The Great Ellipse Solution for Distances and Headings to Steer between Waypoints

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1. INTRODUCTION. The normal Great Circle method of computing the shortest distance between two positions on the Earth – e.g. from an aircraft's present position (PP) to a waypoint (WP) – is not accurate enough to meet present-day requirements for aircraft Nav–Attack systems.

On the surface of an Ellipsoid (or Spheroid), the true 'shortest distance' is along a geodesic curve between the two points, but the computation of this curve is complex, and as shown by R. Williams at Reference, the difference between the geodesic and Great Ellipse distances between two points is negligible ($< 0{\cdot}01$ nm).

The Great Ellipse through two points on a spheroid is defined as the ellipse that passes through the two points and the centre of the spheroid; it therefore has a major axis equal to the Earth's, and a minor axis that is between the Earth's major axis (for two points on the Equator) and minor axis (for two points on the same, or diametrically opposite, longitudes). Thus the problem of deciding on which Great Ellipse the two points lie is equivalent to determining the magnitude of the minor axis β of the ellipse on which they both lie.

2. THE GREAT ELLIPSE ALGORITHM. Let: A = Earth's (WGS-84) major semi-axis = 6 378 137 m, and B = Earth's (WGS-84) minor semi-axis = 6 356 752·317 m.
Let:

Lat1, Long1 be the aircraft's PP Latitude & Longitude, and
Lat2, Long2 be the waypoint WP's Latitude & Longitude.

Let: the auxiliary angles for these two Latitudes be:

$\Theta 1 = \text{ArcTan}(B/A \,.\, \text{Tan}(\text{Lat1}))$, and
$\Theta 2 = \text{ArcTan}(B/A \,.\, \text{Tan}(\text{Lat2}))$.

Then the (WGS-84 ECEF) position components are given by:

$X1 = A \,.\, \text{Cos}(\Theta 1) \,.\, \text{Cos}(\text{Long1})$: $X2 = A \,.\, \text{Cos}(\Theta 2) \,.\, \text{Cos}(\text{Long2})$
$Y1 = A \,.\, \text{Cos}(\Theta 1) \,.\, \text{Sin}(\text{Long1})$: $Y2 = A \,.\, \text{Cos}(\Theta 2) \cdot \text{Sin}(\text{Long2})$
$Z1 = B \,.\, \text{Sin}(\Theta 1)$: $Z2 = B \,.\, \text{Sin}(\Theta 2)$

The Radial distances of these two points from Earth's Centre are:

$$R1 = \sqrt{(X1^2 + Y1^2 + Z1^2)}: \quad R2 = \sqrt{(X2^2 + Y2^2 + Z2^2)}.$$

The Cosine of the Great Ellipse Angle (GEA) between these two radii is:

$$\text{Cos GEA} = (X1 \,.\, X2 + Y1 \,.\, Y2 + Z1 \,.\, Z2)/(R1 \,.\, R2).$$

Thus GEA can be derived by either an Arc Cos function, or if this is not available:

$$\text{GEA} = 2 \,.\, \text{ArcTan}((1 - \text{Cos GEA})/\sqrt{(1 - \text{Cos GEA}^2)}).$$

Note:

(i) GEA is in $0 - \pi$ format, since $-1 \leqslant \text{Cos GEA} \leqslant 1$.
(ii) If Cos GEA = 1 then the PP and WP are coincident!
If Cos GEA = −1 then the PP and WP lie opposite to each other on a meridional ellipse.

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If $0 < \text{GEA} < \pi$, then the problem becomes one of finding a value for β such that the two positions lie on the same ellipse with semi-major and semi-minor axes of A and β, and the angles $\mu 1$ and $\mu 2$ of P1 and P2.

3. DERIVE EARTH-CENTRED ANGLES OF PP AND WP AND THE SEMI-MAJOR AXIS β. Let the Earth-centered angles of P1 and P2 on the Great Ellipse through them be $\mu 1$ and $\mu 2$, and β be the semi-minor axis of this ellipse.
Then:

$$\beta = \frac{R1 . \text{Sin}(\mu 1)}{\sqrt{(1-(R1/A)^2 . \text{Cos}^2(\mu 1))}} = \frac{R2 . \text{Sin}(\mu 2)}{\sqrt{(1-(R2/A)^2 . \text{Cos}^2(\mu 2))}},$$

$$=> Q1 = \frac{\text{Sin}(\mu 2)}{\text{Sin}(\mu 1)} = \frac{R1 . \sqrt{(1-(R2/A)^2)}}{R2 . \sqrt{(1-(R1/A)^2)}}.$$

But:

$$\text{Sin}(\mu 2) = \text{Sin}(Q2-\mu 1) = \text{Sin}(Q2) . \text{Cos}(\mu 1) - \text{Cos}(Q2) . \text{Sin}(\mu 1),$$

where:

$Q2 = (\pi - \text{GEA})$, if both points have North or South Latitudes, and
$Q2 = \text{GEA}$, if points have opposite Latitudes,

hence:

$\mu 1 = \text{ArcTan}(\text{Sin}(Q2)/(Q1 + \text{Cos}(Q2)))$,
If $\mu 1 < 0$ then $\mu 1 = \pi + \mu 1$,
$\mu 2 = Q2 - \mu 1$.

β can be derived from either of the equations above.

Having found the semi-minor axis, β, of the Great Ellipse that passes through the PP, the WP and the Earth's centre, the arc length (metres) along this ellipse between the *auxiliary* angles $\Theta 1$ and $\Theta 2$ (derived from $\mu 1$ and $\mu 2$ by $\text{Tan}(\Theta) = A/\beta . \text{Tan}(\mu)$) can be calculated, plus the (initial) Heading to Steer Ψ_{WP}.

4. COMPUTATION OF THE ARC LENGTH ALONG THE GREAT ELLIPSE. The rate of change of distance with respect to the *auxiliary* angle Θ along the arc of an ellipse with semi-axes A and β is:

$$dS/d\Theta = \sqrt{(A^2 . \text{Sin}^2 \Theta + \beta^2 . \text{Cos}^2 \Theta)}.$$

This can be expressed:

$$dS = \beta . \sqrt{(1 + p . \text{Sin}^2 \Theta)} . d\Theta, \quad \text{where} \quad p = (A^2/\beta^2 - 1).$$

The arc length along the ellipse from $\Theta 1$ to $\Theta 2$ is:

$$S_{1,2} = \int_{\Theta 1}^{\Theta 2} dS = \beta . \int_{\Theta 1}^{\Theta 2} \sqrt{(1 + p . \text{Sin}^2 \Theta)} . d\Theta.$$

But:

$$\text{Sin}^2 \Theta = (1 - \text{Cos}(2\Theta))/2 => dS = \beta . \sqrt{(1 + p/2 - p/2 . \text{Cos}(2\Theta))} . d\Theta.$$

Expanding this by the Binomial Expansion with $k = p/(2+p)$ gives:

$$dS = \beta . \sqrt{(1 + p/2)} . (1 - k^2/16 - k/2 . \text{Cos}(2\Theta) - k^2/16 . \text{Cos}(4\Theta) \ldots) . d\Theta.$$

Integrating with respect to Θ gives:

$$S_{1,2} = \beta . \sqrt{(1 + p/2)} . [(1 - k^2/16) . \Theta - k/4 . \text{Sin}(2\Theta) - k^2/64 . \text{Sin}(4\Theta)]_{\Theta_1}^{\Theta_2}.$$

Table 1 below shows the distances in international nautical miles (1 inm = 1852 m) along the seven Great Ellipses 'tilted' at $\epsilon = 0°$ to $\epsilon = 90°$ by 15° to the Earth's equatorial plane. The

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maximum error compared with the exact (4th Order Runge-Kutta) value is ≈ 5 mm (!!). For a comparison of the Great Ellipse distance with the Great Circle distance $= R_c$.GCA, where R_c is the Earth's equatorial radius in inm, $R_c = A/1852 = 3443.91847$ inm (i.e. 60·1077 inm per 1°, rather than 60).

Table 1. Distances along Great Ellipses from $\epsilon = 0°$ to 90°

Theta°	$\epsilon = 0{\cdot}0°$	$\epsilon = 15{\cdot}0°$	$\epsilon = 30{\cdot}0°$	$\epsilon = 45{\cdot}0°$	$\epsilon = 60{\cdot}0°$	$\epsilon = 75{\cdot}0°$	$\epsilon = 90{\cdot}0°$
0	0·000	0·000	0·000	0·000	0·000	0·000	0·000
6	360·646	360·565	360·344	360·042	359·742	359·522	359·442
12	721·293	721·132	720·695	720·098	719·503	719·068	718·909
18	1081·939	1081·703	1081·058	1080·179	1079·303	1078·663	1078·429
24	1442·585	1442·278	1441·440	1440·298	1439·158	1438·326	1438·022
30	1803·231	1802·860	1801·846	1800·463	1799·085	1798·077	1797·709
36	2163·878	2163·449	2162·279	2160·684	2159·093	2157·931	2157·506
42	2524·524	2524·046	2522·742	2520·964	2519·191	2517·896	2517·423
48	2885·170	2884·652	2883·236	2881·307	2879·384	2877·979	2877·465
54	3245·817	3245·266	3243·762	3241·714	3239·670	3238·178	3237·633
60	3606·463	3605·888	3604·319	3602·180	3600·047	3598·490	3597·920
66	3967·109	3966·517	3964·902	3962·701	3960·506	3958·903	3958·317
72	4327·756	4327·153	4325·509	4323·269	4321·035	4319·404	4318·807
78	4688·402	4687·794	4686·135	4683·875	4681·620	4679·974	4679·372
84	5049·048	5048·438	5046·774	5044·506	5042·243	5040·591	5039·988
90	5409·694	5409·084	5407·419	5405·150	5402·886	5401·234	5400·629
96	5770·341	5769·730	5768·064	5765·794	5763·529	5761·876	5761·271
102	6130·987	6130·374	6128·703	6126·425	6124·153	6122·494	6121·887
108	6491·633	6491·015	6489·328	6487·030	6484·738	6483·063	6482·452
114	6852·280	6851·651	6849·936	6847·598	6845·267	6843·564	6842·942
120	7212·926	7212·280	7210·519	7208·119	7205·726	7203·978	7203·339
126	7573·572	7572·903	7571·075	7568·586	7566·102	7564·289	7563·626
132	7934·219	7933·517	7931·602	7928·992	7926·389	7924·489	7923·794
138	8294·865	8294·122	8292·096	8289·335	8286·582	8284·571	8283·836
144	8655·511	8654·719	8652·559	8649·616	8646·680	8644·536	8643·753
150	9016·157	9015·308	9012·992	9009·836	9006·688	9004·390	9003·550
156	9376·804	9375·890	9373·398	9370·001	9366·614	9364·141	9363·237
162	9737·450	9736·466	9733·780	9730·120	9726·470	9723·805	9722·830
168	10098·096	10097·036	10094·143	10090·201	10086·270	10083·399	10082·350
174	10458·743	10457·603	10454·494	10450·257	10446·031	10442·945	10441·817
180	10819·389	10818·168	10814·838	10810·299	10805·773	10802·467	10801·259

5. DERIVATION OF THE INITIAL HEADING TO STEER. The heading to steer is the included angle between the meridian through PP and the plane of the Great Ellipse passing through PP and WP. Since the heading to steer will be continuously changing as the PP changes (unless the WP is due North or South of the PP, or their longitudes are exactly 180° apart), it *may* be sufficiently accurate to use the spherical geometry equation for the heading to steer, using the current PP and WP Latitudes, ϕP and ϕW, and the Great Circle Angle (GCA).

$$\mathrm{Cos}\,(\mathrm{GCA}) = \mathrm{Cos}\,\phi P.\mathrm{Cos}\,\phi W.\mathrm{Cos}\,(\mathrm{dLong}) + \mathrm{Sin}\,\phi P.\mathrm{Sin}\,\phi W,$$

$$\mathrm{Cos}\,\Psi C_{\mathrm{WP}} = \frac{\mathrm{Sin}\,(\phi W) - \mathrm{Sin}\,(\phi P).\mathrm{Cos}\,(\mathrm{GCA})}{\mathrm{Cos}\,(\phi P)\cdot\mathrm{Sin}\,(\mathrm{GCA})},$$

$$=> \Psi C_{\mathrm{WP}} = 2 * \mathrm{ArcTan}\,((1 - Cos\,\Psi C_{\mathrm{WP}})/\sqrt{(1 - \mathrm{Cos}^2\,\Psi C_{\mathrm{WP}})}).$$

If the WP is West of the PP, then $\Psi C_{\mathrm{WP}} = 2\pi - \Psi C_{\mathrm{WP}}$.

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6. THE GREAT ELLIPSE HEADING TO STEER. The more accurate heading ΨE_{WP} can, if required, be determined by deriving the rate of change of Northerly distance at the PP for a change in the distance along the Great Ellipse through PP and WP, i.e.:

$$\mathrm{Cos}\,(\Psi E_{WP}) = dN/dS \quad \text{at} \quad \mathrm{PP} = d\phi_C/d\mu 1, \text{ where } \phi_C = \text{the Geocentric Latitude.}$$

This heading is independent of the actual direction of the Great Ellipse equatorial base line, since it is an included angle between two planes.

The 'Unit Vector' position of PP in ECEF X, Y, Z coordinates relative to the 'baseline' direction of the Great Ellipse through PP and WP is:

$$X = \mathrm{Sin}\,\mu 1 \,.\, \mathrm{Cos}\,\epsilon,$$

$$Y = -\mathrm{Cos}\,\mu l, \text{ and}$$

$$Z = \mathrm{Sin}\,\mu 1 \,.\, \mathrm{Sin}\,\epsilon.$$

where: ϵ is the 'tilt' of the Great Ellipse to the equatorial plane, given by:

$$\mathrm{Sin}^2\,\epsilon = ((B/\beta)^2 - (B/A)^2)/(1-(B/A)^2).$$

Let $P = \sqrt{(X^2 + Y^2)}$ = the projection of PP onto the equatorial plane
Hence:

$$(P^2 + Z^2) = X^2 + Y^2 + Z^2 = 1,$$

$$\mathrm{Tan}\,\phi_C = Z/P.$$

Hence:

$$d\phi_C/d\mu 1 = \mathrm{Cos}\,\Psi E_{WP} = (P \,.\, dZ/d\mu 1 - Z \,.\, dP/d\mu 1),$$

$$dZ/d\mu 1 = \mathrm{Cos}\,\mu 1 \,.\, \mathrm{Sin}\,\epsilon,$$

$$P = \sqrt{(\mathrm{Sin}^2\,\mu 1 \,.\, \mathrm{Cos}^2\,\epsilon + \mathrm{Cos}^2\,\mu l)},$$

$$=> dP/d\mu 1 = -\mathrm{Sin}\,\mu 1 \,.\, \mathrm{Cos}\,\mu 1 \,.\, \mathrm{Sin}^2\,\epsilon/P.$$

Hence:

$$\mathrm{Cos}\,\Psi E_{WP} = d\phi_C/d\mu 1 = \mathrm{Cos}\,\mu 1 \,.\, \mathrm{Sin}\,\epsilon \,.\, (P + (\mathrm{Sin}^2\,\mu 1 \,.\, \mathrm{Sin}^2\,\epsilon)/P),$$

and

$$\Psi E_{WP} \text{ in } 0 \text{ to } \pi \text{ format is } 2 \,.\, \mathrm{ArcTan}\,((1 - \mathrm{Cos}\,\Psi E_{WP})/\sqrt{(1 - \mathrm{Cos}^2\,\Psi E_{WP})}).$$

7. RESOLUTION OF HEADING AMBIGUITIES. There are eight possible conditions for which the correct heading must be considered, i.e. whether the WP is North or South and East or West of the PP, and whether they are in the same or opposite hemispheres.

REFERENCE

Williams, R. (1996). The Great Ellipse on the surface of the spheroid. This *Journal*, **49**, 229.

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