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STOCHASTIC ORDER OF SAMPLE RANGE FROM HETEROGENEOUS EXPONENTIAL RANDOM VARIABLES

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Let $X_1, \ldots, X_n$ be independent exponential random variables with their respective hazard rates $\lambda_1, \ldots, \lambda_n$, and let $Y_1, \ldots, Y_n$ be independent exponential random variables with common hazard rate λ. Denote by $X_{n:n}$, $Y_{n:n}$ and $X_{1:n}$, $Y_{1:n}$ the corresponding maximum and minimum order statistics. $X_{n:n} - X_{1:n}$ is proved to be larger than $Y_{n:n} - Y_{1:n}$ according to the usual stochastic order if and only if $\lambda \geq \left(\bar{\lambda}^{-1} \prod_{i=1}^{n} \lambda_i\right)^{1/(n-1)}$ with $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i / n$. Further, this usual stochastic order is strengthened to the hazard rate order for $n = 2$. However, a counterexample reveals that this can be strengthened neither to the hazard rate order nor to the reversed hazard rate order in the general case. The main result substantially improves those related ones obtained in Kochar and Rojo [16] and Khaledi and Kochar [13].

1. INTRODUCTION

Order statistics and those statistics having a close relation to order statistics play an important role in reliability theory, data analysis, goodness-of-fit, statistical inference, information science, and other applied probability areas. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of random variables $X_1, X_2, \ldots, X_n$. It is well known that the lifetime of a $(n - k + 1)$-out-of-n system can be characterized by the kth order statistic $X_{k:n}$; that is, the study of order statistics is equivalent to the study of k-out-of-n systems. In particular, the lifetimes of parallel and series systems correspond to the order statistics $X_{n:n}$ and $X_{1:n}$. In the literature, these statistics, which are closely related to order statistics (e.g., sample range and general sample spacings), have been extensively investigated in the case that observations are independent and identically distributed

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(i.i.d.). However, in some practical situations, observations are often non-i.i.d. Due to the complicated expression of the distribution in the non-i.i.d. case, only a smaller amount of results can be found in the literature. One can refer to David and Nagaraja [7] and Balakrishnan and Rao [2,3] for a comprehensive discussion on this topic.

Due to the nice mathematical form and the unique memoryless property, the exponential distribution has been widely applied in statistics, reliability, operations research, life testing, and other areas. Readers can refer to Barlow and Proschan [4] and Balakrishnan and Basu [1] for many concerned works in the literature. Pledger and Proschan [22] was among the first to stochastically compared the order statistics and its spacings of non-i.i.d. exponential random variables with those corresponding to i.i.d. exponential random variables. From then on, many researchers devoted themselves to this topic—for example, Proschan and Sethuraman [23], Kochar and Korwar [14], Kochar and Rojo [16], Dykstra, Kochar, and Rojo [8], Kochar and Ma [15], Bon and Păltănea [5,6], Khaledi and Kochar [10–13], Kochar and Xu [17], and *Păltănea* [21]. It is worthwhile mentioning that Hu [9] investigated conditions on the parameters to compare order statistics from heterogenous random variables and results in some articles mentioned above are special consequences of this article.

Before proceeding to the statement of those existing related conclusions in the literature, let us first recall some stochastic orders that are closely related to the main results to be developed. Throughout this article, the term *increasing* stands for *monotone nondecreasing* and the term *decreasing* stands for *monotone nonincreasing*.

DEFINITION 1.1: *For two random variables X and Y with their densities f and g and distribution functions F and G, respectively, let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. As the ratios in the following statements are well defined, X is said to be smaller than Y in the following*

(a) likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in x

(b) hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x

(c) reversed hazard rate order (denoted by $X \leq_{\mathrm{rh}} Y$) if $G(x)/F(x)$ is increasing in x

(d) stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\bar{G}(x) \geq \bar{F}(x)$

For a comprehensive discussion on stochastic orders, please refer to Shaked and Shanthikumar [24] and Müller and Stoyan [20].

Suppose $X_1, \ldots, X_n$ are independent exponential random variables, with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Z_1, \ldots, Z_n$ be a random sample of size n from an exponential distribution with common hazard rate $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$, the arithmetic mean of $\lambda_i's$, and let $W_1, \ldots, W_n$ be another random sample of size n from an exponential distribution with hazard rate $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of $\lambda_i's$. Kochar and Rojo [16] built

$$X_{n:n} - X_{1:n} \geq_{\mathrm{st}} Z_{n:n} - Z_{1:n}. \tag{1}$$

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Subsequently, Khaledi and Kochar [13] proved

$$X_{n:n} - X_{1:n} \geq_{\mathrm{st}} W_{n:n} - W_{1:n}, \tag{2}$$

which actually improved upon (1). Recently, Kochar and Xu [17] strengthened this result to the reserved hazard rate order,

$$X_{n:n} - X_{1:n} \geq_{\mathrm{rh}} Z_{n:n} - Z_{1:n}.$$

In this article, we further investigate the stochastic order between the sample range from nonidentical independent exponential observations and that from i.i.d. exponential ones. First, Theorem 2.4 proves the stochastic order between the sample range from independent exponential observations with hazard vector $(\lambda_1, \ldots, \lambda_n)$ and that from i.i.d. exponential ones with the hazard rate $\hat{\lambda} = ((\prod_{i=1}^n \lambda_i)/\bar{\lambda})^{1/(n-1)}$ which is less than the geometric mean of those heterogeneous hazard rates. Second, we further show in Theorem 2.8 a necessary and sufficient condition to the stochastic order between these two sample ranges. These main results substantially improve upon (1) and (2). Finally, counterexamples are also presented to confirm that the stochastic order in Theorem 2.4 cannot be strengthened either to the hazard rate order or to the reversed hazard rate order in the general case, although it can be strengthened to the hazard rate order in the case $n = 2$.

2. MAIN RESULTS

For the sake of convenience, we first list several useful lemmas before proceeding to the main result.

LEMMA 2.1: (Mitrinović and Vasić [19]) *For real numbers α_i, β_i, and λ_i, $i = 1, \ldots, n$, if $\beta_i > 0$, and $\lambda_i > 0$, then*

$$\min_{1\leq i\leq n}\left\{\frac{\alpha_i}{\beta_i}\right\} \leq \frac{\sum_{i=1}^n \alpha_i\lambda_i}{\sum_{i=1}^n \beta_i\lambda_i} \leq \max_{1\leq i\leq n}\left\{\frac{\alpha_i}{\beta_i}\right\}.$$

Equalities hold if and only if $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are proportional.

LEMMA 2.2: *For any $(y_1, \ldots, y_n) \in (0, \infty)^n$ with $y_p = \min\{y_1, \ldots, y_n\}$ and $y_q = \max\{y_1, \ldots, y_n\}$, if $y_p < y_q$, then it holds that*

$$\frac{\sum_{i\neq p}^n y_i/(1 - e^{-y_i})}{\sum_{i\neq p}^n y_i} < \frac{\sum_{i\neq q}^n y_i/(1 - e^{-y_i})}{\sum_{i\neq q}^n y_i}.$$

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PROOF: Since $1/(1-e^{-x})$ is positive and decreasing in $x>0$, by Lemma 2.1,

$$\frac{1}{1-e^{-y_q}}=\frac{y_q/(1-e^{-y_q})}{y_q}\le\frac{\sum_{i\neq p,q}^{n}y_i/(1-e^{-y_i})}{\sum_{i\neq p,q}^{n}y_i}\le\frac{y_p/(1-e^{-y_p})}{y_p}=\frac{1}{1-e^{-y_p}}. \tag{3}$$

On the other hand, it is easy to verify that if $a,b,c,$ and d are positive such that $\frac{a}{b}\le\frac{c}{d}\le\frac{e}{f}$ and $\frac{a}{b}<\frac{e}{f}$, then

$$\frac{a+c}{b+d}<\frac{c+e}{d+f}. \tag{4}$$

Based on (3) and (4), we conclude that

$$\frac{\sum_{i\neq p}^{n}y_i/(1-e^{-y_i})}{\sum_{i\neq p}^{n}y_i}<\frac{\sum_{i\neq q}^{n}y_i/(1-e^{-y_i})}{\sum_{i\neq q}^{n}y_i},$$

which is the desired result. ■

The following lemma, which plays an important role in proving our main result, presents a sufficient condition to reach the minimum point of a symmetrical function in a compact set; we will prove it in a manner similar to that in Bon and Păltănea [6].

LEMMA 2.3: *For a symmetric and continuously differentiable mapping* $\phi:\Re_+^n\to\Re_+$, *if for any* $n(>1)$*-dimensional vector* $\mathbf{y}=(y_1,\ldots,y_n)\in\Re_+^n$ *with* $y_p=\min\{y_1,\ldots,y_n\}$ *and* $y_q=\max\{y_1,\ldots,y_n\}$,

$$\frac{\partial\phi(\mathbf{y})/\partial y_q}{\sum_{i\neq q}^{n}(y_i)/y_q}-\frac{\partial\phi(\mathbf{y})/\partial y_p}{\sum_{i\neq p}^{n}(y_i)y_p}>0\quad \textit{for } y_p\neq y_q. \tag{5}$$

Then it holds that

$$\phi(y_1,\ldots,y_n)\ge\phi(\underbrace{m(\mathbf{y}),\ldots,m(\mathbf{y})}_{n}), \tag{6}$$

where

$$m(\mathbf{y})=\left(\frac{\prod_{i=1}^{n}y_i}{\bar{y}}\right)^{1/(n-1)}\quad \textit{and}\quad \bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i.$$

PROOF: Denote $m=m(\mathbf{y})$ and $\mathbf{m}=(\underbrace{m,\ldots,m}_{n})$, for any $\mathbf{y}=(y_1,\ldots,y_n)\in\Re_+^n$. Since (6) holds for $y_p=y_q$, we only need to prove the case with $y_p<y_q$, which tends to $m\in(y_p,y_q)$. Consider the compact set

$$S=\{\mathbf{s}=(s_1,\ldots,s_n)\in[y_p,y_q]^n|m(\mathbf{s})=m\}\in\Re_+^n,$$

which contains $\mathbf{y}$ and $\mathbf{m}$ as its two components. By Weierstrass' extreme value theorem, the mapping ϕ attains the absolute minimum value on the compact set S at some

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point $\mathbf{t} = (t_1, \ldots, t_n)$. Now, suppose $\mathbf{t} \neq \mathbf{m}$. Then there exists t_p and t_q such that $y_p \leq t_p = \min\{t_1, \ldots, t_n\} < \max\{t_1, \ldots, t_n\} = t_q \leq y_q$ and $m(\mathbf{t}) = m$, which can be rewritten as

$$n \prod_{i=1}^{n} t_i = m^{n-1} \sum_{i=1}^{n} t_i$$

or, equivalently,

$$nt_p t_q \prod_{i \neq p,q}^{n} t_i = m^{n-1} \left(t_p + t_q + \sum_{i \neq p,q}^{n} t_i \right).$$

It can be easily verified that

$$\left(n \prod_{i \neq p,q}^{n} t_i \right) z^2 - 2m^{n-1} z - m^{n-1} \left(\sum_{i \neq p,q}^{n} t_i \right) = 0$$

has a positive root $z_0 \in (t_p, t_q)$. Denote, for $t \in [t_p, z_0)$,

$$g(t) = \frac{m^{n-1} \left(t + \sum_{i \neq p,q}^{n} t_i \right)}{nt \prod_{i \neq p,q}^{n} t_i - m^{n-1}},$$

it holds that $g(t_p) = t_q$ and

$$g'(t) = -\frac{m^{n-1} \left(m^{n-1} + \left(n \prod_{i \neq p,q}^{n} t_i \right) \sum_{i \neq p,q}^{n} t_i \right)}{\left(nt \prod_{i \neq p,q}^{n} t_i - m^{n-1} \right)^2} < 0.$$

Thus, $g(t)$ is strictly decreasing on $[t_p, z_0)$. For $t \in [t_p, z_0)$, let $\mathbf{t}(t)$ be the vector with the components $t_p(t) = t, t_q(t) = g(t)$ and $t_i(t) = t_i$ for $i \neq p, q$. Then $\mathbf{t}(t) \in S$. Since, for $t \in [t_p, z_0)$,

$$g'(t) = -\frac{g(t) n \prod_{i \neq p,q}^{n} t_i - m^{n-1}}{tn \prod_{i \neq p,q}^{n} t_i - m^{n-1}} = -\frac{(g(t) + \sum_{i \neq p,q}^{n} t_i)/t}{(t + \sum_{i \neq p,q}^{n} t_i)/g(t)},$$

we have

$$\begin{aligned} \phi'(\mathbf{t}(t)) &= \frac{\partial \phi(\mathbf{t}(t))}{\partial t_p(t)} + \frac{\partial \phi(\mathbf{t}(t))}{\partial t_q(t)} g'(t) \\ &= \frac{g(t) + \sum_{i \neq p,q}^{n} t_i}{t} \cdot \left[\frac{\partial \phi(\mathbf{t}(t))/\partial t_p(t)}{(g(t) + \sum_{i \neq p,q}^{n} t_i)/t} - \frac{\partial \phi(\mathbf{t}(t))/\partial t_q(t)}{(t + \sum_{i \neq p,q}^{n} t_i)/g(t)} \right]. \end{aligned}$$

Note that (5) guarantees that for $\phi'(\mathbf{t}(t_p)) < 0$, there exists some $\varepsilon > 0$ such that $t_p + \varepsilon < z_0$, and $\phi(\mathbf{t}(t)) < \phi(\mathbf{t}(t_p)) = \phi(\mathbf{t})$ for any $t \in (t_p, t_p + \varepsilon)$, which contradicts

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with the assumption that ϕ has an absolute minimum value on the compact set S at the point $\mathbf{t} = (t_1, \ldots, t_n)$. As a result, $\mathbf{m}$ is the unique minimum point of ϕ on the compact set S and, hence, the desired inequality (6) is valid. ■

It should be remarked here that

$$(y_i - y_j) \cdot \left[\frac{\partial \phi(\mathbf{y})/\partial y_i}{\left(\sum_{s\neq i}^{n} y_s\right)/y_i} - \frac{\partial \phi(\mathbf{y})/\partial y_j}{\left(\sum_{s\neq j}^{y_j} y_s\right)/y_j} \right] > 0 \quad \text{for } y_i \neq y_j,\ i,j \in \{1,\ldots,n\}$$

is sufficient to the assumption in (5). Lemma 2.3 under this assumption can also be employed to prove our main result.

Now, we are ready to present the main result.

THEOREM 2.4: *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate $\hat{\lambda} = ((\prod_{i=1}^{n} \lambda_i)/\bar{\lambda})^{1/(n-1)}$. Then*

$$X_{n:n} - X_{1:n} \geq_{\text{st}} Y_{n:n} - Y_{1:n}.$$

PROOF: According to David and Nagaraja [7, p. 26], the sample range $R(X) = X_{n:n} - X_{1:n}$ has its distribution function as follows, for $x \geq 0$:

$$\begin{aligned} F_{R(X)}(x) &= \sum_{i=1}^{n} \left[\frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} \prod_{j\neq i}^{n} \left(1 - e^{-\lambda_j x}\right) \right] \\ &= \left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - e^{-\lambda_i x}} \right) \frac{\prod_{i=1}^{n} \left(1 - e^{-\lambda_i x}\right)}{\sum_{i=1}^{n} \lambda_i} \\ &= \psi(\lambda_1 x, \ldots, \lambda_n x), \end{aligned}$$

where the symmetric function $\psi : \Re_+^n \to (0,1)$ is defined as

$$\psi(y_1, \ldots, y_n) = \left(\sum_{i=1}^{n} \frac{y_i}{1 - e^{-y_i}} \right) \frac{\prod_{i=1}^{n} (1 - e^{-y_i})}{\sum_{i=1}^{n} y_i}.$$

Likewise, the sample range $R(Y) = Y_{n:n} - Y_{1:n}$ has its distribution function

$$F_{R(Y)}(x) = \psi\big(\underbrace{\hat{\lambda}\, x, \ldots, \hat{\lambda}\, x}_{n}\big),$$

To reach the desired result $R(X) \geq_{\text{st}} R(Y)$, it suffices to prove

$$\psi(y_1, \ldots, y_n) \leq \psi\big(\underbrace{m(\mathbf{y}), \ldots, m(\mathbf{y})}_{n}\big),$$

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where $m(\mathbf{y}) = ((\prod_{i=1}^n y_i)/\bar{y})^{1/(n-1)}$ for any vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re_+^n$; that is,

$$\left(\sum_{i=1}^n \frac{y_i}{1-e^{-y_i}}\right) \frac{\prod_{i=1}^n (1-e^{-y_i})}{\sum_{i=1}^n y_i} \le (1-e^{-m(\mathbf{y})})^{n-1}, \tag{7}$$

which is actually equivalent to

$$\begin{aligned}\phi(y_1, \ldots, y_n) &= \left(n \prod_{i=1}^n \frac{y_i}{1-e^{-y_i}}\right) \Big/ \left(\sum_{i=1}^n \frac{y_i}{1-e^{-y_i}}\right)^{-1} \\ &\ge \left(\frac{m(\mathbf{y})}{1-e^{-m(\mathbf{y})}}\right)^{n-1} = \phi\big(\underbrace{m(\mathbf{y}), \ldots, m(\mathbf{y})}_{n}\big).\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial \phi(\mathbf{y})}{\partial y_s} &= \left[\left(n \prod_{i=1}^n \frac{y_i}{1-e^{-y_i}}\right) \Big/ \left(\sum_{i=1}^n \frac{y_i}{1-e^{-y_i}}\right)^2\right] \\ &\quad \times \left[\left(\sum_{i\ne s}^n \frac{y_i}{1-e^{-y_i}}\right) \Big/ \left(\frac{y_s}{1-e^{-y_s}}\right)^{-1}\right] \frac{1-e^{-y_s}-y_s e^{-y_s}}{(1-e^{-y_s})^2},\end{aligned}$$

we have

$$\begin{aligned}\frac{\partial \phi(\mathbf{y})}{\partial y_s} \Big/ \left(\frac{\sum_{i\ne s}^n y_i}{y_s}\right)^{-1} &= \left[\left(n \prod_{i=1}^n \frac{y_i}{1-e^{-y_i}}\right) \Big/ \left(\sum_{i=1}^n \frac{y_i}{1-e^{-y_i}}\right)^2\right] \\ &\quad \times \left[\left(\sum_{i\ne s}^n \frac{y_i}{1-e^{-y_i}}\right) \Big/ \left(\sum_{i\ne s}^n y_i\right)^{-1}\right] \frac{1-e^{-y_s}-y_s e^{-y_s}}{1-e^{-y_s}}.\end{aligned}$$

Denote $y_p = \min\{y_1, \ldots, y_n\}$, and $y_q = \max\{y_1, \ldots, y_n\}$. Since $y_p < y_q$, it holds that

$$\begin{aligned}&\frac{\partial \phi(\mathbf{y})}{\partial y_p} \Big/ \left(\frac{\sum_{i\ne p}^n y_i}{y_p}\right)^{-1} - \frac{\partial \phi(\mathbf{y})}{\partial y_q} \Big/ \left(\frac{\sum_{i\ne q}^n y_i}{y_q}\right)^{-1} \\ &\overset{\text{sgn}}{=} \left[\left(\sum_{i\ne p}^n \frac{y_i}{1-e^{-y_i}}\right) \Big/ \left(\sum_{i\ne p}^n y_i\right)^{-1}\right] \frac{1-e^{-y_p}-y_p e^{-y_p}}{1-e^{-y_p}} \\ &\quad - \left[\left(\sum_{i\ne q}^n \frac{y_i}{1-e^{-y_i}}\right) \Big/ \left(\sum_{i\ne q}^n y_i\right)^{-1}\right] \frac{1-e^{-y_q}-y_q e^{-y_q}}{1-e^{-y_q}}.\end{aligned}$$

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By traditional calculus, it can be verified that the function $(1 - e^x - xe^x)/(1 - e^x)$ is nonnegative and strictly increasing in $x > 0$. So,

$$\frac{1 - e^{-y_p} - y_p e^{-y_p}}{1 - e^{-y_p}} < \frac{1 - e^{-y_q} - y_q e^{-y_q}}{1 - e^{-y_q}}.$$

This in combination with Lemma 2.2, can be use to conclude that

$$\frac{\partial \phi(\mathbf{y})}{\partial y_p} \Big/ \left(\frac{\sum_{i \neq p}^n y_i}{y_p} \right)^{-1} - \frac{\partial \phi(\mathbf{y})}{\partial y_q} \Big/ \left(\frac{\sum_{i \neq q}^n y_i}{y_q} \right)^{-1} < 0.$$

This validates (7) and thus we complete the proof. ■

Remark 2.5: Independent heterogeneous exponential random variables belong to the well-known *proportional hazard rates* (PHR) model. $X_1, \ldots, X_n$ follow the PHR model if they are independent and X_i has the survival function

$$\bar{F}_i(x) = [\bar{F}(x)]^{\lambda_i}, \qquad i = 1, \ldots, n,$$

where $\bar{F}(x)$ is the baseline survival function. Let $Y_1, \ldots, Y_n$ be random variables having common survival function $\bar{F}^{\bar{\lambda}}$ with $\bar{\lambda} = \sum_{i=1}^n \lambda_i / n$. Kochar and Xu [17,18] built $X_{n:n} - X_{1:n} \geq_{\mathrm{st}} Y_{n:n} - Y_{1:n}$. Although $X_{n:n} - X_{1:n} \geq_{\mathrm{rh}} Y_{n:n} - Y_{1:n}$ holds for the exponential case, it remains an open problem for the PHR model.

As an immediate consequence of Theorem 2.4, we can get a simple upper bound of the distribution function $P(X_{n:n} - X_{1:n} < x)$ based on the quantity $\hat{\lambda} = ((\prod_{i=1}^n \lambda_i)/\bar{\lambda})^{1/(n-1)}$.

COROLLARY 2.6: *Under the condition of Theorem* 2.4*, for* $x \geq 0$*,*

$$P(X_{n:n} - X_{1:n} \leq x) \leq (1 - e^{-\hat{\lambda} x})^{n-1}.$$

It can be easily established that $\hat{\lambda} \leq \tilde{\lambda} \leq \bar{\lambda}$. Considering that $(1 - e^{-\lambda x})^{n-1}$ is increasing in λ for any $x > 0$, it follows immediately that the bound established in Corollary 2.6 is smaller than and hence improves the one in terms of the geometric mean $\tilde{\lambda}$ due to Khaledi and Kochar [13].

In view of the chain of implications

$$X \leq_{\mathrm{lr}} Y \Longrightarrow X \leq_{\mathrm{hr\,(rh)}} Y \Longrightarrow X \leq_{\mathrm{st}} Y,$$

one naturally wonders whether the stochastic order in Theorem 2.4 can be strengthened to the hazard rate order, or reserved hazard rate order. The following counterexample serves as a negative answer to this question.

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Example 2.7: Consider (X_1, X_2, X_3), an independent exponential random vector with hazard rate vector $(\lambda_1, \lambda_2, \lambda_3) = (5.5, 5.5, 40)$, and (Y_1, Y_2, Y_3), independent and identical exponential random variables with the hazard rate vector $\hat{\lambda} = \sqrt{1210/17}$.

Denote $F_{R_3(X)}(x)$ and $F_{R_3(X)}(x)$ to be the distribution functions of $R_3(X) = X_{3:3} - X_{1:3}$ and $R_3(Y) = Y_{3:3} - Y_{1:3}$, respectively; it holds that

$$\frac{F_{R_3(X)}(0.05)}{F_{R_3(Y)}(0.05)} \approx 0.761353 > 0.742095 \approx \frac{F_{R_3(X)}(0.06)}{F_{R_3(Y)}(0.06)}.$$

Thus, the ratio $F_{R_3(X)}(x)/F_{R_3(Y)}(x)$ is not increasing with respect to $x \geq 0$, which implies $X_{3:3} - X_{1:3} \not\geq_{\text{rh}} Y_{3:3} - Y_{1:3}$.

Let $\bar{F}_{R_3(X)} = 1 - F_{R_3(X)}$ and $\bar{F}_{R_3(X)} = 1 - F_{R_3(X)}$ be the reliability functions of $R_3(X)$ and $R_3(Y)$, respectively. It can also be verified that

$$\frac{\bar{F}_{R_3(X)}(4.02)}{\bar{F}_{R_3(Y)}(4.02)} \approx 118122 > 117474 \approx \frac{\bar{F}_{R_3(X)}(1.989)}{\bar{F}_{R_3(Y)}(1.989)}.$$

As a result, $\bar{F}_{R_3(X)}(x)/\bar{F}_{R_3(Y)}(x)$ is not increasing in $x \geq 0$ and, hence, $X_{3:3} - X_{1:3} \not\geq_{hr} Y_{3:3} - Y_{1:3}$.

The next result presents a characterization of the usual stochastic order of sample range of heterogeneous exponential random variables in terms of their parameters.

THEOREM 2.8: *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$ and let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential population with hazard rate λ. Then*

$$X_{n:n} - X_{1:n} \geq_{\text{st}} Y_{n:n} - Y_{1:n} \iff \lambda \geq \hat{\lambda} = \left(\frac{\prod_{i=1}^{n} \lambda_i}{\bar{\lambda}} \right)^{1/(n-1)}.$$

PROOF: $\Longleftarrow$ Let $V_1, \ldots, V_n$ be a random sample of size n from the exponential distribution with the hazard rate $\hat{\lambda}$. Theorem 2.4 guarantees $X_{n:n} - X_{1:n} \geq_{\text{st}} V_{n:n} - V_{1:n}$. On the other hand, $Y_{n:n} - Y_{1:n}$ has its probability function $\left(1 - e^{-\lambda x}\right)^{n-1}$, which tells that it is stochastically decreasing with respect to $\lambda > 0$. Then $\lambda \geq \hat{\lambda}$ implies $Y_{n:n} - Y_{1:n} \leq_{\text{st}} V_{n:n} - V_{1:n}$. Now, due to transitivity, we immediately have $X_{n:n} - X_{1:n} \geq_{\text{st}} Y_{n:n} - Y_{1:n}$.

$\Longrightarrow$ The distribution function of $R(X)$ can be written as (see David and Nagaraja [7]), for $x \geq 0$,

$$F_{R(X)}(x) = \frac{1}{n\bar{\lambda}} \sum_{i=1}^{n} \lambda_i \prod_{j \neq i}^{n} (1 - e^{-\lambda_j x}),$$

where $n\bar{\lambda} = \sum_{i=1}^{n} \lambda_i$. By Taylor's expansion, we have, for $x \geq 0$,

$$1 - e^{-\lambda_j x} = \lambda_j \left(x - \frac{\lambda_j}{2} x^2 \right) + o(x^2)$$

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and

$$\prod_{j\neq i}^{n}\left(1-e^{-\lambda_j x}\right)=\prod_{j\neq i}^{n}\lambda_j\left(x^{n-1}-\frac{\sum_{j\neq i}^{n}\lambda_j}{2}x^n\right)+o(x^n).$$

As a result, for $x \geq 0$,

$$\begin{aligned}F_{R(X)}(x)&=\frac{1}{n\bar{\lambda}}\sum_{i=1}^{n}\prod_{j=1}^{n}\lambda_j\left(x^{n-1}-\frac{\sum_{j\neq i}^{n}\lambda_j}{2}x^n\right)+o(x^n)\\&=\frac{\prod_{i=1}^{n}\lambda_i}{n\bar{\lambda}}\left[nx^{n-1}-\frac{(n-1)n\bar{\lambda}}{2}x^n\right]+o(x^n)\\&=\frac{\prod_{i=1}^{n}\lambda_i}{\bar{\lambda}}x^{n-1}+o(x^{n-1}).\end{aligned}$$

Likewise, for $x \geq 0$,

$$F_{R(Y)}(x)=\lambda^{n-1}x^{n-1}+o(x^{n-1}).$$

Now, $X_{n:n}-X_{1:n} \geq_{st} Y_{n:n}-Y_{1:n}$ implies, for $x \geq 0$,

$$\bar{F}_{R(X)}(x)-\bar{F}_{R(Y)}(x)=\left(\lambda^{n-1}-\frac{\prod_{i=1}^{n}\lambda_i}{\bar{\lambda}}\right)x^{n-1}+o(x^{n-1})\geq 0.$$

Therefore, it holds that $\lambda \geq ((\prod_{i=1}^{n}\lambda_i)/\bar{\lambda})^{1/(n-1)}=\hat{\lambda}$. ■

As is discussed in Example 2.7, the hazard rate order of the sample range does not hold in the general case under the condition of Theorem 2.4. However, we prove in the next theorem that it indeed holds for the case $n = 2$. It should be remarked here that when $n = 2$,

$$\hat{\lambda}=\frac{2}{1/\lambda_1+1/\lambda_2};$$

that is, $\hat{\lambda}$ coincides with the harmonic mean of (λ_1, λ_2).

THEOREM 2.9: *Let X_1 and X_2 be independent exponential random variables with their respective hazard rates λ_1 and λ_2 and let Y_1 and Y_2 be independent and of common exponential distribution with the hazard rate $2\lambda_1\lambda_2/(\lambda_1+\lambda_2)$. Then*

$$X_{2:2}-X_{1:2} \geq_{\mathrm{hr}} Y_{2:2}-Y_{1:2}.$$

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PROOF: The reliability function of $R_2(X) = X_{2:2} - X_{1:2}$ is, for $x \geq 0$,

$$\bar{F}_{R_2(X)}(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_2 x} + \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 x},$$

and the reliability function of $R_2(Y) = Y_{2:2} - Y_{1:2}$ is, for $x \geq 0$,

$$\bar{F}_{R_2(Y)}(x) = \exp\left\{-\frac{2\lambda_1\lambda_2}{\lambda_1 + \lambda_2} x\right\}.$$

Denote

$$\begin{aligned}\Delta(x) &= \frac{\bar{F}_{R_2(X)}(x)}{\bar{F}_{R_2(Y)}(x)} \\ &= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_2 x} + \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 x}\right) / (e^{-[2\lambda_1\lambda_2/(\lambda_1+\lambda_2)]x})^{-1}.\end{aligned}$$

Since, for $x \geq 0$,

$$\begin{aligned}&\Delta'(x)\, e^{-\frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2}x} \\ &= -\left(\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_2 x} + \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 x}\right) e^{-[2\lambda_1\lambda_2/(\lambda_1+\lambda_2)]x} \\ &\quad + \frac{2\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_2 x} + \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 x}\right) e^{-[2\lambda_1\lambda_2/(\lambda_1+\lambda_2)]x} \\ &= \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_2 x} + \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_1 x}\right) e^{-[2\lambda_1\lambda_2/(\lambda_1+\lambda_2)]x} \\ &= \frac{\lambda_1\lambda_2(\lambda_2 - \lambda_1)}{(\lambda_1 + \lambda_2)^2}\left(e^{-\lambda_1 x} - e^{-\lambda_2 x}\right) \\ &\geq 0,\end{aligned}$$

the ratio $\Delta(x)$ is increasing in $x \geq 0$. Hence, $R_2(X) \geq_{\text{st}} R_2(Y)$. ■

The following example reveals that the reserved hazard rate order and hence the likelihood ratio order does not hold under the same assumption of Theorem 2.9.

Example 2.10: Consider the independent exponential random variables X_1 and X_2 with their respective hazard rates $\lambda_1 = 2$ and $\lambda_2 = 5$. Let Y_1 and Y_2 be independent exponential random variables with common hazard rate $\hat{\lambda} = 20/7$. Denote $F_{R_2(X)}(x)$ and $F_{R_2(Y)}(x)$ as the distribution functions of $R_2(X) = X_{2:2} - X_{1:2}$ and $R_2(Y) = Y_{2:2} - Y_{1:2}$, respectively. It can be verified that,

$$\frac{F_{R_2(X)}(0.2)}{F_{R_2(Y)}(0.2)} \approx 0.955913 > 0.940229 \approx \frac{F_{R_2(X)}(0.4)}{F_{R_2(X^*)}(0.4)};$$

that is, $F_{R_2(X)}(x)/F_{R_2(Y)}(x)$ is not increasing in $x > 0$. Hence, $X_{2:2} - X_{1:2} \not\geq_{\text{rh}} Y_{2:2} - Y_{1:2}$.

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We present the following corollary, which can be directly derived from Theorems 2.8 and 2.9.

COROLLARY 2.11: *Let X_1 and X_2 be independent exponential random variables with their respective hazard rates λ_1 and λ_2, and let Y_1 and Y_2 be independent exponential random variables with common hazard rate λ. Then the following three statements are equivalent.*

(i) $X_{2:2} - X_{1:2} \geq_{\mathrm{hr}} Y_{2:2} - Y_{1:2}$

(ii) $X_{2:2} - X_{1:2} \geq_{\mathrm{st}} Y_{2:2} - Y_{1:2}$

(iii) $\lambda \geq 2\lambda_1\lambda_2/(\lambda_1 + \lambda_2)$.

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