Canad. Math. Bull. 2026, pp. 1–16
http://dx.doi.org/10.4153/S0008439526102641
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

Discrete harmonic extensions and a discrete Hilbert transform

Ljupcho Petrov

Abstract. The classical Hilbert transform of a function on the real line arises as the boundary value of the function's conjugate-harmonic extension. In this article, harmonic extensions to the upper half-integer lattice $\mathbb{Z} \times \mathbb{N}$ are constructed for a given boundary sequence on the integers; these serve as discrete analogs of the Poisson and conjugate-Poisson integrals. The construction is characterized by: (i) discrete harmonicity with respect to a two-dimensional Laplacian, (ii) a Cauchy–Riemann system, and (iii) boundary values involving a discrete Hilbert transform. This new discrete Hilbert transform is compared to the Riesz–Titchmarsh transform, and Lebesgue bounds are proved. The construction is also extended to the higher dimensional case. These results provide a discrete harmonic-analytic model analogous to the classical theory.

1 Introduction

1.1 Main contributions of this article

On the real line, the Hilbert transform Hf arises as the boundary value of the conjugate-harmonic function $Q[f] = \operatorname{Im} F[f]$, where

$$F[f](z) = \frac{\mathrm{i}}{\pi} \int_{\mathbb{R}} \frac{f(t)}{z-t}\, dt, \quad z = x + \mathrm{i}y \in \mathbb{C}_+.$$

Precisely, with the Poisson extension $\operatorname{Re} F[f] = P[f]$, for a.e. $x \in \mathbb{R}$, we have [14]

$$\lim_{y \to 0^+} P[f](x + \mathrm{i}y) = f(x), \qquad \lim_{y \to 0^+} Q[f](x + \mathrm{i}y) = Hf(x).$$

In this article, the following definition of a discrete Hilbert transform arises. We are not aware of any previous work that realizes a discrete Hilbert transform by means of extensions to $\mathbb{Z} \times \mathbb{N}$ as described below.

Definition 1.1 The discrete Hilbert transform H_{d} of a sequence $a \in \ell^p(\mathbb{Z})$ with $p \geq 1$ is defined as

$$(H_{\mathrm{d}} a)_n = \mathcal{F}^{-1}\big[(-\mathrm{i} \operatorname{sgn} \theta)\, \rho(\theta)^{1/2} \mathrm{e}^{\mathrm{i}\theta/2} \widehat{a}(\theta)\big](n),$$

where $\rho + \rho^{-1} = 4 - 2\cos\theta$.

Received by the editors May 8, 2026 UTC; revised October 10, 2026 UTC; accepted October 14, 2026 UTC.

AMS subject classification: **39A12**, **42A50**.

Keywords: Discrete harmonic extensions, discrete Hilbert transform, conjugate-harmonic functions, discrete Cauchy–Riemann equations, discrete harmonic analysis.

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

In contrast to the continuous model and the semi-discrete settings described below, we impose the fully discrete five-point Laplace equation on $\mathbb{Z} \times \mathbb{N}$. For a given boundary sequence $a = (a_n)_{n\in\mathbb{Z}}$, we build a pair of functions $U, V : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{R}$ that play the role of the Poisson and conjugate-Poisson extensions on the fully discrete upper half-lattice $\mathbb{Z} \times \mathbb{N}$.

Although discrete harmonic extensions have been considered earlier, the present work advances the analysis further. The conjugate extension is forced by requiring a lattice Cauchy–Riemann system to hold and gives rise to the discrete Hilbert transform H_{d}. By analyzing the asymptotics of the associated kernel, we show that H_{d} behaves like the Riesz–Titchmarsh transform at large scales, and we obtain $\ell^p(w)$ bounds. These results, not present in previous papers, show that the resulting Hilbert transform exhibits the same key properties as its classical continuous analog. They provide a discrete harmonic-analytic model analogous to the classical theory and offer a new framework for problems in discrete settings.

More precisely, our main result is the following (definitions appear in Section 2).

Theorem 1.1 *Let $p > 1$ and $a \in \ell^p(\mathbb{Z})$. Define*

$$U(n,k) = \sum_{m\in\mathbb{Z}} P_k(n-m)\, a_m, \qquad V(n,k) = \sum_{m\in\mathbb{Z}} Q_k(n-m)\, a_m.$$

Then,

(1) *(Discrete harmonicity). Both U and V are harmonic with respect to the two-dimensional five-point Laplacian*

$$(\Delta F)(n,k) = F(n+1,k) + F(n-1,k) + F(n,k+1) + F(n,k-1) - 4F(n,k), \quad k \geq 1.$$

(2) *(Boundary values at height $k = 0$). At height $k = 0$, the extensions satisfy*

$$U(n,0) = a_n, \qquad V(n,0) = (H_{\mathrm{d}}a)_n,$$

where H_{d} is the discrete Hilbert transform defined above.

(3) *(Cauchy–Riemann relations). U and V satisfy the discrete Cauchy–Riemann relations of Proposition 3.2.*

This construction yields the operator H_{d} with Fourier multiplier

$$m_{\mathrm{d}}(\theta) = (-\mathrm{i}\,\mathrm{sgn}\,\theta)\,\rho(\theta)^{1/2} e^{\mathrm{i}\theta/2},$$

where $\rho + \rho^{-1} = 4 - 2\cos\theta$. Whereas discrete harmonic extensions were examined in earlier work [1], we associate the extension with a discrete Hilbert operator obtained as the boundary trace of the corresponding conjugate-harmonic function. We also note that the additional power of $\rho(\theta)$ is unavoidable whenever we want to consider a fully discrete lattice, even if points are weighed differently than the standard weights in the discrete Laplacian considered here.

As proven in Section 2, the extensions U and V are real-valued when the sequence is real-valued.

We also extend the constructions to harmonic extensions on $\mathbb{Z}^s \times \mathbb{N}$ in Section 5.

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

1.2 Discrete Hilbert transforms

On the integers, there is an extensive history of authors studying different forms of discrete Hilbert transforms and their properties. The discrete Hilbert transform $\mathcal{H}_0$ maps the sequence $(a_n)_{n\in\mathbb{Z}}$ to

$$(\mathcal{H}_0 a)_n = \frac{1}{\pi} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{a_{n-m}}{m}$$

and is therefore defined as convolution with the kernel $\mathcal{K}_0(n) = \frac{1}{\pi n}$, for $n \neq 0$.

The kernels of the Riesz–Titchmarsh transforms $H^{\pm}$ [13, 15, 16] are given by

$$K^+(n) = \frac{1}{\pi}\frac{1}{n+\frac{1}{2}}, \qquad K^-(n) = \frac{1}{\pi}\frac{1}{n-\frac{1}{2}}.$$

In recent years, the operator norms of these transforms have attracted considerable attention. The long-standing conjecture that the operator norm of the Hilbert transform H on $L^p(\mathbb{R})$ is equal to the operator norm of the discrete Hilbert transform $\mathcal{H}_0$ on $\ell^p(\mathbb{Z})$ has been resolved positively in [4]. Results about the operator norms of the Riesz–Titchmarsh transform have been considered in [5] and additional results in [3].

1.3 Discrete continuous settings on $\mathbb{Z}\times\mathbb{R}_+$

In [6], Ciaurri et al. show that the discrete Riesz–Titchmarsh transforms arise as a limit of conjugate-harmonic extensions for the discrete fractional Laplacian on $\mathbb{Z}\times\mathbb{R}_+$. They start from the one-dimensional discrete Laplacian

$$\Delta_x f(n) = f(n+1) - 2f(n) + f(n-1), \qquad n \in \mathbb{Z},$$

and construct the heat semigroup $\{W_y\}_{y>0}$ with explicit kernel

$$W_y f(n) = \sum_{m\in\mathbb{Z}} e^{-2y} I_{n-m}(2y)\, f(m),$$

where I_k is the modified Bessel function. The function $u(n,y) := W_y f(n)$ solves the semi-discrete heat equation $\partial_y u(n,y) = \Delta_x u(n,y)$ with the boundary sequence on $\mathbb{Z}$ being $u(\cdot,0) = f$. Here, ∂_y is the continuous derivative in y. From the heat semigroup, they obtain the Poisson semigroup by subordination and use it to define the fractional powers $(-\Delta_x)^\sigma$ for $0<\sigma<1$. They utilize the difference factorization of the Laplacian: $\Delta_x = \delta_h^+\,\delta_h^-$, with

$$\delta_h^+ f(n) = f(n+1) - f(n) \quad \text{and} \quad \delta_h^- f(n) = f(n) - f(n-1).$$

Since $(-\Delta_x)^{-1/2}$ is not well defined on $\mathbb{Z}$, they set

$$R^+ = \lim_{\alpha\to 1/2^+} \delta_h^+ (-\Delta_x)^{-\alpha}, \qquad R^- = \lim_{\alpha\to 1/2^+} \delta_h^- (-\Delta_x)^{-\alpha},$$

and identify these as convolution operators with the kernels of the Riesz–Titchmarsh transforms $K^{\pm}$. The associated conjugate-harmonic functions $Q_y f := R^+ P_y f$ and $Q_y^* f := R^- P_y f$ satisfy Cauchy–Riemann-type identities involving δ_h^+, δ_h^-, and the continuous derivative ∂_y. The boundary limits recover R^+ and R^-.

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

In [2], Arcozzi et al. develop a probabilistic and semigroup-based account of the centered Hilbert transform

$$H_c = \tfrac{1}{2}\,(H^+ + H^-),$$

built from the transforms $H^{\pm}$ determined by the discrete derivatives δ_h^+ and δ_h^- via the factorization

$$H^+ \circ \sqrt{-\Delta_x} = \delta_h^+, \qquad H^- \circ \sqrt{-\Delta_x} = \delta_h^-.$$

On the upper half-space $\mathbb{Z} \times \mathbb{R}_+$, they take $A = \sqrt{-\Delta_x}$ and define the semi-discrete Poisson semigroup $P_y = e^{-yA}$, $y > 0$. The Poisson extension $u_f(y,x) := (P_y f)(x)$ solves the semi-discrete Laplace equation

$$\Delta_x u_f + \partial_y^2 u_f = 0 \quad \text{on } \mathbb{Z} \times \mathbb{R}_+.$$

They also present Cauchy–Riemann identities relating u_f to the Poisson extension of $H_c f$.

1.4 Discrete analytic functions

A classical approach in discrete complex analysis is to define a notion of "analyticity" on lattices via discrete Cauchy–Riemann relations. On the integer lattice, one can define a pair of real-valued functions whose finite differences satisfy relations mimicking the continuous Cauchy–Riemann equations. This was pursued early by Ferrand [9] and Isaacs [11], and later by Duffin [7, 8].

1.5 Preliminaries

(1) For a sequence $a \in \ell^1(\mathbb{Z})$, we define its horizontal Fourier transform on the torus $\mathbb{T} = (-\pi, \pi]$ by

$$\mathcal{F}(a)(\theta) = \widehat{a}(\theta) = \sum_{n \in \mathbb{Z}} a_n\, e^{-\mathrm{i}n\theta}, \qquad \theta \in \mathbb{T}.$$

For $g \in L^1(\mathbb{T})$, the inverse Fourier transform is given by

$$\mathcal{F}^{-1}[g](n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, e^{\mathrm{i}n\theta}\, d\theta, \qquad n \in \mathbb{Z}.$$

(2) For a function $F : \mathbb{Z} \times \mathbb{N} \to \mathbb{C}$, we define the discrete two-dimensional Laplacian as follows:

$$(\Delta F)(n,k) = F(n+1,k) + F(n-1,k) + F(n,k+1) + F(n,k-1) - 4F(n,k), \quad k \geq 1.$$

(3) We call F (discrete) harmonic if $\Delta F \equiv 0$. Boundary values are evaluated at the height $k = 0$.

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

2 The Poisson and conjugate-Poisson extensions

2.1 Solutions of the Laplace equation

Seeking solutions of $\Delta U = 0$, we fix k and take the horizontal Fourier transform in n:

$$\widehat{U}(\theta, k) = \sum_{n \in \mathbb{Z}} U(n, k)\, \mathrm{e}^{-\mathrm{i}n\theta}.$$

We repeatedly use the shift rule:

$$\mathcal{F}\big[U(n \pm 1, k)\big] = \mathrm{e}^{\pm \mathrm{i}\theta}\, \widehat{U}(\theta, k). \tag{2.1}$$

Using (2.1) and linearity,

$$0 = \mathcal{F}[\Delta U](\theta, k) = \big(2\cos\theta - 4\big)\, \widehat{U}(\theta, k) + \widehat{U}(\theta, k+1) + \widehat{U}(\theta, k-1).$$

We look for separated solutions of the form

$$\widehat{U}(\theta, k) = C(\theta)\, \rho(\theta)^{\,k},$$

with $C(\theta)$ independent of k. Substituting gives a quadratic equation with two real reciprocal roots:

$$\rho + \rho^{-1} + 2\cos\theta - 4 = 0. \tag{2.2}$$

We select the root in $(0, 1]$:

$$\rho(\theta) = 2 - \cos\theta - \sqrt{(2 - \cos\theta)^2 - 1}. \tag{2.3}$$

This choice yields normalized, positive kernels $P_k \in \ell^1(\mathbb{Z})$ with $\sum_n P_k(n) = 1$ and produces the unique ℓ^p-bounded solution of the Dirichlet problem on $\mathbb{Z} \times \mathbb{N}$ (Lemma 2.1).

2.2 The Poisson extension

Let the boundary sequence be $U(n, 0) = a_n$, so the Fourier transform at $k = 0$ is $\widehat{U}(\theta, 0) = \widehat{a}(\theta)$. Evaluating $\widehat{U}(\theta, k) = C(\theta)\, \rho(\theta)^{\,k}$ at $k = 0$ fixes $C(\theta) = \widehat{a}(\theta)$; hence,

$$\widehat{U}(\theta, k) = \rho(\theta)^{\,k}\, \widehat{a}(\theta), \qquad k \geq 0.$$

By taking the inverse transform in n,

$$U(n, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\theta)^{\,k}\, \widehat{a}(\theta)\, \mathrm{e}^{\mathrm{i}n\theta}\, d\theta. \tag{2.4}$$

Definition 2.1 (Discrete Poisson kernel) For $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, let

$$P_k(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\theta)^{\,k}\, \mathrm{e}^{\mathrm{i}n\theta}\, d\theta. \tag{2.5}$$

For $a \in \ell^p(\mathbb{Z})$ with $p \geq 1$, we have

$$U(\cdot, k) = P_k * a, \qquad U(n, k) = \sum_{m \in \mathbb{Z}} P_k(n - m)\, a_m.$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

Indeed, by Lemma 2.1 below, the convolution is well-defined and in $\ell^p(\mathbb{Z})$ for every $a \in \ell^p(\mathbb{Z})$ with $p \geq 1$.

Lemma 2.1 (Elementary properties of P_k) *The discrete Poisson kernel is positive, symmetric, and normalized:*

(i) $P_k(n) \in [0,1]$ *for all* $n \in \mathbb{Z}$, $k \geq 1$.
(ii) $P_k(-n) = P_k(n)$.
(iii) $\sum_{n\in\mathbb{Z}} P_k(n) = 1$ *for every* $k \geq 1$. *In particular,* $P_k \in \ell^1(\mathbb{Z})$.

Proof

(i) Using $\rho(-\theta) = \rho(\theta)$, we get that

$$P_k(n) = \frac{1}{\pi}\int_0^\pi \rho(\theta)^k \cos(n\theta)\, d\theta. \tag{2.6}$$

Therefore, $P_k(n)$ is real. Let $a_n \geq 0$ and let U solve $\Delta U = 0$ on $\mathbb{Z}\times\mathbb{N}$ with boundary $U(\cdot,0) = a$. By the maximum principle, $U \geq 0$. Taking $a = \delta_0$ (the unit Dirac delta mass) gives $P_k(n) = U(n,k) \geq 0$ for all n. Now, the fact that $P_k(n) \in [0,1]$ follows from part (iii).

(ii) This follows immediately from (2.6).

(iii) Let the Fejér kernel be $F_N(\theta) = \sum_{|n|\leq N}(1 - |n|/(N+1))e^{in\theta}$. Because ρ is continuous with $\rho(0) = 1$, by Fejér's theorem [12],

$$\sum_{|n|\leq N}\Big(1 - \frac{|n|}{N+1}\Big)P_k(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(\theta)^k F_N(\theta)\, d\theta \xrightarrow[N\to\infty]{} \rho(0)^k = 1.$$

If we denote

$$\sigma_N = \frac{1}{N+1}\sum_{r=0}^{N} S_r,$$

with $S_r = \sum_{|n|\leq r} P_k(n)$, then we just proved that $\lim_{N\to\infty}\sigma_N = 1$.

By the nonnegativity of $P_k(n)$, we have $S_r \leq S_N$ for $r \leq N$, so $\sigma_N \leq S_N$; hence, $\liminf_N S_N \geq \lim_N \sigma_N = 1$. Next, fix M and take $N \geq M$. Then,

$$\sigma_N = \frac{1}{N+1}\Big(\sum_{r=0}^{M-1} S_r + \sum_{r=M}^{N} S_r\Big) \geq \frac{N-M+1}{N+1} S_M.$$

Letting $N \to \infty$ gives $1 \geq S_M$. Since M is arbitrary, $\limsup_N S_N \leq 1$.
We conclude that $\lim_{N\to\infty} S_N = \sum_{n\in\mathbb{Z}} P_k(n) = 1$. ∎

2.3 The conjugate-Poisson kernel

Definition 2.2 On the Fourier side, we define the conjugate-Poisson kernel as follows:

$$\widehat{V}(\theta,k) = \big(-\mathrm{i}\,\mathrm{sgn}\,\theta\big)\rho(\theta)^{k+\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\theta/2}\,\widehat{a}(\theta), \qquad k \geq 0. \tag{2.7}$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

The factors $e^{i\theta/2}$ and $\rho(\theta)^{1/2}$ shift V half a unit horizontally and vertically, respectively, locating V at $(n+\frac{1}{2}, k+\frac{1}{2})$. Let $V(n+\frac{1}{2}, k+\frac{1}{2}) = \mathcal{F}^{-1}[\widehat{V}(\cdot, k)](n)$, and let us define

$$Q_k(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(-i\operatorname{sgn}\theta\big)\,\rho(\theta)^{k+\frac{1}{2}}\,e^{i(n+\frac{1}{2})\theta}\,d\theta.$$

We note that $Q_k(n)$ is real-valued, which follows from $\rho(-\theta) = \rho(\theta)$. As a convolution formula, for $a \in \ell^p(\mathbb{Z})$ with $p > 1$, we have

$$V\big(n+\tfrac{1}{2}, k+\tfrac{1}{2}\big) = \sum_{m\in\mathbb{Z}} Q_k(n-m)\,a_m.$$

For notational convenience, we may relabel this as $V(n,k)$, so that both U and V are indexed on $\mathbb{Z}\times\mathbb{N}$. As we will see, this definition is imposed by the Laplacian and our requirement that U and V satisfy a Cauchy–Riemann system.

Lemma 2.2 (Harmonicity) *For each fixed sequence $a \in \ell^p(\mathbb{Z})$ with $p > 1$, the extensions*

$$U(n,k) = (P_k * a)_n, \qquad V(n,k) = (Q_k * a)_n$$

satisfy $\Delta U = \Delta V = 0$ on $\mathbb{Z}\times\mathbb{N}$.

Proof U is harmonic by construction. Taking horizontal Fourier transforms of $\Delta V = 0$ reduces to the same recurrence relation (2.2). ∎

3 Discrete Cauchy–Riemann system

To prove that U and V form a Cauchy–Riemann system, we will need the following algebraic identity.

Lemma 3.1 *For every $\theta \in (-\pi, \pi]$, the quantity $\rho(\theta)$ from (2.2) satisfies*

$$\rho(\theta) - 1 \;=\; -\big(e^{i\theta} - 1\big)\big(-i\operatorname{sgn}\theta\big)\,e^{-i\theta/2}\,\rho(\theta)^{1/2}. \tag{3.1}$$

Proof We rewrite $\rho + \rho^{-1} = 4 - 2\cos\theta$ as follows:

$$\begin{aligned}(\rho-1) + \big(\rho^{-1}-1\big) &= 2(1-\cos\theta)\\ &= 4\sin^2\big(\tfrac{\theta}{2}\big).\end{aligned}$$

Thus, $\dfrac{(\rho-1)^2}{\rho} = 4\sin^2\big(\frac{\theta}{2}\big)$, which implies that

$$\big(\rho-1\big)^2 \;=\; 4\sin^2\big(\tfrac{\theta}{2}\big)\,\rho. \tag{3.2}$$

Moreover, $\sin(\theta/2)$ has the same sign as θ, (3.2) becomes

$$\rho - 1 = -2\sin\tfrac{\theta}{2}\,\big(\operatorname{sgn}\theta\big)\,\rho^{1/2}.$$

Using

$$e^{i\theta} - 1 = 2i\sin\tfrac{\theta}{2}\,e^{i\theta/2},$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

we get

$$\rho - 1 = -\left(e^{i\theta} - 1\right)\left(-i \operatorname{sgn} \theta\right)\rho^{1/2} e^{-i\theta/2}.$$

■

Define the horizontal and vertical difference operators as

$$\begin{aligned}
\delta_h^+ F(n,k) &= F(n+1,k) - F(n,k),\\
\delta_h^- F(n,k) &= F(n,k) - F(n-1,k),\\
\partial_k^+ F(n,k) &= F(n,k+1) - F(n,k),\\
\partial_k^- F(n,k) &= F(n,k) - F(n,k-1).
\end{aligned}$$

Using the properties of the Fourier transform, we obtain the following proposition.

Proposition 3.2 (Discrete Cauchy–Riemann system) *For every $n \in \mathbb{Z}$ and $k \geq 1$,*

$$\delta_h^+ U(n,k) = \partial_k^- V(n,k), \tag{3.3}$$

$$\partial_k^+ U(n,k) = -\delta_h^- V(n,k). \tag{3.4}$$

Proof After taking the Fourier transform $\mathcal{F}$, (3.3) becomes

$$\left(e^{i\theta} - 1\right)\rho^k \widehat{a} = \left(1 - \rho^{-1}\right)\left(-i \operatorname{sgn} \theta\right)\rho^{k+\frac{1}{2}} e^{i\theta/2}\widehat{a},$$

which reduces to the identity from Lemma 3.1. The same happens when we take the Fourier transform of (3.4):

$$\left(\rho - 1\right)\rho^k \widehat{a} = -\left(1 - e^{-i\theta}\right)\left(-i \operatorname{sgn} \theta\right)\rho^{k+\frac{1}{2}} e^{i\theta/2}\,\widehat{a}.$$

■

In the first Cauchy–Riemann equation, the two values of U in

$$\delta_h^+ U(n,k) = U(n+1,k) - U(n,k)$$

are taken at $(n+1,k)$ and (n,k), and their midpoint is $\left(n+\frac{1}{2},k\right)$. Also, even in the original, non-relabeled notation (see the comment after (2.7)), the two values of V in

$$\partial_k^- V(n,k) = V\left(n+\tfrac{1}{2},k+\tfrac{1}{2}\right) - V\left(n+\tfrac{1}{2},k-\tfrac{1}{2}\right)$$

are taken at $\left(n+\frac{1}{2},k+\frac{1}{2}\right)$ and $\left(n+\frac{1}{2},k-\frac{1}{2}\right)$, and their midpoint is $\left(n+\frac{1}{2},k\right)$. Therefore, in (3.3), both sides are centered at the same point: $\left(n+\frac{1}{2},k\right)$.

Similarly, both sides of (3.4) are centered at the same point: $\left(n,k+\frac{1}{2}\right)$. Thus, the two Cauchy–Riemann equations express the equality of perpendicular differences centered at the same corresponding points.

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

4 A discrete Hilbert transform and properties

4.1 A discrete Hilbert transform

The above constructions lead us to the following definition of a discrete Hilbert transform, H_{d}, for a sequence $(a_n)_{n\in\mathbb{Z}}$ (Definition 1.1):

$$(H_{\mathrm{d}}a)_n = \mathcal{F}^{-1}\big[(-\mathrm{i}\,\mathrm{sgn}\,\theta)\,\rho(\theta)^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}\widehat{a}(\theta)\big](n).$$

We have thus proven all assertions of Theorem 1.1, and in this section, we will consider several properties of H_{d}.

The discrete Hilbert transform we are considering has the Fourier multiplier:

$$m_d(\theta) = -\mathrm{i}\,\mathrm{sgn}\,\theta\,\rho(\theta)^{1/2}\,\mathrm{e}^{\mathrm{i}\theta/2}.$$

This multiplier was forced by the Cauchy–Riemann system we considered. Nevertheless, since the Fourier multiplier is bounded, we immediately get the boundedness of the discrete Hilbert transform on $\ell^2(\mathbb{Z})$. We prove stronger boundedness properties below.

4.2 Properties of the discrete Hilbert transform

The kernel of our discrete Hilbert transform model is given by

$$\begin{aligned} K_{\mathrm{d}}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(-\mathrm{i}\,\mathrm{sgn}\,\theta)\,\rho(\theta)^{1/2}\,\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\theta}\,d\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi}\rho(\theta)^{1/2}\,\sin\big((n+\tfrac{1}{2})\theta\big)\,d\theta, \end{aligned} \tag{4.1}$$

so $H_{\mathrm{d}}a = K_{\mathrm{d}} * a$.

By using the Taylor expansion of $\rho(\theta)$, it is possible to integrate and express K_{d} explicitly as an infinite series in n. However, the above integral representation of K_{d} is more useful when exploring properties and weighted bounds of H_{d}, and we will not give the more complicated formula for it in terms of a series representation.

Proposition 4.1 (Kernel antisymmetry and cancellation) *The kernel K_{d} is real-valued. Moreover, it is antisymmetric:*

$$K_{\mathrm{d}}(-n-1) = -K_{\mathrm{d}}(n), \quad n\in\mathbb{Z},$$

and in particular, $\sum_{n\in\mathbb{Z}} K_{\mathrm{d}}(n) = 0$ (understood as pairwise cancellation between pairs n and $-n-1$).

Proof Formula (4.1) shows that the kernel is real-valued. Let $m_d(\theta) = (-\mathrm{i}\,\mathrm{sgn}\,\theta)\,\rho^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}$ be the Fourier multiplier. Then,

$$\begin{aligned} m_d(-\theta) &= (-\mathrm{i}\,\mathrm{sgn}(-\theta))\rho^{1/2}\mathrm{e}^{-\mathrm{i}\theta/2} \\ &= \mathrm{i}\,\mathrm{sgn}\,\theta\,\rho^{1/2}\mathrm{e}^{-\mathrm{i}\theta/2} \\ &= -\,\mathrm{e}^{-\mathrm{i}\theta}\,m_d(\theta). \end{aligned}$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

Thus, we get that

$$\begin{aligned} K_{\mathrm{d}}(-n-1) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} m_d(\theta)\,\mathrm{e}^{-\mathrm{i}(n+1)\theta}\,d\theta \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} m_d(-\theta)\,e^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{e}^{-\mathrm{i}\theta}\,d\theta \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} m_d(\theta)\,\mathrm{e}^{\mathrm{i}n\theta}\,d\theta \\ &= -K_{\mathrm{d}}(n). \end{aligned}$$

The pairwise cancellation of the series $\sum_n K_{\mathrm{d}}(n)$ follows. ∎

4.3 Asymptotics of $K_{\mathrm{d}}(n)$ and $\ell^p(w)$ bounds

We next compare H_{d} to the kernel of the Riesz–Titchmarsh transform

$$\begin{aligned} K_{H^+}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(-\mathrm{i}\,\mathrm{sgn}\,\theta)\;\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\theta}\,d\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi}\sin\bigl((n+\tfrac{1}{2})\theta\bigr)\,d\theta \\ &= \frac{1}{\pi}\frac{1}{n+\frac{1}{2}}. \end{aligned}$$

We define the difference kernel $D(n) := K_{\mathrm{d}}(n) - K_{H^+}(n)$ and prove that $D \in \ell^1(\mathbb{Z})$, while giving an explicit bound.

Weighted $\ell^p(w)$ bounds are an immediate corollary. A weight is a function $w : \mathbb{Z} \to (0,\infty)$. For integers $M \le N$, write $I = [M,N] \cap \mathbb{Z}$ and $|I| = N - M + 1$. For $1 < p < \infty$, the Muckenhoupt class A_p [10] consists of all w for which there exists $C < \infty$ such that, for every pair $M \le N$,

$$\Bigl(\sum_{k=M}^{N} w(k)\Bigr)\Bigl(\sum_{k=M}^{N} w(k)^{-1/(p-1)}\Bigr)^{p-1} \;\le\; C\,|I|^p. \tag{4.2}$$

For A_1, one requires that, for all $M \le N$,

$$\Bigl(\sum_{k=M}^{N} w(k)\Bigr)\max_{k\in I}\frac{1}{w(k)} \;\le\; C\,|I|. \tag{4.3}$$

Now, if we set

$$f(\theta) := \rho(\theta)^{1/2} - 1, \qquad \theta \in [0,\pi],$$

the difference kernel is given by

$$D(n) = K_{\mathrm{d}}(n) - K_{H^+}(n) \;=\; \frac{1}{\pi}\int_{0}^{\pi} f(\theta)\,\sin\bigl((n+\tfrac{1}{2})\theta\bigr)\,d\theta. \tag{4.4}$$

We prove that $D(n) = O(n^{-2})$; hence, $K_{\mathrm{d}}(n) = \frac{1}{\pi(n+\frac{1}{2})} + O(n^{-2})$ as $|n| \to \infty$.

Starting from Equation (3.2): $(\rho-1)^2 = 4\sin^2(\frac{\theta}{2})\,\rho$, for $\theta \in [0,\pi]$, we get that

$$\rho(\theta)^{1/2} = \sqrt{1+\sin^2\tfrac{\theta}{2}} \;-\; \sin\tfrac{\theta}{2}.$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

Expanding around $\theta = 0$, we get that as $\theta \to 0$:

$$\begin{aligned}\rho(\theta)^{1/2} &= \left(1 + \tfrac{1}{2}\sin^2\tfrac{\theta}{2} + O(\theta^4)\right) - \sin\tfrac{\theta}{2} \\ &= 1 - \frac{\theta}{2} + \frac{\theta^2}{8} + O(\theta^3),\end{aligned}$$

so that

$$f(\theta) = \rho(\theta)^{1/2} - 1 = -\frac{\theta}{2} + \frac{\theta^2}{8} + O(\theta^3). \tag{4.5}$$

In particular, $f(0) = 0$, $f'(0) = -\frac{1}{2}$, and $f''(0) = \frac{1}{4}$. Using the explicit formula for $f(\theta)$, one also finds that $f'(\pi) = 0$ and that $f'' \in C[0, \pi]$.

Lemma 4.2 *The difference kernel $D = K_{\mathrm{d}} - K_{H^+}$ belongs to $\ell^1(\mathbb{Z})$, the asymptotic formula*

$$K_{\mathrm{d}}(n) = \frac{1}{\pi(n + \frac{1}{2})} + O(n^{-2})$$

is valid as $|n| \to \infty$, and for every $1 \le p \le \infty$,

$$\|H_{\mathrm{d}} - H^+\|_{\ell^p \to \ell^p} \;\le\; \|D\|_{\ell^1} < \infty.$$

Proof Integrating by parts twice in (4.4), we obtain

$$\begin{aligned}|D(n)| &= \frac{1}{\pi}\left|\int_0^\pi f(\theta)\,\sin\!\big((n+\tfrac{1}{2})\theta\big)\,d\theta\right| \\ &\le \frac{1}{\pi\left(n+\frac{1}{2}\right)^2}\int_0^\pi |f''(\theta)|\,d\theta \\ &= \frac{C_0}{\left(n+\frac{1}{2}\right)^2} \sim \frac{1}{n^2}.\end{aligned} \tag{4.6}$$

Here, we used the fact that $f \in C^2[0, \pi]$, so that $\int_0^\pi |f''(\theta)|\,d\theta < \infty$. This proves that $D(n) = O(n^{-2})$, which implies that $K_{\mathrm{d}}(n) = \frac{1}{\pi(n+\frac{1}{2})} + O(n^{-2})$ as $|n| \to \infty$. Moreover, by using $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{\left(n+\frac{1}{2}\right)^2} = \pi^2$, we find that $\|D\|_{\ell^1} < \infty$.

The difference

$$(H_{\mathrm{d}} - H^+)a = D * a$$

is convolution with the ℓ^1 kernel D, and the final claimed bound follows from the discrete Young's inequality. ∎

Using the fact that H^+ is a Calderón–Zygmund operator, and hence bounded on $\ell^p(\mathbb{Z})$ and $\ell^p(\mathbb{Z}; w)$ when $w \in A_p$, we obtain the following results.

Proposition 4.3 (Weak-type $(1,1)$ and ℓ^p bounds) *For $p > 1$, we have*

$$\|H_{\mathrm{d}} a\|_{\ell^p} \;\le\; \|H^+ a\|_{\ell^p} + \|D\|_{\ell^1}\,\|a\|_{\ell^p}.$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

In addition, since H^+ is of weak type $(1,1)$,

$$\#\bigl\{n:\ |H_{\mathrm{d}}a(n)|>\lambda\bigr\}\ \le\ \frac{C\,\|a\|_{\ell^1}}{\lambda}\ +\ \frac{\|D\|_{\ell^1}\,\|a\|_{\ell^1}}{\lambda}\qquad(\lambda>0),$$

so H_{d} is weak type $(1,1)$ and bounded on ℓ^p for $1<p<\infty$.

Proof Write $H_{\mathrm{d}}=H^+ + D*$. Young's inequality gives $\|D*a\|_{\ell^p}\le\|D\|_{\ell^1}\|a\|_{\ell^p}$, and $\|H^+\|_{\ell^p\to\ell^p}<\infty$ for $1<p<\infty$. For weak $(1,1)$, Chebyshev and ℓ^1-boundedness of $D*$ yield $\#\{|D*a|>\lambda\}\le\|D*a\|_{\ell^1}/\lambda\le\|D\|_{\ell^1}\|a\|_{\ell^1}/\lambda$, and we combine with the weak $(1,1)$ bound for H^+. ∎

By the proof of Lemma 4.2, the function $K(n):=C_0\,(|n|+\frac12)^{-2}\in\ell^1(\mathbb{Z})$ is a radially decreasing majorant of D. Then, $(|D|*|a|)(n)\ \le\ C\,Ma(n)$, where M is the discrete Hardy–Littlewood maximal operator. The weighted bounds for M give [14]

$$\|D*a\|_{\ell^p(w)}\lesssim\ [w]_{A_p}^{1/(p-1)}\ \|a\|_{\ell^p(w)},\qquad 1<p<\infty,$$

$$w\bigl(\{|D*a|>\lambda\}\bigr)\lesssim\ \frac{[w]_{A_1}}{\lambda}\,\|a\|_{\ell^1(w)},\qquad\lambda>0.$$

Thus, H_{d} is bounded on $\ell^p(w)$, if $w\in A_p$, strongly for $1<p<\infty$ and weakly for $p=1$.

We will compare H_{d} and H^+ at the level of their Fourier multipliers:

$$m_{\mathrm{d}}(\theta)=(-\mathrm{i}\operatorname{sgn}\theta)\,\rho(\theta)^{1/2}\,\mathrm{e}^{\mathrm{i}\theta/2},\qquad m_{H^+}(\theta)=(-\mathrm{i}\operatorname{sgn}\theta)\,\mathrm{e}^{\mathrm{i}\theta/2}.$$

The expansion (4.5) gives

$$m_{\mathrm{d}}(\theta)-m_+(\theta)=(-\mathrm{i}\operatorname{sgn}\theta)\bigl(\rho^{1/2}(\theta)-1\bigr)\mathrm{e}^{\mathrm{i}\theta/2}=\frac{\mathrm{i}}{2}\,|\theta|+O(\theta^2)\qquad(\theta\to0).$$

Thus, the difference multiplier vanishes to first order at the origin. On the circle, a C^1 multiplier with a simple zero at 0 has Fourier coefficients decaying as $O(|n|^{-2})$. This is precisely (4.6) proved by two integrations by parts, with the vanishing of $f(0)$ removing the $O(1/n)$ boundary term from the first integration by parts.

5 Harmonic extensions to $\mathbb{Z}^s\times\mathbb{N}$

In this section, we extend our construction to $\mathbb{Z}^s\times\mathbb{N}$.

Let e_j denote the jth coordinate vector in $\mathbb{Z}^s$. Write $\delta^{\pm}_{x_j}$ and $\partial^{\pm}_k$ for the horizontal and vertical forward/backward differences, respectively, so that for every $x\in\mathbb{Z}^s$ and $k\ge1$, we have

$$\delta^+_{x_j}F(x,k)=F(x+e_j,k)-F(x,k),\quad \delta^-_{x_j}F(x,k)=F(x,k)-F(x-e_j,k),$$
$$\partial^+_kF(x,k)=F(x,k+1)-F(x,k),\quad \partial^-_kF(x,k)=F(x,k)-F(x,k-1).$$

5.1 The $(2s+3)$-point Laplacian and the Poisson extension

The fully discrete $(2s+3)$-point Laplacian on $\mathbb{Z}^s\times\mathbb{N}$ is

$$(\Delta_sF)(x,k)=\sum_{j=1}^{s}\bigl(F(x+e_j,k)+F(x-e_j,k)\bigr)+F(x,k+1)+F(x,k-1)-2(s+1)F(x,k),$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

so $\Delta_s = \sum_{j=1}^{s} \delta_{x_j}^- \delta_{x_j}^+ + \partial_k^- \partial_k^+$. Given a boundary sequence $a : \mathbb{Z}^s \to \mathbb{C}$ at height $k = 0$, we seek $U : \mathbb{Z}^s \times \mathbb{N} \to \mathbb{C}$ with $U(\cdot, 0) = a$ and $\Delta_s U = 0$.

Let $\mathbb{T}^s = (-\pi, \pi]^s$ and set

$$\widehat{a}(\theta) = \sum_{x \in \mathbb{Z}^s} a(x)\, e^{-\mathrm{i}x\cdot\theta}, \qquad \mathcal{F}_s^{-1}[g](x) = \frac{1}{(2\pi)^s} \int_{\mathbb{T}^s} g(\theta)\, e^{\mathrm{i}x\cdot\theta}\, d\theta, \ \theta = (\theta_1, \dots, \theta_s).$$

Taking the horizontal Fourier transform of $\Delta_s U = 0$ yields, for each fixed $\theta \in \mathbb{T}^s$,

$$\widehat{U}(\theta, k+1) + \widehat{U}(\theta, k-1) + \Big(2\sum_{j=1}^{s} \cos\theta_j - 2(s+1)\Big)\widehat{U}(\theta, k) = 0.$$

Looking for separated solutions $\widehat{U}(\theta, k) = C(\theta)\, \rho(\theta)^k$ gives

$$\rho + \rho^{-1} = 2\Big((s+1) - \sum_{j=1}^{s} \cos\theta_j\Big).$$

We keep the root in $(0, 1]$ to ensure boundedness as $k \to \infty$:

$$\rho(\theta) \;=\; \alpha(\theta) - \sqrt{\alpha(\theta)^2 - 1}, \qquad \alpha(\theta) := (s+1) - \sum_{j=1}^{s} \cos\theta_j \in [1, 2s+1]. \tag{5.1}$$

Imposing $\widehat{U}(\theta, 0) = \widehat{a}(\theta)$ yields

$$\widehat{U}(\theta, k) = \rho(\theta)^k\, \widehat{a}(\theta), \qquad U(x, k) = \mathcal{F}_s^{-1}\big[\rho(\cdot)^k \widehat{a}\big](x) = \sum_{y \in \mathbb{Z}^s} P_k^{(s)}(x - y)\, a(y),$$

where the s-dimensional discrete Poisson kernel is

$$P_k^{(s)}(x) = \frac{1}{(2\pi)^s} \int_{\mathbb{T}^s} \rho(\theta)^k\, e^{\mathrm{i}x\cdot\theta}\, d\theta.$$

Similarly as in Lemma 2.1, $P_k^{(s)} \geq 0$, $P_k^{(s)} \in \ell^1(\mathbb{Z}^s)$, and $\sum_{x \in \mathbb{Z}^s} P_k^{(s)}(x) = 1$ for every $k \in \mathbb{N}$.

5.2 A vector conjugate function and the discrete Cauchy–Riemann system

For the conjugate part, we construct s components $V = (V_1, \dots, V_s)$.

We introduce

$$\omega_j(\theta) = \begin{cases} \dfrac{\operatorname{sgn}\theta_j\, \sin(\theta_j/2)}{\Big(\sum_{j=1}^{s} \sin^2\frac{\theta_j}{2}\Big)^{1/2}}, & \theta \neq (0, 0, \dots, 0), \\ 0, & \theta = 0, \end{cases} \qquad j = 1, \dots, s.$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

The key algebraic identity now reads

$$\rho(\theta)-1 \;=\; -\rho(\theta)^{1/2}\sum_{j=1}^{s}\omega_j(\theta)\left(e^{\mathrm{i}\theta_j}-1\right)\left(-\mathrm{i}\operatorname{sgn}\theta_j\right)e^{-\mathrm{i}\theta_j/2}, \tag{5.2}$$

and it is derived using the identity:

$$\frac{(\rho(\theta)-1)^2}{\rho(\theta)} \;=\; 2\sum_{j=1}^{s}\bigl(1-\cos\theta_j\bigr) \;=\; 4\sum_{j=1}^{s}\sin^2\frac{\theta_j}{2}, \qquad \theta\in\mathbb{T}^s.$$

We define $\widehat{V}_j$ on the Fourier side by

$$\widehat{V}_j(\theta,k)=\left(-\mathrm{i}\operatorname{sgn}\theta_j\right)\omega_j(\theta)\,\rho(\theta)^{\,k+\frac{1}{2}}\,e^{\mathrm{i}\theta_j/2}\,\widehat{a}(\theta), \qquad \theta\in\mathbb{T}^s,\ k\in\mathbb{N}. \tag{5.3}$$

If we let $V_j(x,k)=\mathcal{F}_s^{-1}[\widehat{V}_j(\cdot,k)](x)$, then each V_j is harmonic with respect to Δ_s.

We obtain the following equations, analogous to the generalized Cauchy–Riemann system in higher dimensions [14, p. 120].

Proposition 5.1 (Discrete Cauchy–Riemann system in $\mathbb{Z}^s\times\mathbb{N}$) *With U and $V=(V_1,\dots,V_s)$ defined above, for every $x\in\mathbb{Z}^s$ and $k\ge 1$, we have*

$$\partial_k^+U(x,k)=-\sum_{j=1}^{s}\delta_{x_j}^-V_j(x,k), \tag{5.4}$$

$$\delta_{x_j}^+U(x,k)=\partial_k^-V_j(x,k) \qquad (j=1,\dots,s), \tag{5.5}$$

$$\delta_{x_j}^+V_l(x,k)=\delta_{x_l}^+V_j(x,k) \qquad (j,l=1,\dots,s). \tag{5.6}$$

Proof Using the shift rules and the definitions,

$$\mathcal{F}_s\bigl[\partial_k^+U\bigr]=(\rho-1)\,\rho^k\widehat{a}, \qquad \mathcal{F}_s\bigl[\delta_{x_j}^-V_j\bigr]=(1-\mathrm{e}^{-\mathrm{i}\theta_j})\,(-\mathrm{i}\operatorname{sgn}\theta_j)\,\omega_j\,\rho^{k+\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\theta_j/2}\widehat{a}.$$

Therefore, (5.4) reduces to (5.2). The proof of (5.5) is similar.

To prove (5.6), we use that, for all j and l, the following identity holds:

$$\left(\mathrm{e}^{\mathrm{i}\theta_j}-1\right)\left(-\mathrm{i}\operatorname{sgn}\theta_l\right)\omega_l(\theta)\,\mathrm{e}^{\mathrm{i}\theta_l/2}=\left(\mathrm{e}^{\mathrm{i}\theta_l}-1\right)\left(-\mathrm{i}\operatorname{sgn}\theta_j\right)\omega_j(\theta)\,\mathrm{e}^{\mathrm{i}\theta_j/2}.$$

■

Thus, the complex vector field $F=(U,V_1,\dots,V_s)$ obeys a lattice Cauchy–Riemann system whose elimination produces the $(2s+3)$-point Laplace equation.

5.3 Boundary transforms and properties

At height $k=0$, (5.3) gives the boundary operators T_j with Fourier multipliers

$$\begin{aligned} m_j(\theta) &= \left(-\mathrm{i}\operatorname{sgn}\theta_j\right)\omega_j(\theta)\,\rho(\theta)^{1/2}\,e^{\mathrm{i}\theta_j/2} \\ &= -\mathrm{i}\,\frac{\sin(\theta_j/2)}{\left(\sum_{j=1}^{s}\sin^2\frac{\theta_j}{2}\right)^{1/2}}\,\rho(\theta)^{1/2}\,e^{\mathrm{i}\theta_j/2}. \end{aligned}$$

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

Therefore, for $x = (x_1, \dots, x_s) \in \mathbb{Z}^s$, we have

$$(T_j a)(x) = \sum_{y \in \mathbb{Z}^s} K_{T_j}(x-y)\, a(y), \quad \text{where} \quad K_{T_j}(x) = \frac{1}{(2\pi)^s} \int_{\mathbb{T}^s} m_j(\theta)\, \mathrm{e}^{\mathrm{i}x\cdot\theta}\, d\theta.$$

We get that the kernel K_{T_j} is real-valued. Moreover, for every $x \in \mathbb{Z}^s$,

$$K_{T_j}(-x - e_j) = -K_{T_j}(x),$$

and we have a similar cancellation property as in Proposition 4.1: $\sum_{x\in\mathbb{Z}^s} K_{T_j}(x) = 0$.

Consider the following Riesz transforms R_j^+ on $\mathbb{Z}^s$, defined by their Fourier multipliers

$$\widehat{R_j^+ a}(\theta) = m_{R_j}(\theta)\, \widehat{a}(\theta), \qquad m_{R_j}(\theta) := \big(-\mathrm{i}\,\mathrm{sgn}\,\theta_j\big)\, \omega_j(\theta)\, e^{\mathrm{i}\theta_j/2}.$$

These operators are Calderón–Zygmund operators on the discrete space $\mathbb{Z}^s$. Indeed, the size and smoothness estimates for their corresponding kernels follow from the expansions of the multipliers near $\theta = 0$: $m_{R_j}(\theta) = -\mathrm{i}\,\frac{\theta_j}{|\theta|} + O(|\theta|)$.

As before, we obtain the mapping properties of T_j by comparing them to the transforms R_j^+. For $1 < p < \infty$ and weights $w \in A_p(\mathbb{Z}^s)$, the operators are bounded on $\ell^p(w;\mathbb{Z}^s)$. Moreover, when $w \in A_1(\mathbb{Z}^s)$, each T_j is of weak type $(1,1)$; that is, $T_j : \ell^1(w;\mathbb{Z}^s) \to \ell^{1,\infty}(w;\mathbb{Z}^s)$.

Acknowledgments The author would like to thank María Cristina Pereyra and Brett D. Wick for useful suggestions and discussions, and the anonymous referee for a careful reading of the manuscript and a thoughtful observation.

References

[1] A. C. Allen and B. H. Murdoch, *A note on preharmonic functions*. Proc. Amer. Math. Soc. **4**(1953), 842–852. MR0060650
[2] N. Arcozzi, K. Domelevo, and S. Petermichl, *Discrete Hilbert transform à la Gundy–Varopoulos*. Proc. Amer. Math. Soc. **148**(2020), 2433–2446. MR4080886
[3] R. Bañuelos, D. Kim, and M. Kwaśnicki, *Sharp ℓ^p inequalities for discrete singular integrals on the lattice $\mathbb{Z}^d$*. J. Funct. Anal. **290** (2026), no. 9, 111359. https://doi.org/10.1016/j.jfa.2026.111359
[4] R. Bañuelos and M. Kwaśnicki, *On the ℓ^p norm of the discrete Hilbert transform*. Duke Math. J. **168**(2019), no. 3, 471–504. MR3909902
[5] R. Bañuelos and M. Kwaśnicki, *The ℓ^p norm of the Riesz–Titchmarsh transform for even integer p*. J. Lond. Math. Soc. **109**(2024), no. 4, e12888, 1–21. MR4727420
[6] Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea, and J. L. Varona, *Harmonic analysis associated with a discrete Laplacian*. J. Anal. Math. **132**(2017), 109–131. MR3666807
[7] R. J. Duffin, *Discrete potential theory*. Duke Math. J. **20**(1953), no. 2, 233–251. MR0070031. https://doi.org/10.1215/S0012-7094-53-02023-7
[8] R. J. Duffin, *Basic properties of discrete analytic functions*. Duke Math. J. **23**(1956), 335–363. MR78441
[9] J. Ferrand, *Fonctions préharmoniques et fonctions préholomorphes (French)*. Bull. Sci. Math. (2) **68** (1944), 152–180. MR13411
[10] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*. Trans. Amer. Math. Soc. **176**(1973), 227–251. MR312139
[11] R. P. Isaacs, *A finite difference function theory*. Univ. Nac. Tucumán. Revista A. **2**(1941), 177–201. MR0006391
[12] T. W. Körner, *Fourier analysis*, Cambridge University Press, Cambridge, 1988.
[13] M. Riesz, *Sur les fonctions conjuguées*. Math. Z. **27**(1928), 218–244. MR1544909

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press

[14] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton University Press, Princeton, NJ, 1993.
[15] E. C. Titchmarsh, *Reciprocal formulae involving series and integrals*. Math. Z. **25**(1926), 321–341. Correction: Math. Z. 26 (1927), 496. MR1544814
[16] E. C. Titchmarsh, *An inequality in the theory of series*. J. Lond. Math. Soc. 3(1928), 81–83.

Department of Mathematics, University of Southern California, USA
e-mail: lpetrov@usc.edu

https://doi.org/10.4153/S0008439526102641 Published online by Cambridge University Press