ESAIM: PS 18 (2014) 418–440
DOI: 10.1051/ps/2013044

ESAIM: Probability and Statistics
www.esaim-ps.org

FROM ALMOST SURE LOCAL REGULARITY TO ALMOST SURE HAUSDORFF DIMENSION FOR GAUSSIAN FIELDS

Erick Herbin[1], Benjamin Arras[1] and Geoffroy Barruel[1]

Abstract. Fine regularity of stochastic processes is usually measured in a local way by local Hölder exponents and in a global way by fractal dimensions. In the case of multiparameter Gaussian random fields, Adler proved that these two concepts are connected under the assumption of increment stationarity property. The aim of this paper is to consider the case of Gaussian fields without any stationarity condition. More precisely, we prove that almost surely the Hausdorff dimensions of the range and the graph in any ball $B(t_0, \rho)$ are bounded from above using the local Hölder exponent at t_0. We define the deterministic local sub-exponent of Gaussian processes, which allows to obtain an almost sure lower bound for these dimensions. Moreover, the Hausdorff dimensions of the sample path on an open interval are controlled almost surely by the minimum of the local exponents. Then, we apply these generic results to the cases of the set-indexed fractional Brownian motion on $\mathbf{R}^N$, the multifractional Brownian motion whose regularity function H is irregular and the generalized Weierstrass function, whose Hausdorff dimensions were unknown so far.

Mathematics Subject Classification. 60G15, 60G17, 60G10, 60G22, 60G60.

Received June 5, 2012. Revised February 6, 2013.

1. Introduction

Since the 70's, the regularity of stochastic processes used to be considered in different ways. On one hand, the local regularity of sample paths is usually measured by local moduli of continuity and Hölder exponents (*e.g.* [11,17,29,40]). And on the other hand, the global regularity can be quantified by the global Hölder exponent (*e.g.* [38,39]) or by fractal dimensions (Hausdorff dimension, box-counting dimension, packing dimension,...) and respective measures of the graph of the processes (*e.g.* [9,31,33]).

As an example, if $B^H = \{B_t^H;\ t \in \mathbf{R}_+\}$ is a real-valued fractional Brownian motion (fBm) with self-similarity index $H \in (0,1)$, the pointwise Hölder exponent at any point $t \in \mathbf{R}_+$ satisfy $\boldsymbol{\alpha}_{B^H}(t) = H$ almost surely. Besides, the Hausdorff dimension of the graph of B^H is given by $\dim_{\mathcal{H}}(\mathrm{Gr}_{B^H}) = 2 - H$ almost surely. In this specific case, we observe a connection between the global and local points of view of regularity for fBm. Is it possible to obtain some general result, for some larger class of processes?

In [1], Adler showed that the Hausdorff dimension of the graph of a $\mathbf{R}^d$-valued Gaussian field $X = \{X_t^{(i)};\ 1 \leq i \leq p,\ t \in \mathbf{R}_+^N\}$, made of i.i.d. Gaussian coordinate processes $X^{(i)}$ with stationary increments, can be deduced

Keywords and phrases. Gaussian processes, Hausdorff dimension, (multi)fractional Brownian motion, multiparameter processes, Hölder regularity, stationarity.

[1] Ecole Centrale Paris, Grande Voie des Vignes, 92295 Chatenay-Malabry, France.
erick.herbin@ecp.fr; benjamin.arras@ecp.fr; geoffroy.barruel@student.ecp.fr

Article published by EDP Sciences

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https://doi.org/10.1051/ps/2013044 Published online by Cambridge University Press

from the local behavior of its incremental variance. More precisely, when the quantities $\sigma^2(t) = \mathbf{E}[|X^{(i)}_{t+t_0} - X^{(i)}_{t_0}|^2]$ independent of $1 \le i \le p$ and $t_0 \in \mathbf{R}^N_+$ satisfy

$$\forall \epsilon > 0, \quad |t|^{\alpha+\epsilon} \le \sigma(t) \le |t|^{\alpha-\epsilon} \quad \text{as } t \to 0, \tag{1.1}$$

the Hausdorff dimension of the graph $\mathrm{Gr}_X = \{(t, X_t) : t \in \mathbf{R}^N_+\}$ of X is proved to be

$$\dim_{\mathcal{H}}(\mathrm{Gr}_X) = \min\left\{\frac{N}{\alpha}, N + d(1-\alpha)\right\}.$$

This result followed Yoder's previous works in [41] where the Hausdorff dimensions of the graph and also the range $\mathrm{Rg}_X = \{X_t : t \in \mathbf{R}^N_+\}$ were obtained for a multiparameter Brownian motion in $\mathbf{R}^d$. As an application to Adler's result, the Hausdorff dimension of the graph of fractional Brownian motion can be deduced from the local Hölder exponents of its sample paths. As an extension of this result, Xiao has completely determined in [37] the Hausdorff dimensions of the image $X(K)$ and the graph $\mathrm{Gr}_X(K)$ of a Gaussian field X whose components may have different distributions, for a compact set $K \subset \mathbf{R}^N_+$, in function of $\dim_{\mathcal{H}} K$.

In this paper, we aim at extending Adler's result to Gaussian random fields with non-stationary increments. We will see that this goal requires a localization of Adler's index α along the sample paths. There is a large litterature about local regularity of Gaussian processes. We refer to [2, 20, 23, 26] for a contemporary and detailled review of it. This field of research is still very active, especially in the multiparameter context, and a non-exhaustive list of authors and recent works in this area includes Ayache and Lévy Véhel [3], Ayache, Shieh and Xiao [4], Baraka, Mountford and Xiao [6], Dozzi [10], Herbin and Lévy Véhel [17], Khoshnevisan and Xiao [21], Lawler and Viklund [22], Lind [24] and Meerschaert, Wang and Xiao [27], Tudor and Xiao [35], Xiao [37–39].

Usually the local regularity of an $\mathbf{R}^d$-valued stochastic process X at $t_0 \in \mathbf{R}^N_+$ is measured by the pointwise and local Hölder exponents $\boldsymbol{\alpha}_X(t_0)$ and $\widetilde{\boldsymbol{\alpha}}_X(t_0)$ defined by

$$\begin{aligned}
\boldsymbol{\alpha}_X(t_0) &= \sup\left\{\alpha > 0 : \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{\|X_t - X_s\|}{\rho^\alpha} < +\infty\right\},\\
\widetilde{\boldsymbol{\alpha}}_X(t_0) &= \sup\left\{\alpha > 0 : \lim_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{\|X_t - X_s\|}{\|t-s\|^\alpha} < +\infty\right\},
\end{aligned} \tag{1.2}$$

when the sample path of X is continuous but not differentiable at t_0 (notice that the previous expressions for the pointwise and local Hölder exponents are only valid when they belong to $[0, 1]$).

A general connection between the local structure of a stochastic process and the Hausdorff dimension of its graph has already been studied. In [7], the specific case of local self-similarity (LASS) property has been considered. Here, we do not want to consider any strong distributional condition, such as stationarity or LASS property, and we show how the local Hölder regularity of a Gaussian random field allows to estimate the Hausdorff dimensions of its range Rg_X and its graph Gr_X. However, the two works could be connected by the study of the local regularity of locally self-similar processes. A first element is proved in [14]: any H-index locally self-similar process has a pointwise Hölder exponent equal to H.

Recently in [17], the quantities $\mathbf{E}[|X_t - X_s|^2]$ when s, t are close to $t_0 \in \mathbf{R}^N_+$ are proved to capture a lot of information about the almost sure local regularity of Gaussian processes. More precisely, the almost sure 2-microlocal frontier of X at t_0 can be determined and, as an extension of the deterministic frame, allows to predict the evolution of the local regularity at t_0 under fractional integrations or derivations. Particularly, as special points of the 2-microlocal frontier, both pointwise and local Hölder exponents can be derived from the study of $\mathbf{E}[|X_t - X_s|^2]$. For all $t_0 \in \mathbf{R}^N_+$, we define in Section 2.1 the exponents $\underline{\mathfrak{a}}_X(t_0)$ and $\widetilde{\mathfrak{a}}_X(t_0)$ of a real-valued Gaussian process X as the minimum of $\underline{\mathfrak{a}} > 0$ and maximum of $\widetilde{\mathfrak{a}} > 0$ such that

$$\forall s, t \in B(t_0, \rho_0), \quad \|t-s\|^{2\,\underline{\mathfrak{a}}} \le \mathbf{E}[|X_t - X_s|^2] \le \|t-s\|^{2\,\widetilde{\mathfrak{a}}},$$

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for some $\rho_0 > 0$. The exponents of the components $X^{(i)}$ of a Gaussian random field $X = (X^{(1)}, \dots, X^{(d)})$ allow to get almost sure lower and upper bounds for quantities,

$$\lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0, \rho))) \quad \text{and} \quad \lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0, \rho))).$$

After the statement of the main result in Section 2.2, the almost sure local Hausdorff dimensions are given uniformly in $t_0 \in \mathbf{R}_+^N$ and the global dimensions $\dim_{\mathcal{H}}(\mathrm{Gr}_X(I))$ and $\dim_{\mathcal{H}}(\mathrm{Rg}_X(I))$ are almost surely bounded for any open interval $I \subset \mathbf{R}_+^N$, in function of $\inf_{t \in I} \underline{\alpha}_{X^{(i)}}(t)$ and $\inf_{t \in I} \widetilde{\alpha}_{X^{(i)}}(t)$. Sections 2.3 and 2.4 are devoted to the proofs of the upper bound and lower bound of the Hausdorff dimensions respectively.

In Section 3, the main result is applied to some stochastic processes whose increments are not stationary and whose Hausdorff dimension is still unknown.

The first one is the multiparameter extension of fractional Brownian motion (MpfBm), derived from the set-indexed fractional Brownian motion introduced in [15, 16]. On the contrary to fractional Brownian sheet studied in [5, 36], the MpfBm does not satisfy the increment stationarity property. Then the study of the local regularity of its sample path allows to determine the Hausdorff dimension of its graph in Section 3.1.

The second application is multifractional Brownian motion (mBm), introduced in [8, 30] as an extension of classical fractional Brownian motion where the self-similarity index $H \in (0, 1)$ is substituted with a function $H : \mathbf{R}_+ \to (0, 1)$ in order to allow the local regularity to vary along the sample path. The immediate consequence is the loss of the increment stationarity property. Then, the knowledge of local Hölder regularity implies the Hausdorff dimensions of the graph and the range of the mBm. In the case of a regular function H, the almost sure value of $\lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0, \rho)))$ was already known to be $2 - H(t_0)$ for any fixed $t_0 \in \mathbf{R}_+$. In Section 3.2, this almost sure result is proved uniformly in t_0. The new case of an irregular function H is also considered. These kind of quantities have already been studied by Meerschaert *et al.* [28] for the level sets of mBm when the function H is regular.

The last application of this article concerns the generalized Weierstrass function, defined as a stochastic version of the well-known Weierstrass function, where the index varies along the trajectory. The local Hölder regularity is determined in Section 3.3 and consequentely, the Hausdorff dimension of its sample path.

2. Hausdorff dimension of the sample paths of Gaussian random fields

In this paper, we denote by *multiparameter Gaussian random field* in $\mathbf{R}^d$, a stochastic process $X = \{X_t;\ t \in \mathbf{R}_+^N\}$, where $X_t = (X_t^{(1)}, \dots, X_t^{(d)}) \in \mathbf{R}^d$ for all $t \in \mathbf{R}_+^N$ and the coordinate processes $X^{(i)} = \{X_t^{(i)};\ t \in \mathbf{R}_+^N\}$ are independent real-valued Gaussian processes (not necessarily zero-mean) with the same law.

2.1. A new local exponent

According to [17], the local regularity of a Gaussian process $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ can be obtained by the *deterministic local Hölder exponent*

$$\widetilde{\alpha}_X(t_0) = \sup \left\{ \alpha > 0 : \lim_{\rho \to 0} \sup_{s,t \in B(t_0, \rho)} \frac{\mathbf{E}[|X_t - X_s|^2]}{\|t - s\|^{2\alpha}} < +\infty \right\}. \tag{2.1}$$

More precisely, the local Hölder exponent of X at any $t_0 \in \mathbf{R}_+^N$ is proved to satisfy $\mathbf{P}\big(\widetilde{\boldsymbol{\alpha}}_X(t_0) = \widetilde{\alpha}_X(t_0)\big) = 1$.

In order to get a localized version of (1.1), we need to introduce a new exponent $\underline{\alpha}_X(t_0)$, the *deterministic local sub-exponent* at any $t_0 \in \mathbf{R}_+^N$, defined for any 2nd-order stochastic process X,

$$\begin{aligned} \underline{\alpha}_X(t_0) &= \inf \left\{ \alpha > 0 : \lim_{\rho \to 0} \inf_{s,t \in B(t_0, \rho)} \frac{\mathbf{E}[|X_t - X_s|^2]}{\|t - s\|^{2\alpha}} = +\infty \right\} \\ &= \sup \left\{ \alpha > 0 : \lim_{\rho \to 0} \inf_{s,t \in B(t_0, \rho)} \frac{\mathbf{E}[|X_t - X_s|^2]}{\|t - s\|^{2\alpha}} = 0 \right\}. \end{aligned} \tag{2.2}$$

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Lemma 2.1. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multiparameter 2nd-order stochastic process.*

Consider $\widetilde{\mathfrak{a}}_X(t_0)$ and $\underline{\mathfrak{a}}_X(t_0)$ the deterministic local Hölder exponent and local sub-exponent of X at $t_0 \in \mathbf{R}_+^N$ (as defined in (2.1) and (2.2)).

For any $\epsilon > 0$, there exists $\rho_0 > 0$ such that

$$\forall s,t \in B(t_0,\rho_0), \quad \|t-s\|^{2\,\underline{\mathfrak{a}}_X(t_0)+\epsilon} \leq \mathbf{E}[|X_t - X_s|^2] \leq \|t-s\|^{2\,\widetilde{\mathfrak{a}}_X(t_0)-\epsilon}.$$

The previous result comes directly from definitions of $\underline{\mathfrak{a}}_X(t_0)$ and $\widetilde{\mathfrak{a}}_X(t_0)$. It allows to derive an ordering relation between the deterministic local sub-exponent and the deterministic local Hölder exponent. We have

$$\forall t_0 \in \mathbf{R}_+^N, \quad \widetilde{\mathfrak{a}}_X(t_0) \leq \underline{\mathfrak{a}}_X(t_0). \tag{2.3}$$

2.2. Main results: The Hausdorff dimension of Gaussian random fields

For sake of self-containess of the paper, we recall the basic frame of the Hausdorff dimension definition.

For all $\delta > 0$, we denote by δ-covering of a non-empty subset E of $\mathbf{R}^d$ all collection $A = (A_i)_{i\in\mathbf{N}}$ such that

- $\forall i \in \mathbf{N}, \operatorname{diam}(A_i) < \delta$, where $\operatorname{diam}(A_i)$ denotes $\sup(\|x-y\|;\ x,y \in A_i)$; and
- $E \subseteq \bigcup_{i\in\mathbf{N}} A_i$.

We denote by $\Sigma_\delta(E)$ the set of δ-covering of E and by $\Sigma(E)$ the set of the covering of E. We define

$$\mathcal{H}_\delta^s(E) = \inf_{A\in\Sigma_\delta(E)} \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s \right\},$$

and the Hausdorff measure of E by $\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(E)$. The Hausdorff dimension of E is defined by

$$\dim_{\mathcal{H}}(E) = \inf\{s \in \mathbf{R}_+ : \mathcal{H}^s(E) = 0\} = \sup\{s \in \mathbf{R}_+ : \mathcal{H}^s(E) = +\infty\}.$$

For any random field $X = \{X_t^{(i)};\ 1 \leq i \leq p,\ t \in \mathbf{R}_+^N\}$ made of i.i.d. Gaussian coordinate processes with possibly non-stationary increments, the Hausdorff dimensions of the range $\operatorname{Rg}_X(B(t_0,\rho)) = \{X_t;\ t \in B(t_0,\rho)\}$ and the graph $\operatorname{Gr}_X(B(t_0,\rho)) = \{(t,X_t);\ t \in B(t_0,\rho)\}$ of X in the ball $B(t_0,\rho)$ of center t_0 and radius $\rho > 0$ can be estimated when ρ goes to 0, using the deterministic local Hölder exponent and the deterministic local sub-exponent of $X^{(i)}$ at t_0.

In the following statements and in the sequel of the paper, the deterministic local Hölder exponent $\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)$ and the deterministic local sub-exponent $\underline{\mathfrak{a}}_{X^{(i)}}(t_0)$ of $X^{(i)}$ at any $t_0 \in \mathbf{R}_+^N$ are independent of $1 \leq i \leq d$, since the component $X^{(i)}$ are assumed to be i.i.d.

Theorem 2.2 (Pointwise almost sure result)**.** *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multi-parameter Gaussian random field in $\mathbf{R}^d$. Let $\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)$ be the deterministic local Hölder exponent and $\underline{\mathfrak{a}}_{X^{(i)}}(t_0)$ the deterministic local sub-exponent of $X^{(i)}$ at fixed $t_0 \in \mathbf{R}_+^N$ as defined in (2.1) and (2.2), independent of $1 \leq i \leq d$. Assume that $\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0) > 0$.*

Then, the Hausdorff dimensions of the graph and the range of X satisfy almost surely,

$$\min\left\{\frac{N}{\underline{\mathfrak{a}}_{X^{(i)}}(t_0)}; N + d(1-\underline{\mathfrak{a}}_{X^{(i)}}(t_0))\right\} \leq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\operatorname{Gr}_X(B(t_0,\rho))) \leq \min\left\{\frac{N}{\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)}; N + d(1-\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0))\right\}$$

and

$$\min\left\{\frac{N}{\underline{\mathfrak{a}}_{X^{(i)}}(t_0)}; d\right\} \leq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\operatorname{Rg}_X(B(t_0,\rho))) \leq \min\left\{\frac{N}{\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)}; d\right\}.$$

The proof of Theorem 2.2 relies on Propositions 2.6 and 2.8.

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Theorem 2.3 (Uniform almost sure result)**.** *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multi-parameter Gaussian random field in $\mathbf{R}^d$. Let $\widetilde{\mathfrak{a}}_{X^{(i)}}(t)$ be the deterministic local Hölder exponent and $\underline{\mathfrak{a}}_{X^{(i)}}(t)$ the deterministic local sub-exponent of $X^{(i)}$ at any $t \in \mathbf{R}_+^N$.*
Set $\mathcal{A} = \{t \in \mathbf{R}_+^N : \liminf_{u \to t} \widetilde{\mathfrak{a}}_{X^{(i)}}(u) > 0\}$.

Then, with probability one, for all $t_0 \in \mathcal{A}$,

$$\min\left\{\frac{N}{\liminf\limits_{u\to t_0}\underline{\mathfrak{a}}_{X^{(i)}}(u)}; N + d(1 - \liminf_{u\to t_0}\underline{\mathfrak{a}}_{X^{(i)}}(u))\right\} \le \lim_{\rho\to 0}\dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \le \min\left\{\frac{N}{\liminf\limits_{u\to t_0}\widetilde{\mathfrak{a}}_{X^{(i)}}(u)}; N + d(1 - \liminf_{u\to t_0}\widetilde{\mathfrak{a}}_{X^{(i)}}(u))\right\}.$$

and

$$\min\left\{\frac{N}{\liminf\limits_{u\to t_0}\underline{\mathfrak{a}}_{X^{(i)}}(u)}; d\right\} \le \lim_{\rho\to 0}\dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \le \min\left\{\frac{N}{\liminf\limits_{u\to t_0}\widetilde{\mathfrak{a}}_{X^{(i)}}(u)}; d\right\}.$$

The proof of Theorem 2.3 relies on Proposition 2.6 and Corollary 2.10.

Theorem 2.4 (Global almost sure result)**.** *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multiparameter Gaussian field in $\mathbf{R}^d$. Let $\widetilde{\mathfrak{a}}_{X^{(i)}}(t)$ be the deterministic local Hölder exponent and $\underline{\mathfrak{a}}_{X^{(i)}}(t)$ the deterministic local sub-exponent of $X^{(i)}$ at any $t \in \mathbf{R}_+^N$.*

For any open interval $I \subset \mathbf{R}_+^N$, assume that the quantities $\underline{\mathfrak{a}} = \inf_{t\in I}\underline{\mathfrak{a}}_{X^{(i)}}(t)$ and $\widetilde{\mathfrak{a}} = \inf_{t\in I}\widetilde{\mathfrak{a}}_{X^{(i)}}(t)$ satisfy $0 < \widetilde{\mathfrak{a}} \le \underline{\mathfrak{a}}$. Then, with probability one,

$$\min\{N/\underline{\mathfrak{a}}; N + d(1 - \underline{\mathfrak{a}})\} \le \dim_{\mathcal{H}}(\mathrm{Gr}_X(I)) \le \min\{N/\widetilde{\mathfrak{a}}; N + d(1 - \widetilde{\mathfrak{a}})\}$$

and

$$\min\{N/\underline{\mathfrak{a}}; d\} \le \dim_{\mathcal{H}}(\mathrm{Rg}_X(I)) \le \min\{N/\widetilde{\mathfrak{a}}; d\}.$$

The proof of Theorem 2.4 relies on Corollary 2.7 and Corollary 2.9.

2.3. Upper bound for the Hausdorff dimension

Lemma 2.5. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multiparameter random process with values in $\mathbf{R}^d$. Let $\widetilde{\boldsymbol{\alpha}}_X(t_0)$ be the local Hölder exponent of X at $t_0 \in \mathbf{R}_+^N$.*

For any ω such that $\widetilde{\boldsymbol{\alpha}}_X(t_0) > 0$,

$$\lim_{\rho\to 0}\dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \le \lim_{\rho\to 0}\dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \le \min\left\{\frac{N}{\widetilde{\boldsymbol{\alpha}}_X(t_0)}; N + d(1 - \widetilde{\boldsymbol{\alpha}}_X(t_0))\right\}.$$

Proof. The first inequality follows the fact that the range $\mathrm{Rg}_X(B(t_0,\rho))$ is a projection of the graph $\mathrm{Gr}_X(B(t_0,\rho))$. For the second inequality, we need to localize the argument of Yoder ([41]), who proved the upper bound for the Hausdorff dimensions of the range and the graph of a Hölderian function from $\mathbf{R}^N$ (or $[0,1]^N$) to $\mathbf{R}^d$ (see also [12], Cor. 11.2 p. 161).

Assume that ω is fixed such that $\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega) > 0$. By definition of $\widetilde{\boldsymbol{\alpha}}_X(t_0)$, for all $\epsilon > 0$ there exists $\rho_0 > 0$ such that for all $\rho \in (0,\rho_0]$,

$$\forall s,t \in B(t_0,\rho), \quad \|X_t(\omega) - X_s(\omega)\| \le \|t - s\|^{\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega)-\epsilon}.$$

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There exists a real $0 < \delta_0 < 1$ such that for all $u \in [0,1]^N$, $t_0 + \delta_0.u \in B(t_0, \rho_0)$ and consequently,

$$\forall u, v \in [0,1]^N, \quad \|X_{t_0+\delta_0.u}(\omega) - X_{t_0+\delta_0.v}(\omega)\| \leq (\delta_0 \ \|u-v\|)^{\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega)-\epsilon}.$$

Then, the function $Y_\bullet(\omega) : u \mapsto Y_u(\omega) = X_{t_0+\rho_0.u}(\omega)$ is Hölder-continuous of order $\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega) - \epsilon$ on $[0,1]^N$ and therefore, according to [41],

$$\dim_{\mathcal{H}}(\mathrm{Rg}_{Y_\bullet(\omega)}([0,1]^N)) \leq \dim_{\mathcal{H}}(\mathrm{Gr}_{Y_\bullet(\omega)}([0,1]^N)) \leq \min\left\{\frac{N}{\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega)-\epsilon}; N + d(1-\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega)+\epsilon)\right\}.$$

We can observe that the graph $\mathrm{Gr}_{X_\bullet(\omega)}(t_0+\delta_0.[0,1]^N))$ is an affine transformation of the graph $\mathrm{Gr}_{Y_\bullet(\omega)}([0,1]^N))$, therefore their Hausdorff dimensions are equal. Moreover, there exists $\rho > 0$ such that $B(t_0,\rho) \subset t_0 + \delta_0.[0,1]^N$. By monotony of the function $\rho \mapsto \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(B(t_0,\rho)))$, we can write

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(B(t_0,\rho))) \leq \min\left\{\frac{N}{\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega)-\epsilon}; N + d(1-\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega)+\epsilon)\right\}.$$

Since this inequality stands for all $\epsilon > 0$, we get

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(B(t_0,\rho))) \leq \min\left\{\frac{N}{\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega)}; N + d(1-\widetilde{\boldsymbol{\alpha}}_X(t_0,\omega))\right\}. \qquad \square$$

Lemma 2.5 gives a random upper bound for the Hausdorff dimensions of the (localized) range and graph of the sample path, in function of its local Hölder exponents. When X is a multiparameter Gaussian field in $\mathbf{R}^d$, we prove that this upper bound can be expressed almost surely with the deterministic local Hölder exponent of the Gaussian component processes $X^{(i)}$.

Proposition 2.6. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multiparameter Gaussian field in $\mathbf{R}^d$. Let $\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)$ be the deterministic local Hölder exponent of $X^{(i)}$ at $t_0 \in \mathbf{R}_+^N$ and assume that $\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0) > 0$.*

Then, almost surely

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \leq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \leq \min\left\{N/\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0); N + d(1-\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0))\right\}.$$

Moreover, an uniform result can be stated on the set

$$\mathcal{A} = \{t_0 \in \mathbf{R}_+^N : \liminf_{u\to t_0} \widetilde{\mathfrak{a}}_{X^{(i)}}(u) > 0\}.$$

With probability one, for all $t_0 \in \mathcal{A}$,

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \leq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \leq \min\left\{N/\liminf_{u\to t_0} \widetilde{\mathfrak{a}}_{X^{(i)}}(u); N + d(1-\liminf_{u\to t_0} \widetilde{\mathfrak{a}}_{X^{(i)}}(u))\right\}.$$

Proof. In [17], the local Hölder exponent of any Gaussian process Y at $t_0 \in \mathbf{R}_+^N$ such that $\widetilde{\mathfrak{a}}_Y(t_0) > 0$ is proved to satisfy $\widetilde{\boldsymbol{\alpha}}_Y(t_0) = \widetilde{\mathfrak{a}}_Y(t_0)$ almost surely. Therefore, by definition of $\widetilde{\boldsymbol{\alpha}}_{X^{(i)}}(t_0)$, for all $\epsilon > 0$ there exists $\rho_0 > 0$ such that for all $\rho \in (0,\rho_0]$, we have almost surely

$$\forall s,t \in B(t_0,\rho), \quad |X_t^{(i)} - X_s^{(i)}| \leq \|t-s\|^{\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)-\epsilon},$$

and consequently, almost surely

$$\forall s,t \in B(t_0,\rho), \quad \|X_t - X_s\| \leq K\ \|t-s\|^{\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)-\epsilon}, \tag{2.4}$$

for some constant $K > 0$.

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From (2.4), we deduce that $\widetilde{\boldsymbol{\alpha}}_X(t_0) \geq \widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)$ almost surely. Then Lemma 2.5 implies almost surely

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \leq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \leq \min\left\{N/\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0); N + d(1-\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0))\right\}.$$

For the uniform result on $t_0 \in \mathbf{R}_+^N$, we use the Theorem 3.14 of [17] which states that if Y is a Gaussian process such that the function $t_0 \mapsto \liminf_{u\to t_0} \widetilde{\mathfrak{a}}_Y(u)$ is positive, then with probability one,

$$\forall t_0 \in \mathbf{R}_+^N, \quad \liminf_{u\to t_0} \widetilde{\mathfrak{a}}_Y(u) \leq \widetilde{\boldsymbol{\alpha}}_Y(t_0) \leq \limsup_{u\to t_0} \widetilde{\mathfrak{a}}_Y(u).$$

This inequality yields to the existence of $\Omega_i \in \mathcal{F}$ for all $1 \leq i \leq d$ with $\mathbf{P}(\Omega_i) = 1$ and:
For all $\omega \in \Omega_i$, all $t_0 \in \mathcal{A}$ and all $\epsilon > 0$, there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$,

$$\forall s,t \in B(t_0,\rho), \quad |X_t^{(i)}(\omega) - X_s^{(i)}(\omega)| \leq \|t-s\|^{\liminf_{u\to t_0} \widetilde{\mathfrak{a}}_{X^{(i)}}(u) - \epsilon}.$$

This yields to: for all $\omega \in \bigcap_{1\leq i\leq d} \Omega_i$, all $t_0 \in \mathcal{A}$ and all $\epsilon > 0$, there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$,

$$\forall s,t \in B(t_0,\rho), \quad \|X_t(\omega) - X_s(\omega)\| \leq K\ \|t-s\|^{\liminf_{u\to t_0} \widetilde{\mathfrak{a}}_{X^{(i)}}(u) - \epsilon},$$

for some constant $K > 0$.
With the argument of Lemma 2.5, the result follows. □

Corollary 2.7. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multiparameter Gaussian field in $\mathbf{R}^d$ and $\widetilde{\mathfrak{a}}_{X^{(i)}}(t_0)$ the deterministic local Hölder exponent of $X^{(i)}$ at $t_0 \in \mathbf{R}_+^N$.*
Assume that for some bounded interval $I \subset \mathbf{R}_+^N$, we have $\alpha = \inf_{t_0\in I} \widetilde{\mathfrak{a}}_{X^{(i)}}(t_0) > 0$. Then, with probability one,

$$\dim_{\mathcal{H}}(\mathrm{Rg}_X(I)) \leq \dim_{\mathcal{H}}(\mathrm{Gr}_X(I)) \leq \min\left\{N/\alpha; N + d(1-\alpha)\right\}.$$

Proof. With the same arguments as in the proof of Proposition 2.6, we can claim that, with probability one, $\forall t_0 \in I,\ \alpha \leq \widetilde{\boldsymbol{\alpha}}_X(t_0)$. Then, there exists $\Omega_0 \in \mathcal{F}$ with $\mathbf{P}(\Omega_0) = 1$ and: For all $\omega \in \Omega_0$, all $t_0 \in I$ and all $\epsilon > 0$, there exist $\rho_0 > 0$ and $K > 0$ such that $\forall \rho \in (0, \rho_0]$,

$$\forall s,t \in B(t_0,\rho), \quad \|X_t(\omega) - X_s(\omega)\| \leq K\ \|t-s\|^{\alpha-\epsilon}.$$

Then the continuity of $t \mapsto X_t(\omega)$ on the bounded interval I allows to deduce that, for all $\omega \in \Omega_0$ and all $\epsilon > 0$, there exists a constant $K' > 0$ such that

$$\forall s,t \in I, \quad \|X_t(\omega) - X_s(\omega)\| \leq K'\ \|t-s\|^{\alpha-\epsilon}. \tag{2.5}$$

If the interval I is compact, we can exhibit an affine one-to-one mapping $I \to [0,1]^N$ and conclude with the arguments of Lemma 2.5 that [41] implies

$$\dim_{\mathcal{H}}(\mathrm{Rg}_{X_\bullet(\omega)}(I)) \leq \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(I)) \leq \min\left\{\frac{N}{\alpha-\epsilon}; N + d(1-\alpha+\epsilon)\right\} \qquad \text{a.s.}$$

Since this inequality stands for any $\epsilon > 0$, the result follows in that case.
If I is not closed, we remark that

$$\dim_{\mathcal{H}}(\mathrm{Rg}_{X_\bullet(\omega)}(I)) \leq \dim_{\mathcal{H}}(\mathrm{Rg}_{X_\bullet(\omega)}(\overline{I})) \quad \text{and} \quad \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(I)) \leq \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(\overline{I})).$$

Then, extending the inequality (2.5) to $\overline{I}$ by continuity, the result for the compact interval $\overline{I}$ is proved as previously. □

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2.4. Lower bound for the Hausdorff dimension

Frostman's Theorem constitutes the key argument to prove the lower bound for the Hausdorff dimensions. We recall the basic notions of potential theory, which are used along the proofs of this section. For any Borel set $E \subseteq \mathbf{R}^d$, the β-dimensional energy of a probability measure μ on E is defined by

$$I_\beta(\mu) = \int_{E\times E} \|x-y\|^{-\beta}\ \mu(\mathrm{d}x)\ \mu(\mathrm{d}y).$$

Then, the β-dimensional Bessel–Riesz capacity of E is defined as

$$C_\beta(E) = \sup\left(\frac{1}{I_\beta(\mu)};\ \mu \text{ probability measure on } E\right).$$

According to Frostman's Theorem, the Hausdorff dimension of E is obtained from the capacity of E by the expression

$$\dim_{\mathcal{H}} E = \sup\left(\beta : C_\beta(E) > 0\right) = \inf\left(\beta : C_\beta(E) = 0\right).$$

Consequently, if $I_\beta(\mu) < +\infty$ for some probability measure (or some mass distribution) μ on E, then $\dim_{\mathcal{H}} E \geq \beta$.

Proposition 2.8. *Let $X = \{X_t;\ t \in \mathbf{R}^N_+\}$ be a multiparameter Gaussian field in $\mathbf{R}^d$ and $\underline{\mathbb{o}}_{X^{(i)}}(t_0)$ the deterministic local sub-exponent of $X^{(i)}$ at $t_0 \in \mathbf{R}^N_+$.*

Then, almost surely

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \geq \begin{cases} N/\underline{\mathbb{o}}_{X^{(i)}}(t_0) & \text{if } N \leq d\ \underline{\mathbb{o}}_{X^{(i)}}(t_0); \\ N + d(1-\underline{\mathbb{o}}_{X^{(i)}}(t_0)) & \text{if } N > d\ \underline{\mathbb{o}}_{X^{(i)}}(t_0); \end{cases}$$

and

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \geq \begin{cases} N/\underline{\mathbb{o}}_{X^{(i)}}(t_0) & \text{if } N \leq d\ \underline{\mathbb{o}}_{X^{(i)}}(t_0); \\ d & \text{if } N > d\ \underline{\mathbb{o}}_{X^{(i)}}(t_0). \end{cases}$$

Proof. Following Adler's proof for the lower bound in the case of processes with stationary increments, we distinguish the two cases: $N \leq d\ \underline{\mathbb{o}}_{X^{(i)}}(t_0)$ and $N > d\ \underline{\mathbb{o}}_{X^{(i)}}(t_0)$.

- Assume that $N \leq d\ \underline{\mathbb{o}}_{X^{(i)}}(t_0)$. In that case, we prove that almost surely,

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho)) \geq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho)) \geq \frac{N}{\underline{\mathbb{o}}_{X^{(i)}}(t_0)}. \tag{2.6}$$

For any $\epsilon > 0$, we consider any $\beta < N/(\underline{\mathbb{o}}_{X^{(i)}}(t_0)+\epsilon) \leq d$ and we aim at showing that the β-dimensional capacity $C_\beta(\mathrm{Rg}_X(B(t_0,\rho)))$ is positive almost surely for all $\rho > 0$.
With this intention, for $E = \mathrm{Rg}_X(B(t_0,\rho)) = X(B(t_0,\rho))$, we consider the β-dimensional energy $I_\beta(\mu)$ of the mass distribution $\mu = \lambda|_{B(t_0,\rho)} \circ X^{-1}$ of E, where $\lambda|_{B(t_0,\rho)}$ denotes the restriction of the Lebesgue measure to $B(t_0,\rho)$. As mentioned above (see also Thm. B in [34]), a sufficient condition for the capacity to be positive is that, almost surely

$$\int_{E\times E} \|x-y\|^{-\beta}\ \mu(\mathrm{d}x)\ \mu(\mathrm{d}y) = \int_{B(t_0,\rho)\times B(t_0,\rho)} \|X_t - X_s\|^{-\beta}\ ds\ \mathrm{d}t < +\infty. \tag{2.7}$$

Since the $X^{(i)}$ are independent and have the same distribution, we compute for all $s,t \in \mathbf{R}^N_+$,

$$\mathbf{E}\left[\|X_t - X_s\|^{-\beta}\right] = \frac{1}{[2\pi\sigma^2(s,t)]^{d/2}} \int_{\mathbf{R}^d} \|x\|^{-\beta} \exp\left(-\frac{\|x\|^2}{2\ \sigma^2(s,t)}\right)\ \mathrm{d}x,$$

where $\sigma^2(s,t) = \mathbf{E}[|X^{(i)}_t - X^{(i)}_s|^2]$ is independent of $1 \leq i \leq d$.

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Let us consider the change of variables $(\mathbf{R}_+ \setminus \{0\}, \mathbf{S}^{d-1}) \to \mathbf{R}^d \setminus \{0\}$ defined by $(r,u) \mapsto r.u$, where $\mathbf{S}^{d-1}$ denotes the unit hypersphere of $\mathbf{R}^d$. The previous expression becomes

$$\begin{aligned}\mathbf{E}\left[\|X_t - X_s\|^{-\beta}\right] &= \frac{K_1}{[2\pi\sigma^2(s,t)]^{d/2}} \int_{\mathbf{R}_+} r^{d-1-\beta} \exp\left(-\frac{r^2}{2\ \sigma^2(s,t)}\right)\ \mathrm{d}r \\ &= K_1\ (\sigma(s,t))^{-\beta} \int_{\mathbf{R}_+} z^{d-1-\beta} \exp\left(-\frac{1}{2}z^2\right)\ \mathrm{d}z,\end{aligned}$$

where K_1 is a positive constant and using the change of variables $r = \sigma(s,t)\ z$.
Since the integral is finite when $\beta < d$, we get

$$\forall s,t \in \mathbf{R}_+^N, \quad \mathbf{E}\left[\|X_t - X_s\|^{-\beta}\right] \le K_2\ (\sigma(s,t))^{-\beta}, \tag{2.8}$$

for some positive constant K_2.
By Tonelli's theorem and Lemma 2.1, this inequality implies the existence of $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$,

$$\mathbf{E}\left[\int_{B(t_0,\rho)\times B(t_0,\rho)} \|X_t - X_s\|^{-\beta}\ \mathrm{d}t\ \mathrm{d}s\right] \le \int_{B(t_0,\rho)\times B(t_0,\rho)} K_2\ \|t-s\|^{-\beta(\underline{\mathfrak{a}}_{X^{(i)}}(t_0)+\epsilon)}\ \mathrm{d}t\ \mathrm{d}s < +\infty$$

because $\beta(\underline{\mathfrak{a}}_{X^{(i)}}(t_0) + \epsilon) < N$. Thus (2.7) holds and for all $\rho \in (0, \rho_0]$,

$$\dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \ge \frac{N}{\underline{\mathfrak{a}}_{X^{(i)}}(t_0) + \epsilon} \qquad \text{a.s.}$$

Taking $\rho, \epsilon \in \mathbf{Q}_+$, this yields to

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \ge \frac{N}{\underline{\mathfrak{a}}_{X^{(i)}}(t_0)} \qquad \text{a.s.},$$

which proves (2.6).

- Assume $N > d\ \underline{\mathfrak{a}}_{X^{(i)}}(t_0)$. We use the previous method to prove that almost surely

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \ge d. \tag{2.9}$$

For any $\epsilon > 0$ such that $d < N/(\underline{\mathfrak{a}}_{X^{(i)}}(t_0) + \epsilon)$, consider any real β such that $\beta < d$. As previously, we show that equation (2.7) is verified, which implies that the β-dimensional capacity $C_\beta(\mathrm{Rg}_X(B(t_0,\rho)))$ is positive almost surely for all $\rho > 0$.
Since $\beta < d$, equation (2.8) still holds. As in the previous case, the inequality $\beta(\underline{\mathfrak{a}}_{X^{(i)}}(t_0)+\epsilon) < N$ implies (2.7) for ρ small enough and then

$$\dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \ge d \qquad \text{a.s.}$$

Taking $\rho \in \mathbf{Q}_+$, the inequality (2.9) follows.

- Assume $N > d\ \underline{\mathfrak{a}}_{X^{(i)}}(t_0)$. To prove the lower bound for the Hausdorff dimension of the graph,

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \ge N + d(1 - \underline{\mathfrak{a}}_{X^{(i)}}(t_0)) \qquad \text{a.s.}, \tag{2.10}$$

we use the same arguments of potential theory than for the range.
For any $\epsilon > 0$, consider any real β such that $d < \beta < N + d(1 - \underline{\mathfrak{a}}_{X^{(i)}}(t_0) - \epsilon)$. In order to prove that the β-dimensional capacity $C_\beta(\mathrm{Gr}_X(B(t_0,\rho)))$ is positive almost surely for all $\rho > 0$, it is sufficient to show that

$$\int_{B(t_0,\rho)\times B(t_0,\rho)} \|(t,X_t) - (s,X_s)\|^{-\beta}\ \mathrm{d}s\ \mathrm{d}t < +\infty \qquad \text{a.s.} \tag{2.11}$$

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A same kind of computation as previously (see also [19] pp. 278–279) allows to obtain

$$\mathbf{E}\left[(\|X_t - X_s\|^2 + \|t-s\|^2)^{-\beta/2}\right] \leq K_4 \, \frac{\|t-s\|^{d-\beta}}{\sigma(s,t)^d}.$$

By Tonelli's Theorem and Lemma 2.1, this inequality implies the existence of $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$,

$$\mathbf{E}\left[\int_{B(t_0,\rho)\times B(t_0,\rho)} \|(t,X_t)-(s,X_s)\|^{-\beta} \, \mathrm{d}t \, \mathrm{d}s\right] \leq \int_{B(t_0,\rho)\times B(t_0,\rho)} K_4 \, \frac{\|t-s\|^{d-\beta}}{\sigma(s,t)^d} \, \mathrm{d}s \, \mathrm{d}t$$
$$\leq \int_{B(t_0,\rho)\times B(t_0,\rho)} K_4 \, \|t-s\|^{-\beta+d(1-\underline{\mathbb{o}}_{X^{(i)}}(t_0)-\epsilon)} \, \mathrm{d}s \, \mathrm{d}t < +\infty,$$

because $\beta < N + d(1 - \underline{\mathbb{o}}_{X^{(i)}}(t_0) - \epsilon)$. Thus (2.11) holds and for all $\rho \in (0, \rho_0]$,

$$\dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho)) \geq N + d(1 - \underline{\mathbb{o}}_{X^{(i)}}(t_0) - \epsilon) \qquad \text{a.s.}$$

Taking $\rho, \epsilon \in \mathbf{Q}_+$, this yields to

$$\lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho)) \geq N + d(1 - \underline{\mathbb{o}}_{X^{(i)}}(t_0)) \qquad \text{a.s.},$$

which proves (2.10). □

We now investigate uniform extensions of Proposition 2.8.

Corollary 2.9. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multiparameter Gaussian field in $\mathbf{R}^d$ and $\underline{\mathbb{o}}_{X^{(i)}}(t)$ the deterministic local sub-exponent of $X^{(i)}$ at any $t \in \mathbf{R}_+^N$.*
Assume that for some open subset $I \subset \mathbf{R}_+^N$, we have $\underline{\alpha} = \inf_{t\in I} \underline{\mathbb{o}}_{X^{(i)}}(t) > 0$.

Then, with probability one,

$$\dim_{\mathcal{H}}(\mathrm{Gr}_X(I)) \geq \begin{cases} N/\underline{\alpha} & \text{if } N \leq d\ \underline{\alpha}; \\ N + d(1-\underline{\alpha}) & \text{if } N > d\ \underline{\alpha}; \end{cases}$$

and

$$\dim_{\mathcal{H}}(\mathrm{Rg}_X(I)) \geq \begin{cases} N/\underline{\alpha} & \text{if } N \leq d\ \underline{\alpha}; \\ d & \text{if } N > d\ \underline{\alpha}. \end{cases}$$

Proof. For any open subset $I \subset \mathbf{R}_+^N$, we first prove that for all ω, the Hausdorff dimension of the graph of $X_\bullet(\omega) : t \mapsto X_t(\omega)$ satisfies

$$\dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(I)) \geq \sup_{t_0 \in I} \lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(B(t_0,\rho))). \tag{2.12}$$

Since I is an open subset of $\mathbf{R}_+^N$, for all $t_0 \in I$, there exists $\rho > 0$ such that $B(t_0,\rho) \subset I$. This leads to $\dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(B(t_0,\rho))) \leq \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(I))$ and then

$$\dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(I)) \geq \lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(B(t_0,\rho))),$$

since $\rho \mapsto \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}(B(t_0,\rho)))$ is decreasing. Then (2.12) follows.

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In the same way, we prove that for all ω,

$$\dim_{\mathcal{H}}(\mathrm{Rg}_{X_\bullet(\omega)}(I)) \geq \sup_{t_0\in I} \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_{X_\bullet(\omega)}(B(t_0,\rho))). \tag{2.13}$$

Following the proof of Proposition 2.8, we distinguish the two cases: $N \leq d\ \underline{\alpha}$ and $N > d\ \underline{\alpha}$ with $\underline{\alpha} = \inf_{t\in I} \underline{\mathfrak{a}}_{X^{(i)}}(t)$. The result follows in the same way. □

Corollary 2.10. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multiparameter Gaussian field in $\mathbf{R}^d$ and $\underline{\mathfrak{a}}_{X^{(i)}}(t)$ the deterministic local sub-exponent of $X^{(i)}$ at any $t \in \mathbf{R}_+^N$.*
Set $\underline{\mathcal{A}} = \{t \in \mathbf{R}_+^N : \liminf_{u\to t} \underline{\mathfrak{a}}_{X^{(i)}}(u) > 0\}$.
Then, with probability one, for all $t_0 \in \underline{\mathcal{A}}$,

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \geq \begin{cases} N/\liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t) & \text{if } N \leq d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t); \\ N + d\left(1 - \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t)\right) & \text{if } N > d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t); \end{cases}$$

and

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) \geq \begin{cases} N/\liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t) & \text{if } N \leq d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t); \\ d & \text{if } N > d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t). \end{cases}$$

Proof. Corollary 2.9 implies the existence of $\Omega^* \in \mathcal{F}$ with $\mathbf{P}(\Omega^*) = 1$ such that: for all $\omega \in \Omega^*$ and all $a, b \in \mathbf{Q}_+^N$ with $a \prec b$, such that $\underline{\alpha} = \inf_{t\in(a,b)} \underline{\mathfrak{a}}_{X^{(i)}}(t) > 0$, we have $\dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}((a,b))) \geq N/\underline{\alpha}$ if $N \leq d\ \underline{\alpha}$ and $\geq N + d(1-\underline{\alpha})$ if $N > d\ \underline{\alpha}$ and $\dim_{\mathcal{H}}(\mathrm{Rg}_{X_\bullet(\omega)}((a,b))) \geq N/\underline{\alpha}$ if $N \leq d\ \underline{\alpha}$ and $\geq d$ if $N > d\ \underline{\alpha}$.

Therefore, taking two sequences $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ such that $\forall n \in \mathbf{N}$, $a_n < t_0 < b_n$ and converging to t_0, we get

$$\lim_{n\to\infty} \dim_{\mathcal{H}}(\mathrm{Gr}_{X_\bullet(\omega)}((a_n,b_n))) \geq \begin{cases} N/\liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t) & \text{if } N \leq d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t); \\ N + d(1 - \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t)) & \text{if } N > d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t); \end{cases}$$

and

$$\lim_{n\to\infty} \dim_{\mathcal{H}}(\mathrm{Rg}_{X_\bullet(\omega)}((a_n,b_n))) \geq \begin{cases} N/\liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t) & \text{if } N \leq d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t); \\ d & \text{if } N > d\ \liminf\limits_{t\to t_0} \underline{\mathfrak{a}}_{X^{(i)}}(t). \end{cases}$$

By monotony of the Hausdorff dimension, the result follows. □

3. Applications

In this section, we apply the main results to Gaussian processes whose fine regularity is not completely known: the set-indexed fractional Brownian motion on $\mathbf{R}_+^N$ and the multi-fractional Brownian motion with a regularity function lower than its own regularity. We end this section with the generalized Weierstrass function, which is not a Gaussian process.

3.1. Set-indexed fractional Brownian motion

A multiparameter extension of fractional Brownian motion (MpfBm) $\mathbf{B}^H = \{\mathbf{B}_t^H;\ t \in \mathbf{R}_+^N\}$ of index $H \in (0, 1/2]$ is defined as a particular case of *set-indexed fractional Brownian motion* (see [15,16]), where the indexing collection is $\mathcal{A} = \{[0,t];\ t \in \mathbf{R}_+^N\} \cup \{\emptyset\}$. It is characterized as a real-valued mean-zero Gaussian process with covariance function

$$\forall s,t \in \mathbf{R}_+^N, \quad \mathbf{E}\left[\mathbf{B}_s^H \mathbf{B}_t^H\right] = \frac{1}{2}\left[m([0,s])^{2H} + m([0,t])^{2H} - m([0,s] \triangle [0,t])^{2H}\right],$$

where m denotes a Radon measure in $\mathbf{R}_+^N$.

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In the specific case where $N = 2$ and m is the Lebesgue measure of $\mathbf{R}_+^2$, the covariance structure of the MpfBm is

$$\forall s,t \in \mathbf{R}_+^2, \quad \mathbf{E}\left[\mathbf{B}_s^H \mathbf{B}_t^H\right] = \frac{1}{2}\left[(s_1 s_2)^{2H} + (t_1 t_2)^{2H} - (s_1 s_2 + t_1 t_2 - 2(s_1 \wedge t_1)(s_2 \wedge t_2))^{2H}\right].$$

Then, its incremental variance is

$$\forall s,t \in \mathbf{R}_+^2, \quad \mathbf{E}\left[|\mathbf{B}_t^H - \mathbf{B}_s^H|^2\right] = (s_1 s_2 + t_1 t_2 - 2(s_1 \wedge t_1)(s_2 \wedge t_2))^{2H}. \tag{3.1}$$

Let us remark that the process $\mathbf{B}^H$ is different from the two other multiparameter extensions of fBm, which are Lévy fractional Brownian motion and fractional Brownian sheet.

The stationarity of the increments of the set-indexed fractional Brownian motion on $\mathbf{R}_+^N$ are studied in [16]. Among all the various definitions of the stationarity property for a multiparameter process, the MpfBm does not satisfy the increment stationarity assumption of [1]. Indeed, (3.1) shows that $\mathbf{E}\left[|\mathbf{B}_t^H - \mathbf{B}_s^H|^2\right]$ does not only depend on $t - s$. Since the Hausdorff dimension of its graph does not come directly from [1], we use the generic results of Section 2.2.

Lemma 3.1. *If m is the Lebesgue measure of $\mathbf{R}^N$, for any $a \prec b$ in $\mathbf{R}_+^N \setminus \{0\}$, there exists two positive constants $m_{a,b}$ and $M_{a,b}$ such that*

$$\forall s,t \in [a,b]; \quad m_{a,b}\, d_1(s,t) \leq m([0,s] \triangle [0,t]) \leq M_{a,b}\, d_\infty(s,t)$$

where d_1 and d_∞ are the usual distances of $\mathbf{R}^N$ defined by

$$d_1 : (s,t) \mapsto \|t-s\|_1 = \sum_{i=1}^N |t_i - s_i|$$
$$d_\infty : (s,t) \mapsto \|t-s\|_\infty = \max_{1 \leq i \leq N} |t_i - s_i|.$$

Proof. For all $s,t \in [a,b]$, we write

$$[0,s] \triangle [0,t] = ([0,s] \setminus [0,t]) \cup ([0,t] \setminus [0,s]).$$

Suppose that for all $i \in I \subset \{1,\dots,N\}$, $s_i > t_i$, and that for all $i \in \{1,\dots,N\} \setminus I$, $s_i \leq t_i$. For any subset J of $\{1,\dots,N\}$, we denote by $\prod_{i\in J}[0,s_i]$ the cartesian product of $[0,s_i]$ for $i \in J$.

We have

$$\begin{aligned}[0,s] &= \prod_{i \notin I}[0,s_i] \times \prod_{i \in I}([0,t_i] \cup [t_i,s_i]) \\ &= \left(\prod_{i\notin I}[0,s_i] \times \prod_{i\in I}[0,t_i]\right) \cup \bigcup_{J \subsetneq I}\left(\prod_{i\notin I}[0,s_i] \times \prod_{i\in J}[0,t_i] \times \prod_{i\in I\setminus J}[t_i,s_i]\right),\end{aligned}$$

and then

$$\begin{aligned}[0,s] \setminus [0,t] &= \bigcup_{J\subsetneq I}\left(\prod_{i\notin I}[0,s_i] \times \prod_{i\in J}[0,t_i] \times \prod_{i\in I\setminus J}[t_i,s_i]\right) \\ &= \{x \in [0,s] :\ \exists i \in I;\ t_i < x_i \leq s_i\}.\end{aligned}$$

We deduce

$$m([0,s] \setminus [0,t]) = \prod_{i\notin I}|s_i| \sum_{J\subsetneq I}\left(\prod_{i\in J}|t_i| \prod_{i\in I\setminus J}|t_i - s_i|\right).$$

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In the same way, we get

$$m([0,t]\setminus[0,s]) = \prod_{i\in I}|s_i| \sum_{J\subsetneq I^c}\left(\prod_{i\in J}|t_i| \prod_{i\in I^c\setminus J}|t_i - s_i|\right).$$

For all $1 \le i \le N$, we have $|a| \le |s_i| \le |b|$ and $|a| \le |t_i| \le |b|$. Then,

$$\begin{aligned} m([0,s]\triangle[0,t]) &\le |b|^{\#I^c}\sum_{J\subsetneq I}|b|^{\#J}d_\infty(s,t)^{\#(I\setminus J)} + |b|^{\#I}\sum_{J\subsetneq I^c}|b|^{\#J}d_\infty(s,t)^{\#(I^c\setminus J)} \\ &\le d_\infty(s,t)\ \underbrace{\left[|b|^{\#I^c}\sum_{J\subsetneq I}|b|^{\#J}d_\infty(s,t)^{\#(I\setminus J)-1} + |b|^{\#I}\sum_{J\subsetneq I^c}|b|^{\#J}d_\infty(s,t)^{\#(I^c\setminus J)-1}\right]}_{\text{bounded in } [a,b]} \\ &\le M_{a,b}\ d_\infty(s,t). \end{aligned}$$

For the lower bound, we write

$$m([0,s]\triangle[0,t]) \ge |a|^{\#I^c}\sum_{J\subsetneq I}|a|^{\#J}\prod_{i\in I\setminus J}|t_i - s_i| + |a|^{\#I}\sum_{J\subsetneq I^c}|a|^{\#J}\prod_{i\in I^c\setminus J}|t_i - s_i|$$

Let m_a be the minimum of $|a|^k$ for $1 \le k \le N$. We get

$$m([0,s]\triangle[0,t]) \ge m_a^2\sum_{J\subsetneq I}\prod_{i\in I\setminus J}|t_i - s_i| + m_a^2\sum_{J\subsetneq I^c}\prod_{i\in I^c\setminus J}|t_i - s_i|. \tag{3.2}$$

Let us remark that

$$\sum_{J\subsetneq I}\prod_{i\in I\setminus J}|t_i - s_i| = \prod_{i\in I}(1+|t_i - s_i|) - 1.$$

Using the expansion

$$\log\prod_{i\in I}(1+|t_i - s_i|) = \sum_{i\in I}\log(1+|t_i - s_i|) = \sum_{i\in I}|t_i - s_i| + o(|t_i - s_i|^2),$$

which implies

$$\prod_{i\in I}(1+|t_i - s_i|) = 1 + \sum_{i\in I}|t_i - s_i| + o(|t_i - s_i|^2),$$

the inequality (3.2) becomes

$$m([0,s]\triangle[0,t]) \ge m_a^2\sum_{1\le i\le N}|t_i - s_i| + o(\|t-s\|_\infty).$$

The result follows. □

Lemma 3.2. *Let* $\mathbf{B}^H = \{\mathbf{B}_t^H;\ t \in \mathbf{R}_+^N\}$ *be a set-indexed fractional Brownian motion on* $\mathbf{R}_+^N$ *with index* $H \in (0, 1/2]$. *The deterministic local Hölder exponent and deterministic local sub-exponent of* $\mathbf{B}^H$ *at any* $t_0 \in \mathbf{R}_+^N$ *are given by* $\widetilde{\mathfrak{a}}_X(t_0) = \underline{\mathfrak{a}}_X(t_0) = H$.

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Proof. We prove that $\widetilde{\mathfrak{a}}_X(t_0) \geq H$ and $\underline{\mathfrak{a}}_X(t_0) \leq H$. The result will follow from $\widetilde{\mathfrak{a}}_X(t_0) \leq \underline{\mathfrak{a}}_X(t_0)$.

Since for all $s, t \in \mathbf{R}_+^N$,

$$\frac{\mathbf{E}\left[|\mathbf{B}_t^H - \mathbf{B}_t^H|^2\right]}{\|t-s\|^{2H}} = \left(\frac{m([0,s] \triangle [0,t])}{d_2(s,t)}\right)^{2H},$$

Lemma 3.1 implies that for all s, t in any interval $[a,b]$,

$$M_1 \left(\frac{d_1(s,t)}{d_2(s,t)}\right)^{2H} \leq \frac{\mathbf{E}\left[|\mathbf{B}_t^H - \mathbf{B}_t^H|^2\right]}{\|t-s\|^{2H}} \leq M_2 \left(\frac{d_\infty(s,t)}{d_2(s,t)}\right)^{2H}, \tag{3.3}$$

for some positive constants M_1 and M_2.

Since the distances d_1, d_2 and d_∞ are equivalent, the inequality (3.3) implies that the quantity $\mathbf{E}\left[|\mathbf{B}_t^H - \mathbf{B}_t^H|^2\right] / \|t-s\|^{2H}$ is bounded on any interval $[a,b]$. Consequently, for all $t_0 \in \mathbf{R}_+^N$, $\widetilde{\mathfrak{a}}_X(t_0) \geq H$ and $\underline{\mathfrak{a}}_X(t_0) \leq H$, by definition of the deterministic local Hölder exponent and the deterministic local sub-exponent. □

A direct consequence from Lemma 3.2 is the local regularity of the sample paths of the set-indexed fractional Brownian motion on $\mathbf{R}_+^N$. In [17], Corollary 3.15 states that for any Gaussian process X such that the function $t \mapsto \widetilde{\mathfrak{a}}_X(t)$ is continuous and positive, the local Hölder exponents satisfy with probability one: $\widetilde{\boldsymbol{\alpha}}_X(t) = \widetilde{\mathfrak{a}}_X(t)$ for all $t \in \mathbf{R}_+^N$. Since the deterministic local Hölder exponents of the MpfBm are constant and positive, the following result comes directly.

Corollary 3.3. *The local Hölder exponent of the set-indexed fractional Brownian motion* $\mathbf{B}^H = \{\mathbf{B}_t^H;\ t \in \mathbf{R}_+^N\}$ *(with* $0 < H \leq 1/2$*) satisfies with probability one,* $\widetilde{\boldsymbol{\alpha}}_{\mathbf{B}^H}(t_0) = H$ *for all* $t_0 \in \mathbf{R}_+^N$.

As an application of Theorem 2.4, the property of constant local regularity of set-indexed fractional Brownian motion on $\mathbf{R}_+^N$ yields to sharp results about the Hausdorff dimensions of its graph and its range.

Proposition 3.4. *Let* $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ *be a multiparameter fractional Brownian field with index* $H \in (0, 1/2]$, i.e. *whose coordinate processes* $X^{(1)}, \ldots, X^{(d)}$ *are i.i.d. set-indexed fractional Brownian motions on* $\mathbf{R}_+^N$ *with index* H.

With probability one, the Hausdorff dimensions of the graph and the range of the sample paths of X *are*

$$\forall I = (a,b) \subset \mathbf{R}_+^N, \quad \begin{aligned} \dim_{\mathcal{H}}(\mathrm{Gr}_X(I)) &= \min\{N/H; N + d(1-H)\}, \\ \dim_{\mathcal{H}}(\mathrm{Rg}_X(I)) &= \min\{N/H; d\}. \end{aligned}$$

Corollary 3.5. *Let* $\mathbf{B}^H = \{\mathbf{B}_t^H;\ t \in \mathbf{R}_+^N\}$ *be a set-indexed fractional Brownian motion with index* $H \in (0, 1/2]$. *With probability one, the Hausdorff dimensions of the graph and the range of the sample paths of* $\mathbf{B}^H$ *are*

$$\forall I = (a,b) \subset \mathbf{R}_+^N, \quad \begin{aligned} \dim_{\mathcal{H}}(\mathrm{Gr}_{\mathbf{B}^H}(I)) &= N + 1 - H, \\ \dim_{\mathcal{H}}(\mathrm{Rg}_{\mathbf{B}^H}(I)) &= 1. \end{aligned}$$

Proposition 3.4 and Corollary 3.5 should be compared to Theorem 1.3 of [5] which states the Hausdorff dimensions of the range and the graph of the fractional Brownian sheet (result extended by Prop. 1 and Thm. 3 of [36]). In particular, the Hausdorff dimensions of the sample path (range and graph) of the multiparameter fractional Brownian motion are equal to the respective quantities for the fractional Brownian sheet, when the Hurst index is the same along each axis.

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3.2. Irregular multifractional Brownian motion

The multifractional Brownian motion (mBm) is an extension of the fractional Brownian motion, where the self-similarity index $H \in (0,1)$ is substituted with a function $H : \mathbf{R}_+ \to (0,1)$ (see [8,30]). More precisely, it can be defined as a zero mean Gaussian process $\{X_t^{(i)};\ t \in \mathbf{R}_+\}$ $(i = 1,2)$ with

$$X_t^{(1)} = \int_{\mathbf{R}} \frac{\mathrm{e}^{\mathrm{i}t\xi} - 1}{|\xi|^{H(t)+1/2}} . \widehat{\mathbb{W}}(\mathrm{d}u), \tag{3.4}$$

or

$$X_t^{(2)} = \int_{-\infty}^{t} \left[(t-u)^{H(t)-1/2} - (-u)^{H(t)-1/2} \right] . \mathbb{W}(\mathrm{d}u) + \int_0^t (t-u)^{H(t)-1/2} . \mathbb{W}(\mathrm{d}u),$$

where $\mathbb{W}$ is a Gaussian measure in $\mathbf{R}$ and $\widehat{\mathbb{W}}$ is the Fourier transform of a Gaussian measure in $\mathbf{C}$. The variety of the class of multifractional Brownian motions is described in [32].

In the first definitions of the mBm, the different groups of authors used to consider the assumption: H is a β-Hölder function and $H(t) < \beta$ for all $t \in \mathbf{R}_+$. Under this so-called (H_β)-assumption, the local regularity of the sample paths was described by

$$\boldsymbol{\alpha}_X(t_0) = \widetilde{\boldsymbol{\alpha}}_X(t_0) = H(t_0) \qquad \text{a.s.}$$

where $\boldsymbol{\alpha}_X(t_0)$ and $\widetilde{\boldsymbol{\alpha}}_X(t_0)$ denote the pointwise and local Hölder exponents of X at any $t_0 \in \mathbf{R}_+$. A localization of the Hausdorff dimension of the graph were also proved: for any $t_0 \in \mathbf{R}_+$,

$$\lim_{\rho \to 0} \dim_{\mathcal{H}} \left[\mathrm{Gr}_X \left(B(t_0, \rho) \right) \right] = 2 - H(t_0) \qquad \text{a.s.}$$

Let us notice that this result could not be a direct consequence of Adler's earlier work [1] since multifractional Brownian motion does not have stationary increments, on the contrary to classical fractional Brownian motion.

The local Hausdorff dimension of level sets of mBm (as a particular case of multi-fractional Brownian sheet) have been considered in [28], under regularity Assumption (H_β). Denoting $\Gamma_x = \{t \in \mathbf{R}_+ : X_t = x\}$, for any $t_0 \in \mathbf{R}_+$, there exists $\rho_0 > 0$, such that almost surely,

$$\lim_{\rho \to 0} \dim_{\mathcal{H}} \left(\Gamma_{X_t} \cap B(t_0, \rho) \right) = 1 - H(t_0) \quad \text{for a.e. } t \in B(t_0, \rho_0).$$

In [13,17], the fine regularity of the multifractional Brownian motion has been studied in the irregular case, *i.e.* when the function H is only assumed to be β-Hölder continuous with $\beta > 0$. In this more general case, the pointwise and local Hölder exponents of X at any $t_0 \in \mathbf{R}_+$ satisfy respectively

$$\boldsymbol{\alpha}_X(t_0) = H(t_0) \wedge \alpha_H(t_0) \qquad \text{a.s.}$$
$$\widetilde{\boldsymbol{\alpha}}_X(t_0) = H(t_0) \wedge \widetilde{\alpha}_H(t_0) \qquad \text{a.s.},$$

where the exponents $\alpha_H(t_0)$ and $\widetilde{\alpha}_H(t_0)$ are given by the following expressions when the function H is continuous but not differentiable at t_0,

$$\alpha_H(t_0) = \sup \left\{ \alpha > 0 : \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|H(t) - H(s)|}{\rho^\alpha} < +\infty \right\},$$
$$\widetilde{\alpha}_H(t_0) = \sup \left\{ \alpha > 0 : \lim_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|H(t) - H(s)|}{|t-s|^\alpha} < +\infty \right\}$$

and $\alpha_H(t_0) = \widetilde{\alpha}_H(t_0) = 1$ when H is differentiable at t_0.

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Roughly speaking, when the function H is irregular, it transmits its local regularity to the sample paths of the mBm. But in that case, nothing is known about the Hausdorff dimension of the range or the graph of the process.

In this section, the main results of the paper stated in Section 2.2 are applied to derive informations on these Hausdorff dimensions, without regularity (H_β)-assumption on the function H. As for Gaussian processes, we define the *local sub-exponent* of H at $t_0 \in \mathbf{R}_+$ by

$$\begin{aligned}\underline{\alpha}_H(t_0) &= \inf\left\{\alpha > 0 : \lim_{\rho\to 0} \inf_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{|t-s|^\alpha} = +\infty\right\}\\ &= \sup\left\{\alpha > 0 : \lim_{\rho\to 0} \inf_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{|t-s|^\alpha} = 0\right\}.\end{aligned}$$

Proposition 3.6. *Let $X = \{X_t;\ t \in \mathbf{R}_+\}$ be the multifractional Brownian motion of integral representation (3.4), with regularity function $H : \mathbf{R}_+ \to (0,1)$ assumed to be β-Hölder-continuous with $\beta > 0$. Let $\widetilde{\alpha}_H(t_0)$ and $\underline{\alpha}_H(t_0)$ be respectively the local Hölder exponent and sub-exponent of H at $t_0 \in \mathbf{R}_+$.*

In the three following cases, the Hausdorff dimension of the graph of the sample path of X satisfies:

(i) *If $H(t_0) < \widetilde{\alpha}_H(t_0) \le \underline{\alpha}_H(t_0)$ for $t_0 \in \mathbf{R}_+$, then*

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) = 2 - H(t_0) \qquad a.s.$$

(ii) *If $\widetilde{\alpha}_H(t_0) < H(t_0) \le \underline{\alpha}_H(t_0)$ for $t_0 \in \mathbf{R}_+$, then*

$$2 - H(t_0) \le \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \le 2 - \widetilde{\alpha}_H(t_0) \qquad a.s.$$

(iii) *If $\widetilde{\alpha}_H(t_0) \le \underline{\alpha}_H(t_0) < H(t_0)$ for $t_0 \in \mathbf{R}_+$, then*

$$2 - \underline{\alpha}_H(t_0) \le \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \le 2 - \widetilde{\alpha}_H(t_0) \qquad a.s.$$

With probability one, the Hausdorff dimension of the range of the sample path of X satisfies:

$$\forall t_0 \in \mathbf{R}_+, \quad \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) = 1.$$

Moreover if the (H_β)-assumption holds then, with probability one,

$$\forall t_0 \in \mathbf{R}_+, \quad \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) = 2 - H(t_0).$$

Proof. In [13], an asymptotic behaviour of the incremental variance of the multifractional Brownian motion, in a neighborhood $B(t_0,\rho)$ of any $t_0 \in \mathbf{R}_+$ as ρ goes to 0, is given by: $\forall s,t \in B(t_0,\rho)$,

$$\mathbf{E}[|X_t - X_s|^2] \sim K(t_0)\ |t-s|^{H(t)+H(s)} + L(t_0)\ [H(t)-H(s)]^2, \tag{3.5}$$

where $K(t_0)$ and $L(t_0)$ are positive constants.

From (3.5), for any $t_0 \in \mathbf{R}_+$, for all $\alpha > 0$ and for all $s,t \in B(t_0,\rho)$,

$$\frac{\mathbf{E}[|X_t - X_s|^2]}{|t-s|^{2\alpha}} \sim K(t_0)\ |t-s|^{H(t)+H(s)-\alpha} + L(t_0)\ \left[\frac{H(t)-H(s)}{|t-s|^\alpha}\right]^2, \tag{3.6}$$

when $\rho \to 0$. This expression allows to evaluate the exponents $\widetilde{\mathfrak{a}}_X(t_0)$ (and consequently $\widetilde{\boldsymbol{\alpha}}_X(t_0)$) and $\underline{\mathfrak{a}}_X(t_0)$, in function of the respective exponents of the function H.

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The local behaviour of H around t_0 is described by one of the two following situations:

- Either there exists $\rho > 0$ such that the restriction $H|_{B(t_0,\rho)}$ is increasing or decreasing. In that case, $\underline{\alpha}_H(t_0) \in \mathbf{R}_+ \cup \{+\infty\}$.
- Or for all $\rho > 0$, there exist $s, t \in B(t_0, \rho)$ such that $H(t) = H(s)$.
  In that case, for all $\alpha > 0$ and for all $\rho > 0$, $\inf_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{|t-s|^\alpha} = 0$ and therefore, $\underline{\alpha}_H(t_0) = +\infty$.

Since $\widetilde{\alpha}_H(t_0) \leq \underline{\alpha}_H(t_0)$ for all $t_0 \in \mathbf{R}_+$ as noticed in Section 2.1, we distinguish the three following cases:

(i) If $H(t_0) < \widetilde{\alpha}_H(t_0) \leq \underline{\alpha}_H(t_0)$ for some $t_0 \in \mathbf{R}_+$, then for all $0 < \epsilon < \widetilde{\alpha}_H(t_0) - H(t_0)$, there exists $\rho_0 > 0$ such that

$$\forall t \in B(t_0, \rho_0), \quad H(t_0) - \epsilon < H(t) < H(t_0) + \epsilon,$$

and thus

$$\forall s, t \in B(t_0, \rho_0), \quad |t-s|^{2H(t_0)+2\epsilon} \leq |t-s|^{H(s)+H(t)} \leq |t-s|^{2H(t_0)-2\epsilon}. \tag{3.7}$$

Then, expression (3.6) implies $H(t_0) - \epsilon \leq \widetilde{\mathfrak{a}}_X(t_0)$ and $\underline{\mathfrak{a}}_X(t_0) \leq H(t_0) + \epsilon$, by definition of the exponents. Letting ϵ tend to 0, and using $\widetilde{\mathfrak{a}}_X(t_0) \leq \underline{\mathfrak{a}}_X(t_0)$, we get $\widetilde{\mathfrak{a}}_X(t_0) = \underline{\mathfrak{a}}_X(t_0) = H(t_0)$.
Then, Theorem 2.2 (with $N > d\ \underline{\mathfrak{a}}_X(t_0)$) implies:

$$\lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) = 2 - H(t_0) \qquad \text{a.s.}$$

(ii) If $\widetilde{\alpha}_H(t_0) < H(t_0) \leq \underline{\alpha}_H(t_0)$ for some $t_0 \in \mathbf{R}_+$, then as previously, we consider any $0 < \epsilon < H(t_0) - \widetilde{\alpha}_H(t_0)$ and we show that expression (3.6) and inequalities (3.7) imply $\widetilde{\mathfrak{a}}_X(t_0) = \widetilde{\alpha}_H(t_0)$ and $\underline{\mathfrak{a}}_X(t_0) = H(t_0)$. Theorem 2.2 (with $N > d\ \underline{\mathfrak{a}}_X(t_0)$) implies:

$$2 - H(t_0) \leq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \leq 2 - \widetilde{\alpha}_H(t_0) \qquad \text{a.s.}$$

(iii) If $\widetilde{\alpha}_H(t_0) \leq \underline{\alpha}_H(t_0) < H(t_0)$ for some $t_0 \in \mathbf{R}_+$, then as previously, we consider any $0 < \epsilon < H(t_0) - \underline{\alpha}_H(t_0)$ and we show that expression (3.6) and inequalities (3.7) imply $\widetilde{\mathfrak{a}}_X(t_0) = \widetilde{\alpha}_H(t_0)$ and $\underline{\mathfrak{a}}_X(t_0) = \underline{\alpha}_H(t_0)$. Theorem 2.2 (with $N > d\ \underline{\mathfrak{a}}_X(t_0)$) implies:

$$2 - \underline{\alpha}_H(t_0) \leq \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) \leq 2 - \widetilde{\alpha}_H(t_0) \qquad \text{a.s.}$$

Since H is β-Hölder-continuous with $\beta > 0$, Theorem 2.3 can be applied with $\mathcal{A} = \mathbf{R}_+$. In the three previous case, we observe that $\underline{\mathfrak{a}}_X(u) < 1$ for all $u \in \mathbf{R}_+$. Consequently, $N > d\ \liminf_{u\to t_0} \underline{\mathfrak{a}}_X(u)$ and, with probability one,

$$\forall t_0 \in \mathbf{R}_+, \quad \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) = 1.$$

When the (H_β)-assumption holds, $\widetilde{\mathfrak{a}}_X(t_0) = \widetilde{\alpha}_H(t_0) = \underline{\mathfrak{a}}_X(t_0)$ for all $t_0 \in \mathbf{R}_+$, and by continuity of H,

$$\liminf_{u\to t_0} \widetilde{\mathfrak{a}}_X(u) = \liminf_{u\to t_0} \underline{\mathfrak{a}}_X(u) = H(t_0).$$

Then, Theorem 2.3 implies: With probability one,

$$\forall t_0 \in \mathbf{R}_+, \quad \lim_{\rho\to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) = 2 - H(t_0). \qquad \square$$

According to Proposition 3.6, the general theorems of Section 2.2 fail to derive sharp values for the Hausdorff dimensions of the sample paths of the multifractional Brownian motion when the (H_β)-assumption for the function H is not satisfied. This is due to the fact that the irregularity of H is not completely controlled by the exponents $\widetilde{\alpha}_H(t_0)$ and $\underline{\alpha}_H(t_0)$. A deeper analysis of the function H is required in order to determine the exact Hausdorff dimensions of the mBm.

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3.3. Generalized Weierstrass function

The local regularity of the Weierstrass function W_H, defined by

$$t \mapsto W_H(t) = \sum_{j=1}^{\infty} \lambda^{-jH} \ \sin \lambda^j t,$$

where $\lambda \geq 2$ and $H \in (0,1)$, has been deeply studied in the literature (*e.g.* see [12]). When λ is large enough, the box-counting dimension of the graph of W_H is known to be $2-H$. Nevertheless the exact value of the Hausdorff dimension remains unknown at this stage.

Different stochastic versions of the Weierstrass function have been considered in [3, 12, 17, 18, 25] and their geometric properties have been investigated. In this section, we consider the *generalized Weierstrass function (GW)*, defined as the 2nd-order stochastic process $X = \{X_t;\ t \in \mathbf{R}_+\}$,

$$\forall t \in \mathbf{R}_+, \quad X_t = \sum_{j=1}^{\infty} Z_j \ \lambda^{-jH(t)} \ \sin(\lambda^j t + \theta_j) \tag{3.8}$$

where

- $\lambda \geq 2$,
- $t \mapsto H(t)$ takes values in $(0,1)$,
- $(Z_j)_{j\geq 1}$ is a sequence of $\mathcal{N}(0,1)$ i.i.d. random variables,
- and $(\theta_j)_{j\geq 1}$ is a sequence of uniformly distributed on $[0, 2\pi)$ random variables independent of $(Z_j)_{j\geq 1}$.

In the specific case of $\theta_j = 0$ for all $j \geq 1$, Theorem 4.9 of [17] determines the local regularity of the sample path of the GW through its 2-microlocal frontier, when the function H is β-Hölder continuous with $\beta > 0$ and when assumption (H_β) holds, *i.e.* $H(t) < \beta$ for all $t \in \mathbf{R}_+$. In particular, the deterministic local Hölder exponent is proved to be $\widetilde{\mathfrak{a}}_X(t_0) = H(t_0)$ for all $t_0 \in \mathbf{R}_+$ and the local Hölder exponent satisfies, with probability one,

$$\forall t_0 \in \mathbf{R}_+, \quad \widetilde{\boldsymbol{\alpha}}_X(t_0) = H(t_0).$$

Moreover, when H is constant and $\theta_j = 0$ for all $j \geq 1$, the Hausdorff dimension of the graph of the sample path of the GW is proved to be equal to $2-H$, as a particular case of Theorem 5.3.1 of [25]. In the sequel, we use Theorem 2.3 to extend this result when H is no longer constant and the θ_j's are not equal to 0.

The two following lemmas are the key results to determine the deterministic local Hölder exponent and sub-exponent of the GW, in the general case. Their proofs of are sketched in [12] when $(\theta_j)_{j\geq 1}$ are independent and uniformly distributed on $[0, 2\pi)$; for sake of completeness, we detail them in this section without requiring the independence of the θ_j's, before considering the case of a non-constant function H.

Lemma 3.7. *Let $\{X_t;\ t \in \mathbf{R}_+\}$ be the stochastic Weierstrass function defined by (3.8). Then, the incremental variance between $u, v \in \mathbf{R}_+$ is given by*

$$\mathbf{E}[|X_u - X_v|^2] = 2\sum_{j\geq 1} \lambda^{-2jH(u)} \sin^2\left(\lambda^j \frac{u-v}{2}\right) + \sum_{j\geq 1} \left(\lambda^{-jH(v)} - \lambda^{-jH(u)}\right)^2. \tag{3.9}$$

Proof. For all $u, v \in \mathbf{R}_+$, we compute

$$X_u - X_v = \sum_{j\geq 1} Z_j \ \lambda^{-jH(u)} \left[\sin(\lambda^j u + \theta_j) - \sin(\lambda^j v + \theta_j)\right] + \sum_{j\geq 1} Z_j \ \left[\lambda^{-jH(v)} - \lambda^{-jH(u)}\right] \sin(\lambda^j v + \theta_j)$$

$$= 2\sum_{j\geq 1} Z_j \ \lambda^{-jH(u)} \sin\left(\lambda^j \frac{u-v}{2}\right) \cos\left(\lambda^j \frac{u+v}{2} + \theta_j\right) + \sum_{j\geq 1} Z_j \ \left[\lambda^{-jH(v)} - \lambda^{-jH(u)}\right] \sin(\lambda^j v + \theta_j).$$

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In the expression of $\mathbf{E}[|X_u - X_v|^2]$, the three following terms appear:

- $\mathbf{E}\left[Z_j Z_k \cos\left(\lambda^j \frac{u+v}{2} + \theta_j\right) \cos\left(\lambda^k \frac{u+v}{2} + \theta_k\right)\right]$,
- $\mathbf{E}\left[Z_j Z_k \sin\left(\lambda^j v + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right)\right]$
- and $\mathbf{E}\left[Z_j Z_k \cos\left(\lambda^j \frac{u+v}{2} + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right)\right]$,

where $j, k \geq 1$.

The first two terms are treated in the same way. For the second one, we have

$$\begin{aligned}\mathbf{E}\left[Z_j Z_k \ \sin\left(\lambda^j v + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right)\right] \\ = \mathbf{E}\left(\mathbf{E}\left[Z_j Z_k \sin\left(\lambda^j v + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right) \mid Z_j, Z_k\right]\right) \\ = \mathbf{E}[Z_j Z_k]\ \mathbf{E}\left[\sin\left(\lambda^j v + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right)\right],\end{aligned}$$

using the independence of (θ_j, θ_k) with (Z_j, Z_k). Then, since $\mathbf{E}[Z_j Z_k] = \mathbb{1}_{j=k}$ and

$$\mathbf{E}[\sin^2(\lambda^j v + \theta_j)] = \frac{1}{2\pi} \int_{[0,2\pi)} \sin^2(\lambda^j v + x)\ \mathrm{d}x = \frac{1}{2},$$

we get

$$\mathbf{E}\left[Z_j Z_k \sin\left(\lambda^j v + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right)\right] = \frac{1}{2}.\mathbb{1}_{j=k}.$$

In the same way, we prove that

$$\mathbf{E}\left[Z_j Z_k \cos\left(\lambda^j \frac{u+v}{2} + \theta_j\right) \cos\left(\lambda^k \frac{u+v}{2} + \theta_k\right)\right] = \frac{1}{2}.\mathbb{1}_{j=k}.$$

For the third term, we compute as previously

$$\begin{aligned}\mathbf{E}\left[Z_j Z_k \cos\left(\lambda^j \frac{u+v}{2} + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right)\right] \\ = \mathbf{E}[Z_j Z_k]\ \mathbf{E}\left[\cos\left(\lambda^j \frac{u+v}{2} + \theta_j\right) \sin\left(\lambda^k v + \theta_k\right)\right] \\ = \mathbb{1}_{j=k}.\mathbf{E}\left[\cos\left(\lambda^j \frac{u+v}{2} + \theta_j\right) \sin\left(\lambda^j v + \theta_j\right)\right] \\ = \mathbb{1}_{j=k}.\frac{1}{2\pi} \int_{[0,2\pi)} \cos\left(\lambda^j \frac{u+v}{2} + x\right) \sin\left(\lambda^j v + x\right)\ \mathrm{d}x = 0,\end{aligned}$$

by a parity argument. The result follows. □

Lemma 3.8. *Let $\{X_t;\ t \in \mathbf{R}_+\}$ be the stochastic Weierstrass function defined by (3.8), where the function H is assumed to be constant.*

Then, for all compact subset $I \subset \mathbf{R}_+$, there exists two constants $C_1 > 0$ and $C_2 > 0$ such that for all $u, v \in I$,

$$0 < C_1 \leq \frac{\mathbf{E}[|X_u - X_v|^2]}{|u-v|^{2H}} \leq C_2 < +\infty. \tag{3.10}$$

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Proof. According to Lemma 3.7, the incremental variance of X is given by

$$\mathbf{E}[|X_u - X_v|^2] = 2\sum_{j\geq 1} \lambda^{-2jH} \sin^2\left(\lambda^j \frac{u-v}{2}\right). \tag{3.11}$$

Let N be the integer such that $\lambda^{-(N+1)} \leq |u-v| < \lambda^{-N}$.

For all $j \leq N$, $\lambda^j \frac{u-v}{2} \leq \frac{1}{2}$. Since $x^2 - \frac{x^4}{3} \leq \sin^2 x \leq x^2$ for all $x \in [0,1]$, expression (3.11) implies

$$\begin{aligned}\mathbf{E}[|X_u - X_v|^2] &\leq 2\sum_{j=1}^{N} \lambda^{-2jH}\lambda^{2j}\left(\frac{u-v}{2}\right)^2 + 2\sum_{j\geq N+1} \lambda^{-2jH} \\ &\leq 2\sum_{j=1}^{N} \lambda^{2j(1-H)}\left(\frac{u-v}{2}\right)^2 + \frac{2\,|u-v|^{2H}}{1-\lambda^{-2H}}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\mathbf{E}[|X_u - X_v|^2] &\geq 2\sum_{j=1}^{N} \lambda^{-2jH}\lambda^{2j}\left(\frac{u-v}{2}\right)^2 - \frac{2}{3}\sum_{j=1}^{N} \lambda^{-2jH}\lambda^{4j}\left(\frac{u-v}{2}\right)^4 \\ &\geq 2\sum_{j=1}^{N} \lambda^{2j(1-H)}\left(\frac{u-v}{2}\right)^2 - \frac{1}{24}\lambda^{-4N}\sum_{j=1}^{N} \lambda^{j(4-2H)} \\ &\geq 2\sum_{j=1}^{N} \lambda^{2j(1-H)}\left(\frac{u-v}{2}\right)^2 - \frac{1}{24}\frac{\lambda^4}{\lambda^{4-2H}-1}|u-v|^{2H}.\end{aligned} \tag{3.13}$$

Now, it remains to compare the term $\sum_{j=1}^{N} \lambda^{2j(1-H)}\left(\frac{u-v}{2}\right)^2$ with $|u-v|^{2H}$.

By definition of the integer N, we have

$$\frac{\lambda^{-2(N+1)}}{4}\sum_{j=1}^{N} \lambda^{2j(1-H)} \leq \sum_{j=1}^{N} \lambda^{2j(1-H)}\left(\frac{u-v}{2}\right)^2 \leq \frac{\lambda^{-2N}}{4}\sum_{j=1}^{N} \lambda^{2j(1-H)}. \tag{3.14}$$

But

$$\begin{aligned}\frac{\lambda^{-2N}}{4}\sum_{j=1}^{N} \lambda^{2j(1-H)} &= \frac{\lambda^{-2N}}{4}\lambda^{2(1-H)}\frac{\lambda^{2N(1-H)}-1}{\lambda^{2(1-H)}-1} \\ &= \frac{\lambda^{2(1-H)}}{4(\lambda^{2(1-H)}-1)}\left(\lambda^{-2NH} - \lambda^{-2N}\right).\end{aligned}$$

Using the definition of N, we get

$$|u-v|^{2H} - \lambda^2\,|u-v|^2 \leq \lambda^{-2NH} - \lambda^{-2N} \leq \lambda^{2H}\,|u-v|^{2H} - |u-v|^2.$$

Then there exists two constants $c_1 > 0$ and $c_2 > 0$ such that for all $u, v \in I$,

$$c_1\,|u-v|^{2H} \leq \frac{\lambda^{-2N}}{4}\sum_{j=1}^{N} \lambda^{2j(1-H)} \leq c_2\,|u-v|^{2H}.$$

Then, the result follows from (3.12), (3.13) and (3.14). □

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When the function $H : \mathbf{R}_+ \to (0,1)$ is β-Hölder continuous (and no longer constant), the double inequality (3.10) can be improved by the following result.

Proposition 3.9. *Let $X = \{X_t;\ t \in \mathbf{R}_+\}$ be a generalized Weierstrass function defined by (3.8), where the function H is assumed to be β-Hölder-continuous with $\beta > 0$.*

Then, for any $t_0 \in \mathbf{R}_+$, for all $\epsilon > 0$, there exist $\rho_0 > 0$ and positive constants c_1, c_2, c_3, c_4 such that for all $u, v \in B(t_0, \rho_0)$,

$$c_1\ |u-v|^{2H(t_0)+\epsilon} + c_3\ [H(u)-H(v)]^2 \leq \mathbf{E}[|X_u - X_v|^2] \tag{3.15}$$

$$\text{and} \qquad \mathbf{E}[|X_u - X_v|^2] \leq c_2\ |u-v|^{2H(t_0)-\epsilon} + c_4\ [H(u)-H(v)]^2. \tag{3.16}$$

Proof. Since the function $H : \mathbf{R}_+ \to (0,1)$ is continuous, for all $t_0 \in \mathbf{R}_+$ and all $\epsilon > 0$, there exists $\rho_0 > 0$ such that

$$\forall u, v \in B(t_0, \rho_0), \quad H(u), H(v) \in (H(t_0) - \epsilon; H(t_0) + \epsilon).$$

Then, the first term of the expression (3.9) for $\mathbf{E}[|X_u - X_v|^2]$ satisfies

$$2\sum_{j\geq 1} \lambda^{-2jH(u)}\ \sin^2\left(\lambda^j \frac{u-v}{2}\right) \leq 2\sum_{j\geq 1} \lambda^{-2j(H(t_0)-\epsilon)}\ \sin^2\left(\lambda^j \frac{u-v}{2}\right)$$

and

$$2\sum_{j\geq 1} \lambda^{-2jH(u)}\ \sin^2\left(\lambda^j \frac{u-v}{2}\right) \geq 2\sum_{j\geq 1} \lambda^{-2j(H(t_0)+\epsilon)}\ \sin^2\left(\lambda^j \frac{u-v}{2}\right).$$

Then, according to Lemma 3.8, there exist two constants $c_1 > 0$ and $c_2 > 0$ such that for all $u, v \in B(t_0, \rho_0)$,

$$c_1\ |u-v|^{2(H(t_0)+\epsilon)} \leq 2\sum_{j\geq 1} \lambda^{-2jH(u)}\ \sin^2\left(\lambda^j \frac{u-v}{2}\right) \leq c_2\ |u-v|^{2(H(t_0)-\epsilon)}. \tag{3.17}$$

For the second term of the expression (3.9) for $\mathbf{E}[|X_u - X_v|^2]$, we consider the function $\psi_{\lambda,j} : x \mapsto \lambda^{-jx} = \mathrm{e}^{-jx\ln\lambda}$ of derivative $\psi'_{\lambda,j}(x) = -j\ln\lambda\ \lambda^{-jx}$.

From the finite increment theorem, for all $u, v \in B(t_0, \rho_0)$, there exists h_{uv} between $H(u)$ and $H(v)$ (*i.e.* in either $(H(u), H(v))$ or $(H(v), H(u))$) such that

$$|\lambda^{-jH(u)} - \lambda^{-jH(v)}| = |H(u) - H(v)|\ j\ln\lambda\ \lambda^{-jh_{uv}}.$$

Using the fact that $H(u)$ and $H(v)$ belong to the interval $(H(t_0)-\epsilon, H(t_0)+\epsilon)$ implies $H(t_0)-\epsilon < h_{uv} < H(t_0)+\epsilon$, we get

$$|H(u) - H(v)|\ j\ln\lambda\ \lambda^{-j(H(t_0)+\epsilon)} \leq |\lambda^{-jH(u)} - \lambda^{-jH(v)}|$$

$$\text{and} \quad |\lambda^{-jH(u)} - \lambda^{-jH(v)}| \leq |H(u) - H(v)|\ j\ln\lambda\ \lambda^{-j(H(t_0)-\epsilon)}.$$

Since $\sum_{j\geq 1} j\lambda^{-j(H(t_0)-\epsilon)} < +\infty$ and $\sum_{j\geq 1} j\lambda^{-j(H(t_0)+\epsilon)} < +\infty$, the second term of (3.9) is bounded by

$$c_3\ [H(u)-H(v)]^2 \leq \sum_{j\geq 1} \left[\lambda^{-jH(u)} - \lambda^{-jH(v)}\right]^2 \leq c_4\ [H(u)-H(v)]^2. \tag{3.18}$$

The result follows from (3.9), (3.17) and (3.18). □

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The following result shows that Theorem 2.3 allows to derive the Hausdorff dimensions of the graph of the generalized Weierstrass function.

Corollary 3.10. *Let $X = \{X_t;\ t \in \mathbf{R}_+\}$ be a generalized Weierstrass function defined by (3.8), where the function H is assumed to be β-Hölder-continuous with $\beta > 0$ and satisfies the (H_β)-assumption.*

Then, the local Hölder exponents and sub-exponents of X are given by

$$\forall t_0 \in \mathbf{R}_+, \quad \widetilde{\mathfrak{a}}_X(t_0) = \underline{\mathfrak{a}}_X(t_0) = H(t_0).$$

Consequently, the Hausdorff dimensions of the graph and the range of the sample path of X satisfy: With probability one,

$$\begin{aligned} \forall t_0 \in \mathbf{R}_+, \quad & \lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) = 2 - H(t_0), \\ & \lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) = 1. \end{aligned}$$

Proof. According to the (H_β)-assumption, $H(t_0) < \beta$ for all $t_0 \in \mathbf{R}_+$.

Let us fix $t_0 \in \mathbf{R}_+$ and consider any $0 < \epsilon < 2(\beta - H(t_0))$. From Proposition 3.9 and the fact that H is β-Hölder continuous with $2H(t_0) - \epsilon < 2H(t_0) + \epsilon < 2\beta$, there exist $\rho_0 > 0$ and two constants $C_1 > 0$ and $C_2 > 0$ such that for all $u, v \in B(t_0, \rho_0)$,

$$C_1\ |u - v|^{2H(t_0)+\epsilon} \leq \mathbf{E}[|X_u - X_v|^2] \leq C_2\ |u - v|^{2H(t_0)-\epsilon}.$$

From the definitions of the deterministic local Hölder exponent and sub-exponent $\widetilde{\mathfrak{a}}_X(t_0)$ and $\underline{\mathfrak{a}}_X(t_0)$, we get

$$\begin{aligned} \forall 0 < \epsilon < 2(\beta - H(t_0)), \quad & \widetilde{\mathfrak{a}}_X(t_0) \geq H(t_0) - \epsilon/2, \\ & \underline{\mathfrak{a}}_X(t_0) \leq H(t_0) + \epsilon/2 \end{aligned}$$

and therefore, $H(t_0) \leq \widetilde{\mathfrak{a}}_X(t_0) \leq \underline{\mathfrak{a}}_X(t_0) \leq H(t_0)$ leads to $\widetilde{\mathfrak{a}}_X(t_0) = \underline{\mathfrak{a}}_X(t_0) = H(t_0)$.

Since the generalized Weierstrass function X is not strictly a Gaussian process, Theorem 2.3 cannot be applied directly. However, the proofs of upper and lower bounds for the Hausdorff dimensions, which rely on the comparison of $\mathbf{E}\big[|X_t - X_s|^\alpha\big]$ with $\big(\mathbf{E}\big[|X_t - X_s|^2\big]\big)^{\alpha/2}$, can be extended to X (which is Gaussian, conditionally on the filtration generated by the phases).

Consequently, by continuity of the function H, we can conclude: With probability one,

$$\begin{aligned} \forall t_0 \in \mathbf{R}_+, \quad & \lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Gr}_X(B(t_0,\rho))) = 2 - H(t_0), \\ & \lim_{\rho \to 0} \dim_{\mathcal{H}}(\mathrm{Rg}_X(B(t_0,\rho))) = 1. \end{aligned}$$

□

Remark 3.11. Proposition 3.10 should be compared to Theorem 1 of [18], where the Hausdorff dimension of the graph of the process $\{Y_t;\ t \in \mathbf{R}_+\}$ defined by

$$\forall t \in \mathbf{R}_+, \quad Y_t = \sum_{n=1}^{+\infty} \lambda^{-nH} \sin(\lambda^n t + \theta_n),$$

where $\lambda \geq 2$, $H \in (0,1)$ and $(\theta_n)_{n \geq 1}$ are independent random variables uniformly distributed on $[0, 2\pi)$, is proved to be $D = 2 - H$.

The generalized Weierstrass function X differs from the process Y, in the form of the random serie (the θ_n's in the definition of Y_t cannot be all equal) and in the fact that the exponent H is constant in the definition of Y, on the contrary to X.

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