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A NOTE ON THE SECOND MOMENT OF AUTOMORPHIC L-FUNCTIONS

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Abstract. We obtain the formula for the twisted harmonic second moment of the L-functions associated with primitive Hecke eigenforms of weight 2. A consequence of our mean-value theorem is reminiscent of recent results of Conrey and Young on the reciprocity formula for the twisted second moment of Dirichlet L-functions.

§1. *Introduction.* In this paper, we study the twisted second moment of the family of L-functions arising from $\mathcal{S}_2^*(q)$, the set of primitive Hecke eigenforms of weight 2, lever q (q prime). For $f(z) \in \mathcal{S}_2^*(q)$, f has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{1/2} \lambda_f(n) e(nz),$$

where the normalization is such that $\lambda_f(1) = 1$. The L-function associated to f has an Euler product

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \left(1 - \frac{\lambda_f(q)}{q^s}\right)^{-1} \prod_{\substack{h \text{ prime} \\ h \neq q}} \left(1 - \frac{\lambda_f(h)}{h^s} + \frac{1}{h^{2s}}\right)^{-1}.$$

The series is absolutely convergent when $\Re s > 1$, and admits analytic continuation to all of $\mathbb{C}$. The functional equation for $L(f, s)$ is

$$\Lambda(f, s) := \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma\left(s + \frac{1}{2}\right) L(f, s) = \varepsilon_f \Lambda(f, 1 - s),$$

where $\varepsilon_f = -q^{1/2} \lambda_f(q) = \pm 1$. We define the harmonic average as

$$\sum_f^h A_f := \sum_{f \in \mathcal{S}_2^*(q)} \frac{A_f}{4\pi(f, f)},$$

where (f, g) is the Petersson inner product on the space $\Gamma_0(q) \backslash \mathbb{H}$.

We are interested in the twisted second moment of this family of L-functions. We define

$$S(p, q) = \sum_{f \in \mathcal{S}_2^*(q)}^h L\left(f, \frac{1}{2}\right)^2 \lambda_f(p).$$

Our main theorem is the following.

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THEOREM 1. *Suppose q is prime and $0 < p \leq Cq$, for some fixed $C < 1$. Then we have*

$$S(p, q) = \frac{d(p)}{\sqrt{p}} \log \frac{q}{4\pi^2 p} + O(p^{1/2} q^{-1+\varepsilon}).$$

Remark 1. The twisted harmonic fourth moment has been considered by Kowalski *et al* [6], where they gave an asymptotic formula for the fourth power mean value provided that $p \ll q^{1/9-\varepsilon}$.

Remark 2. In a similar setting, Iwaniec and Sarnak [3] have given the exact formula for the twisted second moment of the automorphic L-functions arising from $\mathcal{H}_k(1)$, the set of newforms in $\mathcal{S}_k(1)$, where $\mathcal{S}_k(1)$ is the linear space of holomorphic cusp forms of weight k. Precisely, they showed that for $k > 2$, $k \equiv 0 \pmod 2$, and, for any $m \geq 1$, we have

$$\begin{aligned}
&\frac{12}{k-1} \sum_{f \in \mathcal{H}_k(1)} w_f L\left(f, \frac{1}{2}\right)^2 \lambda_f(m) \\
&\quad = 2(1 + i^k) \frac{d(m)}{\sqrt{m}} \left(\sum_{0 < l \leq k/2} \frac{1}{l} - \log 2\pi\sqrt{m} \right) \\
&\qquad - \frac{2\pi i^k}{\sqrt{m}} \sum_{h \neq m} d(h) d(h-m) p_k\left(\frac{h}{m}\right) \\
&\qquad + \frac{2\pi i^k}{\sqrt{m}} \sum_{h} d(h) d(h+m) q_k\left(\frac{h}{m}\right),
\end{aligned}$$

where $p_k(x)$ and $q_k(x)$ are Hankel transforms of Bessel functions

$$p_k(x) = \int_0^\infty Y_0(y\sqrt{x}) J_{k-1}(y)\, dy,$$

and

$$q_k(x) = \frac{2}{\pi} \int_0^\infty K_0(y\sqrt{x}) J_{k-1}(y)\, dy.$$

Here the weight $w_f = \zeta(2) L(\mathrm{sym}^2(f), 1)^{-1}$, where the symmetric square L-function $L(\mathrm{sym}^2(f), s)$ corresponding to f is defined by

$$L(\mathrm{sym}^2(f), s) = \zeta(2s) \sum_{n=1}^\infty \frac{\lambda_f(n^2)}{n^s}.$$

In the context of Dirichlet L-functions, consider

$$M(p, q) = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\ q)} \left| L\left(\frac{1}{2}, \chi\right) \right|^2 \chi(p),$$

where $\sum^*$ denotes summation over all primitive characters $\chi \pmod q$, and $\varphi^*(q)$ is the number of primitive characters. This is the twisted second moment of

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Dirichlet L-functions. In a recent paper, Conrey [1] proved that there is a kind of reciprocity formula relating $M(p, q)$ and $M(-q, p)$ when p and q are distinct prime integers. Precisely, Conrey showed that

$$M(p, q) = \frac{\sqrt{p}}{\sqrt{q}} M(-q, p) + \frac{1}{\sqrt{p}} \left(\log \frac{q}{p} + A \right) + \frac{B}{2\sqrt{q}} + O\left(\frac{p}{q} + \frac{\log q}{q} + \frac{\log pq}{\sqrt{pq}} \right),$$

where A and B are some explicit constants. This provides an asymptotic formula for $M(p, q) - \sqrt{p/q} M(-q, p)$ under the condition that $p \ll q^{2/3-\varepsilon}$. The error term above was improved by Young [7] so that the asymptotic formula holds for $p \ll q^{1-\varepsilon}$.

We now take p to be prime and, similarly as before, $S(q, p)$ is defined as the harmonic second moment, twisted by $\lambda_g(q)$, of the family of L-functions arising from $g(z) \in \mathcal{S}_2^*(p)$. We note that as q is prime, the Ramanujan bound $|\lambda_f(n)| \le d(n)$ [2] yields

$$S(q, p) \ll \sum_{g \in \mathcal{S}_2^*(p)}^{h} L\left(g, \frac{1}{2} \right)^2 \ll \log p.$$

Thus, as a trivial consequence of Theorem 1, for $p \le q$ we have

$$S(p, q) - \sqrt{p/q} S(q, p) = \frac{2}{\sqrt{p}} \log \frac{q}{4\pi^2 p} + O(p^{1/2+\varepsilon} q^{-1/2}).$$

This leads to an asymptotic formula for $S(p, q) - \sqrt{p/q} S(q, p)$, at least for p as large as $q^{1/2-\varepsilon}$. The results in the Dirichlet L-functions case [1, 7] suggest that the asymptotic formula should hold for $p \ll q^{\theta}$, for any $\theta < 1$. However, our technique fails to extend the range to any power $\theta > 1/2$. For that purpose, we need more refined estimates for the off-diagonal terms of $S(p, q)$ and $S(q, p)$. The intricate calculations seem to suggest that there is a large cancellation between these two expressions. The nature of this is not well-understood.

§2. *Preliminary lemmas.* We require some lemmas. We begin with Hecke's formula for primitive forms.

LEMMA 1. *For $m, n \ge 1$,*

$$\lambda_f(m) \lambda_f(n) = \sum_{\substack{d | (m,n) \\ (d,q)=1}} \lambda_f\left(\frac{mn}{d^2} \right).$$

The next lemma is a particular case of Petersson's trace formula.

LEMMA 2. *For $m, n \ge 1$, we have*

$$\sum_{f \in \mathcal{S}_2^*(q)}^{h} \lambda_f(m) \lambda_f(n) = \delta_{m,n} - J_q(m, n),$$

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where $\delta_{m,n}$ is the Kronecker symbol and

$$J_q(m,n) = 2\pi \sum_{c=1}^{\infty} \frac{S(m,n;cq)}{cq} J_1\left(\frac{4\pi\sqrt{mn}}{cq}\right).$$

Here $J_1(x)$ is the Bessel function of order one, and $S(m,n;c)$ is the Kloosterman sum

$$S(m,n;c) = \sum_{a \,(\mathrm{mod}\ c)}^{*} e\left(\frac{ma+n\bar{a}}{c}\right).$$

Moreover we have

$$J_q(m,n) \ll (m,n,q)^{1/2}(mn)^{1/2+\varepsilon} q^{-3/2}.$$

The above estimate follows easily from the bound $J_1(x) \ll x$ and Weil's bound on Kloosterman sums.

We mention a result of Jutila [**4**, Theorem 1.7], which is an extension of the Voronoi summation formula.

LEMMA 3. *Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a C^∞ function which vanishes in the neighbourhood of 0 and is rapidly decreasing at infinity. Then, for $c \geq 1$ and $(a,c) = 1$,*

$$\begin{aligned} c\sum_{m=1}^{\infty} d(m) e\left(\frac{am}{c}\right) f(m) &= 2\int_0^{\infty} \left(\log\frac{\sqrt{x}}{c} + \gamma\right) f(x)\,dx \\ &- 2\pi \sum_{m=1}^{\infty} d(m) e\left(\frac{-\bar{a}m}{c}\right) \int_0^{\infty} Y_0\left(\frac{4\pi\sqrt{mx}}{c}\right) f(x)\,dx \\ &+ 4\sum_{m=1}^{\infty} d(m) e\left(\frac{\bar{a}m}{c}\right) \int_0^{\infty} K_0\left(\frac{4\pi\sqrt{mx}}{c}\right) f(x)\,dx. \end{aligned}$$

The next lemma concerns the approximate functional equation for L-functions.

LEMMA 4. *Let $G(s)$ be an even entire function satisfying $G(0) = 1$ and G has a double zero at each $s \in \mathbb{Z}$. Furthermore let us assume that $G(s) \ll_{A,B} (1+|s|)^{-A}$ for any $A > 0$ in any strip $-B \leq \Re s \leq B$. Then, for $f \in \mathcal{S}_2^*(q)$,*

$$L\left(f, \frac{1}{2}\right)^2 = 2\sum_{n=1}^{\infty} \frac{d(n)\lambda_f(n)}{\sqrt{n}} W_q\left(\frac{4\pi^2 n}{q}\right),$$

where

$$W_q(x) = \frac{1}{2\pi i} \int_{(1)} G(s)\Gamma(s+1)^2 \zeta_q(2s+1) x^{-s} \frac{ds}{s}.$$

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Here $\zeta_q(s)$ is defined by

$$\zeta_q(s) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-s} \quad (\sigma > 1).$$

Proof. From Lemma 1 we first note that

$$L(f, s)^2 = \zeta_q(2s) \sum_{n=1}^{\infty} \frac{d(n)\lambda_f(n)}{n^s} \quad (\sigma > 1).$$

Consider

$$A(f) := \frac{1}{2\pi i} \int_{(1)} \frac{G(s)\Lambda\left(f, s + \frac{1}{2}\right)^2}{(\sqrt{q}/2\pi)} \frac{ds}{s}.$$

Moving the line of integration to $\Re s = -1$, and applying Cauchy's theorem and the functional equation, we derive that $A(f) = L(f, \frac{1}{2})^2 - A(f)$. Expanding $\Lambda(f, s + \frac{1}{2})^2$ in a Dirichlet series and integrating termwise we obtain the lemma. □

For our purpose, W_q is basically a "cut-off" function. Indeed, we have the following.

LEMMA 5. *The function W_q satisfies*

$$W_q^{(j)}(x) \ll_{j,N} x^{-N} \quad \text{for } x \geq 1 \text{ and all } j, N \geq 0,$$

$$x^i W_q^{(j)}(x) \ll_{i,j} |\log x| \quad \text{for } 0 < x < 1 \text{ and all } i \geq j \geq 0,$$

and

$$W_q(x) = -\left(1 - \frac{1}{q}\right)\frac{\log x}{2} + \frac{\log q}{q} + O_N(x^N) \quad \text{for } 0 < x < 1 \text{ and all } N \geq 0.$$

The implicit constants are independent of q.

Proof. The first estimate is a direct consequence of Stirling's formula after differentiating under the integral sign and shifting the line of integration to $\Re s = N$. The only difference in the other two estimates is that one has to move the line of integration to $\Re s = -N$. □

§3. *Proof of Theorem 1.* Our argument in this section follows closely [5]. From Lemmas 4 and 2 we obtain

$$S(p, q) = 2\frac{d(p)}{\sqrt{p}} W_q\left(\frac{4\pi^2 p}{q}\right) - 2R(p, q),$$

where

$$R(p, q) = \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} J_q(n, p) W_q\left(\frac{4\pi^2 n}{q}\right).$$

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Using Lemma 5, the first term is

$$\frac{d(p)}{\sqrt{p}} \log \frac{q}{4\pi^2 p} + O(p^{-1/2}q^{-1+\varepsilon} + p^{1/2+\varepsilon}q^{-1}).$$

Thus, we are left to consider $R(p, q)$. We have

$$R(p, q) = 2\pi \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} \sum_{c=1}^{\infty} \frac{S(n, p; cq)}{cq} J_1\left(\frac{4\pi\sqrt{np}}{cq}\right) W_q\left(\frac{4\pi^2 n}{q}\right).$$

Using Weil's bound for Kloosterman sums and $J_1(x) \ll x$, the contribution from the terms $c \geq q$ is bounded by

$$p^{1/2}q^{-3/2} \sum_{n=1}^{\infty} (n, p)^{1/2} d(n) \left| W_q\left(\frac{4\pi^2 n}{q}\right) \right| \sum_{c \geq q} \frac{d(c)}{c^{3/2}} \ll p^{1/2}q^{-1+\varepsilon}.$$

Thus we need to study

$$\frac{2\pi}{q} \sum_{c<q} \frac{1}{c} \sideset{}{^*}\sum_{a (\mathrm{mod}\ cq)} e\left(\frac{\overline{a}p}{cq}\right) \sum_{n=1}^{\infty} d(n) e\left(\frac{an}{cq}\right) \frac{J_1((4\pi\sqrt{np})/cq) W_q((4\pi^2 n)/q)}{\sqrt{n}}.$$

We fix a C^∞ function $\xi : \mathbb{R}^+ \to [0, 1]$, which satisfies $\xi(x) = 0$ for $0 \leq x \leq 1/2$ and $\xi(x) = 1$ for $x \geq 1$, and attach the weight $\xi(n)$ to the innermost sum. Using Lemma 3, this is equal to

$$\frac{4\pi}{q^2} \sum_{c<q} \frac{1}{c^2} S(0, p; cq) \int_0^{\infty} \left(\log \frac{\sqrt{t}}{cq} + \gamma\right) J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right) \times W_q\left(\frac{4\pi^2 t}{q}\right) \xi(t) \frac{dt}{\sqrt{t}} - Y + K,$$

where

$$Y = \frac{4\pi^2}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n=1}^{\infty} d(n) S(0, p - n; cq) \times \int_0^{\infty} Y_0\left(\frac{4\pi\sqrt{nt}}{cq}\right) J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right) W_q\left(\frac{4\pi^2 t}{q}\right) \xi(t) \frac{dt}{\sqrt{t}}, \tag{1}$$

and

$$K = \frac{8\pi}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n=1}^{\infty} d(n) S(0, p + n; cq) \times \int_0^{\infty} K_0\left(\frac{4\pi\sqrt{nt}}{cq}\right) J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right) W_q\left(\frac{4\pi^2 t}{q}\right) \xi(t) \frac{dt}{\sqrt{t}}. \tag{2}$$

We will deal with Y and K in the next three lemmas. For the first sum, since $S(0, p; cq) = \mu(q) S(0, p\overline{q}; c)$ and $J_1(x) \ll x$, this is bounded by

$$p^{1/2}q^{-3} \sum_{c<q} \frac{1}{c^2} \int_{1/2}^{\infty} \left| W_q\left(\frac{4\pi^2 t}{q}\right) \right| (\log tcq)\, dt \ll p^{1/2}q^{-2+\varepsilon}.$$

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LEMMA 6. *For K defined as in (2), we have*

$$K \ll p^{1/2}q^{\varepsilon}(q-p)^{-1+\varepsilon}.$$

Hence $K \ll p^{1/2}q^{-1+\varepsilon}$, given that $p \le Cq$ for some fixed $C < 1$.

Remark 3. This is the only place where the condition $p \le Cq$ for some constant $C < 1$ is used.

Proof. The integral involving K_0, using $K_0(y) \ll y^{-1/2}\mathrm{e}^{-y}$, is

$$\begin{aligned}
&\int_0^\infty K_0\left(\frac{4\pi\sqrt{nt}}{cq}\right)J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right)W_q\left(\frac{4\pi^2 t}{q}\right)\xi(t)\,\frac{dt}{\sqrt{t}}\\
&\quad=\frac{cq}{2\pi\sqrt{n}}\int_0^\infty K_0(y)J_1\left(\sqrt{\frac{p}{n}}y\right)W_q\left(\frac{c^2qy^2}{4n}\right)\xi\left(\frac{c^2q^2y^2}{16\pi^2 n}\right)dy\\
&\quad\ll\frac{cp^{1/2}q^{1+\varepsilon}}{n}\int_{\sqrt{n}/cq}^\infty y^{1/2}\mathrm{e}^{-y}\,dy\ll\frac{cp^{1/2}q^{1+\varepsilon}}{n}\,\mathrm{e}^{-\sqrt{n}/2cq}.
\end{aligned}$$

Thus, as $S(0,\,p+n;\,cq)=S(0,\,(p+n)\overline{q};\,c)S(0,\,p+n;\,q)$ and $|S(0,\,(p+n)\overline{q};\,c)|\le\sum_{l|(p+n,c)}l$,

$$\begin{aligned}
K&\ll p^{1/2}q^{-1+\varepsilon}\sum_{n=1}^\infty\frac{d(n)}{n}\,\mathrm{e}^{-\sqrt{n}/2q^2}|S(0,\,p+n;\,q)|\sum_{c<q}\frac{\sum_{l|(p+n,c)}l}{c}\\
&\ll p^{1/2}q^{-1+\varepsilon}\sum_{n=1}^\infty\frac{d(n)d(p+n)}{n}\,\mathrm{e}^{-\sqrt{n}/2q^2}|S(0,\,p+n;\,q)|.
\end{aligned}$$

We break the sum over n according to whether $q|(p+n)$ or $q\nmid(p+n)$. The contribution of the latter is $O(p^{1/2}q^{-1+\varepsilon})$. That of the former is bounded by

$$\begin{aligned}
&p^{1/2}q^{\varepsilon}\sum_{l=1}^\infty\frac{d(l)d(ql-p)}{ql-p}\exp(-\sqrt{ql-p}/2q^2)\\
&\ll p^{1/2}q^{\varepsilon}(q-p)^{-1+\varepsilon}+p^{1/2}q^{-1+\varepsilon}.
\end{aligned}$$

The lemma follows. □

The case of Y is more complicated as Y_0 is an oscillating function. For that we need the following standard lemma (for example, see [5]).

LEMMA 7. *Let $v \ge 0$ and J be a positive integer. If f is a compactly supported C^∞ function on $[Y,\,2Y]$, and there exists $\beta > 0$ such that*

$$y^j f^{(j)}(y) \ll_j (1+\beta Y)^j$$

for $0 \le j \le J$, then, for any $\alpha > 1$, we have

$$\int_0^\infty Y_v(\alpha y)f(y)\,dy \ll \left(\frac{1+\beta Y}{1+\alpha Y}\right)^J Y.$$

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LEMMA 8. *For Y defined as in (1), we have*

$$Y \ll p^{1/2} q^{-1+\varepsilon}.$$

Proof. We have

$$Y = \frac{4\pi^2}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n=1}^{\infty} d(n) S(0, p-n; cq) y(n), \tag{3}$$

where

$$y(n) = \int_0^{\infty} Y_0\left(\frac{4\pi\sqrt{nt}}{cq}\right) J_1\left(\frac{4\pi\sqrt{tp}}{cq}\right) W_q\left(\frac{4\pi^2 t}{q}\right) \xi(t) \frac{dt}{\sqrt{t}}. \tag{4}$$

We make a smooth dyadic partition of unity that $\xi = \sum_k \xi_k$, where each ξ_k is a compactly supported C^∞ function on the dyadic interval $[X_k, 2X_k]$. Moreover, ξ_k satisfies $x^j \xi_k^{(j)}(x) \ll 1$, for all $j \geq 0$. We work on each ξ_k individually, but we write ξ instead of ξ_k and, accordingly, X rather than X_k.

By the change of variable $x := 2\sqrt{t}/cq$, we have

$$y(n) = cq \int_0^{\infty} Y_0(2\pi\sqrt{n}x) J_1(2\pi\sqrt{p}x) W_q(\pi^2 c^2 q x^2) \xi\left(\frac{c^2 q^2 x^2}{4}\right) dx.$$

We define

$$f(x) := J_1(2\pi\sqrt{p}x) W_q(\pi^2 c^2 q x^2) \xi\left(\frac{c^2 q^2 x^2}{4}\right).$$

This is a C^∞ function compactly supported on $[\rho, 2\rho]$, where $\rho = 2\sqrt{X}/cq$.

We first treat the case $1/2 \leq X \leq q$. We note that this involves $O(\log q)$ dyadic intervals. From Lemma 5 we have $x^j W^{(j)}(x) \ll_j \log q$ for $1/q \ll x \ll 1$. This, together with the recurrence relation $(x^v J_v(x))' = x^v J_{v-1}(x)$, gives

$$x^j f^{(j)}(x) \ll_j (1 + \sqrt{p}x)^j \log q. \tag{5}$$

We are in a position to apply Lemma 7 to f with $\alpha = 2\pi\sqrt{n}$, $\beta = \sqrt{p}$ and $Y = \rho = 2\sqrt{X}/cq$. The lemma yields, for any positive integer J,

$$y(n) \ll cq\rho \left(\frac{1 + \sqrt{p}\rho}{1 + \sqrt{n}\rho}\right)^J \log q. \tag{6}$$

Later, we will break the sum over n in (3) in the following way

$$\sum_{n \geq 1} = \sum_{n \leq \rho^{-\kappa}} + \sum_{n > \rho^{-\kappa}},$$

where $\kappa > 2$ will be chosen later. The estimate (6) will be used for $n > \rho^{-\kappa}$. We need another estimate for the range $n \leq \rho^{-\kappa}$. For this we go back to (4), using $Y_0(x) \ll 1 + |\log x|$ and $J_1(x) \ll x$, to derive

$$y(n) \ll \frac{\sqrt{p}X}{cq} (\log q)^2. \tag{7}$$

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We denote by Y_1 and Y_2 the corresponding split sums ($Y = Y_1 + Y_2$). For the first sum, using (7), we have

$$\begin{aligned} Y_1 &= \frac{4\pi^2}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n \le \rho^{-\kappa}} d(n) S(0, p-n; cq) y(n) \\ &\ll p^{1/2} X q^{-3+\varepsilon} \sum_{n \le \rho^{-\kappa}} d(n) |S(0, p-n; q)| \sum_{c<q} \frac{1}{c^3} \sum_{l|(p-n,c)} l \\ &\ll p^{1/2} X q^{-3+\varepsilon} \sum_{n \le (q^2/(2\sqrt{X}))^{\kappa}} d(n) |S(0, p-n; q)| \\ &\quad \times \sum_{(2\sqrt{X}/q) n^{1/\kappa} \le c < q} \sum_{l|(p-n,c)} \frac{l}{c^3} \\ &\ll p^{1/2} q^{-1+\varepsilon} \sum_{n \le (q^2/(2\sqrt{X}))^{\kappa}} \frac{d(n) d(p-n)}{n^{2/\kappa}} |S(0, p-n; q)| \\ &\ll p^{1/2} q^{2\kappa - 5 + \varepsilon}. \end{aligned} \tag{8}$$

For the second sum, we note that $\sqrt{p}\rho \ll 1$ in this range. Using (6), we have

$$y(n) \ll \sqrt{X} (\log q) n^{-J/2} \rho^{-J}.$$

Similarly to above, we deduce that

$$\begin{aligned} Y_2 &= \frac{4\pi^2}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n > \rho^{-\kappa}} d(n) S(0, p-n; cq) y(n) \\ &\ll \sqrt{X} q^{-2+\varepsilon} \sum_{n > \rho^{-\kappa}} \frac{d(n)}{n^{J/2}} |S(0, p-n; q)| \sum_{c<q} \frac{1}{c^2} \sum_{l|(p-n,c)} l \rho^{-J} \\ &\ll X^{-(J-1)/2} q^{J-2} \sum_{n} \frac{d(n)}{n^{J/2}} |S(0, p-n; q)| \\ &\quad \times \sum_{l|p-n} l^{J-1} \sum_{c \le (2\sqrt{X}/ql) n^{1/\kappa}} c^{J-2} \\ &\ll q^{-1+\varepsilon} \sum_{n} \frac{d(n) d(p-n)}{n^{J/2-(J-1)/\kappa}} |S(0, p-n; q)|. \end{aligned} \tag{9}$$

To this end, we choose $\kappa = 2 + \varepsilon/2$ and J large enough so that $J/2 - (J-1)/\kappa > 1$. We hence obtain $Y_1 \ll p^{1/2} q^{-1+\varepsilon}$ and, since the sum over n in (9) converges, $Y_2 \ll q^{-1+\varepsilon}$.

For $X > q$, similarly to (5), using the bound $x^j W^{(j)}(x) \ll_j x^{-2}$, we have

$$x^j f^{(j)}(x) \ll_j (1 + \sqrt{p}x)^j q^{-2} (cx)^{-4}.$$

Lemma 7 then gives

$$y(n) \ll cq\rho \left(\frac{1 + \sqrt{p}\rho}{1 + \sqrt{n}\rho} \right)^J \frac{q^2}{X^2}.$$

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For the range $n \leq \rho^{-\kappa}$, a better bound than (7) in this case is

$$y(n) \ll \frac{\sqrt{p}X}{cq}(\log q)\frac{q^2}{X^2}.$$

Since $q^2/X^2 \ll 1$, all the previous estimates remain valid. The only place where this is not the case is the sum over $n \ll (q^2/2\sqrt{X})^{\kappa}$ in (8). However, this sum is void for $X > q^4/4$ and the former estimate still works in the larger interval $X \leq q^4/4$. Also, the quantity saved q^2/X^2 is sufficient to allow the sum over the dyadic values of X involved to converge. The lemma follows. □

The proof of the theorem is complete.

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