n \in S_{m,r}$ that are b^ℓ-Niven for all ℓ with $1 \leq \ell \leq k$. Moreover, for every $s_0 \geq 1$, there exists such an n with*

$$\mathsf{s}_b(n) = \mathsf{s}_{b^2}(n) = \cdots = \mathsf{s}_{b^k}(n) \geq s_0.$$

PROOF. We fix $s_0 \geq 1$ and choose an admissible $s \geq s_0$ as in Lemma 5.1, so that $s \equiv r \pmod{m}$ and $\gcd(s, b) = 1$.

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We set $\overline{B} := b^K$ and define

$$\Omega_s = \operatorname{ord}_{ms}(\overline{B}), \quad N_s := \sum_{j=0}^{s-1} \overline{B}^{j\Omega_s}$$

(note that $\operatorname{ord}_{ms}(\overline{B})$ exists because $\gcd(\overline{B}, ms) = 1$). By construction, $\overline{B}^{\Omega_s} \equiv 1 \pmod{ms}$, so the same argument as in the proof of Proposition 4.1 gives $N_s \equiv s \equiv r \pmod{m}$ and $s \mid N_s$.

Now, we fix $\ell \in \{1, 2, \ldots, k\}$. Since $\ell \mid K$, we can write

$$\overline{B}^{j\Omega_s} = b^{Kj\Omega_s} = (b^\ell)^{(K/\ell)j\Omega_s}.$$

Thus, we see that

$$N_s = \sum_{j=0}^{s-1} (b^\ell)^{(K/\ell)j\Omega_s}$$

is a sum of distinct powers of b^ℓ, each with coefficient 1; hence, the base-b^ℓ expansion of N_s has exactly s digits equal to 1 and no carries, so $\mathsf{s}_{b^\ell}(N_s) = s$. Since also $s \mid N_s$, it follows that $\mathsf{s}_{b^\ell}(N_s) \mid N_s$; this holds for every $\ell = 1, \ldots, k$, as desired.

Finally, distinct admissible s chosen in this way yield distinct integers N_s since $\mathsf{s}_b(N_s) = s$; the infinitude claim then follows by taking larger and larger s_0. □

7.3. Potential for further extensions. Our construction relies on choosing ω_s so that $B^{\omega_s} \equiv 1 \pmod{m}$ and, hence, $B^{j\omega_s} \equiv 1 \pmod{m}$ for all j. Corollary 7.1 shows that the same construction can simultaneously handle several bases that are all powers of a common base b, provided the exponents are taken from a finite set. However, if a prime p divides both m and b, then $p \mid B$ and so $B^\ell \equiv 0 \pmod{p}$ for every $\ell \geq 1$, making it impossible for us to attain $B^{\omega_s} \equiv 1 \pmod{m}$. Thus, the assumption $\gcd(m, b) = 1$ is not merely technical: it is required for the particular congruence-forcing mechanism used here.

Harrington *et al.* [7] proved that *every* arithmetic progression—not just those with $\gcd(m, b) = 1$—contains infinitely many b-Niven numbers for any single base $b \geq 2$; this suggests it is at least plausible that the relative primality hypothesis in Theorem 1.1 could be relaxed. We also might hope to simultaneously attain Niven-ness across sets of *non*-power-related bases, but this would seem to require new ideas for linking the structure across bases.

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