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ABSOLUTE CONTINUITY OF DISTRIBUTIONS OF ONE-DIMENSIONAL LÉVY PROCESSES

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Abstract

In this paper we study the existence of Lebesgue densities of one-dimensional Lévy processes. Equivalently, we show the absolute continuity of the distributions of one-dimensional Lévy processes. Compared with the previous literature, we consider Lévy processes with Lévy symbols of a logarithmic behavior at ∞.

Keywords: Lévy process; Lebesgue density; subordinator

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1. Introduction

Let $\{P_t,\ t > 0\}$ be a distribution of the Lévy process X_t whose Fourier transform $\mathcal{F}(P_t)$ (characteristic function of X_t) is written as $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi y} P_t(\mathrm{d}y) = \mathrm{e}^{-t\Psi(\xi)}$. We call Ψ the Lévy symbol of X_t. By the Lévy–Khintchine formula, Ψ is represented in the form

$$\Psi(\xi) = -\mathrm{i}b\xi + \frac{1}{2}A\xi^2 - \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\xi y} - 1 - \mathrm{i}\xi y\, \mathbf{1}_{(-1,1)}(y))\nu(\mathrm{d}y),$$

where b is a constant, A is a nonnegative constant, and ν is the Lévy measure, that is,

$$\int_{\mathbb{R}} \min(1, |y|^2)\nu(\mathrm{d}y) < \infty.$$

The Lebesgue densities of Lévy processes have been used as an important tool in theoretical probability, physics, and finance; see [1], [10], and [12] and the references therein. It is natural to consider Lévy processes whose characteristic function is slowly decaying at ∞, and to ask whether its distribution possesses a density.

For a rotation invariant α-stable Lévy process, Blumenthal and Getoor [2] showed that the density function p_t satisfies $p_t(x) \approx \min(t^{d/\alpha}, t|x|^{-d-\alpha})$. Pruitt and Taylor [11] investigated multivariate stable densities in a more general setting. They obtained the upper bound $p_t(x) \le ct^{-d/\alpha}(1 + t^{-1/\alpha}|x|)^{-1-\alpha}$ using Fourier analytic methods. Using the perturbation formula, Głowacki and Hebisch [5], [6] proved that if $\nu(\mathrm{d}x) = |x|^d g(x/|x|)\,\mathrm{d}x$ and g is bounded then $p_t(x) \le c\min(t^{d/\alpha}, t|x|^{d-\alpha})$. When g is continuous on unit sphere $\mathbb{S}$, we even have $\lim_{r\to\infty} r^{d+\alpha} p_1(r\theta) = cg(\theta)$, $\theta \in \mathbb{S}$, and if $g(\theta) = 0$ then, additionally, $\lim_{r\to\infty} r^{d+2\alpha} p_1(r\theta) = c_\theta > 0$, which was proved by Dziubański [3]. Zaigraev [14] obtained further asymptotic expansions of the α-stable density for sufficiently regular g.

The classic paper by Hartman and Wintner [7] gives sufficient conditions in terms of the Lévy symbol Ψ under which there exists a Lebesgue density $p_t(x)$ of X_t; these conditions

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guarantee that $p_t \in C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ denotes the set of all continuous functions which vanish at ∞. More precisely, if

$$\liminf_{|\xi|\to\infty} \frac{\operatorname{Re}\Psi(\xi)}{1+\ln|\xi|} = \infty \tag{1.1}$$

then $P_t(\mathrm{d}x) = p_t(x)\,\mathrm{d}x$ for all $t > 0$, and $p_t \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. Also, if

$$\liminf_{|\xi|\to\infty} \frac{\operatorname{Re}\Psi(\xi)}{1+\ln|\xi|} > \frac{1}{t} \tag{1.2}$$

then $P_s(\mathrm{d}x) = p_s(x)\,\mathrm{d}x$ for all $s > t$ and $p_s \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. In fact, under assumption (1.1), $(1+|\xi|)^n \mathrm{e}^{-t\Psi(\cdot)} \in L^1(\mathbb{R})$ for all $n \in \mathbb{R}$ and so $\mathcal{F}^{-1}(\mathrm{e}^{-t\Psi})$ (the inverse Fourier transform of $\mathrm{e}^{-t\Psi}$) is a function of $C_0(\mathbb{R})$.

Kallenberg [8] found sufficient conditions that the Lévy processes have a Lebesgue density function. Kallenberg showed that if

$$\lim_{\varepsilon\to 0} \frac{\int_{-\varepsilon}^{\varepsilon} y^2 \nu(\mathrm{d}y)}{\varepsilon^2 |\ln\varepsilon|} = \infty$$

then $P_t(\mathrm{d}x) = p_t(x)\,\mathrm{d}x$ for all $t > 0$ and $p_t \in C_b^\infty(\mathbb{R}) \cap C_0(\mathbb{R})$, and if

$$\liminf_{\varepsilon\to 0} \frac{\int_{-\varepsilon}^{\varepsilon} y^2 \nu(\mathrm{d}y)}{\varepsilon^2 |\ln\varepsilon|} > \frac{1}{t}$$

then $P_s(\mathrm{d}x) = p_s(x)\,\mathrm{d}x$ for all $s > t > 0$ and $p_s \in C_b^\infty(\mathbb{R}) \cap C_0(\mathbb{R})$.

Tucker [13] provided necessary and sufficient criteria for the existence of a transition density.

Recently, Knopova and Schilling [9] showed that condition (1.1) is equivalent to $p_t, \nabla p_t \in L^1(\mathbb{R}^d)$ for all $t > 0$.

In those studies, the Lévy symbols with behavior at ∞ as in (1.1) and (1.2) were considered, and this resulted in Lebesgue densities with desirable continuity and boundedness. If the behavior of a Lévy symbol at ∞ is not sufficiently good then the corresponding characteristic function decays slower and so it is possible that $\mathrm{e}^{-t\Psi(\cdot)} \notin L^1(\mathbb{R})$ and we cannot guarantee the existence of a Lebesgue density function.

In this paper we study the existence of the Lebesgue densities of one-dimensional Lévy processes with Lévy symbols that have logarithmic behavior at ∞.

To show the existence of Lebesgue densities, we use Fourier analytic methods. Since the differentiability of f is closely related to the behavior of $\mathcal{F}^{-1}(f)$ (the inverse Fourier transform of f) at ∞, and vice versa, the Lévy symbol Ψ with slow decay behavior at ∞ does not give enough information for the continuity of the distribution P_t. This could be avoided by assuming, additionally, that $\Psi \in C^2(\mathbb{R}\setminus\{0\})$. More precisely, we consider the Lévy symbol Ψ written as $\Psi(\xi) = \Psi_1(\xi) + i\Psi_2(\xi)$, where $\Psi_1, \Psi_2 \in C^2(\mathbb{R}\setminus\{0\}) \cap C(\mathbb{R})$ satisfy the following assumptions.

(A) There are $n \in \mathbb{N}$, $0 < \varepsilon_2, \varepsilon \le 1$, $0 < \varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \alpha > 0$, and $c > 0$ such that

$$\Psi_1(\xi) \ge \alpha s_n^\varepsilon(\xi), \qquad |\Psi_2(\xi)| \le c \begin{cases} |\xi|^{\varepsilon_{11}} & \text{if } |\xi| \le 1, \\ s_n^{\varepsilon_2 - 1}(\xi) h(|\xi|) & \text{if } |\xi| \ge 1, \end{cases}$$

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and

$$|\Psi_1'(\xi)|,\ |\Psi_2'(\xi)| \le c \begin{cases} |\xi|^{\varepsilon_{12}-1} & \text{if } |\xi| \le 1, \\ s_n^{\varepsilon_2-1}(\xi) h(|\xi|)|\xi|^{-1} & \text{if } |\xi| \ge 1, \end{cases}$$

$$|\Psi_1''(\xi)|,\ |\Psi_2''(\xi)| \le c \begin{cases} |\xi|^{\varepsilon_{13}-2} & \text{if } |\xi| \le 1, \\ s_n^{\varepsilon_2-1}(\xi) h(|\xi|)|\xi|^{-2} & \text{if } |\xi| \ge 1. \end{cases}$$

Here,

$$s_1(\xi) := \ln(1+|\xi|), \qquad s_n(\xi) := \ln(1+s_{n-1}(\xi)), \quad n \ge 2, \tag{1.3}$$

and $h\colon [1,\infty) \to [0,\infty)$ is a nonincreasing function satisfying the following:

(A$_1$) $h'(\xi) = -h(\xi)h_1(\xi)$ for a nonnegative function h_1 satisfying $\lim_{\xi\to\infty} h_1(\xi) = 0$.

Remark 1.1. The trivial examples of h satisfying (A$_1$) are $h = 1$ and $h = r_{n-1}^{-1}$, where

$$r_0(\xi) := 1, \qquad r_n(\xi) := s_n(\xi) r_{n-1}(\xi), \quad n \ge 1.$$

Remark 1.2. Compared with Hartman and Wintner' assumptions (1.1) and (1.2), our assumption (A) contains the case of $\mathrm{e}^{-t\Psi} \notin L^1(\mathbb{R})$. In fact, $\Psi = s_n$, $n \ge 2$, defined by (1.3) satisfies $\mathrm{e}^{-ts_n} \notin L^1(\mathbb{R})$ for $n \ge 2$. In Section 5 we will show that s_n, $n \ge 2$, satisfies assumption (A).

The following states the main result of this paper.

Theorem 1.1. *Suppose that Ψ satisfies assumption (A) for fixed $0 < \varepsilon_2, \varepsilon \le 1$, $0 < \varepsilon_{11}, \varepsilon_{12}$, ε_{13}, and $n \ge 2$. Then the measures $P_t, t > 0$, are absolutely continuous with respect to the Lebesgue measure, and their densities p_t satisfy*

(i) *that*

$$p_t(x) \le c(t)\frac{1}{|x|} \quad \text{if } 0 < |x| \le 1,$$

$$p_t(x) \le c(t) \begin{cases} \max(|x|^{-1-\varepsilon_1}, |x|^{-2}), & \varepsilon_1 \ne 1, \\ |x|^{-2}\ln(1+|x|), & \varepsilon_1 = 1, \end{cases} \quad \text{if } |x| \ge 1, \tag{1.4}$$

where $\varepsilon_1 = \min(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) > 0$. If $\limsup_{\xi\to\infty} \xi h_1(\xi) < 1$ then $c(t)$ satisfies $c(t) \le ct$ for $t \le 1$.

(ii) *If Ψ is symmetric with respect to 0 then*

$$p_t(x) \le c(t) q_t(x), \qquad x \ne 0, \tag{1.5}$$

where

$$q_t(x) := \frac{1}{|x|}\mathrm{e}^{-\alpha t s_n^{\varepsilon}(1/x)} s_n^{\varepsilon_2-1}\left(\frac{1}{x}\right) h\left(\frac{1}{|x|}\right) \quad \text{if } 0 < |x| \le 1, \tag{1.6}$$

$$q_t(x) := \begin{cases} \max(|x|^{-1-\varepsilon_1}, |x|^{-2}), & \varepsilon_1 \ne 1, \\ |x|^{-2}\ln(1+|x|), & \varepsilon_1 = 1, \end{cases} \quad \text{if } |x| \ge 1. \tag{1.7}$$

Remark 1.3. Let Ψ, h, ε, ε_i, and n satisfy the same assumptions as for Theorem 1.1. In (1.7), since $\varepsilon_1 > 0$, q_t is integrable in $(1,\infty)$. If h satisfies the condition

(A$_2$) $h(\xi) \le c r_{n-1}^{-1}(\xi)$

then $0 < \varepsilon, \varepsilon_2 \le 1$ says that $q_t, t > 0$, defined by (1.6) is a function in $L^1(\mathbb{R})$; see Appendix D.

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Theorem 1.2. *Suppose that Ψ is symmetric with respect to 0 and satisfies assumption (A) for the same h, ε, ε_i, and n as for Theorem 1.1. Additionally, we assume that h satisfies (A_2). Then $p_t = \mathcal{F}^{-1}(\mathrm{e}^{-t\Psi}) \in L^1(\mathbb{R})$ and p_t satisfies (1.5).*

Theorem 1.3. *Suppose that Ψ is a symmetric real-valued function satisfying the assumption*

$$\Psi(\xi) \le \alpha_0 s_n^{\varepsilon}(\xi), \qquad -\Psi''(\xi) \ge c \begin{cases} s_{n-1}^{\varepsilon_2-1}(\xi)h(\xi)|\xi|^{-2} & \text{if } |\xi| \ge 1, \\ |\xi|^{\varepsilon_1-2} & \text{if } |\xi| \le 1. \end{cases} \tag{1.8}$$

Then

$$p_t(x) \ge c(t)q_t(x). \tag{1.9}$$

Remark 1.4. From the definition of q_t in (1.6), p_t is unbounded near $x = 0$.

We denote by $\mathcal{F}(f)$ the Fourier transform of a function f, that is,

$$\mathcal{F}(f)(\xi) = \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi y} f(y)\,\mathrm{d}y,$$

and $\mathcal{F}^{-1}(f)$ the inverse Fourier transform of a function f, that is,

$$\mathcal{F}^{-1}(f)(x) = \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\mathrm{i}\xi} f(\xi)\,\mathrm{d}\xi.$$

Throughout this paper we denote by c various generic positive constants and by $c(*, \dots, *)$ the constants depending only on the quantities appearing in the parenthesis.

2. Proof of Theorem 1.1

We introduce three lemmas which will be useful in proving Theorems 1.1 and 1.3; see Appendices A and B for proofs of the lemmas.

Lemma 2.1. *Let f be a real-value function defined in $\mathbb{R}$ and H, G be nonnegative and nonincreasing functions defined in $(0, \infty)$ such that*

$$|f'(\xi)| \le cH(|\xi|), \qquad |f''(\xi)| \le cG(|\xi|).$$

Then, for $x \in \mathbb{R} \setminus \{0\}$,

$$\left|\int_{\mathbb{R}} f(\xi)\cos(x\xi)\,\mathrm{d}\xi\right| \le c\left(\frac{1}{|x|^2}\int_{1/|x|}^{\infty} G(y)\,\mathrm{d}y + \frac{1}{|x|^3}G\left(\frac{1}{|x|}\right) + \int_0^{1/|x|} G(y)y^2\,\mathrm{d}y\right),$$

$$\left|\int_{\mathbb{R}} f(\xi)\sin(x\xi)\,\mathrm{d}\xi\right| \le c\left(\frac{1}{|x|^2}\int_{1/|x|}^{\infty} G(y)\,\mathrm{d}y + \frac{1}{|x|^2}\int_0^1 H\left(\frac{\xi}{|x|}\right)\mathrm{d}\xi\right).$$

Lemma 2.2. *Let $0 < \varepsilon, \varepsilon_2 \le 1$, and $n \ge 2$. Let $h\colon [1, \infty) \to [0, \infty)$ be a nonincreasing function such that $h'(z) = -h(z)h_1(z)$.*

(i) *If $\lim_{z\to\infty} h_1(z) = 0$ then, for $\alpha_0 > -1$ and $a \ge 1$,*

$$\int_1^a z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)} s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z \le c(\alpha_0, t)a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)} s_n^{\varepsilon_2-1}(a)h(a).$$

(ii) *Let $\alpha_0 > -1$ and $t \le 1$. Let $z > a_0$, $zh_1(z) < 1/(\alpha_0+1) - \varepsilon_0$ for some $\varepsilon_0 > 0$ and $(1 - \varepsilon_2 + \varepsilon t\alpha)\ln(1 + a_0)^{-1} \le \frac{1}{2}\varepsilon_0$. Then, for $a \ge a_0$,*

$$\int_{a_0}^a z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)} s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z \le c(\alpha_0)a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)} s_n^{\varepsilon_2-1}(a)h(a).$$

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(iii) *For $\alpha_0 < -1$ and $a \geq 1$,*

$$\int_a^\infty z^{\alpha_0} \mathrm{e}^{-t\alpha s_n^\varepsilon(z)} s_n^{\varepsilon_2 - 1}(z) h(z)\,\mathrm{d}z \leq c(\alpha_0) a^{\alpha_0+1} \mathrm{e}^{-t\alpha s_n^\varepsilon(a)} s_n^{\varepsilon_2-1}(a) h(a).$$

Lemma 2.3. *Let μ_t be a finite measure on $\mathbb{R}$ such that $\mathcal{F}(\mu_t)(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}\mu_t(x) = \mathrm{e}^{-t\Psi(\xi)}$. Suppose that Ψ satisfies assumption (A) for fixed $n \geq 1$. Then μ_t is absolutely continuous with respect to the Lebesgue measure.*

Proof of Equation (1.4). Let $\Psi = \Psi_1 + \mathrm{i}\Psi_2$ satisfy assumption (A). Set $f_t := \mathrm{e}^{-t\Psi}$.

Let $f_{t1}(\xi) = \mathrm{e}^{-t\Psi_1(\xi)} \cos t\Psi_2(\xi)$, $f_{t2}(\xi) = \mathrm{e}^{-t\Psi_1(\xi)} \sin t\Psi_2(\xi)$ such that $f_t = f_{t1}(\xi) + \mathrm{i} f_{t2}(\xi)$.

By assumption (A), we have

$$\begin{aligned} &\Psi_1(\xi) \geq g_1(|\xi|), \qquad |\Psi_2(\xi)| \leq g_2(|\xi|), \\ &|\Psi_1'(\xi)|,\ |\Psi_2'(\xi)| \leq g_3(|\xi|), \qquad |\Psi_1''(\xi)|,\ |\Psi_2''(\xi)| \leq g_4(|\xi|), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} g_1(\xi) &:= \alpha s_n^\varepsilon(\xi), \qquad g_2(\xi) := c \begin{cases} |\xi|^{\varepsilon_{11}}, & |\xi| \leq 1, \\ s_n^{\varepsilon_2-1}(\xi) h(|\xi|), & |\xi| \geq 1, \end{cases} \\ g_3(\xi) &:= c \begin{cases} |\xi|^{\varepsilon_{12}-1}, & |\xi| \leq 1, \\ s_n^{\varepsilon_2-1}(\xi) h(|\xi|)|\xi|^{-1}, & |\xi| \geq 1, \end{cases} \\ g_4(\xi) &:= c \begin{cases} |\xi|^{\varepsilon_{13}-2}, & |\xi| \leq 1, \\ s_n^{\varepsilon_2-1}(\xi) h(|\xi|)|\xi|^{-2}, & |\xi| \geq 1, \end{cases} \end{aligned} \tag{2.2}$$

with the same constant $c > 0$ that appears in (A). Let $\varepsilon_1 := \min(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}) > 0$.

Since $2|\Psi_1'(\xi)\Psi_2'(\xi)| \leq |\Psi_1'(\xi)|^2 + |\Psi_2'(\xi)|^2$, from (2.1), for $i = 1, 2$, we have

$$\begin{aligned} |f_{ti}'(\xi)| &\leq c\mathrm{e}^{-tg_1(\xi)}(|t\Psi_1'(\xi)| + |t\Psi_2'(\xi)|) \leq c\mathrm{e}^{-tg_1(\xi)} g_3(\xi) := cH(t, |\xi|), \\ |f_{ti}''(\xi)| &\leq c\mathrm{e}^{t\Psi_1(\xi)}((t\Psi_1'(\xi))^2 + |t\Psi_1''(\xi)| + |t\Psi_2''(\xi)| + (-t\Psi_2'(\xi))^2) \\ &\leq c\mathrm{e}^{-tg_1(|\xi|)}(t^2 g_3(|\xi|)^2 + t g_4(|\xi|)) \\ &:= cG(t, |\xi|), \end{aligned} \tag{2.3}$$

where $H(t, \xi) := t\mathrm{e}^{-tg_1(\xi)} g_3(\xi)$ and $G(t, \xi) := \mathrm{e}^{-tg_1(\xi)}(t^2 g_3(\xi)^2 + t g_4(\xi))$. From Lemma 2.1, we have

$$\begin{aligned} |\mathcal{F}^{-1}(\mathrm{e}^{-t\Psi})(x)| &= \left| \int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)} \mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi \right| \\ &\leq \left| \int_{\mathbb{R}} f_{t1}(\xi) \cos x\xi\,\mathrm{d}\xi \right| + \left| \int_{\mathbb{R}} f_{t1}(\xi) \sin x\xi\,\mathrm{d}\xi \right| + \left| \int_{\mathbb{R}} f_{t2}(\xi) \cos x\xi\,\mathrm{d}\xi \right| \\ &\quad + \left| \int_{\mathbb{R}} f_{t2}(\xi) \sin x\xi\,\mathrm{d}\xi \right| \\ &\leq c\left(\frac{1}{|x|^2} \int_{1/|x|}^\infty G(t, y)\,\mathrm{d}y + \frac{1}{|x|^3} G\left(t, \frac{1}{|x|}\right) + \int_0^{1/|x|} G(t, y) y^2\,\mathrm{d}y \right. \\ &\quad \left. + \frac{1}{|x|^2} \int_0^1 H\left(t, \frac{y}{x}\right) \mathrm{d}y \right). \end{aligned} \tag{2.4}$$

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Note that, from (2.2), we have

$$H(t,\xi) \le c \begin{cases} t\mathrm{e}^{-t\alpha s_n^{\varepsilon}(\xi)}|\xi|^{\varepsilon_{12}-1}, & |\xi| \le 1, \\ t\mathrm{e}^{-t\alpha s_n^{\varepsilon}(\xi)} s_n^{\varepsilon_2-1}(\xi) h(|\xi|)|\xi|^{-1}, & |\xi| \ge 1, \end{cases} \tag{2.5}$$

$$G(t,\xi) \le c \begin{cases} \mathrm{e}^{-\alpha t s_n^{\varepsilon}(\xi)}(t^2|\xi|^{2\varepsilon_1-2} + t|\xi|^{\varepsilon_1-2}), & |\xi| \le 1, \\ t(t+1)\mathrm{e}^{-t\alpha s_n^{\varepsilon}(\xi)} s_n^{\varepsilon_2-1}(\xi) h(\xi)|\xi|^{-2}, & |\xi| \ge 1. \end{cases} \tag{2.6}$$

The $|x| \ge 1$ case. Since $\mathrm{e}^{-t\alpha s_n^{\varepsilon}(\xi)} \le 1$, using (2.6), we have

$$\begin{aligned} \frac{1}{|x|^2}\int_{1/|x|}^{\infty} G(t,\xi)\,\mathrm{d}\xi &\le c\frac{1}{|x|^2}\int_1^{\infty} t(t+1)\mathrm{e}^{-t\alpha s_n^{\varepsilon}(\xi)} s_n^{\varepsilon_2-1}(\xi) h(\xi)\xi^{-2}\,\mathrm{d}\xi \\ &\quad + c\frac{1}{|x|^2}\int_{1/|x|}^{1} \mathrm{e}^{-t\alpha s_n^{\varepsilon}(\xi)}(t^2\xi^{2\varepsilon_1-2} + t\xi^{\varepsilon_1-2})\,\mathrm{d}\xi \\ &\le c\frac{1}{|x|^2}\int_1^{\infty} t(t+1)\xi^{-2}\,\mathrm{d}\xi + c\frac{1}{|x|^2}\int_{1/|x|}^{1}(t^2\xi^{2\varepsilon_1-2} + t\xi^{\varepsilon_1-2})\,\mathrm{d}\xi \\ &\le ct(t+1)\begin{cases} \max\left(\dfrac{1}{|x|^2}, \dfrac{1}{|x|^{1+\varepsilon_1}}\right), & \varepsilon_1 \ne 1, \\ \dfrac{\ln(1+|x|)}{|x|^2}, & \varepsilon_1 = 1. \end{cases} \end{aligned} \tag{2.7}$$

Since $1/|x| \le 1$, using (2.6), we have

$$\begin{aligned} \frac{1}{|x|^3} G\left(t, \frac{1}{|x|}\right) &\le c\frac{1}{|x|^3}\mathrm{e}^{-\alpha t s_n^{\varepsilon}(1/x)}\left(t^2\left|\frac{1}{x}\right|^{2\varepsilon_1-2} + t\left|\frac{1}{x}\right|^{\varepsilon_1-2}\right) \\ &\le ct(t+1)\frac{1}{|x|^3}\left|\frac{1}{x}\right|^{\varepsilon_1-2} \\ &\le ct(t+1)\frac{1}{|x|^{1+\varepsilon_1}} \end{aligned} \tag{2.8}$$

and

$$\int_0^{1/|x|} G(t,\xi)\xi^2\,\mathrm{d}\xi \le ct(t+1)\int_0^{1/|x|}\xi^{\varepsilon_1}\,\mathrm{d}\xi = ct(t+1)\frac{1}{|x|^{1+\varepsilon_1}}. \tag{2.9}$$

From (2.5), we have

$$\frac{1}{|x|^2}\int_0^1 H\left(t, \frac{\xi}{x}\right)\mathrm{d}\xi \le ct\frac{1}{|x|}\int_0^{1/|x|}\xi^{\varepsilon_1-1}\,\mathrm{d}\xi \le ct\frac{1}{|x|^{1+\varepsilon_1}}. \tag{2.10}$$

Applying (2.4)–(2.10) to Lemma 2.1, we can conclude that the estimate (1.4) holds for $|x| \ge 1$.

The $|x| \le 1$ case. Taking $a = 2\pi/|x|$ and $\alpha_0 = -2$ in Lemma 2.2(iii), for $|x| \le 1$, we have

$$\begin{aligned} \frac{1}{|x|^2}\int_{1/|x|}^{\infty} G(t,\xi)\,\mathrm{d}\xi &\le c\frac{1}{|x|^2}\int_{1/|x|}^{\infty} t(t+1)\mathrm{e}^{-\alpha t s_n^{\varepsilon}(\xi)} s_n^{\varepsilon_2-1}(\xi) h(\xi)\xi^{-2}\,\mathrm{d}\xi \\ &\le c\frac{1}{|x|^2}t(t+1)\mathrm{e}^{-\alpha t s_n^{\varepsilon}(1/|x|)} s_n^{\varepsilon_2-1}\left(\frac{1}{|x|}\right)h\left(\frac{1}{|x|}\right)\left|\frac{1}{x}\right|^{-1} \\ &\le c\frac{1}{|x|}t(t+1)\mathrm{e}^{-\alpha t s_n^{\varepsilon}(1/|x|)} s_n^{\varepsilon_2-1}\left(\frac{1}{|x|}\right)h\left(\frac{1}{|x|}\right) \end{aligned} \tag{2.11}$$

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with $c(t) = ct(t+1) \le 2ct$ for $t \le 1$. By Lemma 2.2(i), we have

$$\begin{aligned}\int_0^{1/|x|} G(t,\xi)\xi^2\,\mathrm{d}\xi &\le ct(t+1)\int_0^1 \xi^{\varepsilon_1}\,\mathrm{d}\xi + ct(t+1)\int_1^{1/|x|} \mathrm{e}^{-t\alpha s_n^\varepsilon(\xi)} s_n^{\varepsilon_2-1}(\xi)h(\xi)\,\mathrm{d}\xi \\ &\le c(t)\frac{1}{|x|}t(t+1)\mathrm{e}^{-t\alpha s_n^\varepsilon(1/|x|)}s_n^{\varepsilon_2-1}\left(\frac{1}{|x|}\right)h\left(\frac{1}{|x|}\right). \end{aligned} \tag{2.12}$$

Moreover, if $t \le 1$ then, by Lemma 2.2(ii), we have

$$\begin{aligned}\int_0^{1/|x|} G(t,\xi)\xi^2\,\mathrm{d}\xi &\le ct(t+1)\int_0^{a_0} \xi^{\varepsilon_1}\,\mathrm{d}\xi + ct(t+1)\int_{a_0}^{1/|x|} \mathrm{e}^{-\alpha t s_n^\varepsilon(\xi)} s_n^{\varepsilon_2-1}(\xi)h(\xi)\,\mathrm{d}\xi \\ &\le ct\frac{1}{|x|}\mathrm{e}^{-t\alpha s_n^\varepsilon(1/|x|)}s_n^{\varepsilon_2-1}\left(\frac{1}{|x|}\right)h\left(\frac{1}{|x|}\right), \end{aligned} \tag{2.13}$$

where a_0 is a constant as defined in Lemma 2.2.

By direct calculation, we obtain

$$\frac{1}{|x|^3}G\left(t,\frac{1}{|x|}\right) \le ct(t+1)\frac{1}{|x|}\mathrm{e}^{-t\alpha s_n^\varepsilon(1/|x|)}s_n^{\varepsilon_2-1}\left(\frac{1}{|x|}\right)h\left(\frac{1}{|x|}\right). \tag{2.14}$$

By Lemma 2.2(i), we have

$$\begin{aligned}\frac{1}{|x|^2}\int_0^1 H\left(t,\frac{\xi}{|x|}\right)\mathrm{d}\xi &\le \frac{t}{|x|^2}\int_0^1 \mathrm{e}^{-tg_1(\xi/|x|)}g_3\left(\frac{\xi}{|x|}\right)\mathrm{d}\xi \\ &\le ct\frac{1}{|x|}\int_0^1 \xi^{\varepsilon_1-1}\,\mathrm{d}\xi + ct\frac{1}{|x|}\int_1^{1/|x|} \mathrm{e}^{-\alpha t s_n^\varepsilon(\xi)}s_n^{\varepsilon_2-1}(\xi)|\xi|^{-1}h(\xi)\,\mathrm{d}\xi \\ &\le ct\frac{1}{|x|}. \end{aligned} \tag{2.15}$$

Applying (2.11)–(2.15) to Lemma 2.1, we can conclude that estimate (1.4) holds for $|x| \le 1$, completing the proof. □

Proof of Equation (1.5). Since Ψ is a symmetric function, we have $\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\sin(x\xi)\,\mathrm{d}\xi = 0$. From Lemma 2.1, we have

$$\begin{aligned}|\mathcal{F}^{-1}(\mathrm{e}^{-t\Psi})(x)| &= \left|\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi\right| \\ &\le \left|\int_{\mathbb{R}} f_{t1}(\xi)\cos x\xi\,\mathrm{d}\xi\right| + \left|\int_{\mathbb{R}} f_{t2}(\xi)\cos x\xi\,\mathrm{d}\xi\right| \\ &\le c\left(\frac{1}{|x|^2}\int_{1/|x|}^\infty G(t,y)\,\mathrm{d}y + \frac{1}{|x|^3}G\left(t,\frac{1}{|x|}\right) + \int_0^{1/|x|} G(t,y)y^2\,\mathrm{d}y\right). \end{aligned}$$

Hence, from the proof of (1.4), we obtain (1.5). □

The remainder of the proof of Theorem 1.1 is the absolutely continuity of P_t. But, from Lemma 2.3, P_t is absolutely continuous with respect to the Lebesgue measure. Hence, we complete the proof of Theorem 1.1. □

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3. Proof of Theorem 1.2

We split the proof into three steps.

Step 1. Let $\phi \in C_c^\infty((-2,2))$ such that $\phi(\xi) = 1$ for $|\xi| \le 1$ and such that ϕ is symmetric. Set $\phi_k(\xi) := \phi(2^{-k}\xi)$, $k \ge 0$. Note that $|D^m \phi_k(\xi)| \le c_m 2^{-mk} \mathbf{1}_{2^{k-1} \le |\xi| \le 2^{k+1}}(\xi)$ for $m \in \mathbb{N}$, where $\mathbf{1}$ is a characteristic function. Set

$$f_{t1}^k(\xi) := \phi(2^{-k}\xi)\mathrm{e}^{-t\Psi_1(\xi)} \cos t\Psi_2(\xi), \qquad f_{t2}^k(\xi) := \phi(2^{-k}\xi)\mathrm{e}^{-t\Psi_1(\xi)} \sin t\Psi_2(\xi),$$
$$f_t^k(\xi) := \phi(2^{-k}\xi)\mathrm{e}^{-t\Psi(\xi)}.$$

Since Ψ and ϕ are symmetric functions, we have

$$\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)} \sin(x\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} f_t^k(\xi) \sin(x\xi)\,\mathrm{d}\xi = 0.$$

By the same argument as in the proof of Theorem 1.1(ii), we obtain

$$|\mathcal{F}^{-1}(\mathrm{e}^{-t\Psi})(x)|, \qquad |\mathcal{F}^{-1}(f_t^k)(x)| \le c(t) q_t(x), \qquad x \ne 0, \tag{3.1}$$

and $q_t \in L^1(\mathbb{R})$ by Remark 1.3. Since the f_t^k are continuous and compactly supported, we have $f_t^k \in L^1(\mathbb{R})$ for each k. This implies that their inverse Fourier transforms $\mathcal{F}^{-1}(f_t^k)$ are continuous in $\mathbb{R}$ and so with (3.1), we conclude that $\mathcal{F}^{-1}(f_t^k) \in L^1(\mathbb{R})$ and, thus, $\mathcal{F}(\mathcal{F}^{-1})(f_t^k) = f_t^k$; see Proposition 8.26 in [4].

Step 2. Now we will show that $\{\mathcal{F}^{-1}(f_t^k)\}$ converges to some $p_t \in L^1(\mathbb{R})$.

Let $l > k$. From (2.1), for $i = 1, 2$, we obtain

$$\begin{aligned}|(f_{ti}^l - f_{ti}^k)''(\xi)| &\le c\,\mathbf{1}_{2^{k-1} \le |\xi| \le 2^{k+1}}(G(t,\xi) + \mathrm{e}^{-tg_1(|\xi|)}(t2^{-k} g_3(|\xi|) + 2^{-2k})) \\ &\quad + c\,\mathbf{1}_{2^{k+1} \le |\xi| \le 2^{l+1}}(G(t,\xi) + \mathrm{e}^{-tg_1(|\xi|)}(t2^{-l} g_3(|\xi|) + 2^{-2l})) \\ &:= cG_{1kl}(t, |\xi|).\end{aligned} \tag{3.2}$$

From Lemma 2.1, for $x \ne 0$, we have

$$\begin{aligned}&|\mathcal{F}^{-1}(f_t^l - f_t^k)(x)| \\ &\quad \le \left| \int_{\mathbb{R}} (f_{t1}^l(\xi) - f_{t1}^k(\xi)) \cos x\xi\,\mathrm{d}\xi \right| + \left| \int_{\mathbb{R}} (f_{t2}^l(\xi) - f_{t2}^k(\xi)) \sin x\xi\,\mathrm{d}\xi \right| \\ &\quad \le c\left(\frac{1}{|x|^2} \int_{1/|x|}^\infty G_{1kl}(t,y)\,\mathrm{d}y + \frac{1}{|x|^3} G_{1kl}\left(t, \frac{1}{|x|}\right) + \int_0^{1/|x|} G_{1kl}(t,y) y^2\,\mathrm{d}y \right).\end{aligned} \tag{3.3}$$

Note that, from (2.2), (2.3), (3.2), and assumption (A_2), we obtain

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{|x|^2} \int_{1/|x|}^\infty G_{1kl}(t,y)\,\mathrm{d}y\,\mathrm{d}x &= 2\int_0^\infty \frac{1}{|x|^2} \int_{1/|x|}^\infty G_{1kl}(t,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= 2\int_0^\infty y G_{1kl}(t,y)\,\mathrm{d}y \\ &\le c\int_{2^{k-1}}^{2^{k+1}} y(G(t,y) + \mathrm{e}^{-tg_1(|y|)}(t2^{-k} g_3(|y|) + 2^{-2k}))\,\mathrm{d}y \\ &\quad + c\int_{2^{k+1}}^{2^{l+1}} y(G(t,y) + \mathrm{e}^{-tg_1(|y|)}(t2^{-l} g_3(|y|) + 2^{-2l}))\,\mathrm{d}y\end{aligned}$$

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$$\leq c(t)\int_{2^{k-1}}^{2^{k+1}} \mathrm{e}^{-t\alpha_n^{\varepsilon}(y)}(y^{-1}+2^{-k})\,\mathrm{d}y + c(t)\int_{2^{k+1}}^{2^{l+1}} \mathrm{e}^{-t\alpha_n^{\varepsilon}(y)} s_n^{\varepsilon}(y) s_n^{\varepsilon_2-1}(y) r_{n-1}^{-1}(y) y^{-1}\,\mathrm{d}y + c(t)\int_{2^{k+1}}^{2^{l+1}} \mathrm{e}^{-t\alpha_n^{\varepsilon}(y)} 2^{-l}\,\mathrm{d}y.$$

By direct calculation, the first and the third terms go to 0 as $k,\ l \to \infty$. From the proof of Remark 1.3 (see Appendix D), the integrand of the second term is integrable in $(1,\infty)$ and, thus, goes to 0 as $k,\ l \to \infty$. Therefore, we have

$$\int_{\mathbb{R}} \frac{1}{|x|^2}\int_{1/|x|}^{\infty} G_{1kl}(t,y)\,\mathrm{d}y\,\mathrm{d}x \to 0 \quad \text{as } k,\ l \to \infty.$$

Similarly, we have

$$\int_{\mathbb{R}} \frac{1}{|x|^2} G_{1kl}\left(t,\frac{1}{|x|}\right)\mathrm{d}x = 2\int_0^{\infty} y G_{1kl}(t,y)\,\mathrm{d}y \to 0 \quad \text{as } k,\ l \to \infty,$$
$$\int_{\mathbb{R}}\int_0^{1/|x|} G_{1kl}(t,y)y^2\,\mathrm{d}y\,\mathrm{d}x = 2\int_0^{\infty} y G_{1kl}(t,y)\,\mathrm{d}y \to 0 \quad \text{as } k,\ l \to \infty.$$

Hence, by (3.3), we have

$$\|\mathcal{F}^{-1}(f_t^l - f_t^k)\|_{L^1(\mathbb{R})} \to 0 \quad \text{as } k,\ l \to \infty.$$

Therefore, $\{\mathcal{F}^{-1}(f_t^k)\}$ is a Cauchy sequence in $L^1(\mathbb{R})$. By the completeness of $L^1(\mathbb{R})$, we conclude that $\{\mathcal{F}^{-1}(f_t^k)\}$ converges to some $p_t \in L^1(\mathbb{R})$.

Step 3. Let $p_t \in L^1(\mathbb{R})$ be the limit of $\{\mathcal{F}^{-1}(f_t^k)\}$ obtained in the previous step. By the well known property of Fourier transform, this implies that $\mathcal{F}\mathcal{F}^{-1}(f_t^k) = f_t^k$ uniformly converges to $\mathcal{F}(p_t)$. On the other hand, note that f_t^k uniformly converges to f_t in $\mathbb{R}$. Therefore, we conclude that $\mathcal{F}(p_t) = f_t$. Moreover, for all $\psi \in C_c^{\infty}(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}} p_t(x)\psi(x)\,\mathrm{d}x &= \lim_{k\to\infty}\int_{\mathbb{R}} \mathcal{F}^{-1}(f_t^k)(x)\psi(x)\,\mathrm{d}x \\ &= \lim_{k\to\infty}\int_{\mathbb{R}} f_t^k(\xi)\mathcal{F}(\psi)(\xi)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}} f_t(\xi)\mathcal{F}(\psi)(\xi)\,\mathrm{d}\xi.\end{aligned}$$

This implies that $\mathcal{F}^{-1}(f_t) = p_t$, and estimate (3.1) implies that p_t satisfies (1.5). We complete the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

In the proof of Theorem 1.1, we have seen that $\mathcal{F}^{-1}(\mathrm{e}^{-t\Psi}) := p_t$ is in $L^1(\mathbb{R})$ such that p_t satisfies (1.5) and $\mathcal{F}(p_t) = \mathrm{e}^{-t\Psi}$. Since Ψ is a symmetric, real-valued function, $\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\sin(x\xi)\,\mathrm{d}\xi = 0$. Hence, p_t is also a real-valued function.

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Now it remains to show that $p_t \geq 0$, and p_t satisfies (1.9). Let $f(x) := \mathrm{e}^{-t\Psi(x)}$. Then

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}\xi &= \int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\cos(x\xi)\,\mathrm{d}\xi \\ &= \frac{1}{|x|}\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi/x)}\cos\xi\,\mathrm{d}\xi \\ &= \sum_{-\infty}^{\infty}\frac{1}{|x|}\int_{2k\pi}^{2(k+1)\pi} f\left(\frac{\xi}{x}\right)\cos\xi\,\mathrm{d}\xi.\end{aligned}$$

Note that

$$\begin{aligned}f''(\xi) &= \mathrm{e}^{-t\Psi(\xi)}((-t\Psi'(\xi))^2 + (-t\Psi''(\xi))) \\ &\geq \mathrm{e}^{-t\Psi(\xi)}(-t\Psi''(\xi)) \\ &\geq ct\begin{cases}\mathrm{e}^{-c_1 t|\xi|^{\varepsilon_1-2}}, & |\xi| \leq 1, \\ \mathrm{e}^{-t\alpha_0 s_n^{\varepsilon}(\xi)}s_n^{\varepsilon_2-1}(\xi)h(\xi)(1+|\xi|)^{-2}, & |\xi| \geq 1,\end{cases} \\ &:= g(t, |\xi|),\end{aligned}$$

where c is a positive constant such that g is nonincreasing. Then, for $k \in \mathbb{Z}$, as in the proof of Lemma 2.1, we have

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}\xi &= \sum_{-\infty}^{\infty}\frac{1}{|x|^3}\int_0^{\pi/2}\int_{\xi}^{\pi-\xi}\int_{\theta_1}^{\pi+\theta_1} f''\left(\frac{2\pi k+\theta_2}{x}\right)\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\ &\geq c\sum_{-\infty}^{\infty}\frac{1}{|x|^3}\int_{\pi/8}^{3\pi/8}\int_{\xi}^{\pi-\xi}\int_{\theta_1}^{\pi+\theta_1} g\left(\frac{|2\pi k+\theta_2|}{|x|}\right)\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\ &\geq c\sum_{0\leq k<\infty}\frac{1}{|x|^3}g\left(\frac{|2\pi k+\pi/8|}{|x|}\right) \\ &\geq c\frac{1}{|x|^2}\int_{1/8\pi|x|}^{\infty} g(t, y)\,\mathrm{d}y.\end{aligned}$$

Hence, for $|x| \geq 1$, we have

$$\begin{aligned}&\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}\xi \\ &\quad\geq c\frac{t}{|x|^2}\int_1^{\infty}\mathrm{e}^{-t\alpha_0 s_n^{\varepsilon}(\xi)}s_n^{\varepsilon_2-1}(\xi)h(\xi)|\xi|^{-2}\,\mathrm{d}\xi + c\frac{t}{|x|^2}\int_{1/|x|}^{1}\mathrm{e}^{-c_1 t|\xi|^{\varepsilon_1-2}}\,\mathrm{d}\xi \\ &\quad\geq c\begin{cases}\dfrac{t}{|x|^2} + t\mathrm{e}^{-c_1 t}\max\left(\dfrac{1}{|x|^{1+\varepsilon_1}}, \dfrac{1}{|x|^2}\right), & \varepsilon_1 \neq 1, \\ \dfrac{t}{|x|^2}(1+\mathrm{e}^{-c_1 t}\ln(1+|x|)), & \varepsilon_1 = 1.\end{cases}\end{aligned}$$

Applying Lemma 2.2(iii), for $\alpha_0 = -2$ and $a = 1/2\pi|x|$ for $|x| \leq 1$, we have

$$\int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}\xi \geq c(t)\frac{1}{|x|}\mathrm{e}^{-t\alpha_0 s_n^{\varepsilon}(1/x)}s_n^{\varepsilon_2-1}\left(\frac{1}{x}\right)h\left(\frac{1}{x}\right),$$

completing the proof. □

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5. Example: subordinators

A typical example of the one-dimensional Lévy processes is the subordinator. Let the Lévy process S_t be a subordinator, that is,

$$S_t \geq 0 \quad \text{almost surely for each } t > 0, \qquad S_{t_1} \leq S_{t_2} \quad \text{almost surely whenever } t_1 < t_2.$$

The Lévy symbol Ψ of subordinator S_t has the form

$$\Psi(\xi) = \mathrm{i}b\xi - \int_0^\infty (\mathrm{e}^{\mathrm{i}\xi y} - 1)\lambda(\mathrm{d}y), \tag{5.1}$$

where $b \geq 0$ and the Lévy measure λ satisfies the additional requirements

$$\lambda(-\infty, 0) = 0, \qquad \int_0^\infty \min(y, 1)\lambda(\mathrm{d}y) < \infty. \tag{5.2}$$

Conversely, any mapping $\Psi : \mathbb{R} \to \mathbb{C}$ of the form (5.1) is the Lévy symbol of a subordinator.

If S_t is a subordinator then, for each $t \geq 0$, the function $f(\xi) = \mathrm{e}^{-t\Psi(\xi)}$ can be analytically extended to the region $\{v + \mathrm{i}\xi \mid v \in \mathbb{R},\ \xi > 0\}$. Let $F(z) = \mathrm{e}^{-t\Psi(z)}$ be an analytical extension of $\mathrm{e}^{-t\Psi(\xi)}$ over $\{v + \mathrm{i}\xi \mid v \in \mathbb{R},\ \xi > 0\}$. Then we obtain

$$F(\mathrm{i}\xi) = \mathrm{e}^{-t\Psi(\mathrm{i}\xi)} := \mathrm{e}^{-t\psi(\xi)}, \qquad \xi \geq 0.$$

Here,

$$\psi(\xi) = \Psi(\mathrm{i}\xi) = -b\xi - \int_0^\infty (\mathrm{e}^{-\xi y} - 1)\lambda(\mathrm{d}y) \quad \text{for each } \xi > 0. \tag{5.3}$$

The function ψ is usually called the Laplace exponent of the subordinator.

The Laplace exponents of subordinators are characterized by the Bernstein functions. We say that the continuous function $\psi : [0, \infty) \to [0, \infty)$ is a Bernstein function if $(-1)^k \psi^{(k)} \leq 0$ for all $k \in \mathbb{N}$. If ψ is the Laplace exponent of the subordinator then, from (5.3), ψ is a Bernstein function.

Conversely, if ψ is a Bernstein function then there are a nonnegative real number b and a measure λ defined in $\mathbb{R}$ satisfying (5.2) such that (5.3) holds. Moreover, there is a subordinator S_t such that

$$\Psi(\xi) = \psi(-\mathrm{i}\xi) \tag{5.4}$$

is the Lévy symbol of S_t.

Let $\psi^{\varepsilon,n}(\xi) = (s_n(\xi))^\varepsilon$, $0 < \varepsilon \leq 1$, $1 \leq n$. Note that, for $n \geq 2$, $0 < \varepsilon \leq 1$, $\psi^{\varepsilon,n}$ do not satisfy the Hartman–Wintner conditions (1.1) and (1.2).

Clearly, $\psi^{\varepsilon,n}$ is a Bernstein function and so $\psi^{\varepsilon,n}$ is a Laplace exponent of some subordinators. Then, by (5.4), $\Psi^{\varepsilon,n}(\xi) := \psi^{\varepsilon,n}(-\mathrm{i}\xi)$ is a Lévy symbol.

Theorem 5.1. *Let $n \geq 1$ be a fixed natural number. Then*

(i) $\Psi^{\varepsilon,n}(\xi) := \psi^{\varepsilon,n}(-\mathrm{i}\xi)$ *satisfies assumption (A) for $\varepsilon = \varepsilon_2$, $\varepsilon_{11} = \varepsilon_{12} = 1$, $\varepsilon_{13} = 2$, and $h(z) = r_{n-1}^{-1}(z)$;*

(ii) $\psi^{\varepsilon,n}(\xi) = (s_n(\xi))^\varepsilon$ *satisfies assumption (1.8) for $\varepsilon = \varepsilon_2$, $\varepsilon_{12} = 1$, $\varepsilon_{13} = 2$, and $h(z) = r_{n-1}^{-1}(z)$.*

Remark 5.1. (i) Let X_t be a Lévy processes whose Lévy symbol is $\Psi^{\varepsilon,n}$. Then, by Theorem 1.2, X_t has Lebesgue density p_{t1} satisfying (1.5).

(ii) By Theorem 1.3, $\psi^{\varepsilon,n}$ is a Lévy symbol with Lévy process Y_t. Moreover, Y_t has Lebesgue density p_{t2} satisfying (1.9).

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Proof of Theorem 5.1(i). First, we consider the case of $\varepsilon = 1$ and $n = 1$. By (5.4), we have

$$\Psi^{1,1}(\xi) = \psi^{1,1}(-\mathrm{i}\xi) = \ln(1 - \mathrm{i}\xi) = \tfrac{1}{2}\ln(1+\xi^2) - \mathrm{i}\arctan\xi := \Psi_1^{1,1}(\xi) + \mathrm{i}\Psi_2^{1,1}(\xi).$$

Observe that $\Psi_1^{1,1}$ and $\Psi_2^{1,1}$ satisfy (A) for $n = 1$. Suppose that $\Psi^{1,n}(\xi)$ satisfies (A). Note that

$$\begin{aligned}\Psi^{1,n+1}(\xi) &= \psi^{1,n+1}(-\mathrm{i}\xi)\\ &= \ln(1 + s_n(-\mathrm{i}\xi))\\ &= \ln(1 + \Psi_1^n(\xi) - \mathrm{i}\Psi_2^n(\xi))\\ &= \frac{1}{2}\ln((1+\Psi_1^n(\xi))^2 + (\Psi_2^n(\xi))^2) - \mathrm{i}\arctan\frac{\Psi_2^n(\xi)}{1+\Psi_1^n(\xi)}\\ &:= \Psi_1^{1,n+1}(\xi) + \mathrm{i}\Psi_2^{1,n+1}(\xi),\end{aligned}$$

where

$$\Psi_1^{1,n+1}(\xi) := \tfrac{1}{2}\ln((1+\Psi_1^n(\xi))^2 + (\Psi_2^n(\xi))^2), \qquad \Psi_2^{1,n+1}(\xi) := \arctan\frac{\Psi_2^n(\xi)}{1+\Psi_1^n(\xi)}.$$

Under the assumption that $\Psi^{1,n} = \Psi_1^{1,n} + \mathrm{i}\Psi_2^{1,n}$ satisfies (A), it is easy to show that $\Psi^{1,n+1} = \Psi_1^{1,n+1} + \Psi_2^{1,n+1}$ satisfies (A). By mathematical induction, we conclude that $\Psi^{1,n}$ satisfies (A) for all $n \geq 1$.

Next, let $0 < \varepsilon < 1$. From the Euler identity of complex numbers, note that if $Z = \ln(x+\mathrm{i}y)$ then $Z = \ln r + \mathrm{i}\theta$, where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan y/x$, so that $\ln Z = \ln\sqrt{\ln^2 r + \theta^2} + \mathrm{i}\arctan\theta/\ln r$, and that

$$\begin{aligned}Z^\varepsilon &= \exp(\varepsilon \ln Z)\\ &= \exp\left(\varepsilon\left(\ln\sqrt{\ln^2 r + \theta^2} + \mathrm{i}\arctan\frac{\theta}{\ln r}\right)\right)\\ &= (\ln^2 r + \theta^2)^{\varepsilon/2}\left(\cos\left(\varepsilon\arctan\frac{\theta}{\ln r}\right) + \mathrm{i}\sin\left(\varepsilon\arctan\frac{\theta}{\ln r}\right)\right).\end{aligned}$$

Let $Z = \ln(1 + s_{n-1}(-\mathrm{i}\xi)) = \ln(1 + \Psi_1^{1,n-1}(\xi) - \mathrm{i}\Psi_2^{1,n-1}(\xi))$. Then

$$r = \sqrt{(1+\Psi_1^{1,n-1}(\xi))^2 + (\Psi_2^{1,n-1}(\xi))^2} \quad \text{and} \quad \theta = -\arctan\frac{\Psi_2^{1,n-1}(\xi)}{1+\Psi_1^{1,n-1}(\xi)}.$$

Hence, we have

$$\begin{aligned}\Psi^{\varepsilon,n}(\xi) &:= \psi^{\varepsilon,n}(-\mathrm{i}\xi)\\ &= \ln^\varepsilon(1 + s_{n-1}(-\mathrm{i}\xi)\\ &= (\ln^2 r + \theta^2)^{\varepsilon/2}\left(\cos\left(\varepsilon\arctan\frac{\theta}{\ln r}\right) + \mathrm{i}\sin\left(\varepsilon\arctan\frac{\theta}{\ln r}\right)\right)\\ &:= \Psi_1^{\varepsilon,n}(\xi) + \mathrm{i}\Psi_2^{\varepsilon,n}(\xi),\end{aligned}$$

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where

$$\begin{aligned}\Psi_1^{\varepsilon,n}(\xi) &= (\ln^2 r+\theta^2)^{\varepsilon/2}\cos\left(\varepsilon\arctan\frac{\theta}{\ln r}\right)\\ &= \left(\frac{1}{4}\ln^2((1+\Psi_1^{1,n-1}(\xi))^2+(\Psi_2^{1,n-1}(\xi))^2)+\arctan^2\frac{\Psi_2^{1,n-1}(\xi)}{1+\Psi_1^{1,n-1}(\xi)}\right)^{\varepsilon/2}\\ &\quad\times\cos\left(\varepsilon\arctan\frac{\arctan(\Psi_2^{1,n-1}(\xi)/(1+\Psi_1^{1,n-1}(\xi)))}{\sqrt{(1+\Psi_1^{1,n-1}(\xi))^2+(\Psi_2^{1,n-1}(\xi))^2}}\right),\\ \Psi_2^{\varepsilon,n}(\xi) &= (\ln^2 r+\theta^2)^{\frac{\varepsilon}{2}}\sin\left(\varepsilon\arctan\frac{\theta}{\ln r}\right)\\ &= -\left(\frac{1}{4}\ln^2((1+\Psi_1^{1,n-1}(\xi))^2+(\Psi_2^{1,n-1}(\xi))^2)+\arctan^2\frac{\Psi_2^{1,n-1}(\xi)}{1+\Psi_1^{1,n-1}(\xi)}\right)^{\varepsilon/2}\\ &\quad\times\sin\left(\varepsilon\arctan\frac{\arctan(\Psi_2^{1,n-1}(\xi)/(1+\Psi_1^{1,n-1}(\xi)))}{\sqrt{(1+\Psi_1^{1,n-1}(\xi))^2+(\Psi_2^{1,n-1}(\xi))^2}}\right).\end{aligned}$$

Since $\Psi^{1,n-1}=\Psi_1^{1,n-1}+\mathrm{i}\Psi_2^{1,n-1}$ satisfies (A), it is easy to show that $\Psi^{\varepsilon,n}=\Psi_1^{\varepsilon,n}+\mathrm{i}\Psi_2^{\varepsilon,n}$ satisfies (A).

(ii) First, we consider the $\varepsilon=1$ case. Let $s_0=0$. Direct computation shows that

$$(\psi^{1,n}(\xi))'=s_n'(\xi)=A_{n-1}(\xi),\qquad (\psi^{1,n}(\xi))''=s_n''(\xi)=-\sum_{0\le k\le n-1}B_k(\xi),$$

where, for $0\le k\le n-1$,

$$\begin{aligned}A_k(\xi)&:=(1+s_k(\xi))^{-1}(1+s_{k-1}(\xi))^{-1}\cdots(1+s_1(\xi))^{-1}(1+s_0(\xi))^{-1}(1+\xi)^{-1},\\ B_k(\xi)&:=A_{n-1}(\xi)A_k(\xi).\end{aligned}$$

Since $A_k(\xi)\le c$ for $0\le k\le n-1$ and $A_k(\xi)\le r_k^{-1}(\xi)|\xi|^{-1}$, we have

$$|(\psi^{1,n}(\xi))'|\le\begin{cases}1, & |\xi|\le 1,\\ r_{n-1}^{-1}(\xi)|\xi|^{-1}, & |\xi|\ge 1,\end{cases}\qquad |(\psi^{1,n}(\xi))''|\le n\begin{cases}1, & |\xi|\le 1,\\ r_{n-1}^{-1}(\xi)|\xi|^{-2}, & |\xi|\ge 1.\end{cases}$$

Hence, $\psi^{1,n}$ satisfies (A).

Since $B_k(\xi)>0$ for all $0\le k\le n-1$, we have

$$-(\psi^{1,n}(\xi))''=\sum_{0\le k\le n-1}B_k(\xi)>B_0(\xi).\tag{5.5}$$

Note that $A_k\ge 1$ for $|\xi|\le 1$. Hence, from (5.5), we obtain

$$-(\psi^{1,n}(\xi))''=\sum_{0\le k\le n-1}B_k(\xi)\ge 1.$$

Since $(1+s_k(\xi))^{-1}\ge cs_k^{-1}(\xi)$, from (5.5), for $1\le|\xi|$, we have

$$-(\psi^{1,n}(\xi))''=\sum_{0\le k\le n-1}B_k(\xi)\ge cr_{n-1}^{-1}(\xi)|\xi|^{-2},\qquad |\xi|\ge 1.$$

Hence, $\psi^{1,n}$ satisfies (1.8).

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Next, we consider the $0 < \varepsilon < 1$ case. Direct computation shows that

$$-(\psi^{\varepsilon,n}(\xi))'' = -(\varepsilon-1)s_n^{\varepsilon-2}(\xi)A_{n-1}^2(\xi) + s_n^{\varepsilon-1}(\xi)\sum_{1\le k\le n-1} B_k(\xi) \ge s_{n-1}^{\varepsilon-1}(\xi)B_0(\xi).$$

By the above calculation, we can show that $\psi^{\varepsilon,n}$ satisfies (A) and (1.8). □

Appendix A. Proof of Lemma 2.1

We start by proving the first inequality of Lemma 2.1.

By changing variables, we have

$$\left|\int_{\mathbb{R}} f(\xi)\cos(x\xi)\,\mathrm{d}\xi\right| = \frac{1}{|x|}\left|\int_{\mathbb{R}} f\left(\frac{\xi}{x}\right)\cos\xi\,\mathrm{d}\xi\right| \le \frac{1}{|x|}\sum_{-\infty}^{\infty}|I_k^1(x)|, \tag{A.1}$$

where $I_k^1(x) = \int_{2\pi k}^{2\pi(k+1)} f(\xi/x)\cos\xi\,\mathrm{d}\xi$. The term I_k^1 can be written as

$$I_k^1(x) = \int_0^{\pi/2}\left(f\left(\frac{2\pi k+\xi}{x}\right) - f\left(\frac{2\pi k+\pi-\xi}{x}\right) - f\left(\frac{2\pi k+\pi+\xi}{x}\right) + f\left(\frac{2\pi k+2\pi-\xi}{x}\right)\right)\cos\xi\,\mathrm{d}\xi, \tag{A.2}$$

since

$$\begin{aligned}
&\int_{2\pi k}^{2\pi(k+1)} f\left(\frac{\xi}{x}\right)\cos\xi\,\mathrm{d}\xi\\
&\quad= \int_{2\pi k}^{2\pi k+\pi/2} f\left(\frac{\xi}{x}\right)\cos\xi\,\mathrm{d}\xi + \int_{2\pi k+\pi/2}^{2\pi k+\pi} f\left(\frac{\xi}{x}\right)\cos\xi\,\mathrm{d}\xi\\
&\qquad+ \int_{2\pi k+\pi}^{2\pi k+3\pi/2} f\left(\frac{\xi}{x}\right)\cos\xi\,\mathrm{d}\xi + \int_{2\pi k+3\pi/2}^{2\pi k+2\pi} f\left(\frac{\xi}{x}\right)\cos\xi\,\mathrm{d}\xi\\
&\quad= \int_0^{\pi/2} f\left(\frac{2\pi k+\xi}{x}\right)\cos(2\pi k+\xi)\,\mathrm{d}\xi\\
&\qquad+ \int_0^{\pi/2} f\left(\frac{2\pi k+\pi-\xi}{x}\right)\cos(2\pi k+\pi-\xi)\,\mathrm{d}\xi\\
&\qquad+ \int_0^{\pi/2} f\left(\frac{2\pi k+\pi+\xi}{x}\right)\cos(2\pi k+\pi+\xi)\,\mathrm{d}\xi\\
&\qquad+ \int_0^{\pi/2} f\left(\frac{2\pi k+2\pi-\xi}{x}\right)\cos(2\pi k+2\pi-\xi)\,\mathrm{d}\xi\\
&\quad= \int_0^{\pi/2}\left(f\left(\frac{2\pi k+\xi}{x}\right) - f\left(\frac{2\pi k+\pi-\xi}{x}\right) - f\left(\frac{2\pi k+\pi+\xi}{x}\right)\right.\\
&\qquad\left.+ f\left(\frac{2\pi k+2\pi-\xi}{x}\right)\right)\cos\xi\,\mathrm{d}\xi.
\end{aligned}$$

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Observe that

$$\begin{aligned} &f\left(\frac{2\pi k+\xi}{x}\right)-f\left(\frac{2\pi k+\pi-\xi}{x}\right)-f\left(\frac{2\pi k+\pi+\xi}{x}\right)+f\left(\frac{2\pi k+2\pi-\xi}{x}\right)\\ &\quad=-\int_{\xi}^{\pi-\xi}\frac{1}{x}f'\left(\frac{2\pi k+\theta_1}{x}\right)\mathrm{d}\theta_1+\int_{\xi}^{\pi-\xi}\frac{1}{x}f'\left(\frac{2\pi k+\pi+\theta_1)}{x}\right)\mathrm{d}\theta_1\\ &\quad=\frac{1}{x^2}\int_{\xi}^{\pi-\xi}\int_{\theta_1}^{\pi+\theta_1}f''\left(\frac{2\pi k+\theta_2}{x}\right)\mathrm{d}\theta_2\,\mathrm{d}\theta_1\quad\text{for }\xi\in\left(0,\tfrac{1}{2}\pi\right). \end{aligned}\tag{A.3}$$

Since f'' is dominated by the nonnegative and nonincreasing function G, using Fubini's theorem twice, we have

$$\begin{aligned} |I_k^1(x)|&\le\frac{1}{|x|^2}\int_0^{\pi/2}\int_{\xi}^{\pi-\xi}\int_{\theta_1}^{\pi+\theta_1}\left|f''\left(\frac{2\pi k+\theta_2}{x}\right)\right|\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\xi\\ &\le c\frac{1}{|x|^2}\int_0^{\pi/2}\int_0^{\pi-\xi}\int_{\theta_1}^{\pi+\theta_1}G\left(\frac{|2\pi k+\theta_2|}{|x|}\right)\mathrm{d}\theta_2\,\mathrm{d}\theta_1\,\mathrm{d}\xi\\ &=c\frac{1}{|x|^2}\int_0^{\pi/2}\theta_1\int_{\theta_1}^{\pi+\theta_1}G\left(\frac{|2\pi k+\theta_2|}{|x|}\right)\mathrm{d}\theta_2\,\mathrm{d}\theta_1\\ &\quad+c\frac{1}{|x|^2}\int_{\pi/2}^{\pi}(\pi-\theta_1)\int_{\theta_1}^{\pi+\theta_1}G\left(\frac{|2\pi k+\theta_2|}{|x|}\right)\mathrm{d}\theta_2\,\mathrm{d}\theta_1\,\mathrm{d}\xi\\ &\le c\frac{1}{|x|^2}\int_0^{\pi/2}G\left(\frac{|2\pi k+\theta_2|}{|x|}\right)\int_0^{\theta_2}\theta_1\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\\ &\quad+c\frac{1}{|x|^2}\int_{\pi/2}^{3\pi/2}G\left(\frac{|2\pi k+\theta_2|}{|x|}\right)\mathrm{d}\theta_2\\ &\quad+c\frac{1}{|x|^2}\int_{3\pi/2}^{2\pi}G\left(\frac{|2\pi k+\theta_2|}{|x|}\right)\int_{-\pi+\theta_2}^{\pi}(\pi-\theta_1)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \end{aligned}\tag{A.4}$$

For $k\ge1$ or $k\le-2$, we have

$$|I_k^1(x)|\le c\begin{cases}\dfrac{1}{|x|^2}G\left(\dfrac{2\pi k}{|x|}\right), & k\ge1,\\[2ex] \dfrac{1}{|x|^2}G\left(\dfrac{2\pi|k+1|}{|x|}\right), & k\le-2.\end{cases}$$

Since G is a nonincreasing function, we have $G(2\pi k/|x|)\le2\int_{k-1/2}^{k}G(2\pi y/|x|)\,\mathrm{d}y$ for $k\ge1$. Hence, we have

$$\begin{aligned} \sum_{1\le k}|I_k^1(x)|+\sum_{k\le-2}|I_k^1(x)|&\le c\frac{1}{x^2}\left(\sum_{1\le k}G\left(\frac{2\pi|k|}{|x|}\right)+\sum_{k\le-2}G\left(\frac{2\pi|k+1|}{|x|}\right)\right)\\ &\le c\frac{1}{x^2}\sum_{1\le k<\infty}G\left(\frac{2\pi k}{|x|}\right)\\ &\le2c\frac{1}{x^2}\int_{1/2}^{\infty}G\left(\frac{2\pi y}{|x|}\right)\mathrm{d}y\\ &\le c\frac{1}{\pi|x|}\int_{1/|x|}^{\infty}G(y)\,\mathrm{d}y. \end{aligned}\tag{A.5}$$

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Since G is a nonincreasing and nonnegative function, using a change of variables, we obtain $\int_0^{2a} G(y)y^2\,\mathrm{d}y = 2^3\int_0^a G(2y)y^2\,\mathrm{d}y \le 2^3\int_0^a G(y)y^2\,\mathrm{d}y$ for $a > 0$. Hence, for $k = 0$ in (A.4), we have

$$\begin{aligned}|I_0^1(x)| &\le c\bigg(\frac{1}{|x|^2}\int_0^{\pi/2} G\bigg(\frac{\theta_2}{|x|}\bigg)\theta_2^2\,\mathrm{d}\theta_2 + \frac{1}{|x|^2}\int_{\pi/2}^{2\pi} G\bigg(\frac{\theta_2}{|x|}\bigg)\,\mathrm{d}\theta_2\bigg)\\ &\le c\bigg(|x|\int_0^{\pi/2|x|} G(\theta_2)\theta_2^2\,\mathrm{d}\theta_2 + \frac{1}{|x|^2}G\bigg(\frac{1}{|x|}\bigg)\bigg)\\ &\le c\bigg(|x|\int_0^{1/|x|} G(\theta_2)\theta_2^2\,\mathrm{d}\theta_2 + \frac{1}{|x|^2}G\bigg(\frac{1}{|x|}\bigg)\bigg).\end{aligned} \tag{A.6}$$

Similarly, for $k = -1$ in (A.4), we have

$$\begin{aligned}|I_{-1}^1(x)| &\le c\bigg(\frac{1}{|x|^2}\int_0^{3\pi/2} G\bigg(\frac{|-2\pi+\theta_2|}{|x|}\bigg)\,\mathrm{d}\theta_2\\ &\qquad + \frac{1}{|x|^2}\int_{3\pi/2}^{2\pi} G\bigg(\frac{|-2\pi+\theta_2|}{|x|}\bigg)(2\pi-\theta_2)\,\mathrm{d}\theta_2\bigg)\\ &\le c\bigg(|x|\int_0^{\pi/2|x|} G(\theta_2)\theta_2^2\,\mathrm{d}\theta_2 + \frac{1}{|x|^2}G\bigg(\frac{1}{|x|}\bigg)\bigg)\\ &\le c\bigg(|x|\int_0^{1/|x|} G(\theta_2)\theta_2^2\,\mathrm{d}\theta_2 + \frac{1}{|x|^2}G\bigg(\frac{1}{|x|}\bigg)\bigg).\end{aligned} \tag{A.7}$$

Combining (A.1) and (A.5)–(A.7), we obtain the first inequality of Lemma 2.1. Next we prove the second inequality of the lemma.

Changing variables, we have

$$\begin{aligned}\int_{\mathbb{R}} f(\xi)\sin(x\xi)\,\mathrm{d}\xi &= \frac{1}{|x|}\int_{\mathbb{R}} f\bigg(\frac{\xi}{x}\bigg)\sin\xi\,\mathrm{d}\xi\\ &= \frac{1}{|x|}\int_{-\pi/2}^{\pi/2} f\bigg(\frac{\xi}{x}\bigg)\sin\xi\,\mathrm{d}\xi + \frac{1}{|x|}\sum_{0<k<\infty}(I_k^{2,1}+I_k^{2,2}),\end{aligned} \tag{A.8}$$

where

$$\begin{aligned}I_k^{2,1}(x) &= \int_{2\pi k+\pi/2}^{2\pi(k+1)+\pi/2} f\bigg(\frac{\xi}{x}\bigg)\sin\xi\,\mathrm{d}\xi, \qquad k > 0,\\ I_k^{2,2}(x) &= \int_{2\pi(-k-1)-\pi/2}^{2\pi(-k)-\pi/2} f\bigg(\frac{\xi}{x}\bigg)\sin\xi\,\mathrm{d}\xi, \qquad k > 0.\end{aligned}$$

By the same argument as for (A.2), $I_k^{2,1}$ can be written as

$$\begin{aligned}I_k^{2,1}(x) = \int_0^{\pi/2}\bigg(&f\bigg(\frac{2\pi k+\pi/2+\xi}{x}\bigg) - f\bigg(\frac{2\pi k+3\pi/2-\xi}{x}\bigg)\\ &- f\bigg(\frac{2\pi k+3\pi/2+\xi}{x}\bigg) + f\bigg(\frac{2\pi k+5\pi/2-\xi}{x}\bigg)\bigg)\sin\xi\,\mathrm{d}\xi.\end{aligned}$$

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Observe that

$$\begin{aligned} &f\left(\frac{2\pi k+\pi/2+\xi}{x}\right)-f\left(\frac{2\pi k+3\pi/2-\xi}{x}\right)-f\left(\frac{2\pi k+3\pi/2+\xi}{x}\right)\\ &\quad+f\left(\frac{2\pi k+5\pi/2-\xi}{x}\right)\\ &\quad=\frac{1}{x}\int_{\pi/2+\xi}^{3\pi/2-\xi}f'\left(\frac{2\pi k+\theta_1}{x}\right)\mathrm{d}\theta_1-\frac{1}{x}\int_{\pi/2+\xi}^{3\pi/2-\xi}f'\left(\frac{2\pi k+\pi+\theta_1}{x}\right)\mathrm{d}\theta_1\\ &\quad=\frac{1}{x^2}\int_{\pi/2+\xi}^{3\pi/2-\xi}\int_{\theta_1}^{\pi+\theta_1}f''\left(\frac{2\pi k+\theta_2}{x}\right)\mathrm{d}\theta_2\,\mathrm{d}\theta_1. \end{aligned}$$

Hence, we have

$$\begin{aligned} |I_k^{2,1}(x)|&\le c\int_0^{\pi/2}\frac{1}{x^2}\int_{\pi/2+\xi}^{3\pi/2-\xi}\int_{\theta_1}^{\pi+\theta_1}\left|f''\left(\frac{2\pi k+\theta_2}{x}\right)\right|\mathrm{d}\theta_2\,\mathrm{d}\theta_1\sin\xi\,\mathrm{d}\xi\\ &\le c\frac{1}{|x|^2}\int_0^{\pi/2}G\left(\frac{2\pi k+\pi/2}{|x|}\right)\mathrm{d}\xi\\ &\le c\frac{1}{|x|^2}G\left(\frac{2\pi k+\pi/2}{|x|}\right). \end{aligned}\tag{A.9}$$

Similar computations show that

$$|I_k^{2,2}(x)|\le c\frac{1}{|x|^2}G\left(\frac{2\pi|k|+\pi/2}{|x|}\right).\tag{A.10}$$

From (A.9) and (A.10), we have

$$\begin{aligned} \sum_{0<k<\infty}(|I_k^{2,1}(x)|+|I_k^{2,2}(x)|)&\le c\frac{1}{|x|^2}\sum_{1\le k<\infty}G\left(\frac{2\pi k+\pi/2}{|x|}\right)\\ &\le\frac{1}{|x|^2}\int_0^{\infty}G\left(\frac{2\pi y+\pi/2}{|x|}\right)\mathrm{d}y\\ &=c\frac{1}{|x|}\int_{\pi/2|x|}^{\infty}G(y)\,\mathrm{d}y\\ &\le c\frac{1}{|x|}\int_{1/|x|}^{\infty}G(y)\,\mathrm{d}y. \end{aligned}\tag{A.11}$$

Finally, note that

$$\begin{aligned} \left|\frac{1}{x}\int_{-\pi/2}^{\pi/2}f\left(\frac{\xi}{x}\right)\sin\xi\,\mathrm{d}\xi\right|&=\left|\frac{1}{x}\int_0^{\pi/2}\left(f\left(\frac{\xi}{x}\right)-f\left(-\frac{\xi}{x}\right)\right)\sin\xi\,\mathrm{d}\xi\right|\\ &\le\frac{1}{|x|^2}\int_0^{\pi/2}\int_{-\xi}^{\xi}\left|f'\left(\frac{\theta}{x}\right)\right|\mathrm{d}\theta\,\mathrm{d}\xi\\ &\le 2\frac{1}{|x|^2}\int_0^{\pi/2}H\left(\frac{\theta}{x}\right)\int_{\theta}^{\pi/2}\mathrm{d}\xi\,\mathrm{d}\theta \end{aligned}$$

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$$\leq 2\frac{1}{|x|^2}\int_0^{\pi/2} H\left(\frac{\theta}{x}\right)\left(\frac{1}{2}\pi-\theta\right)\mathrm{d}\theta$$
$$\leq c\frac{t}{|x|}\int_0^1 H\left(\frac{\theta}{x}\right)\mathrm{d}\theta. \tag{A.12}$$

From (A.8), (A.11), and (A.12), we obtain the second inequality of Lemma 2.1. This completes the proof of Lemma 2.1. □

Appendix B

Proof of Lemma 2.2(i). It is sufficient to show that

$$\lim_{a\to\infty}\frac{\int_1^a z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z}{a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}(a)s_n^{\varepsilon_2-1}(a)h(a)}=(\alpha_0+1)^{-1}.$$

Using L' Hospital's theorem, we have

$$\lim_{a\to\infty}\frac{\int_1^a z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z}{a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a)}=\lim_{a\to\infty}\frac{a^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a)}{T(a)},$$

where

$$\begin{aligned}T(a)&=\frac{\mathrm{d}}{\mathrm{d}a}(a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a))\\&=(\alpha_0+1)a^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a)-\varepsilon t\alpha a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}h(a)(s_n(a))'\\&\quad+(\varepsilon_2-1)a^{\alpha_0+1}e^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a)(s_n(a))'\\&\quad+a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h'(a).\end{aligned}$$

Let $s_0(a)=0$. Let $s_0(a)=0$. Note that, for $n\geq 1$, we have

$$s_n'(a)=\prod_{k=0}^{n-1}(1+s_k)^{-1}(a)(1+a)^{-1}.$$

Direct computation shows that, for $n\geq 2$,

$$\begin{aligned}T(a)&=(a^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a))\\&\quad\times\left(\alpha_0+1+\frac{a}{1+a}(\varepsilon_2-1-\varepsilon t\alpha s_n^{\varepsilon-1}(a))\prod_{k=0}^{n-1}(1+s_k(a))^{-1}-h_1(a)\right).\end{aligned}$$

Since $\lim_{a\to\infty}s_n(a)=\infty$ and $\lim_{a\to\infty}h_1(a)=0$, we obtain

$$\lim_{a\to\infty}\frac{\int_1^a z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z}{a^{\alpha_0+1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a)}=(\alpha_0+1)^{-1}.$$

This completes the proof of Lemma 2.2(i).

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(ii) Using integration by parts, we obtain

$$\int_{a_0}^{a} z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z = (1+\alpha_0)^{-1}z^{(1+\alpha_0)}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)|_{a_0}^{a} - (1+\alpha_0)^{-1}\int_{a_0}^{a} z^{(1+\alpha_0)}\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}(z)s_n^{\varepsilon_2-1}(z)h(z))\,\mathrm{d}z.$$

Here

$$-\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)) = (z^{-1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)) \times\left(\frac{z}{1+z}(1-\varepsilon_2+\varepsilon t\alpha s_n^{\varepsilon-1}(z))\prod_{k=0}^{n-1}(1+s_k(z))^{-1}+zh_1(z)\right) = z^{-1}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)R(z),$$

where

$$R(z) := \frac{z}{1+z}(1-\varepsilon_2+\varepsilon t\alpha s_n^{\varepsilon-1}(z))\prod_{k=0}^{n-1}(1+s_k(z))^{-1}+zh_1(z).$$

Since $zh_1(z) < 1/(1+\alpha_0)-\varepsilon_0$ and $(1-\varepsilon_2+\varepsilon t\alpha)\ln(1+a_0)^{-1} \le \frac{1}{2}\varepsilon_0$ for $z \ge a_0$, we obtain

$$R(z) \le (1-\varepsilon_2+\varepsilon t\alpha)\ln(1+a_0)^{-1}+h_1(z)z < \frac{1}{2}\varepsilon_0+\frac{1}{1+\alpha_0}-\varepsilon_0 = \frac{1}{1+\alpha_0}-\frac{1}{2}\varepsilon_0.$$

Hence, for $z \ge a_0$, we have

$$-(1+\alpha_0)^{-1}\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{-t\alpha s_{n-1}^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)) \le \left(1-\frac{\varepsilon_0}{2(1+\alpha_0)}\right)z^{-1}\mathrm{e}^{-t\alpha s_{n-1}^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z).$$

Hence, we have

$$\int_{a_0}^{a} z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z \le c(\alpha_0)z^{(1+\alpha_0)}\mathrm{e}^{-t\alpha s_{n-1}^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)|_{a_0}^{a} \le c(\alpha_0)a^{1+\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(a)}s_n^{\varepsilon_2-1}(a)h(a).$$

This completes the proof of Lemma 2.2(ii).

(iii) Changing variables and integrating by parts sequentially, we obtain

$$\int_{a}^{\infty} z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z = (1+\alpha_0)^{-1}z^{(1+\alpha_0)}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)|_{a}^{\infty} - (1+\alpha_0)^{-1}\int_{a}^{\infty} z^{(1+\alpha_0)}\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z))\,\mathrm{d}z.$$

Since $\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}h(z)$ is a nonincreasing function, $(\mathrm{d}/\mathrm{d}z)(\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z))$ is a nonpositive function. Since $\alpha_0 < -1$, we obtain

$$\int_{a}^{\infty} z^{\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)\,\mathrm{d}z \le (1+\alpha_0)^{-1}z^{(1+\alpha_0)}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(z)h(z)|_{a}^{\infty} = -(1+\alpha_0)^{-1}a^{1+\alpha_0}\mathrm{e}^{-t\alpha s_n^{\varepsilon}(z)}s_n^{\varepsilon_2-1}(a)h(a).$$

This completes the proof of Lemma 2.2(iii) and concludes the proof of the lemma. □

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Appendix C

Proof of Lemma 2.3. Let $\phi \in C_c^\infty(\mathbb{R} \setminus (-r_0, r_0))$ for some $r_0 > 0$. Note that, by Theorem 1.1(i), we have $\tilde{p}_t := \mathcal{F}^{-1}(\mathrm{e}^{-t\Psi})\mathbf{1}_{\mathbb{R}\setminus(-r_0/2, r_0/2)} \in L^1(\mathbb{R})$, where $\mathbf{1}$ is a characteristic function. Let $\psi, \eta \in C_c^\infty(-2, 2)$ such that $\psi \equiv 1$ for $|x| \le 1$ and $\int_{\mathbb{R}} \eta(x)\,\mathrm{d}x = 1$. Let $\psi_\delta(x) = \psi(\delta x)$ and $\eta_\delta(x) = \delta^{-1}\eta(x/\delta)$ and $\tilde{p}_{t\delta}(x) = \psi_{t\delta}(x)(\eta_\delta * \tilde{p}_t)(x)$, where '$*$' denotes a one-dimensional convolution. It is clear that $\tilde{p}_{t\delta} \to \tilde{p}_t$ in $L^1(\mathbb{R})$ as $\delta \to 0$. Moreover, from Proposition 9.5 in [4], $\tilde{p}_{t\delta} \to \tilde{p}_t$ in the sense of a distribution. Hence, we have

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{F}^{-1}(\mathrm{e}^{-t\Psi})(x)\phi(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \tilde{p}_t(x)\phi(x)\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\int_{\mathbb{R}} \tilde{p}_{t\delta}(x)\phi(x)\,\mathrm{d}x \\ &= \lim_{\delta\to 0}\langle \tilde{p}_{t\delta}, \phi\rangle \\ &= \langle \tilde{p}_t, \phi\rangle \\ &= \langle \mathcal{F}^{-1}(\mathrm{e}^{-t\Psi}), \phi\rangle, \end{aligned} \tag{C.1}$$

where '$\langle\cdot,\cdot\rangle$' denotes a dual pairing between $C_c^\infty(\mathbb{R})$ and its dual space. Hence, from (C.1), for $\phi \in C_c^\infty(\mathbb{R}\setminus\{0\})$, we have

$$\int_{\mathbb{R}} \phi(x)\,\mathrm{d}\mu_t(x) = \int_{\mathbb{R}} \mathrm{e}^{-t\Psi(\xi)}\mathcal{F}(\phi)(\xi)\,\mathrm{d}\xi = \langle \mathcal{F}^{-1}(\mathrm{e}^{-t\Psi}), \phi\rangle = \int_{\mathbb{R}} \mathcal{F}^{-1}(\mathrm{e}^{-t\Psi})(x)\phi(x)\,\mathrm{d}x. \tag{C.2}$$

It is well known that μ_t is uniquely decomposed by $\mu_t = \mu_{t,d} + \mu_{t,ac} + \mu_{t,cs}$, where $\mu_{t,d}$ is discrete, $\mu_{t,ac}$ is absolutely continuous, and $\mu_{t,cs}$ is singular continuous with respect to the Lebesgue measure; see Section 27 in [12]. But, (C.2) implies that $\mu_{t,cs} = 0$ and $\mu_{t,d} = a\delta_0$ for some $a \in \mathbb{R}$, where δ_0 is the discrete measure at 0. Since $\mathcal{F}(\mu_t)(\xi) = \mathcal{F}(\mu_{t,ac})(\xi) + \mathcal{F}(a\delta_0)(\xi) = \mathcal{F}(\mu_{t,ac})(\xi) + a = \mathrm{e}^{-t\Psi(\xi)}$ and $\lim_{|\xi|\to\infty}\mathcal{F}(\mu_{t,ac})(\xi) = \lim_{|\xi|\to\infty}\mathrm{e}^{-t\Psi(\xi)} = 0$, we have $a = 0$. Therefore, we complete the proof of Lemma 2.3. □

Appendix D

Proof of Remark 1.3. It is clear that $\int_{|x|\ge 1} q_t(x)\,\mathrm{d}x < \infty$.

Note that $s_n(1/x) \ge c_n$ for $|x| \le 1$ and $0 < \varepsilon_2 \le 1$. Using the change of variables $(x = y^{-1})$, we obtain

$$\begin{aligned}\int_{|x|\le 1} \frac{1}{|x|}\mathrm{e}^{-\alpha t s_n^\varepsilon(1/x)} s_n^{\varepsilon_2 - 1}\left(\frac{1}{x}\right) r_{n-1}^{-1}\left(\frac{1}{x}\right)\mathrm{d}x &\le \int_0^1 \frac{1}{x}\mathrm{e}^{-\alpha t s_n^\varepsilon(1/x)} r_{n-1}^{-1}\left(\frac{1}{x}\right)\mathrm{d}x \\ &\le c\int_1^\infty \frac{1}{x}\mathrm{e}^{-\alpha t s_n^\varepsilon(x)} r_{n-1}^{-1}(x)\,\mathrm{d}x.\end{aligned}$$

Again, using the change of variables $(\ln(x+1) = y)$, we obtain

$$c\int_{\ln 2}^\infty \frac{1}{x}\mathrm{e}^{-\alpha t s_{n-1}^\varepsilon(x)} r_{n-2}^{-1}(x)\,\mathrm{d}x \le c\int_{s_{n-1}(1)}^\infty \frac{1}{x}\mathrm{e}^{-\alpha t(\ln(x+1))^\varepsilon}\,\mathrm{d}x \le c\int_{s_n(1)}^\infty \mathrm{e}^{-\alpha t x^\varepsilon}\,\mathrm{d}x < \infty.$$

This implies that $q_t \in L^1(\mathbb{R})$. □

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