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ON THE EIGENVALUES AND THE NODAL POINTS OF THE EIGENFUNCTIONS OF SOME EIGENVALUE PROBLEMS WITH EIGENPARAMETER-DEPENDENT BOUNDARY CONDITIONS

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Abstract. Consider the following two eigenvalue problems:

$$\begin{cases} y''(x) + [\lambda^2 - q(x)]y(x) = 0,\ 0 \le x \le \pi, \\ y(0) = 0,\ ay'(\pi) + \lambda y(\pi) = 0, \end{cases} \tag{0.1}$$

and

$$\begin{cases} z''(x) + [\mu^2 - q(x)]z(x) = 0,\ 0 \le x \le \pi, \\ z'(0) = 0,\ az'(\pi) + \mu z(\pi) = 0, \end{cases} \tag{0.2}$$

where $q(x)$ is real-valued and integrable on $[0, \pi]$. Let $\{\lambda_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ and $\{\mu_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ denote the eigenvalues of equations (0.1) and (0.2), respectively. Then

$$\cdots < \mu_{-3} < \lambda_{-2} < \mu_{-2} < \lambda_{-1} < \mu_{-1} < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \mu_3 < \cdots .$$

Moreover, the number of zeros of the eigenfunctions of (0.1) ((0.2), respectively) corresponding to λ_n (μ_n, respectively) in $(0, \pi)$ is equal to $|n| - 1$.

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1. Introduction. In recent years, the eigenvalue problems of second-order differential equations with eigenparameter-dependent boundary conditions had been studied by some mathematicians (see [**2**, **4**, **5**, **9**, **12**]). These kinds of eigenvalue problems are very interesting and different from the classical eigenvalue problems. In [**9**], Hochstadt considered the following eigenvalue problem:

$$\begin{cases} y''(x) + [\lambda^2 - q(x)]y(x) = 0,\ 0 \le x \le \pi, \\ y(0) = 0,\ ay'(\pi) + \lambda y(\pi) = 0, \end{cases} \tag{1.1}$$

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where $a \neq 0$, $a \in \mathbb{R}$, and $q(x)$ is real-valued and integrable on $[0, \pi]$. Let $y(x; \nu)$ denote the solution of the initial-value problem

$$y''(x) + [\nu - q(x)]y(x) = 0, y(0) = 0, y'(0) = 1.$$

Denote

$$C_1(\lambda) = ay'(\pi; \lambda^2) + \lambda y(\pi; \lambda^2),$$

and call it the *characteristic function* of equation (1.1). Then, λ_* is an eigenvalue of equation (1.1) if and only if λ_* is a zero of $C_1(\lambda)$. The number of linearly independent solutions corresponding to λ_* is called the *geometric multiplicity* of λ_*, and the multiplicity of λ_* as a zero of $C_1(\lambda)$ is called the *algebraic multiplicity* of λ_*.

Under some assumptions, Hochstadt used the idea of *inverse Liouville transformation* to transform equation (1.1) into a string equation of the form

$$\begin{cases} z''(x) + \lambda^2 M^4(x) z(x) = 0, \ 0 \leq x \leq c, \\ z(0) = 0, \ az'(c) + \lambda z(c) = 0, \end{cases} \tag{1.2}$$

where $c = \int_0^\pi \frac{1}{M^2(t)} dt$. Then, he transformed equation (1.2) into an integral equation and applied the properties of the integral operator corresponding to this integral equation to obtain the following result.

THEOREM 1.1. *All eigenvalues of equation* (1.1) *are real, geometrically simple, and algebraically simple.*

In [**5**], under the assumption that $q(x) \geq 0$ in $[0, \pi]$, we proved Theorem 1.1 by an approach which is different from Hochstadt's. In Section 3 of this paper, we consider another kind of eigenvalue problem:

$$\begin{cases} z''(x) + [\mu^2 - q(x)]z(x) = 0, \ 0 \leq x \leq \pi, \\ z'(0) = 0, \ az'(\pi) + \mu z(\pi) = 0, \end{cases} \tag{1.3}$$

where $a \neq 0$, $a \in \mathbb{R}$, and $q(x)$ is real-valued and integrable on $[0, \pi]$. Also under the assumption $q(x) \geq 0$ in $[0, \pi]$, we prove that the eigenvalues of equation (1.3) are real, geometrically simple, and algebraically simple by the method used in [**5**] (see Theorems 3.1 and 3.3).

It is natural to whether there is an order between the eigenvalues of equations (1.1) and (1.3), which have different boundary conditions. The problem about comparing the eigenvalues of equations with different boundary conditions had been thoroughly investigated for the classical Sturm–Liouville problems.

Consider the following Sturm–Liouville eigenvalue problems:

$$\begin{cases} y''(x) + [\lambda - q(x)]y(x) = 0, \ 0 \leq x \leq \pi, \\ y(0) = 0, \ y(\pi) = 0, \end{cases} \tag{1.4}$$

$$\begin{cases} z''(x) + [\mu - q(x)]z(x) = 0, \ 0 \leq x \leq \pi, \\ z'(0) = 0, \ z'(\pi) = 0, \end{cases} \tag{1.5}$$

where $q(x)$ is real-valued and integrable on $[0, \pi]$. We call equation (1.4) the *Dirichlet* boundary value problem and equation (1.5) the *Neumann* boundary value problem. Let $\{\sigma_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=0}^\infty$ denote the eigenvalues of equations (1.4) and (1.5), respectively. Then, $\{\sigma_n\}_{n=1}^\infty$ and $\{\nu_n\}_{n=0}^\infty$ are interlacing as shown below (see [**7**]):

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$$\nu_0 < \{\sigma_1, \nu_1\} < \{\sigma_2, \nu_2\} < \{\sigma_3, \nu_3\} < \cdots .$$

Following the classical cases, we shall call equations (1.1) and (1.3) the *Dirichlet-type* eigenvalue problem and the *Neumann-type* eigenvalue problem, respectively. By observing a special example (see the beginning of Section 5 for related discussions), we find there is an interlacing relation between the eigenvalues of equations (1.1) and (1.3) for the case $q(x) \equiv 0$.

To investigate whether there is an interlacing relation between the eigenvalues of equations (1.1) and (1.3) for the general case, we use the following approach: First, in Section 4, we construct a function $E_1(\lambda)$, which is similar to that constructed in [**1**, Chapter 8] for studying the classical Sturm–Liouville problem, and apply this $E_1(\lambda)$ to show that the eigenvalues of equation (1.1) form an infinite sequence which is neither bounded above nor bounded below. We also apply $E_1(\lambda)$ to find the number of zeros of the corresponding eigenfunctions of equation (1.1). Then, in Section 5, we prove for the case $q(x) \geq 0$, there is an interlacing property between the eigenvalues of equations (1.1) and (1.3) (see Theorem 5.3); as a consequence of this result, we find the eigenvalues of the *Neumann-type* eigenvalue problem (1.3) form an infinite sequence which is neither bounded above nor bounded below. We also find the number of zeros of the eigenfunctions of equation (1.3) in the interval $[0, \pi]$ (see Theorem 5.6). Some related notations and basic properties are presented in Section 2.

2. Preliminaries and notations. For $q(x)$ being a real-valued integrable function defined in $[0, \pi]$, let $y(x; \nu)$ and $z(x; \nu)$ denote the solutions of the following initial-value problems, respectively,

$$y''(x) + [\nu - q(x)]y(x) = 0,\ y(0) = 0,\ y'(0) = 1, \tag{2.1}$$

$$z''(x) + [\nu - q(x)]z(x) = 0,\ z(0) = 1,\ z'(0) = 0. \tag{2.2}$$

We have the following basic information about these functions (see [**11**]):

LEMMA 2.1. *For fixed $x \in [0, \pi]$, the functions $y(x; \lambda^2)$, $y'(x; \lambda^2)$, $z(x; \lambda^2)$, and $z'(x; \lambda^2)$ are entire functions of order 1 with respect to variable λ, and*

$$y(x; \lambda^2) = \frac{\sin \sqrt{\lambda^2}x}{\sqrt{\lambda^2}} + O\left(\frac{e^{|Im\sqrt{\lambda^2}|x}}{|\lambda|^2}\right), |\lambda| \to \infty, \tag{2.3}$$

$$y'(x; \lambda^2) = \cos \sqrt{\lambda^2}x + O\left(\frac{e^{|Im\sqrt{\lambda^2}|x}}{|\lambda|}\right), |\lambda| \to \infty, \tag{2.4}$$

$$z(x; \lambda^2) = \cos \sqrt{\lambda^2}x + O\left(\frac{e^{|Im\sqrt{\lambda^2}|x}}{|\lambda|}\right), |\lambda| \to \infty, \tag{2.5}$$

$$z'(x; \lambda^2) = -\sqrt{\lambda^2} \sin \sqrt{\lambda^2}x + O\left(e^{|Im\sqrt{\lambda^2}|x}\right), |\lambda| \to \infty. \tag{2.6}$$

Let $a \neq 0$, $a \in \mathbb{R}$, and $q(x)$ be a real-valued integrable function defined in $[0, \pi]$. Consider the eigenvalue problem

$$\begin{cases} y''(x) + [\lambda^2 - q(x)]y(x) = 0, \\ y(0) = 0,\ ay'(\pi) + \lambda y(\pi) = 0, \end{cases} \tag{2.7}$$

and define

$$C_1(\lambda) = ay'(\pi; \lambda^2) + \lambda y(\pi; \lambda^2).$$

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It is easy to see that λ_* is an eigenvalue of equation (2.7) if and only if λ_* is a zero of $C_1(\lambda)$ (see [9]). The function $C_1(\lambda)$ shall be called the *characteristic function* of equation (2.7). It is an entire function of order 1 in the variable λ. By Lemma 3.2 in [5], we know that 0 is not an eigenvalue of equation (2.7). Let λ_n denote the eigenvalues of equation (2.7), arranged in ascending order, with $n\lambda_n > 0$, where $n \in \mathbb{Z} - \{0\}$, i.e.,

$$\cdots \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots .$$

Under the same conditions on a and $q(x)$, we consider another eigenvalue problem

$$\begin{cases} z''(x) + [\mu^2 - q(x)]z(x) = 0, \\ z'(0) = 0,\ az'(\pi) + \mu z(\pi) = 0, \end{cases} \tag{2.8}$$

and define

$$C_2(\mu) = az'(\pi;\, \mu^2) + \mu z(\pi;\, \mu^2).$$

Similarly, μ_* is an eigenvalue of equation (2.8) if and only if μ_* is a zero of $C_2(\mu)$. And the function $C_2(\mu)$ shall be called the *characteristic function* of equation (2.8). It is an entire function of order 1 in the variable μ. Let μ_n denote the eigenvalues of equations (2.8), arranged in ascending order, with $n\mu_n \ge 0$, where $n \in \mathbb{Z} - \{0\}$, i.e.,

$$\cdots \le \mu_{-2} \le \mu_{-1} \le 0 \le \mu_1 \le \mu_2 \le \cdots .$$

According to the classical Sturm–Liouville eigenvalue problems, we may call equation (2.7) the *Dirichlet-type* eigenvalue problem and call equation (2.8) the *Neumann-type* eigenvalue problem in this paper.

3. The *Neumann-type* eigenvalue problems. In this section, we prove some properties of the eigenvalues of equation (2.8) under the assumption that $q(x) \ge 0$ in $[0, \pi]$.

THEOREM 3.1. *All eigenvalues of equation* (2.8) *are real.*

Proof. Suppose μ is an eigenvalue of equation (2.8), and $z(x)$ is a corresponding eigenfunction. Then

$$z''(x)\bar{z}(x) + [\mu^2 - q(x)]|z(x)|^2 = 0. \tag{3.1}$$

Integrating both sides of (3.1) from 0 to π and using integration by parts, we obtain

$$z'(x)\bar{z}(x)\big|_0^\pi - \int_0^\pi |z'(x)|^2 dx + \mu^2 \int_0^\pi |z(x)|^2 dx - \int_0^\pi q(x)|z(x)|^2 dx = 0. \tag{3.2}$$

Note that

$$z'(0) = 0,\ az'(\pi) + \mu z(\pi) = 0,\ a \ne 0,$$

then by (3.2), we have

$$\mu^2 \int_0^\pi |z(x)|^2 dx - \frac{\mu}{a}|z(\pi)|^2 - \left(\int_0^\pi |z'(x)|^2 dx + \int_0^\pi q(x)|z(x)|^2 dx \right) = 0. \tag{3.3}$$

As $q(x) \ge 0$ in $[0, \pi]$,

$$\int_0^\pi |z'(x)|^2 dx + \int_0^\pi q(x)|z(x)|^2 dx \ge 0,$$

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Because (3.3) is a quadratic equation of μ with discriminant

$$\frac{1}{a^2}|z(\pi)|^4 + 4\left(\int_0^\pi |z(x)|^2 dx\right)\left(\int_0^\pi |z'(x)|^2 dx + \int_0^\pi q(x)|z(x)|^2 dx\right) \geq 0,$$

we find that μ must be real. □

Using (3.3), it is easy to see the following result holds.

THEOREM 3.2. *Zero is an eigenvalue of equation* (2.8) *if and only if* $q(x) \equiv 0$ *in* $[0, \pi]$.

Now, let $z_1(x)$ and $z_2(x)$ be two eigenfunctions of equation (2.8) corresponding to the eigenvalue μ_*. Since

$$z_1''(x) + [\mu_*^2 - q(x)]z_1(x) = 0,$$
$$z_2''(x) + [\mu_*^2 - q(x)]z_2(x) = 0,$$

we have

$$z_1''(x)z_2(x) - z_1(x)z_2''(x) = 0.$$

Thus for all x in $[0, \pi]$,

$$z_1'(x)z_2(x) - z_1(x)z_2'(x) = z_1'(0)z_2(0) - z_1(0)z_2'(0) = 0.$$

This tells us that $z_1(x)$ and $z_2(x)$ are linearly dependent, i.e., μ_* is *geometrically* simple. Moreover, the eigenvalues of equation (2.8) are *algebraically* simple, as shown below.

THEOREM 3.3. *All eigenvalues of equation* (2.8) *are algebraically simple.*

Proof. We only need to show that if μ_* is an eigenvalue of equation (2.8), then

$$C_2'(\mu_*) \neq 0.$$

Note that

$$C_2'(\mu) = 2\mu[az_\nu'(\pi; \mu^2) + \mu z_\nu(\pi; \mu^2)] + z(\pi; \mu^2). \tag{3.4}$$

Since $z(x; \mu^2)$ satisfies equation (2.2), we have

$$\begin{cases} z''\left(x; \mu_*^2\right) + \left[\mu_*^2 - q(x)\right] z\left(x; \mu_*^2\right) = 0, \\ z\left(0; \mu_*^2\right) = 1, z'\left(0; \mu_*^2\right) = 0. \end{cases} \tag{3.5}$$

We also have the following relation obtained from differentiating (2.2) corresponding to μ:

$$\begin{cases} 2\mu_* z_\nu''\left(x; \mu_*^2\right) + 2\mu_*\left[\mu_*^2 - q(x)\right] z_\nu\left(x; \mu_*^2\right) + 2\mu_* z\left(x; \mu_*^2\right) = 0, \\ z_\nu'\left(0; \mu_*^2\right) = z_\nu\left(0; \mu_*^2\right) = 0. \end{cases} \tag{3.6}$$

As μ_* being real (see Theorem 3.1), equations (3.5) and (3.6) imply that

$$2\mu_*\left[z'\left(x; \mu_*^2\right) z_\nu\left(x; \mu_*^2\right) - z\left(x; \mu_*^2\right) z_\nu'\left(x; \mu_*^2\right)\right]\Big|_0^\pi = 2\mu_* \int_0^\pi \left|z\left(x; \mu_*^2\right)\right|^2 dx. \tag{3.7}$$

Then applying the boundary conditions for $z(x; \mu_*^2)$ and $z_\mu(x; \mu_*^2)$, (3.7) can be written as

$$2\mu_*\left[z'\left(\pi; \mu_*^2\right) z_\nu\left(\pi; \mu_*^2\right) - z\left(\pi; \mu_*^2\right) z_\nu'\left(\pi; \mu_*^2\right)\right] = 2\mu_* \int_0^\pi \left|z\left(x; \mu_*^2\right)\right|^2 dx. \tag{3.8}$$

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Because $az'(\pi;\mu_*^2)+\mu_* z(\pi;\mu_*^2)=0$, (3.8) implies

$$-\frac{2\mu_*}{a}z\left(\pi;\mu_*^2\right)\left[az'_\nu\left(\pi;\mu_*^2\right)+\mu_* z_\nu\left(\pi;\mu_*^2\right)\right]=2\mu_*\int_0^\pi |z\left(x;\mu_*^2\right)|^2dx. \tag{3.9}$$

Note that $z(\pi;\mu_*^2)\neq 0$ (otherwise $z'(\pi;\mu_*^2)=z(\pi;\mu_*^2)=0$), which implies $z(x;\mu_*^2)\equiv 0$. Thus, we have

$$2\mu_*\left[az'_\nu\left(\pi;\mu_*^2\right)+\mu_* z_\nu\left(\pi;\mu_*^2\right)\right]=\frac{-2a\mu_*}{z\left(\pi;\mu_*^2\right)}\int_0^\pi |z\left(x;\mu_*^2\right)|^2dx.$$

Applying this equation in (3.4), we find

$$C_2'(\mu_*)=\frac{1}{z\left(\pi;\mu_*^2\right)}\left[|z\left(\pi;\mu_*^2\right)|^2-2a\mu_*\int_0^\pi |z\left(x;\mu_*^2\right)|^2dx\right]. \tag{3.10}$$

On the other hand, multiplying (3.5) by $z(x;\mu_*^2)$, integrating the resulting equation from 0 to π, and using integration by parts, we have

$$\begin{aligned}&\left[z\left(x;\mu_*^2\right)z'\left(x;\mu_*^2\right)\right]|_0^\pi-\int_0^\pi |z'\left(x;\mu_*^2\right)|^2dx\\&\quad+\mu_*^2\int_0^\pi |z\left(x;\mu_*^2\right)|^2dx-\int_0^\pi q(x)|z\left(x;\mu_*^2\right)|^2dx=0.\end{aligned} \tag{3.11}$$

By means of the boundary conditions satisfied by $z(x;\mu_*^2)$, (3.11) can be written as

$$\begin{aligned}&-\frac{\mu_*}{a}\left[|z\left(\pi;\mu_*^2\right)|^2-a\mu_*\int_0^\pi |z\left(x;\mu_*^2\right)|^2dx\right]\\&\quad=\int_0^\pi |z'\left(x;\mu_*^2\right)|^2dx+\int_0^\pi q(x)|z\left(x;\mu_*^2\right)|^2dx.\end{aligned} \tag{3.12}$$

Now, we return to the evaluation of $C_2'(\mu_*)$. First, if 0 is an eigenvalue of equation (2.8), then by (3.4),

$$C_2'(0)=z(\pi;0)\neq 0.$$

If μ_* is an nonzero eigenvalue, then (3.10) and (3.12) imply

$$\begin{aligned}C_2'(\mu_*)&=\frac{1}{z\left(\pi;\mu_*^2\right)}\left[-\frac{a}{\mu_*}\left(\int_0^\pi |z'\left(x;\mu_*^2\right)|^2dx+\int_0^\pi q(x)|z\left(x;\mu_*^2\right)|^2dx\right)\right.\\&\quad\left.-a\mu_*\int_0^\pi |z\left(x;\mu_*^2\right)|^2dx\right]\\&=-\frac{a}{\mu_* z\left(\pi;\mu_*^2\right)}\left[\int_0^\pi |z'\left(x;\mu_*^2\right)|^2dx+\int_0^\pi q(x)|z\left(x;\mu_*^2\right)|^2dx\right.\\&\quad\left.+\mu_*^2\int_0^\pi |z\left(x;\mu_*^2\right)|^2dx\right]\neq 0.\end{aligned} \tag{3.13}$$

Hence, $C_2'(\mu_*)\neq 0$ for any eigenvalue of equation (2.8). □

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4. On the determination of the number of zeros of the eigenfunctions of the *Dirichlet-type* eigenvalue problem. In this section, we try to prove that the eigenvalues of equation (2.7) form an infinite sequence, which is neither bounded above nor bounded below, and to find the number of zeros of the corresponding eigenfunctions in $[0, \pi]$. For our purpose, we modify the construction presented in [**1**, Chapter 8] to construct the following function. Define

$$E_1(x; \lambda) = \frac{y'(x; \lambda^2) + i\lambda y(x; \lambda^2)}{y'(x; \lambda^2) - i\lambda y(x; \lambda^2)} = e^{i\theta_1(x;\lambda)}, \, x \in [0, \pi], \, \lambda \in \mathbb{R}. \tag{4.1}$$

Since for each $\lambda \in \mathbb{R}$, $(y'(x; \lambda^2))^2 + y^2(x; \lambda^2) \neq 0$ for all x in $[0, \pi]$, we have

$$(y'(x; \lambda^2))^2 + \lambda^2 y^2(x; \lambda^2) \neq 0$$

for all $x \in [0, \pi]$, $\lambda \in \mathbb{R} \setminus \{0\}$. Hence, $E_1(x; \lambda)$ is well-defined and is continuously differentiable with respect to x and λ by Lemma 2.1. Furthermore, using the equation that $y(x; 0)$ satisfies, we find

$$y(x; 0)y''(x; 0) = q(x)y^2(x; 0), \tag{4.2}$$

which implies

$$y(x; 0)y'(x; 0) = \int_0^x |y'(t; 0)|^2 dt + \int_0^x q(t)|y(t; 0)|^2 dt > 0, \, \forall x \in (0, \pi], \tag{4.3}$$

because $y(0; 0) = 0$. Applying (4.3), which implies that $y(x; 0) > 0$ in $(0, \pi]$, and $y'(0; 0) = 1$, we obtain

$$y'(x; 0) = y'(0; 0) + \int_0^x q(x)y(t; 0)dt \geq 1, \, \forall x \in [0, \pi]. \tag{4.4}$$

It is easy to see that λ_* is an eigenvalue of equation (2.7) if and only if

$$E_1(\pi; \lambda_*) = \frac{1 - ai}{1 + ai}.$$

The previous equation is equivalent to saying that

$$\theta_1(\pi; \lambda_*) = -2 \tan^{-1} a + 2k\pi, \, k \in \mathbb{Z}. \tag{4.5}$$

Calculating the derivatives of $E_1(x; \lambda)$ with respect to x and λ, respectively, we obtain

$$\begin{aligned} E_1'(x; \lambda) &= i\theta_1'(x; \lambda)E_1(x; \lambda) \\ &= \frac{2i\lambda}{[y'(x; \lambda^2) - i\lambda y(x; \lambda^2)]^2} \left[(y'(x; \lambda^2))^2 - y(x; \lambda^2)y''(x; \lambda^2)\right], \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} E_{1\lambda}(x; \lambda) &= i\theta_{1\lambda}(x; \lambda)E_1(x; \lambda) \\ &= \frac{2i[y(x; \lambda^2)y'(x; \lambda^2) + 2\lambda^2(y_\nu(x; \lambda^2)y'(x; \lambda^2) - y_\nu'(x; \lambda^2)y(x; \lambda^2))]}{[y'(x; \lambda^2) - i\lambda y(x; \lambda^2)]^2}. \end{aligned} \tag{4.7}$$

Equations (4.6) and (4.7) imply that for $x, \lambda \in \mathbb{R}$,

$$\theta_1'(x; \lambda) = \frac{2\lambda[(y'(x; \lambda^2))^2 - y(x; \lambda^2)y''(x; \lambda^2)]}{(y'(x; \lambda^2))^2 + \lambda^2 y^2(x; \lambda^2)}, \tag{4.8}$$

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and

$$\theta_{1\lambda}(x;\lambda)=\frac{2[y(x;\lambda^2)y'(x;\lambda^2)+2\lambda^2(y_\nu(x;\lambda^2)y'(x;\lambda^2)-y'_\nu(x;\lambda^2)y(x;\lambda^2))]}{(y'(x;\lambda^2))^2+\lambda^2y^2(x;\lambda^2)}. \tag{4.9}$$

The following lemma simplifies the expression of $\theta_{1\lambda}(x;\lambda)$.

LEMMA 4.1.

$$\theta_{1\lambda}(x;\lambda)=\frac{2\left[\int_0^x|y'(t;\lambda^2)|^2dt+\int_0^x(q(t)+\lambda^2)|y(t;\lambda^2)|^2dt\right]}{(y'(x;\lambda^2))^2+\lambda^2y^2(x;\lambda^2)},\ \forall x,\lambda\in\mathbb{R}.$$

Proof. Note that for $\lambda\in\mathbb{C}$, we have shown in [5] that

$$\begin{aligned}&y_\nu(x;\lambda^2)\bar{y}'(x;\lambda^2)-y'_\nu(x;\lambda^2)\bar{y}(x;\lambda^2)+(\bar{\lambda}^2-\lambda^2)\int_0^x y_\nu(t;\lambda^2)\bar{y}(t;\lambda^2)dt\\&\quad=\int_0^x|y(t;\lambda^2)|^2dt,\end{aligned} \tag{4.10}$$

and

$$\bar{y}(x;\lambda^2)y'(x;\lambda^2)+\lambda^2\int_0^x|y(t;\lambda^2)|^2dt=\int_0^x|y'(t;\lambda^2)|^2dt+\int_0^x q(t)|y(t;\lambda^2)|^2dt. \tag{4.11}$$

Since for $\lambda\in\mathbb{R}$,

$$y(x;\lambda^2)=\bar{y}(x;\lambda^2),$$

equations (4.10) and (4.11) can be rewritten as

$$y_\nu(x;\lambda^2)y'(x;\lambda^2)-y'_\nu(x;\lambda^2)y(x;\lambda^2)=\int_0^x|y(t;\lambda^2)|^2dt, \tag{4.12}$$

and

$$y(x;\lambda^2)y'(x;\lambda^2)+\lambda^2\int_0^x|y(t;\lambda^2)|^2dt=\int_0^x|y'(t;\lambda^2)|^2dt+\int_0^x q(t)|y(t;\lambda^2)|^2dt. \tag{4.13}$$

for $\lambda\in\mathbb{R}$. By (4.9), (4.12), and (4.13), we can rewrite $\theta_{1\lambda}(x;\lambda)$ as follows:

$$\theta_{1\lambda}(x;\lambda)=\frac{2\left[\int_0^x|y'(t;\lambda^2)|^2dt+\int_0^x(q(t)+\lambda^2)|y(t;\lambda^2)|^2dt\right]}{(y'(x;\lambda^2))^2+\lambda^2y^2(x;\lambda^2)},\ \forall x,\lambda\in\mathbb{R}. \tag{4.14}$$

□

Applying Lemma 4.1, we can find the asymptotic behaviors of $\theta_1(\pi;\lambda)$, as shown below.

LEMMA 4.2.

$$\lim_{\lambda\to\infty}\theta_1(\pi;\lambda)=\infty,\ \lim_{\lambda\to-\infty}\theta_1(\pi;\lambda)=-\infty.$$

Proof. Note that Lemma 4.1 implies that $\theta_1(\pi;\lambda)$ is a strictly increasing function in λ. Suppose

$$\lim_{\lambda\to\infty}\theta_1(\pi;\lambda)=C<\infty.$$

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Then for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|\theta_1(\pi; \lambda) - C| < \epsilon, \forall \lambda > N.$$

This implies

$$\left| \frac{\theta_1(\pi; n+1) - \theta_1(\pi; n)}{(n+1) - n} | = |\theta_{1\lambda}(\pi; a_n) \right| < 2\epsilon, \forall n > N, \tag{4.15}$$

where $n < a_n < n + 1$. Using (4.14) and the asymptotic formulae (2.3) and (2.4), we have

$$\lim_{\lambda \to \infty} \theta_{1\lambda}(\pi; \lambda) = 2\pi,$$

which leads to a contradiction if we take $\epsilon = \frac{\pi}{4}$. So, we find

$$\lim_{\lambda \to \infty} \theta_1(\pi; \lambda) = \infty.$$

A similar argument implies that

$$\lim_{\lambda \to -\infty} \theta_1(\pi; \lambda) = -\infty. \qquad \square$$

Now, we start to prove the main theorem of this section.

THEOREM 4.3. *$\{\lambda_n\}_{n \in \mathbb{Z} \setminus \{0\}}$ is an infinite sequence and $y(x; \lambda_n^2)$ has exactly $|n| - 1$ zeros in $(0, \pi)$.*

Proof. Applying (4.5), Lemma 4.2, and the results in [**5**], we know that $\{\lambda_n\}_{n \in \mathbb{Z} \setminus \{0\}}$ is a strictly increasing sequence, i.e.,

$$\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \cdots .$$

From the definition of $E_1(x; \lambda)$, we see that

$$E_1(x; 0) = 1, x \in [0, \pi].$$

Since $\theta_{1\lambda}(0; \lambda) = 0$ for all $\lambda \in \mathbb{R}$, we may assume $\theta_1(0; \lambda) = 0$ for all $\lambda \in \mathbb{R}$. By (4.8), we see $\theta_1'(x; 0) = 0$, which implies that

$$\theta_1(x; 0) = 0, \forall x \in [0, \pi].$$

Since $\theta_{1\lambda}(\pi; \lambda) > 0$ for all $\lambda \in \mathbb{R}$, and $\theta_1(\pi; 0) = 0$, we find

$$\lambda \theta_1(\pi; \lambda) > 0, \forall \lambda \in \mathbb{R} \setminus \{0\}, \tag{4.16}$$

which implies

$$\theta_1(\pi; \lambda_n) > 0, \theta_1(\pi; \lambda_{-n}) < 0, \forall n \in \mathbb{N}. \tag{4.17}$$

As $\theta_1(\pi; \lambda)$ being strictly increasing in λ, (4.5) and (4.17) imply that for all $n \in \mathbb{N}$,

$$2(n-1)\pi < \theta_1(\pi; \lambda_n) < 2n\pi, -2n\pi < \theta_1(\pi; \lambda_{-n}) < 2(1-n)\pi. \tag{4.18}$$

Note that for x_* in $[0, \pi]$, $y(x_*; \lambda_n^2) = 0$ if and only if $E_1(x_*; \lambda_n) = 1$, which is equivalent to saying that $\theta_1(x_*; \lambda_n) = 2k\pi$ for some $k \in \mathbb{Z}$. By (4.8), we find

$$\theta_1'(x_*; \lambda_n) = 2\lambda_n. \tag{4.19}$$

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We want to show for each $n \in \mathbb{N}$,

$$0 < \theta_1(x; \lambda_n) < 2n\pi, \forall x \in (0, \pi], \tag{4.20}$$

$$-2n\pi < \theta_1(x; \lambda_{-n}) < 0, \forall x \in (0, \pi]. \tag{4.21}$$

We only prove (4.20), as (4.21) can be proved by the similar arguments. Suppose there is x_1 in $(0, \pi)$ such that $\theta_1(x_1; \lambda_n) \geq 2n\pi$. If $\theta_1(x_1; \lambda_n) = 2n\pi$, then since $\theta_1'(x_1; \lambda_n) = 2\lambda_n > 0$, there is $0 < \delta < \pi - x_1$ such that $\theta_1(x; \lambda_n) > 2n\pi$ for all x in $(x_1, x_1 + \delta)$. On the other hand, if $\theta_1(x_1; \lambda_n) > 2n\pi$, then there exists $0 < \tilde{\delta} < \pi - x_1$ such that $\theta_1(x; \lambda_n) > 2n\pi$ for all x in $(x_1, x_1 + \tilde{\delta})$. Then, (4.18) implies that there exists x_2 in (x_1, π) such that $\theta_1(x_2; \lambda_n) = 2n\pi$ and $\theta_1(x; \lambda_n) > 2n\pi$ for all x in (x_1, x_2). This implies $\theta_1'(x_2; \lambda_n) < 0$, which contradicts to (4.19). Similarly, we can prove that $0 < \theta_1(x; \lambda_n)$ for all x in $(0, \pi]$. Hence, we prove (4.20). By (4.18), (4.20), and (4.21), we find that $y(x; \lambda_n^2)$ has at least $|n| - 1$ zeros in $(0, \pi)$, where $n \in \mathbb{Z} \setminus \{0\}$. For $n \in \mathbb{Z} \setminus \{0\}$, suppose the number of zeros of $y(x; \lambda_n^2)$ in $(0, \pi)$ is larger than or equal to $|n|$, then by (4.20) and (4.21), there are $x_3 < x_4$ in $(0, \pi)$ such that $\theta_1(x_3; \lambda_n) = \theta_1(x_4; \lambda_n) = 2l_0\pi$ for some $1 \leq |l_0| \leq |n| - 1$, $l_0 \in \mathbb{Z}$. Equation (4.19) implies that

$$\theta_1'(x_3; \lambda_n)\theta_1'(x_4; \lambda_n) > 0. \tag{4.22}$$

According to (4.22), we find that there exists at least one zero x_5 of $y(x; \lambda_n^2)$ in (x_3,x_4) with $\theta_1(x_5; \lambda_n) = 2l_0\pi$. Repeating this process, we can obtain an infinite sequence of zeros of $y(x; \lambda_n^2)$ in (x_3,x_4), which leads to the contradiction that there is x_* in (x_3,x_4) with $y(x_*; \lambda_n^2) = y'(x_*; \lambda_n^2) = 0$. Hence, $y(x; \lambda_n^2)$ has exactly $|n| - 1$ zeros in $(0, \pi)$. □

However, the method of using the function $E_1(x; \lambda)$ to investigate equation (2.7) cannot be extended to investigate equation (2.8). For assuming 0 not in the spectrum of equation (2.8), if we define

$$E(x; \mu) = \frac{z'(x; \mu^2) + i\mu z(x; \mu^2)}{z'(x; \mu^2) - i\mu z(x; \mu^2)}, \tag{4.23}$$

then

$$\lim_{\mu \to 0} \lim_{x \to 0} E(x; \mu) = -1. \tag{4.24}$$

But

$$\lim_{x \to 0} \lim_{\mu \to 0} E(x; \mu) = 1. \tag{4.25}$$

This implies that the function $E(x; \mu)$ is not well-defined for all x, $\lambda \in \mathbb{R}$. We will deal with this problem in the next section.

5. Interlacing property of the eigenvalues of the *Dirichlet-type* eigenvalue problem and the *Neumann-type* eigenvalue problem. We start this section by observing the case $q(x) \equiv 0$. In this case,

$$y(x; \lambda^2) = \frac{\sin \lambda x}{\lambda}, z(x; \mu^2) = \cos \mu x.$$

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By the definition of $C_1(\lambda)$ and $C_2(\mu)$,

$$C_1(\lambda) = a\cos\lambda\pi + \sin\lambda\pi, \tag{5.1}$$

$$C_2(\mu) = \mu(\cos\mu\pi - a\sin\mu\pi). \tag{5.2}$$

Solving $C_1(\lambda) = 0$ and $C_2(\mu) = 0$, we find that $\{-\frac{1}{\pi}\tan^{-1} a + n\}_{n\in\mathbb{Z}}$ are the eigenvalues of equation (2.7) and $\{0\} \cup \{\frac{1}{\pi}\tan^{-1}\frac{1}{a} + n\}_{n\in\mathbb{Z}}$ are the eigenvalues of equation (2.8). It is easy to see that $z(x; 0)$ and $z(x; (\frac{1}{\pi}\tan^{-1}\frac{1}{a})^2)$ have no zeros in $[0, \pi]$ and $z(x; (\frac{1}{\pi}\tan^{-1}\frac{1}{a} + n)^2)$ has $|n|$ zeros in $[0, \pi]$ for $|n| \geq 1$. Since $a \neq 0$, we divide the cases into $a > 0$ and $a < 0$. For $a > 0$,

$$\cdots < \frac{1}{\pi}\tan^{-1}\frac{1}{a} - 1 < -\frac{1}{\pi}\tan^{-1} a < 0 < \frac{1}{\pi}\tan^{-1}\frac{1}{a} < -\frac{1}{\pi}\tan^{-1} a + 1 < \frac{1}{\pi}\tan^{-1}\frac{1}{a} + 1 < \cdots.$$

For $a < 0$,

$$\cdots < \frac{1}{\pi}\tan^{-1}\frac{1}{a} - 1 < -\frac{1}{\pi}\tan^{-1} a - 1 < \frac{1}{\pi}\tan^{-1}\frac{1}{a} < 0 < -\frac{1}{\pi}\tan^{-1} a < \frac{1}{\pi}\tan^{-1}\frac{1}{a} + 1 < \cdots.$$

By previous two inequalities, we see the interlacing property of the eigenvalues of equations (2.7) and (2.8). In order to prove the interlacing property for the general case, we need the following two lemmas.

LEMMA 5.1. *For $n \in \mathbb{Z} \setminus \{0\}$,*

$$-\frac{1}{a}y\left(\pi; \lambda_n^2\right) C_2(\lambda_n) = 1,$$

$$\frac{1}{a}z\left(\pi; \mu_n^2\right) C_1(\mu_n) = 1.$$

Proof. Recall that

$$C_1(\lambda) = ay'(\pi; \lambda^2) + \lambda y(\pi; \lambda^2), \tag{5.3}$$

$$C_2(\lambda) = az'(\pi; \lambda^2) + \lambda z(\pi; \lambda^2). \tag{5.4}$$

Differentiating (5.3) and (5.4) with respect to λ, we have

$$C_1'(\lambda) = 2\lambda\left[ay_\nu'(\pi; \lambda^2) + \lambda y_\nu(\pi; \lambda^2)\right] + y(\pi; \lambda^2), \tag{5.5}$$

$$C_2'(\lambda) = 2\lambda\left[az_\nu'(\pi; \lambda^2) + \lambda z_\nu(\pi; \lambda^2)\right] + z(\pi; \lambda^2). \tag{5.6}$$

Note that by equations (2.1) and (2.2),

$$z(x; \lambda^2)y''(x; \lambda^2) - z''(x; \lambda^2)y(x; \lambda^2) = 0, \forall x \in [0, \pi], \lambda \in \mathbb{C}, \tag{5.7}$$

which leads to

$$z(x; \lambda^2)y'(x; \lambda^2) - z'(x; \lambda^2)y(x; \lambda^2) = 1, \forall x \in [0, \pi], \lambda \in \mathbb{C}. \tag{5.8}$$

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Equation (5.8) implies that for $n \in \mathbb{Z} \setminus \{0\}$,

$$z\left(\pi;\lambda_n^2\right) y'\left(\pi;\lambda_n^2\right) - z'\left(\pi;\lambda_n^2\right) y\left(\pi;\lambda_n^2\right) = 1, \tag{5.9}$$

$$z\left(\pi;\mu_n^2\right) y'\left(\pi;\mu_n^2\right) - z'\left(\pi;\mu_n^2\right) y\left(\pi;\mu_n^2\right) = 1. \tag{5.10}$$

Since $C_1(\lambda_n) = 0$ and $C_2(\mu_n) = 0$, we can rewrite (5.9) and (5.10) as

$$-y\left(\pi;\lambda_n^2\right)\left[z'\left(\pi;\lambda_n^2\right) + \frac{\lambda_n}{a} z\left(\pi;\lambda_n^2\right)\right] = 1, \tag{5.11}$$

$$z\left(\pi;\mu_n^2\right)\left[y'\left(\pi;\mu_n^2\right) + \frac{\mu_n}{a} y\left(\pi;\mu_n^2\right)\right] = 1. \tag{5.12}$$

By (5.3), (5.4), (5.11), and (5.12), we obtain for $n \in \mathbb{Z} \setminus \{0\}$,

$$-\frac{1}{a} y\left(\pi;\lambda_n^2\right) C_2(\lambda_n) = 1, \tag{5.13}$$

$$\frac{1}{a} z\left(\pi;\mu_n^2\right) C_1(\mu_n) = 1. \tag{5.14}$$

□

LEMMA 5.2. *For $n \in \mathbb{Z} \setminus \{0\}$,*

$$\begin{aligned}&\frac{\lambda_n}{a^2} |y\left(\pi;\lambda_n^2\right)|^2 C_1'(\lambda_n) C_2(\lambda_n)\\ &\quad = \int_0^\pi |y'\left(x;\lambda_n^2\right)|^2 dx + \int_0^\pi \left(\lambda_n^2 + q(x)\right) |y\left(x;\lambda_n^2\right)|^2 dx,\end{aligned} \tag{5.15}$$

$$\begin{aligned}&-\frac{\mu_n}{a^2} |z\left(\pi;\mu_n^2\right)|^2 C_1(\mu_n) C_2'(\mu_n)\\ &\quad = \int_0^\pi |z'\left(x;\mu_n^2\right)|^2 dx + \int_0^\pi \left(\mu_n^2 + q(x)\right) |z\left(x;\mu_n^2\right)|^2 dx.\end{aligned} \tag{5.16}$$

Proof. Recall that (4.12) tells us for $n \in \mathbb{Z} \setminus \{0\}$,

$$y'\left(\pi;\lambda_n^2\right) y_\nu\left(\pi;\lambda_n^2\right) - y_\nu'\left(\pi;\lambda_n^2\right) y\left(\pi;\lambda_n^2\right) = \int_0^\pi |y\left(x;\lambda_n^2\right)|^2 dx. \tag{5.17}$$

Since $C_1(\lambda_n) = 0$, (5.17) becomes

$$-\frac{1}{a} y\left(\pi;\lambda_n^2\right)\left[a y_\nu'\left(\pi;\lambda_n^2\right) + \lambda_n y_\nu\left(\pi;\lambda_n^2\right)\right] = \int_0^\pi |y\left(x;\lambda_n^2\right)|^2 dx. \tag{5.18}$$

By (5.5) and (5.18),

$$\begin{aligned}-\frac{1}{a} y\left(\pi;\lambda_n^2\right) C_1'(\lambda_n) &= -\frac{1}{a} y\left(\pi;\lambda_n^2\right)\left[2\lambda_n\left(a y_\nu'\left(\pi;\lambda_n^2\right) + \lambda_n y_\nu\left(\pi;\lambda_n^2\right)\right) + y\left(\pi;\lambda_n^2\right)\right]\\ &= 2\lambda_n \int_0^\pi |y\left(x;\lambda_n^2\right)|^2 dx - \frac{1}{a} |y\left(\pi;\lambda_n^2\right)|^2.\end{aligned} \tag{5.19}$$

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According to (4.13), we have

$$y\left(\pi;\lambda_n^2\right)y'\left(\pi;\lambda_n^2\right)+\lambda_n^2\int_0^\pi |y\left(x;\lambda_n^2\right)|^2dx$$
$$=\int_0^\pi |y'\left(x;\lambda_n^2\right)|^2dx+\int_0^\pi q(x)|y\left(x;\lambda_n^2\right)|^2dx. \tag{5.20}$$

Because $C_1(\lambda_n)=0$, (5.20) can be written as

$$-\frac{\lambda_n}{a}|y\left(\pi;\lambda_n^2\right)|^2+\lambda_n^2\int_0^\pi |y\left(x;\lambda_n^2\right)|^2dx$$
$$=\int_0^\pi |y'\left(x;\lambda_n^2\right)|^2dx+\int_0^\pi q(x)|y\left(x;\lambda_n^2\right)|^2dx. \tag{5.21}$$

Then by (5.19) and (5.21), we obtain the following relation:

$$-\frac{1}{a}y\left(\pi;\lambda_n^2\right)C_1'(\lambda_n)$$
$$=\frac{1}{\lambda_n}\left[\int_0^\pi |y'\left(x;\lambda_n^2\right)|^2dx+\int_0^\pi \left(\lambda_n^2+q(x)\right)|y\left(x;\lambda_n^2\right)|^2dx\right]. \tag{5.22}$$

Equation (5.15) is an immediate consequence of (5.13) and (5.22). Equation (5.16) can be proved by a similar argument. □

Now, we prove the interlacing property of the eigenvalues of equations (2.7) and (2.8) for general case.

THEOREM 5.3. *Suppose 0 is not an eigenvalue of equation* (2.8)*. Then,*

$$\cdots<\lambda_{-3}<\mu_{-3}<\lambda_{-2}<\mu_{-2}<\lambda_{-1}<\mu_{-1}<\mu_1<\lambda_1<\mu_2<\lambda_2<\mu_3<\lambda_3<\cdots. \tag{5.23}$$

Proof. Let

$$\widetilde{C}_2(\lambda)=\lambda C_2(\lambda).$$

Then by the assumption that 0 is not an eigenvalue of equation (2.8) (which implies $C_2(0)\neq 0$), we find that 0 is a simple zero of $\widetilde{C}_2(\lambda)$. Using (5.15) and (5.16), we find for all $n\in\mathbb{Z}\setminus\{0\}$,

$$C_1'(\lambda_n)\widetilde{C}_2(\lambda_n)>0, \tag{5.24}$$
$$C_1(\mu_n)\widetilde{C}_2'(\mu_n)<0. \tag{5.25}$$

Note that

$$C_1(0)\widetilde{C}_2'(0)=a^2y'(\pi;0)z'(\pi;0).$$

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By the equations satisfied by $y(x;0)$ and $z(x;0)$, respectively, we find

$$y'(\pi;0) = y'(0;0) + \int_0^\pi q(x)y(x;0)dx \geq 1 > 0, \tag{5.26}$$

$$z'(\pi;0) = z'(0;0) + \int_0^\pi q(x)z(x;0)dx \geq 0. \tag{5.27}$$

Since 0 is not an eigenvalue of equation (2.8), $z'(\pi;0) \neq 0$, which implies that $z'(\pi;0) > 0$. Hence,

$$C_1(0)\widetilde{C}_2'(0) > 0. \tag{5.28}$$

By Theorem 3.3 in [**5**], we know that λ_n is *algebraically* simple for each $n \in \mathbb{Z} \setminus \{0\}$, which leads to

$$C_1'(\lambda_{-1})C_1'(\lambda_1) < 0, \tag{5.29}$$

and

$$C_1'(\lambda_j)C_1'(\lambda_{j+1}) < 0, j \in \mathbb{Z} \setminus \{-1, 0\}. \tag{5.30}$$

By (5.24), (5.29), (5.30), and the definition of $\widetilde{C}_2(\lambda)$, we find that

$$C_2(\lambda_{-1})C_2(\lambda_1) > 0, \tag{5.31}$$

and

$$C_2(\lambda_j)C_2(\lambda_{j+1}) < 0, j \in \mathbb{Z} \setminus \{-1, 0\}, \tag{5.32}$$

that is, there are odd number of μ_n's lying between any two consecutive eigenvalues λ_j and λ_{j+1}, where $j \in \mathbb{Z} \setminus \{-1, 0\}$, and the number of μ_n's in $(\lambda_{-1}, \lambda_1)$ is even. Combining previous observation with Theorem 4.3, we find that $C_2(\mu)$ has infinitely many zeros, which are our μ_n's, and Theorem 3.3 tells us these eigenvalues are *algebraically* simple. On the other hand, we obtain from (5.25) and (5.28),

$$C_1(\mu_{-1})C_1(0) > 0,\ C_1(0)C_1(\mu_1) > 0, \tag{5.33}$$

and

$$C_1(\mu_l)C_1(\mu_{l+1}) < 0, l \in \mathbb{Z} \setminus \{-1, 0\}, \tag{5.34}$$

Thus, there are odd number of $\lambda_n's$ between any two consecutive eigenvalues μ_l and μ_{l+1}, where $l \in \mathbb{Z} \setminus \{-1, 0\}$, and the numbers of $\lambda_n's$ in $(\mu_{-1}, 0)$ and $(0, \mu_1)$ are even. Suppose there are two $\lambda_n's$ in $(0, \mu_1)$, then by the previous discussion, there must be an eigenvalue of equation (2.8) in $(0, \mu_1)$, which contradicts to μ_1 being the first positive eigenvalue of equation (2.8). Similarly, there is no $\lambda_n's$ in $(\mu_{-1}, 0)$. Hence, we find

$$\lambda_{-1} < \mu_{-1} < \mu_1 < \lambda_1. \tag{5.35}$$

For $j \in \mathbb{Z} \setminus \{-1, 0\}$, suppose there are at least three μ_n's between λ_j and λ_{j+1}. Then, we know that there must exist an eigenvalue of equation (2.7) between λ_j and λ_{j+1}, which is a contradiction. Thus, there is exact one μ_n between any two consecutive eigenvalues λ_j and λ_{j+1}. Similarly, we can show that there is exact one λ_n between any two consecutive eigenvalues μ_l and μ_{l+1}, where $l \in \mathbb{Z} \setminus \{-1, 0\}$. The interlacing relation (5.23) follows from previous discussions. □

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At the end of this paper, we try to find the number of zeros of the eigenfunctions of equation (2.8). First, we observe the case that $q(x) \equiv c$ in $[0, \pi]$, where $c > 0$. In this case,

$$C_2(\mu) = \mu \cos\left((\mu^2 - c)^{1/2}\pi\right) - a(\mu^2 - c)^{1/2} \sin\left((\mu^2 - c)^{1/2}\pi\right). \tag{5.36}$$

For $a > 0$, it is easy to see $C_2(\mu) > 0$ for all μ in $(0, \sqrt{c})$, and

$$C_2(0) = a\sqrt{c} \sinh \sqrt{c}\pi > 0,\ C_2(-\sqrt{c}) = -\sqrt{c} < 0,\ C_2(\sqrt{c}) = \sqrt{c} > 0. \tag{5.37}$$

Equation (5.37) tells us that there is a μ_* in $(-\sqrt{c}, 0)$ such that $C_2(\mu_*) = 0$. We want to claim that μ_* is unique. We get from (5.36) that for $-\sqrt{c} < \mu < 0$,

$$C_2'(\mu) = (1 - a\pi\mu)\cosh((\mu^2 - c)^{1/2}\pi) - \frac{\mu(\pi\mu + a)}{(\mu^2 - c)^{1/2}} \sinh\left((\mu^2 - c)^{1/2}\pi\right). \tag{5.38}$$

Since $C_2(\mu_*) = 0$, we have

$$\mu_* \cosh\left(\left(c - \mu_*^2\right)^{1/2}\pi\right) = -a\left(c - \mu_*^2\right)^{1/2} \sinh\left(\left(c - \mu_*^2\right)^{1/2}\pi\right). \tag{5.39}$$

Formulae (5.38) and (5.39) imply

$$C_2'(\mu_*) = \frac{\cosh((c - \mu_*^2)^{1/2}\pi)}{a(c - \mu_*^2)}\left[\pi(1 + a^2)\mu_*^3 - a^2 c\pi\mu_* + ac\right]. \tag{5.40}$$

Suppose μ_* is not the only zero of $C_2(\mu)$ in $(-\sqrt{c}, 0)$. Since all zeros of $C_2(\mu)$ are simple, there are $\mu^{(1)}$, $\mu^{(2)}$ in $(-\sqrt{c}, 0)$ such that μ_*, $\mu^{(1)}$, and $\mu^{(2)}$ are consecutive zeros of $C_2(\mu)$. Without loss of generality, we may assume $\mu_* < \mu^{(1)} < \mu^{(2)}$. Then,

$$C_2'(\mu_*)C_2'(\mu^{(1)}) < 0,\ C_2'(\mu^{(1)})C_2'(\mu^{(2)}) < 0. \tag{5.41}$$

Note that for $j = 1, 2$,

$$\operatorname{sgn} C_2'(\mu_*) = \operatorname{sgn}\left[\pi(1 + a^2)\mu_*^3 - a^2 c\pi\mu_* + ac\right], \tag{5.42}$$

$$\operatorname{sgn} C_2'(\mu^{(j)}) = \operatorname{sgn}\left[\pi(1 + a^2)\mu^{(j)^3} - a^2 c\pi\mu^{(j)} + ac\right]. \tag{5.43}$$

Consider the following polynomial $p(x)$,

$$p(x) = \pi(1 + a^2)x^3 - a^2 c\pi x + ac.$$

The sum of the roots of the equation $p(x) = 0$ is 0, and the product of the roots of it is $-\frac{ac}{\pi(1+a^2)}$, which is negative since $a, c > 0$. This implies that $p(x) = 0$ has exact one negative root, denoted by x_0. By (5.41), we find

$$\mu_* < x_0 < \mu^{(1)} < \mu^{(2)},$$

or

$$\mu_* < \mu^{(1)} < x_0 < \mu^{(2)}.$$

Hence,

$$C_2'(\mu^{(1)})C_2'(\mu^{(2)}) > 0,$$

or

$$C_2'(\mu_*)C_2'(\mu^{(1)}) > 0,$$

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which contradicts to (5.41). So, such μ_* is the only zero of $C_2(\mu)$ in $(-\sqrt{c}, 0)$. Moreover, the corresponding eigenfunction $z(x; \mu_*^2) = \cosh((c-\mu_*^2)^{1/2}x)$ is positive in $[0, \pi]$.

According to previous discussion, we find for $n \in \mathbb{N}$, $\mu_n^2 > c$, $\mu_{-(n+1)}^2 > c$. Since $a > 0$, $C_2(\mu_n) = 0$, and $C_2(\mu_{-(n+1)}) = 0$, we find

$$\cos\left(\left(\mu_n^2 - c\right)^{1/2}\pi\right)\sin\left(\left(\mu_n^2 - c\right)^{1/2}\pi\right) > 0, \tag{5.44}$$

$$\cos\left(\left(\mu_{-(n+1)}^2 - c\right)^{1/2}\pi\right)\sin\left(\left(\mu_{-(n+1)}^2 - c\right)^{1/2}\pi\right) < 0. \tag{5.45}$$

Similarly, we have

$$C_2'(\mu_n) = -\frac{\cos\left(\left(\mu_n^2 - c\right)^{1/2}\pi\right)}{a\left(\mu_n^2 - c\right)}\left[\pi(1+a^2)\mu_n^3 - a^2c\pi\mu_n + ac\right], \tag{5.46}$$

and

$$C_2'(\mu_{-(n+1)}) = -\frac{\cos\left(\left(\mu_{-(n+1)}^2 - c\right)^{1/2}\pi\right)}{a\left(\mu_{-(n+1)}^2 - c\right)}\left[\pi(1+a^2)\mu_{-(n+1)}^3 - a^2c\pi\mu_{-(n+1)} + ac\right]. \tag{5.47}$$

Note that for $k \in \mathbb{N} \cup \{0\}$,

$$C_2(\pm\sqrt{c+(k+1)^2}) = \pm(-1)^{k+1}\sqrt{c+(k+1)^2}, \tag{5.48}$$

$$C_2\left(\pm\sqrt{c+\left(k+\frac{1}{2}\right)^2}\right) = (-1)^{k+1}a\left(k+\frac{1}{2}\right). \tag{5.49}$$

By an argument similar to that for showing the uniqueness of μ_* in $(-\sqrt{c}, 0)$, we find

$$-\sqrt{c+n^2} < \mu_{-(n+1)} < -\sqrt{c+\left(n-\frac{1}{2}\right)^2}. \tag{5.50}$$

Equation (5.50) implies that the function $z(x; \mu_{-(n+1)}^2) = \cos((\mu_{-(n+1)}^2 - c)^{1/2}x)$ has exactly n zeros in $[0, \pi]$.

From (5.48) and (5.49), we find that for each $l \in \mathbb{N} \cup \{0\}$, there is a $k_l \in \mathbb{N}$ such that

$$\sqrt{c+l^2} < \mu_{k_l} < \sqrt{c+\left(l+\frac{1}{2}\right)^2}. \tag{5.51}$$

We want to show that μ_{k_l} is unique for each $l \in \mathbb{N} \cup \{0\}$. Suppose there exists $l_0 \in \mathbb{N} \cup \{0\}$ such that there are $\mu_{k_{l_0}}$, $\mu_{k_{l_0}+1}$, and $\mu_{k_{l_0}+2}$ in $\left(\sqrt{c+l_0^2}, \sqrt{c+(l_0+\frac{1}{2})^2}\right)$. Then by Theorem 5.3, there are $\lambda_{k_{l_0}}$ and $\lambda_{k_{l_0}+1}$ with

$$\mu_{k_{l_0}} < \lambda_{k_{l_0}} < \mu_{k_{l_0}+1} < \lambda_{k_{l_0}+1} < \mu_{k_{l_0}+2}. \tag{5.52}$$

According to Theorem 4.3, we know

$$\lambda_{k_{l_0}} < \sqrt{c+k_{l_0}^2} < \lambda_{k_{l_0}+1}. \tag{5.53}$$

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(5.52) and (5.53) imply that

$$\sqrt{c+l_0^2} < \sqrt{c+k_{l_0}^2} < \sqrt{c+\left(l_0+\frac{1}{2}\right)^2}, \tag{5.54}$$

which is a contradiction. Hence, such a μ_{k_n} is unique. By the previous result, we find for $n \in \mathbb{N}$,

$$\sqrt{c+(n-1)^2} < \mu_n < \sqrt{c+\left(n-\frac{1}{2}\right)^2}. \tag{5.55}$$

So, we find that the eigenfunction $z(x;\mu_n^2) = \cos((\mu_n^2-c)^{1/2}x)$ has exactly $n-1$ zeros in $[0,\pi]$. For the case $a<0$, an argument similar to that used to deal with the case $a>0$ leads to the same result.

Now, we try to extend the observation on the cases $q(x)\equiv 0$ and $q(x)\equiv c>0$ to deal with the general case. Let

$$C(x;\lambda) = az'\left(x;\lambda^2\right) + \lambda z\left(x;\lambda^2\right), \tag{5.56}$$

then

$$C'(x;\lambda) = az''\left(x;\lambda^2\right) + \lambda z'\left(x;\lambda^2\right), \tag{5.57}$$

$$C_\lambda(x;\lambda) = 2\lambda[az'_\nu\left(x;\lambda^2\right) + \lambda z_\nu\left(x;\lambda^2\right)] + z\left(x;\lambda^2\right). \tag{5.58}$$

As we did in the proof of Theorem 4.3, define for $x,\lambda\in\mathbb{R}$,

$$E_2(x;\lambda) = \frac{z(x;\lambda^2)+iC(x;\lambda)}{z(x;\lambda^2)-iC(x;\lambda)} = e^{i\theta_2(x;\lambda)}. \tag{5.59}$$

Since for each $\lambda\in\mathbb{R}$, $(z'(x;\lambda^2))^2+z^2(x;\lambda^2)\neq 0$ for all x in $[0,\pi]$, we have

$$(z(x;\lambda^2))^2 + C^2(x;\lambda) \neq 0$$

for all $\lambda\in\mathbb{R}\setminus\{0\}$. Hence, $E_2(x;\lambda)$ is well-defined, and Lemma 2.1 implies $E_2(x;\lambda)$ is continuously differentiable with respect to x and λ. It is easy to see that μ_* is an eigenvalue of equation (2.8) if and only if $\theta_2(\pi;\mu_*) = 2k\pi$, where $k\in\mathbb{Z}$. Differentiating $E_2(x;\lambda)$ with respect to x and λ, respectively, we find

$$E_2'(x;\lambda) = \frac{2ai\left[z''(x;\lambda^2)z(x;\lambda^2) - (z'(x;\lambda^2))^2\right]}{\left[z'(x;\lambda^2)-iC(x;\lambda)\right]^2}, \tag{5.60}$$

$$E_{2\lambda}(x;\lambda) = \frac{2i\left[z^2(x;\lambda^2) + 2a\lambda(z'_\nu(x;\lambda^2)z(x;\lambda^2) - z_\nu(x;\lambda^2)z'(x;\lambda^2))\right]}{\left[z'(x;\lambda^2)-iC(x;\lambda)\right]^2}. \tag{5.61}$$

By (5.59)–(5.61), we obtain

$$\theta_2'(x;\lambda) = \frac{2a\left[z''(x;\lambda^2)z(x;\lambda^2) - (z'(x;\lambda^2))^2\right]}{(z'(x;\lambda^2))^2 + C^2(x;\lambda)}, \tag{5.62}$$

$$\theta_{2\lambda}(x;\lambda) = \frac{2\left[z^2(x;\lambda^2) + 2a\lambda(z'_\nu(x;\lambda^2)z(x;\lambda^2) - z_\nu(x;\lambda^2)z'(x;\lambda^2))\right]}{(z'(x;\lambda^2))^2 + C^2(x;\lambda)}. \tag{5.63}$$

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Applying (3.7), (5.63) can be simplified as

$$\theta_{2\lambda}(x;\lambda)=\frac{2\left[z^2(x;\lambda^2)-2a\lambda\int_0^x z^2(t;\lambda^2)dt\right]}{(z'(x;\lambda^2))^2+C^2(x;\lambda)}. \tag{5.64}$$

Define $\theta_2(0;0)=0$. Since $\theta_{2\lambda}(0;\lambda)>0$ by (5.64), we have

$$\lambda\theta_2(0;\lambda)>0,\ \forall\lambda\neq 0. \tag{5.65}$$

Furthermore, we can show the boundedness of function $\theta_2(0;\lambda)$:

LEMMA 5.4.

$$\theta_2(0;\lambda)\in(0,\pi),\ \forall\lambda>0, \tag{5.66}$$

$$\theta_2(0;\lambda)\in(-\pi,0),\ \forall\lambda<0. \tag{5.67}$$

Proof. Suppose there exists $\lambda_0>0$ such that $\theta_2(0;\lambda_0)=\pi$, then

$$E_2(0;\lambda_0)=\frac{1+\lambda_0 i}{1-\lambda_0 i}=-1,$$

which is impossible. This proves (5.66). Equation (5.67) can be shown by the similar argument. □

LEMMA 5.5. *For $a>0$ in equation (2.8),*

$$0<\theta_2(\pi;0)<\pi.$$

Proof. By (3.1),

$$z(x;0)z'(x;0)=\int_0^x|z'(t;0)|^2dt+\int_0^x q(t)|z(t;0)|^2dt. \tag{5.68}$$

Suppose there is x_0 in $(0,\pi]$ such that $z(x_0;0)=0$. Because $z(x;0)$ and $z'(x;0)$ cannot vanish at the same point, $z'(x_0;0)\neq 0$. Hence, we have

$$\int_0^{x_0}|z'(t;0)|^2dt>0.$$

Taking $x=x_0$ in (5.68), we find

$$0=\int_0^{x_0}|z'(t;0)|^2dt+\int_0^{x_0}q(t)|z(t;0)|^2dt>0, \tag{5.69}$$

which is absurd. As $z(0;0)=1$, we know $z(x;0)>0$ for all x in $[0,\pi]$. By (2.2),

$$z'(x;0)=\int_0^x q(t)z(t;0)dt\geq 0,\ \forall x\in[0,\pi]. \tag{5.70}$$

Since 0 is not an eigenvalue of equation (2.8), $q(x)$ is nontrivial in $[0,\pi]$. This implies

$$z'(\pi;0)=\int_0^\pi q(t)z(t;0)dt>0. \tag{5.71}$$

Plugging $x=\pi$ and $\lambda=0$ into (5.59), we find

$$E_2(\pi;0)=\frac{z(\pi;0)+iaz'(\pi;0)}{z(\pi;0)-iaz'(\pi;0)}=e^{i\theta_2(\pi;0)}. \tag{5.72}$$

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By (5.72), we obtain

$$\theta_2(\pi;0) = 2\tan^{-1}\frac{az'(\pi;0)}{z(\pi;0)} + 2l_0\pi, \tag{5.73}$$

for some $l_0 \in \mathbb{Z}$. Because $a, z(\pi;0), z'(\pi;0) > 0$,

$$2l_0\pi < \theta_2(\pi;0) < (2l_0+1)\pi. \tag{5.74}$$

Suppose there exists x_1 in $(0,\pi)$ such that $\theta_2(x_1;0) = (2l_0-1)\pi$ or $(2l_0+1)\pi$, then by (5.59), we find

$$E_2(x_1;0) = \frac{z(x_1;0) + iaz'(x_1;0)}{z(x_1;0) - iaz'(x_1;0)} = -1, \tag{5.75}$$

which is absurd since $z(x_1;0) > 0$. Hence,

$$(2l_0-1)\pi < \theta_2(x;0) < (2l_0+1)\pi, \forall x \in [0,\pi]. \tag{5.76}$$

Because $\theta_2(0;0) = 0$, we find $\theta_2(\pi;0) \in (0,\pi)$. □

By use of Lemmas 5.4 and 5.5, we can find the number of zeros of $z(x;\mu_n^2)$ for a general $q(x)$.

THEOREM 5.6. *$\{\mu_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an infinite sequence, and $z(x;\mu_n^2)$ has exactly $|n|-1$ zeros in $(0,\pi)$.*

Proof. We have seen that $\{\mu_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an infinite sequence. Here, we just want to find the number of zeros of $z(x;\mu_n^2)$ in $(0,\pi)$. We only deal with the case $a > 0$ for the reason that the case $a < 0$ can be proved by the similar way.

Applying (3.12) and taking $x = \pi$ and $\lambda = \mu_n$ in (5.64), we find

$$\theta_{2\lambda}(\pi;\mu_n) = \frac{-2a\left[\int_0^\pi \left(z'\left(x;\mu_n^2\right)\right)^2 + (\mu_n^2 + q(x))z^2\left(x;\mu_n^2\right)dx\right]}{\mu_n\left(z'\left(x;\mu_n^2\right)\right)^2}. \tag{5.77}$$

By (5.77), we have

$$\theta_{2\lambda}(\pi;\mu_n) \begin{cases} < 0, \text{ if } \mu_n > 0, \\ > 0, \text{ if } \mu_n < 0. \end{cases} \tag{5.78}$$

Claim that for $n \in \mathbb{N}$,

$$\theta_2(\pi;\mu_n) = \theta_2(\pi;\mu_{-n}) = 2(1-n)\pi. \tag{5.79}$$

Since $\theta_2(\pi;0) \in (0,\pi)$, $\theta_{2\lambda}(\pi;\mu_1) < 0$, and $\theta_{2\lambda}(\pi;\mu_{-1}) > 0$, we know

$$\theta_2(\pi;\mu_1) = \theta_2(\pi;\mu_{-1}) = 0. \tag{5.80}$$

Suppose $\theta_2(\pi;\mu_k) = 2(1-k)\pi$ for $k = 1, 2, \ldots, n$. Since $\theta_{2\lambda}(\pi;\mu_k) < 0$, $\theta_2(\pi;\mu) < 2(1-k)\pi$ for all $\mu_k < \mu < \mu_{k+1}$. This implies either $\theta_2(\pi;\mu_{k+1}) = -2k\pi$ or $\theta_2(\pi;\mu_{k+1}) = 2(1-k)\pi$. If $\theta_2(\pi;\mu_{k+1}) = 2(1-k)\pi$, then $\theta_{2\lambda}(\pi;\mu_{k+1}) \geq 0$, which contradicts to (5.78). By mathematical induction, we find $\theta_2(\pi;\mu_n) = 2(1-n)\pi$ holds. The other part of (5.79) can be proved by the similar way.

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Finally, we count the number of zeros of $z(x;\,\mu_n^2)$ in $(0,\,\pi)$, where $n \in \mathbb{Z}\setminus\{0\}$. Note that for x_* in $(0,\,\pi)$, $z(x_*;\,\mu_n^2)=0$ if and only if $E_2(x_*;\,\mu_n)=-1$, i.e., $\theta_2(x_*;\,\mu_n)=(2k+1)\pi$ for some $k\in\mathbb{Z}$. By (5.62),

$$\theta_2'(x_*;\,\mu_n)<0. \tag{5.81}$$

Our purpose is to show for $n\in\mathbb{N}$,

$$(-2n+1)\pi<\theta_2(x;\,\mu_n)<\pi, \tag{5.82}$$

$$(-2n+1)\pi<\theta_2(x;\,\mu_{-n})<0. \tag{5.83}$$

Here, we only prove (5.82), since (5.83) can be proved similarly. From (5.66), we know $\theta_2(0;\,\mu_n)\in(0,\,\pi)$. Suppose there is x_2 in $(0,\,\pi]$ such that $\theta_2(x_2;\,\mu_n)=\pi$, then by (5.81), there is $0<\delta_1<x_2$ such that $\theta_2(x;\,\mu_n)>\pi$ for all x in $(x_2-\delta_1,\,x_2)$. Thus, there must be $x_3\in(0,\,x_2)$ such that $\theta_2(x_3;\,\mu_n)=\pi$, and $\theta_2'(x_3;\,\mu_n)>0$, which contradicts to (5.81). If there exists x_4 in $(0,\,\pi]$ such that $\theta_2(x_4;\,\mu_n)=(-2n+1)\pi$, then by (5.81), there is $0<\delta_2<\pi-x_4$ such that $\theta_2(x;\,\mu_n)<(-2n+1)\pi$ for all x in $(x_4,\,x_4+\delta_2)$. According to (5.79), there must be $x_5\in(x_4,\,\pi)$ such that $\theta_2(x_5;\,\mu_n)=(-2n+1)\pi$ and $\theta_2'(x_5;\,\mu_n)>0$, which contradicts to (5.81). By (5.79), (5.82), and (5.83), we find that $z(x;\,\mu_n^2)$ has at least $|n|-1$ zeros in $(0,\,\pi)$, where $n\in\mathbb{Z}\setminus\{0\}$.

For $n\in\mathbb{Z}\setminus\{0\}$, suppose the number of the zeros of $z(x;\,\mu_n^2)$ in $(0,\,\pi)$ is larger than or equal to $|n|$, then by (5.82) and (5.83), there are $x_6<x_7$ in $(0,\,\pi)$ such that $\theta_2(x_6;\,\mu_n)=\theta_2(x_7;\,\mu_n)=(-2|l_0|+1)\pi$ for some $1\le|l_0|\le|n|-1$, $l_0\in\mathbb{Z}$. Equation (5.81) implies that

$$\theta_2'(x_6;\,\mu_n)\theta_2'(x_7;\,\mu_n)>0. \tag{5.84}$$

According to (5.84), we find that there exists at least one zero x_8 of $z(x;\,\mu_n^2)$ in (x_6,x_7) with $\theta_2(x_8;\,\mu_n)=(-2|l_0|+1)\pi$. Repeating this process, we can obtain an infinite sequence of the zeros of $z(x;\,\mu_n^2)$ in (x_6,x_7), which implies that there is x_* in (x_6,x_7) with $z(x_*;\,\mu_n^2)=z'(x_*;\,\mu_n^2)=0$, but it is absurd. Hence, $z(x;\,\mu_n^2)$ has exactly $|n|-1$ zeros in $(0,\,\pi)$. □

REFERENCES

1. F. V. Atkinson, *Discrete and continuous boundary problems* (Academic Press, New York, 1964).

2. P. A. Binding, P. J. Browne and K. Seddighi, Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Proc. Edinburgh Math. Soc.* **37** (1993), 57–72.

3. B. P. Allahverdiev, E. Bairamov and E. Ugurlu, Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions, *J. Math. Anal. Appl.* **401**(1) (2013), 388–396.

4. P. J. Browne and B. D. Sleeman, Inverse nodal problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions, *Inverse Prob.* **12** (1996), 377–381.

5. C.-H. Chan, Some eigenvalue problems for vectorial Sturm-Liouville equations with eigenparameter dependent boundary conditions, *Trans. AMS* **364** (2012), 119–136.

6. C. I. Gheorghiu, On the numerical treatment of the eigenparameter dependent boundary conditions, *Numer Algorithms* **77**(1) (2018), 77–93.

7. M. S. P. Eastham, Q. Kong, H. Wu and A. Zettl, Inequalities among eigenvalues of Sturm-Liouville problems, *J. Inequalities Appl.* **3** (1999), 25–43.

8. F. Gesztesy and B. Simon, A new approach to the inverse spectral theory, II. General real potentials and the connection to the spectral measure, *Ann. Math*. **152** (2000), 593–643.

9. H. Hochstadt, On the inverse problems associated with second-order differential operators, *Acta Math.* **119** (1967), 173–192.

https://doi.org/10.1017/S0017089520000087 Published online by Cambridge University Press

10. K. Aydemir and O. S. Mukhtarov, Variational principles for spectral analysis of one Sturm-Liouville problem with transmission conditions, *Adv. Differ. Equ.* **76** (2016), 1–14.

11. J. Pöschel and E. Trubowitz, *Inverse spectral theory* (Academic Press, New York, 1987).

12. P. A. Binding, P. J. Browne and B. A. Watson, Inverse spectral problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions, *J. London Math. Soc.* **62**(1) (2000), 161–182.

13. C.-L. Shen, Some inverse spectral problems for vectorial Sturm-Liouville equations, *Inverse Prob.* **17** (2001), 1253–1294.

14. B. Simon, A new approach to the inverse spectral theory, I. Fundamental formalism, *Ann. Math.* **150** (1999), 1029–1057.

15. Z. Zheng, J. Cai, K. Li and M. Zhang, A discontinuous Sturm-Liouville problem with boundary conditions rationally dependent on the eigenparameter, *Boundary Value Prob.* **103** (2018).

16. M. Zhang and Y. Wang, Dependence of eigenvalues of Sturm-Liouville problems with interface conditions, *Appl. Math. Comput.* **265** (2015), 31–39.

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