Adv. Appl. Prob. **48,** 947–971 (2016)
*doi:*10.1017/apr.2016.60
© *Applied Probability Trust* 2017

RANDOM FIELDS OF BOUNDED VARIATION AND COMPUTATION OF THEIR VARIATION INTENSITY

BRUNO GALERNE,* *Université Paris Descartes*

Abstract

The main purpose of this paper is to define and characterize random fields of bounded variation, that is, random fields with sample paths in the space of functions of bounded variation, and to study their mean total variation. Simple formulas are obtained for the mean total directional variation of random fields, based on known formulas for the directional variation of deterministic functions. It is also shown that the mean variation of random fields with stationary increments is proportional to the Lebesgue measure, and an expression of the constant of proportionality, called the *variation intensity*, is established. This expression shows, in particular, that the variation intensity depends only on the family of two-dimensional distributions of the stationary increment random field. When restricting to random sets, the obtained results give generalizations of well-known formulas from stochastic geometry and mathematical morphology. The interest of these general results is illustrated by computing the variation intensities of several classical stationary random field and random set models, namely Gaussian random fields and excursion sets, Poisson shot noises, Boolean models, dead leaves models, and random tessellations.

Keywords: Functions of bounded variation; directional variation; variation intensity; specific perimeter; stationary increment random field; germ–grain model

2010 Mathematics Subject Classification: Primary 60G60
Secondary 60G17; 60G51; 60D05

1. Introduction

In this paper we focus on the definition and characterizations of random fields (RFs) of bounded variation, that is, RFs with paths in the space of functions of bounded variation, as well as to provide explicit formulas for the computation of their mean total variation. The motivations for this work stem from the modeling of textures by RFs in image processing, where functions of bounded variation are an important model. The functional space $\mathrm{BV}(U)$ of functions of bounded variation defined over some open set $U \subset \mathbb{R}^d$ is a subspace of $L^1(U)$ for which the functions are weakly differentiable in the sense that their distributional derivative Df can be represented by a finite Radon measure over U [2]. The total variation of $f \in L^1(U)$ is then defined by $|Df|(U)$, the total variation of the vector-valued measure Df, if $f \in \mathrm{BV}(U)$, and by $+\infty$ otherwise.

Since the seminal paper of Rudin *et al.* [23], total variation has been widely used for various tasks in image processing such as denoising, zooming, deconvolution, etc. (see, e.g. [3] and the references therein). Concerning textures, the total variation generally appears in the problem of decomposing an image into a cartoon part (or geometric part) and a textural part following the framework proposed by Meyer [20]. For this problem, we assume that the total variation

Received 21 August 2014; revision received 15 October 2015.
* Postal address: Laboratoire MAP5 (UMR CNRS 8145), Université Paris Descartes, Sorbonne Paris Cité, 45 rue des Saints-Pères, 75006 Paris, France. Email address: bruno.galerne@parisdescartes.fr

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

of a textural part is high or even infinite. Yet, to the best of the author's knowledge, little is known about the total variation of classical texture models, such as Gaussian RFs, shot noise, and dead leaves models. However, for the latter germ–grain models, the RFs correspond to random geometric images with few contours, and intuitively their total variation should depend on the geometry of the grains. Hence, it is natural to ask the following question: what is the mean total variation of classical germ–grain models?

In this paper we provide an answer to this question by introducing general definitions and characterization results for RFs of bounded variation. To the best of the author's knowledge, RFs of bounded variation over $\mathbb{R}^d$ have never been studied for $d \geq 2$, with the exception of [15] and the very recent paper [5] in which the excursion sets of shot noise RFs were studied. While the present work was motivated by image modeling, the results are presented in an arbitrary dimension $d \geq 1$ and are not limited to this field. Indeed, the results are of potential interest in areas of applied probability where piecewise smooth RFs are at hand, such as stochastic geometry (with, e.g. Boolean random functions or colored tessellations; see Section 6.2). In comparison, the notions of regularity that are often considered in applied probability, such as Hölder continuity, do not allow for discontinuities in the RF sample paths.

We now detail the main results of the paper. For an RF f, our general strategy is first to deal with directional variations $|D_u f|$, $u \in S^{d-1}$, and then integrate over all directions $u \in S^{d-1}$ to obtain results on the variation $|Df|$. The advantage of dealing with directional variations is that it gives simple expressions involving only difference quotients, and it also provides additional information on the anisotropy of the RF f. Section 2 is devoted to deterministic functions of bounded directional variation. Relations between the directional variation $|D_u f|$ of a function f and the integral of its difference quotients are emphasized, yielding the fundamental relation

$$|D_u f|(U) = \lim_{r \to 0} \int_{U \ominus [0,ru]} \frac{|f(x+ru) - f(x)|}{|r|} \, \mathrm{d}x$$

for any open set $U \subset \mathbb{R}^d$. After this preliminary study of deterministic functions of bounded directional variation, RFs of (locally) bounded (directional) variation are defined and characterized in Section 3. In particular, we define the *directional variation intensity measure* $\Theta_{V_u}(f, \cdot)$ of an RF f as the expectation of the directional variation of f, and we show that

$$\Theta_{V_u}(f, U) = \mathbb{E}(|D_u f|(U)) = \lim_{r \to 0} \int_{U \ominus [0,ru]} \frac{\mathbb{E}(|f(x+ru) - f(x)|)}{|r|} \, \mathrm{d}x.$$

Particular interest is then given to RFs f with stationary increments having locally bounded (directional) variation in Section 4. If f is such an RF, we prove that the mean directional variation of f on every domain U is proportional to the Lebesgue measure of U. Denoted $\theta_{V_u}(f)$, the constant of proportionality is called the *directional variation intensity* of f and is given by

$$\theta_{V_u}(f) = \lim_{r \to 0} \frac{\mathbb{E}(|f(ru) - f(0)|)}{|r|}. \tag{1}$$

In particular, the directional variation intensity $\theta_{V_u}(f)$ depends only on the family of two-dimensional distributions of the stationary increment RF f.

As mentioned above, we are aware of only one general result on the variation of RFs which is due to Ibragimov [15]. Ibragimov's theorem shows that if an RF is Lipschitz in mean then f has almost surely (a.s.) locally bounded variation. We improve Ibragimov's theorem by giving the optimal bound on the (directional) variation intensity measure and we establish that

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

the converse of Ibragimov's theorem holds when the RF has stationary increments: a locally integrable stationary increment RF has bounded variation with a finite variation intensity if and only if it is Lipschitz in mean.

By definition, the variation of an indicator function is its perimeter [2]. Hence, as a particular case of our results on the mean variation of RFs, in Section 5 we derive formulas for the mean perimeter of random sets first established by the author in [10], and originally due to Matheron [18], [19]. In particular, the specific directional variation of a stationary random set $X \subset \mathbb{R}^d$ is given by

$$\theta_{V_u}(X) = 2 \lim_{r \to 0} \frac{v_X(ru)}{|r|}, \tag{2}$$

where $v_X(y) = \mathbb{P}(y \in X,\ 0 \notin X)$ is the variogram of X. In addition, mean coarea formulas for the mean total variation are established.

Finally, in Section 6 we illustrate the use of (1) and (2) by computing the variation intensity of several classical stationary RF models. We first study the variation of stationary Gaussian RFs and the perimeter of their excursion sets. We show that a Gaussian RF f_G has finite variation intensity if and only if the one-dimensional restrictions of its covariance are twice differentiable at 0. According to [24], this condition also implies that the sample paths of f_G are a.s. in the Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, which shows that functions of bounded variation are not an interesting model for Gaussian RFs. In Section 6.2 we provide expressions for the directional and nondirectional variation intensities of several RFs associated with germ–grain models, namely Poisson shot noise of random sets, Boolean models, colored dead leaves RFs, and colored tessellations. A germ–grain model defines an RF by combining colored random sets according to an interaction principle (addition, supremum, occultation, etc.). These constructions result in RF models that present numerous discontinuities along the geometric contours of the grains. For these models, the obtained formulas explicitly clarify the somewhat intuitive relation between the geometry of the grains X and the total variation of the RF. Moreover, they show that, for all the considered germ–grain models, there are only two geometric features of influence on the variation intensity: the mean perimeter and the mean Lebesgue measure of the grains.

2. Functions of bounded directional variation and difference quotients

In this section we gather several necessary results from the theory of functions of bounded variation, with a particular interest in functions of bounded directional variation. For a general treatment of the subject, we refer the reader to Ambrosio *et al.* [2]. Some basic definitions are recalled and the relations between the directional variation of a function and the integral of its difference quotients are emphasized.

Notation. For any open subset $U \subset \mathbb{R}^d$, $\mathcal{B}(U)$ denotes the set of Borel subsets of U, and we write $V \Subset U$ if $V \subset U$ is open and relatively compact in U. We denote by $L^1(U)$ (respectively $L^1_{\mathrm{loc}}(U)$) as the space of integrable (respectively locally integrable) functions over U. We respectively denote by $\mathcal{C}^0(U, \mathbb{R})$, $\mathcal{C}^1(U, \mathbb{R})$, and $\mathcal{C}^\infty(U, \mathbb{R})$ the space of functions from U to $\mathbb{R}$ that are continuous, continuously differentiable, and infinitely differentiable. The vector field spaces $\mathcal{C}^0(U, \mathbb{R}^d)$, $\mathcal{C}^1(U, \mathbb{R}^d)$, and $\mathcal{C}^\infty(U, \mathbb{R}^d)$ are defined similarly. The spaces $\mathcal{C}^0_c(U, \mathbb{R})$, $\mathcal{C}^\infty_c(U, \mathbb{R}^d)$, etc. denote the spaces of functions or vector fields that have compact support. We denote by S^{d-1} the unit Euclidean sphere in $\mathbb{R}^d$. If $\varphi \in \mathcal{C}^1(U, \mathbb{R})$ and $u \in S^{d-1}$, we write $\partial\varphi(x)/\partial u = \langle \nabla\varphi(x), u \rangle$, $x \in \mathbb{R}^d$, for the directional derivative of φ in the direction u. Finally, let $\mathcal{H}^{d-1}$ denote the Hausdorff measure of dimension $d-1$.

For the remainder of this section, U denotes an open set of $\mathbb{R}^d$.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Definition 1. (*Functions of bounded (directional) variation.*) Let $f \in L^1(U)$. Then f is a *function of bounded variation* in U if the distributional derivative of f is representable by a finite Radon measure, i.e. if there exists an $\mathbb{R}^d$-valued Radon measure, denoted by $Df = (D_1 f, \dots, D_d f)$, such that $|Df|(U) < +\infty$ and, for all $\varphi = (\varphi_1, \dots, \varphi_d) \in \mathcal{C}_c^\infty(U, \mathbb{R}^d)$,

$$\int_U f(x) \operatorname{div} \varphi(x)\, \mathrm{d}x = -\sum_{i=1}^{d} \int_U \varphi_i(x) D_i f(\mathrm{d}x).$$

Let $u \in S^{d-1}$. Then f is a *function of bounded directional variation* in U in the direction u if the directional distributional derivative of f in the direction u is representable by a finite Radon measure, i.e. if there exists a signed Radon measure, denoted by $D_u f$, such that $|D_u f|(U) < +\infty$ and, for $\varphi \in \mathcal{C}_c^\infty(U, \mathbb{R})$,

$$\int_U f(x) \frac{\partial \varphi}{\partial u}(x)\, \mathrm{d}x = -\int_U \varphi(x) D_u f(\mathrm{d}x).$$

The vector space of all functions of bounded variation in U (respectively bounded directional variation in U in the direction u) is denoted by $\mathrm{BV}(U)$ (respectively $\mathrm{BV}_u(U)$).

A function $f \in L^1_{\mathrm{loc}}(U)$ has *locally bounded variation* in U (respectively *locally bounded directional variation* in U in the direction u) if $f \in \mathrm{BV}(V)$ (respectively $f \in \mathrm{BV}_u(V)$) for all open sets $V \Subset U$. The corresponding space is denoted by $\mathrm{BV}_{\mathrm{loc}}(U)$ (respectively $\mathrm{BV}_{u,\mathrm{loc}}(U)$).

If $f \in \mathrm{BV}(U)$ then $|Df|(U)$, is called the *variation* of f in U. Similarly, if $f \in \mathrm{BV}_u(U)$, $|D_u f|(U)$, is called the *directional variation* of f in the direction u in U. There is of course a strong relation between the spaces $\mathrm{BV}(U)$ and $\mathrm{BV}_u(U)$. As discussed in [8] and [10, Proposition 8], for a function $f \in L^1(U)$, there is equivalence between

- $f \in \mathrm{BV}(U)$,
- $f \in \mathrm{BV}_u(U)$ for all $u \in S^{d-1}$, and
- for all vectors e_i of the canonical basis $f \in \mathrm{BV}_{e_i}(U)$.

In addition, if $f \in \mathrm{BV}(U)$ then, for all measurable sets $A \in \mathcal{B}(U)$, we have

$$D_u f(A) = \langle Df(A), u \rangle = \sum_{i=1}^{d} u_i D_i f(A), \qquad u = (u_1, \dots, u_d) \in S^{d-1}, \tag{3}$$

$$|Df|(A) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} |D_u f|(A)\, \mathcal{H}^{d-1}(\mathrm{d}u), \tag{4}$$

where ω_{d-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$.

Remark 1. As a consequence of (3), the maps $u \mapsto D_u f(A)$ and $u \mapsto |D_u f|(A)$ are continuous on S^{d-1} for all fixed $A \in \mathcal{B}(U)$ as soon as $f \in \mathrm{BV}(U)$.

In what follows, given a function $f \in L^1(U)$ and a direction $u \in S^{d-1}$, we will consider difference quotients of the form $x \mapsto (f(x+ru) - f(x))/r$ for some $r \neq 0$. Such functions are not defined on the whole domain U, but on $U \cap (-ru + U)$. Since we will typically let r tend to 0, we will further restrict this domain of definition to the one of points x such that the whole segment $[x, x+ru]$ is included in U. This set is denoted by $U \ominus [0, ru] = \{x \in U,\ [x, x+ru] \subset U\}$.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Notation for difference quotients. Since directional difference quotients will be central in the remainder of the paper, we adopt the shorthand notation:

$$f(x; ru) = \frac{f(x + ru) - f(x)}{r}, \qquad x \in U,\ r \in \mathbb{R} \setminus \{0\},\ u \in S^{d-1},$$

for a function or an RF f.

Lemma 1 and Theorem 1 below extend [10, Proposition 11] to the case of integrable functions defined over an arbitrary domain U (and not only $\mathbb{R}^d$ as in [10]). Because of this general domain U, the use of standard approximation by convolution is discarded, and we thus propose adapted proofs.

Lemma 1. *Let $u \in S^{d-1}$. Then, for all functions $f \in \mathrm{BV}_u(U)$ and $r \neq 0$,*

$$\int_{U \ominus [0,ru]} |f(x; ru)| \, \mathrm{d}x \leq |D_u f|(U).$$

Proof. The proof is an adaptation of Lemma 3.24 of [2, p. 133]. We first suppose that $f \in \mathcal{C}^1(U)$. Then, we have

$$\begin{aligned}
\int_{U \ominus [0,r]} |f(x; ru)| \, \mathrm{d}x &= \int_{U \ominus [0,ru]} \left| \int_0^1 \frac{\partial f}{\partial u}(x + rtu) \, \mathrm{d}t \right| \mathrm{d}x \\
&\leq \int_0^1 \int_{U \ominus [0,ru]} \left| \frac{\partial f}{\partial u}(x + rtu) \right| \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^1 |D_u f|(rtu + U \ominus [0, ru]) \, \mathrm{d}t.
\end{aligned}$$

Now, for all $t \in [0, 1]$, $rtu + U \ominus [0, ru] \subset U$ and, thus, $|D_u f|(rtu + U \ominus [0, ru]) \leq |D_u f|(U)$. Hence, the lemma is valid for $f \in \mathcal{C}^1(U)$. This inequality extends to all functions in $\mathrm{BV}_u(U)$ thanks to the density of smooth functions in $\mathrm{BV}_u(U)$: for all $f \in \mathrm{BV}_u(U)$, there exists a sequence $(f_n)_n$, $f_n \in \mathcal{C}^1(U)$, such that (f_n) tends to f in $L^1(U)$ and the total directional variation $|D_u f_n|(U) = \int_U |\partial f_n / \partial u|$ tends to $|D_u f|(U)$ (using the directional equivalent of [2, Theorem 3.9, p. 122]). □

We recall the definition of weak* convergence for the signed Radon measure [2, pp. 26–27]. Let $\mathcal{C}_0^0(U, \mathbb{R})$ denote the closure of the space of continuous and compactly supported functions $\mathcal{C}_c^0(U, \mathbb{R})$ for the sup norm. Then we say that a sequence $(\mu_n)_{n \in \mathbb{N}}$ of signed Radon measure on U converges in the weak* sense to the signed Radon measure μ if, for every $\varphi \in \mathcal{C}_0^0(U, \mathbb{R})$, $\lim_{n \to +\infty} \int_U \varphi(x) \mu_n(\mathrm{d}x) = \int_U \varphi(x) \mu(\mathrm{d}x)$. In the next theorem we provide a characterization of functions of finite-directional variation in terms of difference quotients.

Theorem 1. *Let $u \in S^{d-1}$ and $f \in L^1(U)$. The following three assertions are equivalent.*

(i) $f \in \mathrm{BV}_u(U)$.

(ii) *The family of signed measures $\mu_r : A \mapsto \int_{U \ominus [0,ru]} f(x; ru) \mathbf{1}_A(x) \, \mathrm{d}x$, $A \in \mathcal{B}(U)$, $r \neq 0$, converges in the three sense to some signed Radon measure μ as $r \to 0$.*

(iii) *It holds that* $\liminf_{r \to 0} \int_{U \ominus [0,ru]} |f(x; ru)| \, \mathrm{d}x < +\infty$.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

If any of these assertions holds then $\mu = D_u f$ *and*

$$|D_u f|(U) = \lim_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x. \tag{5}$$

Besides, (5) is also valid in the degenerate case: f *is not in* $\mathrm{BV}_u(U)$ *if and only if* $\lim_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x = +\infty$.

Proof. We begin with a preliminary observation. For all $\varphi \in \mathcal{C}^1_c(U, \mathbb{R})$,

$$\lim_{r\to 0} \int_U \varphi(x)\mu_r(\mathrm{d}x) = -\int_U \frac{\partial\varphi}{\partial u}(x) f(x)\,\mathrm{d}x. \tag{6}$$

Indeed, for $\varphi \in \mathcal{C}^1_c(U, \mathbb{R})$ and $r \neq 0$,

$$\begin{aligned}\int_U \varphi(x)\mu_r(\mathrm{d}x) &= \int_{U\ominus[0,ru]} \varphi(x)\frac{f(x+ru)-f(x)}{r}\,\mathrm{d}x\\ &= \frac{1}{r}\int_{U\ominus[-ru,0]} \varphi(x-ru) f(x)\,\mathrm{d}x - \frac{1}{r}\int_{U\ominus[0,ru]} \varphi(x) f(x)\,\mathrm{d}x\\ &= \int_U \frac{\varphi(x-ru)\mathbf{1}_{U\ominus[-ru,0]}(x) - \varphi(x)\mathbf{1}_{U\ominus[0,ru]}(x)}{r} f(x)\,\mathrm{d}x.\end{aligned}$$

Hence, since, for all $x \in U$,

$$\lim_{r\to 0} \frac{\varphi(x-ru)\mathbf{1}_{U\ominus[-ru,0]}(x) - \varphi(x)\mathbf{1}_{U\ominus[0,ru]}(x)}{r} = -\frac{\partial\varphi}{\partial u}(x)$$

and, for all r such that $|r|$ is small enough such that $\mathrm{supp}(\varphi) \subset U \ominus [0, ru]$,

$$\begin{aligned}\frac{\varphi(x-ru)\mathbf{1}_{U\ominus[-ru,0]}(x) - \varphi(x)\mathbf{1}_{U\ominus[0,ru]}(x)}{r} f(x) &= \frac{\varphi(x-ru)-\varphi(x)}{r} f(x)\\ &\leq \left\|\frac{\partial\varphi}{\partial u}\right\|_\infty f(x) \in L^1(U),\end{aligned}$$

we obtain (6) by dominated convergence.

We prove the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The implication (i) $\Rightarrow$ (iii) is immediate thanks to Lemma 1. Let us consider the implication (iii) $\Rightarrow$ (ii). Note that, for all $r \neq 0$, $\int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x = |\mu_r|(U)$. Since by the hypothesis, $\liminf_{r\to 0} |\mu_r|(U) < +\infty$, there exists a sequence (r_n) converging to 0 such that $\lim_{n\to+\infty} \int_{U\ominus[0,ru]} |f(x; r_n u)|\,\mathrm{d}x = \liminf_{r\to 0} |\mu_r|(U) < +\infty$. Hence, the sequence $(|\mu_{r_n}(U)|)_{n\in\mathbb{N}}$ is convergent, and, in particular, is bounded. By weak* compactness [2, Theorem 1.59, p. 26], there exists a subsequence $(r_{n_k})_{k\in\mathbb{N}}$ such that $(\mu_{r_{n_k}})$ weak*-converges to some signed Radon measure μ. Hence, by definition of the weak*-convergence, for all $\psi \in \mathcal{C}^0_c(U, \mathbb{R})$,

$$\int_U \psi(x)\mu(\mathrm{d}x) = \lim_{k\to+\infty} \int_U \psi(x)\mu_{r_{n_k}}(\mathrm{d}x).$$

But, according to (6), for all $\varphi \in \mathcal{C}^1_c(U, \mathbb{R})$, $\int_U \varphi(x)\mu(\mathrm{d}x) = -\int_U (\partial\varphi(x)/\partial u) f(x)\,\mathrm{d}x = \lim_{r\to 0} \int_U \varphi(x)\mu_r(\mathrm{d}x)$. By the density of $\mathcal{C}^1_c(U, \mathbb{R})$ in $\mathcal{C}^0_0(U, \mathbb{R})$ for the uniform convergence, the equality $\int_U \varphi\mu = \lim \int_U \varphi\mu_r$ extends to all $\varphi \in \mathcal{C}^0_0(U, \mathbb{R})$, which shows that the whole family (μ_r) converges in the weak* sense to μ.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

We now show the implication (ii) $Rightarrow$ (i). Since (μ_r) converges in the weak* sense to μ and, according to (6), for all $\varphi \in \mathcal{C}_c^1(U, \mathbb{R})$, $\int_U \varphi(x)\mu(\mathrm{d}x) = \lim_{r\to 0} \int_U \varphi(x)\mu_r(\mathrm{d}x) = -\int_U (\partial\varphi(x)/\partial u) f(x)\,\mathrm{d}x$, $f \in \mathrm{BV}_u(U)$ and $D_u f = \mu$.

It remains to show (5). Thanks to the inequality of Lemma 1, it is enough to show that $|D_u f|(U) \le \liminf_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x$. But this inequality translates into $|\mu|(U) \le \liminf |\mu_r|(U)$, which is immediate since the total variation application $\nu \mapsto |\nu|(U)$ is lower semicontinuous with respect to the weak* convergence [2, Theorem 1.59, p. 26]. □

3. Random fields of bounded variation

In this section, U denotes an open set of $\mathbb{R}^d$. Following the generally approved definition, an RF of $\mathbb{R}^d$ is a family of random variables (RVs) $\xi_x : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ indexed by $x \in \mathbb{R}^d$. However, the study of the variation of an RF requires a further constraint, namely that the RF sample paths are integrable functions.

Definition 2. (*Integrable random fields.*) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. An *integrable random field* f defined on U is a random element of $L^1(U)$, that is, a measurable map $f : (\Omega, \mathcal{A}) \to (L^1(U), \mathcal{B}(L^1(U)))$, where $L^1(U)$ is endowed with the Borel σ-algebra of the $L^1(U)$-convergence. Similarly, a *locally integrable random field* f defined on U is an RV of $L^1_{\mathrm{loc}}(U)$.

We will speak of an a.s. integrable random field to allow the function f to be outside $L^1(U)$ with probability 0. We observe that if f is an integrable RF, then the integrals $\omega \mapsto \int_{\mathbb{R}^d} f(\omega, x)\varphi(x)\,\mathrm{d}x$, $\varphi \in \mathcal{C}_c^0(\mathbb{R}^d, \mathbb{R})$, are well-defined RVs and, thus, $f\mathcal{L}^d$ defines a unique signed Radon measure. We also remark that if $f : (\Omega \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a jointly measurable RF such that its sample paths are a.s. integrable, then f defines an integrable RF in the sense of the above definition.

Definition 3. (*Random fields of bounded (directional) variation.*) An a.s. integrable RF $f \in L^1(U)$ is a *random field of bounded variation* in U if there exists some random $\mathbb{R}^d$-valued Radon measure $Df = (D_1 f, \ldots, D_d f)$ such that $|Df|(U)$ is a.s. finite and, for all $\varphi = (\varphi_1, \ldots, \varphi_d) \in \mathcal{C}_c^\infty(U, \mathbb{R}^d)$,

$$\int_U f(x) \operatorname{div} \varphi(x)\,\mathrm{d}x = -\sum_{i=1}^d \int_U \varphi_i(x) D_i f(\mathrm{d}x) \quad \text{a.s.}$$

Let $u \in S^{d-1}$. An a.s. integrable RF $f \in L^1(U)$ is an *RF of bounded directional variation* in U in the direction u if there exists some random signed Radon measure $D_u f$ such that $|D_u f|(U)$ is a.s. finite and, for all $\varphi \in \mathcal{C}_c^\infty(U, \mathbb{R})$,

$$\int_{\mathbb{R}^d} f(x) \frac{\partial \varphi}{\partial u}(x)\,\mathrm{d}x = -\int_{\mathbb{R}^d} \varphi(x) D_u f(\mathrm{d}x).$$

An a.s. locally integrable RF $f \in L^1_{\mathrm{loc}}(U)$ is an RF *of locally bounded variation* in U (respectively *RF of bounded directional variation* in U in the direction u) if, for all $V \Subset U$, the restriction of f to V is an RF of bounded variation in V (respectively of bounded directional variation in V in the direction u).

Definition 4. (*(Directional) variation intensity measures.*) For an RF f of locally bounded variation in U, the intensity measure of the variation $|Df|$, that is, the measure $A \mapsto \mathbb{E}(|Df|(A))$

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

is called the *variation intensity measure* of f and is denoted by $\Theta_V(f, \cdot)$. Similarly, for an RF f of locally bounded directional variation in U in the direction $u \in S^{d-1}$, the intensity measure of the variation $|D_u f|$, that is, the measure $A \mapsto \mathbb{E}(|D_u f|(A))$ is called the *directional variation intensity measure* of f in the direction u and is denoted by $\Theta_{V_u}(f, \cdot)$.

The variation intensity measure is an important characteristic of an RF f of bounded variation: for all measurable sets A, $\Theta_V(f, A)$ is the mean variation of f in A. The mean variation $\Theta_V(f, U)$ on the whole domain U will be called the *mean total variation* of f and $\Theta_{V_u}(f, U)$ will be called the *mean total directional variation* of f in the direction u. We easily establish the analog of (4) for the case of RFs of bounded variation.

Proposition 1. (Integral geometric formula for directional variation intensity measures.) *Let f be an RF of bounded variation in U. Then f is an RF of bounded directional variation in U for all directions $u \in S^{d-1}$, and, for all $A \in \mathcal{B}(U)$,*

$$\Theta_V(f, A) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \Theta_{V_u}(f, A) \mathcal{H}^{d-1}(\mathrm{d}u).$$

Proof. As for Proposition 8 of [10], if Df is a random $\mathbb{R}^d$-valued Radon measure representing the distributional derivative of f then $\langle Df, u\rangle$ is a random signed Radon measure which represents the directional distributional derivative of f. Hence, f is an RF of bounded directional variation in U for all directions $u \in S^{d-1}$. The integral geometric formula is obtained by applying Fubini's theorem to (4). □

In the next proposition we show that the proposed definition of an RF of bounded variation is equivalent with a more instinctive definition: an RF has bounded variation if its sample paths have a.s. bounded variation.

Proposition 2. (Sample paths of RFs of bounded variation.) *Let $f : \Omega \times U \to \mathbb{R}$ be a jointly measurable RF such that $f(\omega, \cdot) \in L^1(U)$ a.s., and let $u \in S^{d-1}$. Then, f is an RF of bounded directional variation in U in the direction u (in the sense of Definition 3) if and only if its sample paths $x \mapsto f(\omega, x)$ are in $\mathrm{BV}_u(U)$ for $\mathbb{P}$-almost everywhere ($\mathbb{P}$-a.e.) $\omega \in \Omega$. Similarly, f is an RF of bounded variation in U if and only if its sample paths $x \mapsto f(\omega, x)$ are in $\mathrm{BV}(U)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. We first show the equivalence for RFs of bounded directional variation. The direct sense is immediate from the definition: since $D_u f$ is a.s. a Radon measure, the sample paths are a.s. in $\mathrm{BV}_u(U)$. Conversely, note that $\Omega' \subset \Omega$ the set of $\omega \in \Omega$ for which $x \mapsto f(\omega, x)$ are in $\mathrm{BV}_u(U)$. Then, for all $\omega \in \Omega'$, there exists a signed Radon measure $\mu(\omega, \cdot)$ such that, for all $\varphi \in \mathcal{C}_c^\infty(U, \mathbb{R})$, $\int_U f(\omega, x)(\partial\varphi(x)/\partial u)\,\mathrm{d}x = -\int_U \varphi(x)\mu(\omega, \mathrm{d}x)$. The only difficulty is to ensure that $\omega \mapsto \mu(\omega, \cdot)$ is a well-defined random signed Radon measure, that is, a measurable map. Let (r_n) be a sequence converging to 0. According to Theorem 1, for all $\omega \in \Omega'$, $\mu(\omega, \cdot)$ is the weak* limit of the sequence of signed Radon measures

$$\mu_{r_n}(\omega, A) = \int_{U \ominus [0, ru]} \frac{f(\omega, x + r_n u) - f(\omega, x)}{r_n} \mathbf{1}_A(x)\,\mathrm{d}x, \qquad A \in \mathcal{B}(U).$$

Hence, μ is measurable since it is the almost sure limit of the weak*-convergent sequence of random signed Radon measures $(\mu_n)_{n\in\mathbb{N}}$. The corresponding equivalence for an RF of bounded variation is straightforward, using the fact that $f(\omega, \cdot) \in \mathrm{BV}(U)$ if and only if $\mathrm{BV}_{e_i}(U)$ for all e_i in the canonical basis. □

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Remark 2. (*Notation.*.) Proposition 2 extends directly to RFs of locally bounded variation and locally bounded directional variation. In view of Proposition 2, we will use the notation $f \in \mathrm{BV}(U)$ a.s., $f \in \mathrm{BV}_{u,\mathrm{loc}}(U)$ a.s., etc. to express that an RF f has bounded variation, locally bounded directional variation, etc.

We write Theorem 1 in the case of RFs to establish an integral expression of the total variation intensity.

Proposition 3. (Characterization of an RF of bounded directional variation.) *Let $f \in L^1(U)$ a.s. and $u \in S^{d-1}$. Then the following three assertions are equivalent.*

(i) $f \in \mathrm{BV}_u(U)$ *a.s.*

(ii) $\liminf_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x < +\infty$ *a.s.*

(iii) $\lim_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x$ *exists and is finite a.s.*

If any of these conditions holds then the total variation intensity is

$$\Theta_{V_u}(f, U) = \lim_{r\to 0} \int_{U\ominus[0,ru]} \mathbb{E}(|f(x; ru)|)\,\mathrm{d}x. \tag{7}$$

Proof. The characterization is a consequence of Theorem 1. We now turn to the expression of the total variation intensity. According to Theorem 1, $\lim_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x = |D_u f|(U)$ a.s. and, by Lemma 1, for all $r \neq 0$, $\int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x \leq |D_u f|(U)$ a.s. Hence, if the RV $|D_u f|(U)$ is integrable, (7) is obtained by dominated convergence and Fubini's theorem. If $\mathbb{E}(|D_u f|(U)) = \Theta_{V_u}(f, U) = +\infty$ then (7) is still valid since, by Fatou's lemma,

$$\begin{aligned} +\infty = \Theta_{V_u}(f, U) &= \mathbb{E}\left(\liminf_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x\right) \\ &\leq \liminf_{r\to 0} \mathbb{E}\left(\int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x\right). \end{aligned}$$

□

Remark 3. (*Degenerate case.*) Equation (7) can be extended to the degenerate case where the limit on the right-hand side is infinite. However, we specify that this degenerate case is more subtle than in the deterministic case (see Theorem 1). Indeed, there are two different cases for which the limit of (7) is infinite: either the RF f is not of bounded variation or f is of bounded variation but its total variation $|D_u f|(U)$ has infinite expectation. In both cases, it is coherent to say that the mean total variation of the process is infinite. This convention will be used in the remainder of the paper.

As mentioned in the introduction, we are aware of only one result dealing with the variation of RFs defined over $\mathbb{R}^d$ for $d \geq 2$. This result is due to Ibragimov [15] and gives a sufficient condition for an RF to be of bounded variation.

Theorem 2. (Ibragimov's theorem [15].) *Let $U \Subset \mathbb{R}^d$, and let $f : \Omega \times U \to \mathbb{R}$ be a measurable and separable random field. Suppose that there exists $x_0 \in U$ such that $\mathbb{E}(f(x_0)) < +\infty$ and that there exists $K > 0$ such that, for any $x, y \in U$, $\mathbb{E}(|f(x) - f(y)|) \leq K|x - y|$. Then the realizations of f are a.s. in $\mathrm{BV}(U)$, and there is a constant $C > 0$ such that, for all $A \in \mathcal{B}(U)$, $\Theta_V(f, A) \leq C\mathcal{L}^d(A)$.*

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

We establish results similar to Ibragimov's theorem for both RFs of bounded variation and RFs of bounded directional variation.

Proposition 4. (A sufficient condition for locally bounded directional variation.) *Let $f\colon \Omega \times U \to \mathbb{R}$ be a jointly measurable RF such that $f(\omega,\cdot) \in L^1_{\mathrm{loc}}(U)$ a.s.*

- *(Directional case.) Let $u \in S^{d-1}$. Suppose that there exists a constant $K > 0$ such that, for all $r \in \mathbb{R}$ and $x \in U \ominus [0, ru]$,*

$$\mathbb{E}(|f(x+ru) - f(x)|) \le K|r|. \tag{8}$$

  Then, f is an RF of locally bounded directional variation in the direction u and for all $W \Subset U$, $\Theta_{V_u}(f, W) \le K\mathcal{L}^d(W)$.

- *(Nondirectional case.) Suppose that there exists a constant $K > 0$ such that, for all $x, y \in U$, $\mathbb{E}(|f(x) - f(y)|) \le K|x-y|$. Then, f is an RF of locally bounded variation, and, for all $W \Subset U$, $\Theta_V(f, W) \le (d\omega_d/2\omega_{d-1})K\mathcal{L}^d(W)$.*

Proof. Let $W \Subset U$. By Fatou's lemma and Fubini's theorem,

$$\begin{aligned}\mathbb{E}\left(\liminf_{r\to 0}\int_{W\ominus[0,ru]}|f(x;ru)|\,\mathrm{d}x\right) &\le \liminf_{r\to 0}\mathbb{E}\left(\int_{W\ominus[0,ru]}|f(x;ru)|\,\mathrm{d}x\right)\\ &\le K\liminf_{r\to 0}\mathcal{L}^d(W\ominus[0,ru])\\ &\le K\mathcal{L}^d(W)\\ &< +\infty.\end{aligned}$$

In particular, $\liminf \int_{W\ominus[0,ru]}|f(x;ru)|\,\mathrm{d}x < +\infty$ a.s. and, thus, $f \in \mathrm{BV}_u(W)$ a.s. by Proposition 3. This is valid for all $W \Subset U$, and, thus, $f \in \mathrm{BV}_{u,\mathrm{loc}}(U)$ a.s. The inequality $\Theta_{V_u}(f, W) \le K\mathcal{L}^d(W)$ is immediate using (7).

Now consider the nondirectional case. If, for all $x, y \in U$, $\mathbb{E}(|f(x) - f(y)|) \le K|x-y|$, then the first part of the proposition shows that f has a.s. bounded directional variation in the d directions of the canonical basis, and, thus, $f \in \mathrm{BV}(U)$ a.s. In addition, for all $W \Subset U$ and all $u \in S^{d-1}$, $\Theta_{V_u}(f, W) \le K\mathcal{L}^d(W)$. Hence, by Proposition 1, for all $U \Subset \mathbb{R}^d$,

$$\Theta_V(f, W) = \frac{1}{2\omega_{d-1}}\int_{S^{d-1}}\Theta_{V_u}(f, W)\mathcal{H}^{d-1}(\mathrm{d}u) \le \frac{d\omega_d}{2\omega_{d-1}}K\mathcal{L}^d(W),$$

where we used the fact that $\mathcal{H}^{d-1}(S^{d-1}) = \mathrm{d}\omega_d$. □

In the following, (8) will be referred to as *Lipschitzness in mean in the direction u*. We specify that the upper bounds given in Proposition 4 are optimal. Indeed, we will show later (see Corollary 1) that any RF f with stationary increments and locally bounded directional variation satisfies $\Theta_{V_u}(f, W) = K\mathcal{L}^d(W)$ for some constant K (and the upper bound on $\Theta_V(f, W)$ is shown to be reached for isotropic RFs with stationary increments).

4. Variation intensity of random fields with stationary increments

Let $f\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a jointly measurable RF f, which is said to have stationary increments or to be a stationary increment random field if, for all $y \in \mathbb{R}^d$, both RFs $x \mapsto f(x+y) - f(y)$ and $x \mapsto f(x) - f(0)$ have the same finite-dimensional distributions. The next theorem, which constitutes the main result of the paper, defines and gives an expression for the directional variation intensity $\theta_{V_u}(f)$ of a stationary increment RF f.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Theorem 3. (Definition and computation of the (directional) variation intensity of stationary increment RF.) *Let $u \in S^{d-1}$, and let $f\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a jointly measurable stationary increment RF such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. The following assertions are equivalent.*

(i) *$f \in \mathrm{BV}_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. and its directional variation intensity measure $\Theta_{V_u}(f,\cdot)$ is locally finite.*

(ii) *$f \in \mathrm{BV}_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. and its directional variation intensity measure $\Theta_{V_u}(f,\cdot)$ is proportional to the Lebesgue measure: there exists a constant $\theta_{V_u}(f) \geq 0$ such that, for all $A \in \mathcal{B}(\mathbb{R}^d)$, $\Theta_{V_u}(f,A) = \theta_{V_u}(f)\mathcal{L}^d(A)$.*

(iii) *$\liminf_{r\to 0} \mathbb{E}(|f(0;ru)|) < +\infty$.*

(iv) *$\lim_{r\to 0} \mathbb{E}(|f(0;ru)|)$ exists and is finite.*

If any of the above assertions holds, the constant of proportionality $\theta_{V_u}(f)$ is given by

$$\theta_{V_u}(f) = \liminf_{r\to 0} \mathbb{E}(|f(0;ru)|) = \lim_{r\to 0} \mathbb{E}(|f(0;ru)|)$$

and is called the directional variation intensity of f in the direction u.

Regarding nondirectional variation, $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. with a locally finite variation intensity measure $\Theta_V(f,\cdot)$ if and only if, for all $u \in S^{d-1}$, the limit $\lim_{r\to 0} \mathbb{E}(|f(0;ru)|)$ exists and is finite. In this case, the variation intensity measure $\Theta_V(f,\cdot)$ is proportional to the Lebesgue measure, and the constant of proportionality $\theta_V(f)$, called the variation intensity of f, is given by

$$\begin{aligned}\theta_V(f) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \theta_{V_u}(f)\mathcal{H}^{d-1}(\mathrm{d}u) &= \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \lim_{r\to 0} \mathbb{E}(|f(0;ru)|)\mathcal{H}^{d-1}(\mathrm{d}u) \\ &= \lim_{r\to 0} \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \mathbb{E}(|f(0;ru)|)\mathcal{H}^{d-1}(\mathrm{d}u). \quad (9)\end{aligned}$$

Proof. We will show the following chain of implications: (ii) $\Leftrightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i). First, note that the implications (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are trivial. Since f has stationary increments, thanks to Fubini's theorem, for all $U \Subset \mathbb{R}^d$ and $r \neq 0$,

$$\mathbb{E}\left(\int_{U\ominus[0,ru]} |f(x;ru)|\,\mathrm{d}x\right) = \mathbb{E}(|f(0;ru)|)\mathcal{L}^d(U\ominus[0,ru]). \quad (10)$$

Let us consider the implications (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (ii). Suppose that $f \in \mathrm{BV}_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. and that $\Theta_{V_u}(f,\cdot)$ is locally finite. Let $U \Subset \mathbb{R}^d$. According to Lemma 1,

$$\int_{U\ominus[0,ru]} |f(x;ru)|\,\mathrm{d}x \leq |D_u f|(U).$$

By hypothesis, $|D_u f|(U)$ is an L^1-RV. Hence, we can apply the reverse Fatou lemma,

$$\begin{aligned}\limsup_{r\to 0} \mathbb{E}(|f(0;ru)|)\mathcal{L}^d(U) &= \limsup_{r\to 0} \mathbb{E}(|f(0;ru)|)\mathcal{L}^d(U\ominus[0,ru]) \\ &= \limsup_{r\to 0} \mathbb{E}\left(\int_{U\ominus[0,ru]} |f(x;ru)|\,\mathrm{d}x\right) \\ &\leq \mathbb{E}\left(\limsup_{r\to 0} \int_{U\ominus[0,ru]} |f(x;ru)|\,\mathrm{d}x\right)\end{aligned}$$

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

$$\begin{aligned}&\leq \mathbb{E}(|D_u f|(U))\\&= \Theta_{V_u}(f, U)\\&< +\infty.\end{aligned}$$

Besides, by Fatou's lemma,

$$\begin{aligned}\Theta_{V_u}(f, U) &= \mathbb{E}(|D_u f|(U))\\&= \mathbb{E}\left(\liminf_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x\right)\\&\leq \liminf_{r\to 0} \mathbb{E}(|f(0; ru)|)\mathcal{L}^d(U \ominus [0, ru])\\&\leq \liminf_{r\to 0} \mathbb{E}(|f(0; ru)|)\mathcal{L}^d(U).\end{aligned}$$

Hence, we have shown that, for all $U \Subset \mathbb{R}^d$,

$$\limsup_{r\to 0} \mathbb{E}(|f(0; ru)|)\mathcal{L}^d(U) \leq \Theta_{V_u}(f, U) \leq \liminf_{r\to 0} \mathbb{E}(|f(0; ru)|)\mathcal{L}^d(U).$$

This shows that $\mathbb{E}(|f(0; ru)|)$ has a finite limit, say $\theta_{V_u}(f)$, and that for all bounded open sets U, $\Theta_{V_u}(f, U) = \theta_{V_u}(f)\mathcal{L}^d(U)$. This equality extends to all Borel sets $A \in \mathcal{B}(\mathbb{R}^d)$ and, thus, Θ_{V_u} is proportional to the Lebesgue measure.

It remains to show that (iii) $\Rightarrow$ (i). Thanks to Fatou's lemma and (10),

$$\begin{aligned}\mathbb{E}\left(\liminf_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x\right) &\leq \liminf_{r\to 0} \mathbb{E}(|f(0; ru)|)\mathcal{L}^d(U \ominus [0, ru])\\&\leq \mathcal{L}^d(U) \liminf_{r\to 0} \mathbb{E}(|f(0; ru)|)\\&< +\infty.\end{aligned}$$

In particular, $\liminf_{r\to 0} \int_{U\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x$ is a.s. finite, and, thus, by Proposition 3, $f \in \mathrm{BV}_u(U)$ a.s. This is valid for all $U \Subset \mathbb{R}^d$, and, thus, $f \in \mathrm{BV}_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. Besides, by (5), the above equation can be written as $\Theta_{V_u}(f, U) \leq \liminf_{r\to 0} \mathbb{E}(|f(0; ru)|)\mathcal{L}^d(U) < +\infty$, and, thus, $\Theta_{V_u}(f, \cdot)$ is locally finite.

Regarding nondirectional variation, the result is a straightforward consequence of the above equivalence and Proposition 1, which becomes (9) in this context. The fact that the limit and the integral commute follows from the bounded convergence theorem using the bound

$$\begin{aligned}\mathbb{E}(|f(0; ru)|) &\leq \theta_{V_u}(f) \frac{\mathcal{L}^d(B, (0, 1))}{\mathcal{L}^d(B(0, 1) \ominus [0, ru])}\\&\leq \theta_{V_u}(f) \frac{1}{(1 - R)^d}, \qquad r \in [-R, R] \setminus \{0\},\ R < 1,\end{aligned}$$

given by Lemma 1 and Theorem 3. □

Remark 4. (*Degenerate case.*) As for Proposition 3, we extend the definition of $\theta_{V_u}(f)$ to the degenerate case, where $\lim_{r\to 0} \mathbb{E}(|f(0; ru)|) = +\infty$.

We remark that the directional variation intensities $\theta_{V_u}(f)$, $u \in S^{d-1}$, and the variation intensity $\theta_V(f)$ depend only on the two-dimensional distributions of the stationary increment

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

RF f. In addition, if the RF f has a.s. $\mathcal{C}^1$ sample paths, then we simply have $\theta_V(f) = \mathbb{E}(\|\nabla f(0)\|)$ and $\theta_{V_u}(f) = \mathbb{E}(|\langle \nabla f(0), u\rangle|)$.

To complete this section, we show that the directional Lipschitzness in mean introduced in Section 3 is a necessary and sufficient condition for a stationary increment RF to have finite-directional variation intensity.

Corollary 1. (Directional Lipschitzness in mean of a stationary increment RF.) *Let $u \in S^{d-1}$ and $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a jointly measurable stationary increment RF such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. Then, for all $x \in \mathbb{R}^d$ and $r \in \mathbb{R}$, $\mathbb{E}(|f(x + ru) - f(x)|) \le \theta_{V_u}(f)|r|$.*

Proof. First, since f has stationary increments, for all $x \in \mathbb{R}^d$ and $r \in \mathbb{R}$, $\mathbb{E}(|f(x + ru) - f(x)|) = \mathbb{E}(|f(ru) - f(0)|)$. Let $r \neq 0$ and $\rho > 0$. By Lemma 1,

$$\int_{B(0,\rho)\ominus[0,ru]} |f(x; ru)|\,\mathrm{d}x \le |D_u f|(B(0, \rho)) \quad \text{a.s.}$$

Hence, by Fubini's theorem, $\mathbb{E}(|f(0; ru)|)\mathcal{L}^d(B(0, \rho) \ominus [0, ru]) \le \theta_{V_u}(f)\mathcal{L}^d(B(0, \rho))$ and, thus, $\mathbb{E}(|f(0; ru)|) \le \theta_{V_u}(f)(\mathcal{L}^d(B(0, \rho))/\mathcal{L}^d(B(0, \rho) \ominus [0, ru]))$. Letting $\rho \to +\infty$ we obtain the announced inequality. □

Remark 5. Combining the results of Proposition 4, and Corollary 1, we find that a stationary increment RF has finite-directional variation intensity if and only if it is directionally Lipschitz in mean. We also mention that a local Lipschitzness in mean in a neighborhood of the origin is sufficient. Indeed, if there exists a constant $K > 0$ and a real $R > 0$ such that, for all $r \in [0, R)$, $\mathbb{E}(|f(ru) - f(0)|) \le K|r|$, then $\theta_{V_u}(f) = \liminf_{r\to 0} \mathbb{E}(|f(0; ru)|) \le K < +\infty$.

As already mentioned in Section 3, Corollary 1 shows that the first upper bound of Proposition 4 becomes an equality for any stationary increment RF of bounded directional variation having a finite-directional variation intensity.

5. The case of random sets

In this section we apply the general results of the previous sections to the special case of random indicator functions. This enables us to obtain all the results of [10] linking the covariogram and the mean perimeter of sets as direct corollaries (recall that these formulas were first established by Matheron and are widely stated in the mathematical morphology literature ([18], [19], [26]).

The classical framework for the study of random sets is that of random closed sets (RACS) [21]. The framework of random integrable functions gives another framework for random sets, namely, that of random measurable sets (RAMS). We refer the reader to [12] for a discussion regarding the links between RAMS and RACS. By definition, a RAMS X is a measurable map $X : (\Omega, \mathcal{A}) \to (\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathcal{B}(\mathbb{R}^d)))$, where $\mathcal{B}(\mathcal{B}(\mathbb{R}^d))$ denotes the Borel σ-algebra induced by the local convergence in measure (the local convergence in measure of a sequence $(A)_n$ towards A simply corresponds to the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ of the indicator functions $\mathbf{1}_{A_n}$ towards $\mathbf{1}_A$). We recall that the (variational) perimeter $\mathrm{Per}(A)$ of a measurable set A is defined as $\mathrm{Per}(A) = |D\mathbf{1}_A|(\mathbb{R}^d)$ if $\mathbf{1}_A \in \mathrm{BV}(\mathbb{R}^d)$, and $\mathrm{Per}(A) + \infty$ otherwise. Similarly, the directional variation $V_u(A)$ in the direction $u \in S^{d-1}$ of A is $V_u(A) = |D_u\mathbf{1}_A|(\mathbb{R}^d)$ if $\mathbf{1}_A \in \mathrm{BV}_u(\mathbb{R}^d)$ and $+\infty$ otherwise.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

We first consider a RAMS X for which $\mathbb{E}(\mathcal{L}^d(X)) < +\infty$. The *mean covariogram* γ_X of X is the function $\gamma_X : \mathbb{R}^d \to [0, \infty)$ defined by

$$\gamma_X(y) = \mathbb{E}(\mathcal{L}^d(X \cap (y + X))) = \int_{\Omega \times \mathbb{R}^d} \mathbf{1}_{X(\omega)}(x)\mathbf{1}_{X(\omega)}(x + y)\mathbb{P}(d\omega)\,\mathrm{d}x.$$

Thanks to the identity $\gamma_X(0) - \gamma_X(y) = \frac{1}{2}\mathbb{E}(\int_{\mathbb{R}^d} |\mathbf{1}_X(x+y) - \mathbf{1}_X(x)|\,\mathrm{d}x)$ due to Matheron [10], the identity of Proposition 3 can be written as

$$\lim_{r \to 0} \frac{\gamma_X(0) - \gamma_X(ru)}{|r|} = \frac{1}{2}\mathbb{E}(V_u(X)). \tag{11}$$

Noting that $(\gamma_X^u)'(0) = \lim_{r \to 0^+}(\gamma_X(ru) - \gamma_X(0))/r$, by integration over all the directions $u \in S^{d-1}$ and by applying the dominated convergence theorem, we have

$$-\frac{1}{\omega_{d-1}} \int_{S^{d-1}} (\gamma_X^u)'(0)\mathcal{H}^{d-1}(\mathrm{d}u) = \mathbb{E}(\mathrm{Per}(X)).$$

Hence, the main results of [10] are also valid in the more general framework of RAMS (see also [12] for equivalent formulas for the local covariogram).

We now discuss the case of stationary random sets. We call *stationary* any random set $X : \Omega \to \mathcal{B}(\mathbb{R}^d)$ such that the map $(\omega, x) \mapsto \mathbf{1}_{X(\omega)}(x)$ is a jointly measurable and stationary RF. A stationary random set is said to be of *locally bounded variation* if the stationary RF $\mathbf{1}_X : (\omega, x) \mapsto \mathbf{1}_{X(\omega)}(x)$ is of locally bounded variation, and we define random sets of *locally bounded directional variation* similarly. We write $\theta_V(X) := \theta_V(\mathbf{1}_X)$, which is referred to as the *variation intensity* or the *specific variation* of the stationary random set X. Similarly, $\theta_{V_u}(X) := \theta_{V_u}(\mathbf{1}_X)$ is called the *directional variation intensity* or the *specific directional variation* in the direction u. Let X be a stationary random set. The *variogram* ν_X of X is the function $\nu_X : \mathbb{R}^d \to \mathbb{R}$ defined by $\nu_X(y) = \mathbb{P}(y \in X,\ 0 \notin X)$. We remark that, since X is a random set, $|\mathbf{1}_X(ru) - \mathbf{1}_X(0)| \in \{0, 1\}$, and, thus, $\mathbb{E}(|\mathbf{1}_X(ru) - \mathbf{1}_X(0)|) = \mathbb{P}(ru \in X,\ 0 \notin X) + \mathbb{P}(ru \notin X,\ 0 \in X) = 2\nu_X(ru)$. By Theorem 3, for all $u \in S^{d-1}$, the limit $(\nu_X^u)'(0) := \lim_{r \to 0}(1/|r|)\nu_X(ru) \in [0, +\infty]$ exists, and the specific directional variation $\theta_{V_u}(X)$ is given by

$$\theta_{V_u}(X) = 2(\nu_X^u)'(0) = 2 \lim_{r \to 0} \frac{1}{|r|}\mathbb{P}(ru \in X,\ 0 \notin X), \tag{12}$$

showing that the specific directional variation is twice the directional derivative of the variogram at the origin [10]. Integrating over all directions, we obtain

$$\theta_V(X) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} (\nu_X^u)'(0)\mathcal{H}^{d-1}(\mathrm{d}u).$$

This shows that Theorem 17 of[10] extends to jointly measurable stationary indicator functions that are not necessarily closed sets.

To complete this section, we establish a mean corea formula for random excursion sets. First, we recall the coarea formula for deterministic functions. In what follows, U is an open subset of $\mathbb{R}^d$. Recall that, for any measurable set A, we define the *perimeter* of A in U as the variation of the indicator function $\mathbf{1}_A$ in U, and we write $\mathrm{Per}(A, U) := |D\mathbf{1}_A|(U)$. Similarly, we define $V_u(A, U) := |D_u\mathbf{1}_A|(U)$, the directional variation of A in U. If $f : U \to \mathbb{R}$ is a measurable function and $t \in \mathbb{R}$, $\{f > t\}$ denotes the set $\{x \in U,\ f(x) > t\}$ and is called the *(upper) level set* of level t of f.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Proposition 5. (Coarea formula.) *Let $f \in L^1(U)$. Then, $f \in \mathrm{BV}(U)$ if and only if the sets $\{f > t\}$ are of finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and the function $t \mapsto \mathrm{Per}(\{f > t\}, U)$ is in $L^1(\mathbb{R})$, and in this case $|Df|(U) = \int_{-\infty}^{+\infty} \mathrm{Per}(\{f > t\}, U)\,\mathrm{d}t$.*

We refer the reader to [2, p. 145] for the proof of the coarea formula. We mention that the coarea formula remains valid if the upper level sets are replaced by other level sets: $\{f \geq t\}$, $\{f < t\}$, or $\{f \leq t\}$. Besides, a coarea formula also holds for directional variation: $|D_u f|(U) = \int_{-\infty}^{+\infty} D_u(\{f > t\}, U)\,\mathrm{d}t$, $u \in S^{d-1}$. Using the coarea formula, we can obtain a relation between the variation intensity $\Theta_{V_u}(f, U) = \mathbb{E}(|D_u f|(U))$ of f and the variation intensity of its level sets.

Proposition 6. (Coarea formula for the total variation intensity.) *Let f be an RF a.s. in $L^1(U)$, and let $u \in S^{d-1}$. Then $f \in \mathrm{BV}_u(U)$ a.s. with finite mean total directional variation $\Theta_{V_u}(f, U)$ if and only if, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, its level sets $\{f > t\}$ have a.s. finite-directional variation in U in the direction u and $t \mapsto \Theta_{V_u}(\{f > t\}, U)$ is in $L^1(\mathbb{R})$, and in this case $\Theta_{V_u}(f, U) = \int_{-\infty}^{+\infty} \Theta_{V_u}(\{f > t\}, U)\,\mathrm{d}t$. Similarly, $f \in \mathrm{BV}(U)$ a.s. with finite mean total variation if and only if, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, its level sets $\{f > t\}$ have a.s. finite variation and $t \mapsto \Theta_V(\{f > t\}, U)$ is in $L^1(\mathbb{R})$, and in this case $\Theta_V(f, U) = \int_{-\infty}^{+\infty} \Theta_V(\{f > t\}, U)\,\mathrm{d}t$.*

Proof. The proof consists of applying the Fubini and Lebesgue theorems. First, we prove that the function $g\colon \Omega \times U \times \mathbb{R} \to \{0, 1\}$ defined by $g(\omega, x, t) = \mathbf{1}_{\{f>t\}}(\omega, x)$ is measurable. Let $(t_n)_{n\in\mathbb{N}}$ be a dense sequence of $\mathbb{R}$. Then we can easily check that

$$g^{-1}(1) = \bigcup_{n\in\mathbb{N}} \{(\omega, x),\ f(\omega, x) > t_n\} \times (t_n, +\infty),$$

which is in the product σ-algebra $\mathcal{A} \otimes \mathcal{B}(U) \otimes \mathcal{B}(\mathbb{R})$ since f is jointly measurable. Second, we have the elementary identity

$$|f(\omega, x + ru) - f(\omega, x)| = \int_{-\infty}^{+\infty} |\mathbf{1}_{\{f>t\}}(\omega, x + ru) - \mathbf{1}_{\{f>t\}}(\omega, x)|\,\mathrm{d}t.$$

Hence, by Fubini's theorem,

$$\begin{aligned}\int_{U\ominus[0,ru]} &\mathbb{E}(|f(x + ru) - f(x)|)\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \int_{U\ominus[0,ru]} \mathbb{E}(|\mathbf{1}_{\{f>t\}}(x + ru) - \mathbf{1}_{\{f>t\}}(x)|)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We now suppose that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the level sets $\{f > t\}$ have a.s. finite-directional variation in the direction u and that $t \mapsto \Theta_{V_u}(\{f > t\}, U)$ is in $L^1(\mathbb{R})$. Then, by Proposition 3, $\lim_{r\to 0} \int_{U\ominus[0,ru]} \mathbb{E}(|\mathbf{1}_{\{f>t\}}(x; ru)|)\,\mathrm{d}x = \Theta_{V_u}(\{f > t\}, U)$ for $\mathcal{L}^1$-a.e. t, and, by Lemma 1, $\int_{U\ominus[0,ru]} \mathbb{E}(|f(x; ru)|)\,\mathrm{d}x \leq \Theta_{V_u}(\{f > t\}, U) \in L^1(\mathbb{R})$. Hence, Lebesgue's theorem applies and $\lim_{r\to 0} \int_{U\ominus[0,ru]} \mathbb{E}(|f(x; ru)|)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \Theta_{V_u}(\{f > t\}, U)\,\mathrm{d}t < +\infty$. By Proposition 3, we deduce that f has a.s. bounded directional variation and that $\Theta_{V_u}(f, U) = \int_{-\infty}^{+\infty} \Theta_{V_u}(\{f > t\}, U)\,\mathrm{d}t$. We now prove the converse implication. Suppose that $f \in \mathrm{BV}_u(U)$ a.s. with

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

$\Theta_{V_u}(f, U) < +\infty$. Then, by Fatou's lemma and Fubini's theorem,

$$\begin{aligned}\int_{-\infty}^{+\infty} \liminf_{r\to 0} \int_{U\ominus[0,ru]} \mathbb{E}(|\mathbf{1}_{\{f>t\}}(x; ru)|)\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{r\to 0} \int_{U\ominus[0,ru]} \mathbb{E}(|f(x; ru)|)\,\mathrm{d}x \\ &= \Theta_{V_u}(f, U) \\ &< +\infty.\end{aligned}$$

In particular, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, $\liminf_{r\to 0} \int_{U\ominus[0,ru]} |\mathbf{1}_{\{f>t\}}(x; ru)|\,\mathrm{d}x < +\infty$ a.s. Hence, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, Proposition 3 ensures that $\{f > t\}$ has a.s. locally bounded variation in U in the direction u, and that $\Theta_{V_u}(\{f > t\}, U) = \lim_{r\to 0} \int_{U\ominus[0,ru]} \mathbb{E}(|\mathbf{1}_{\{f>t\}}(x; ru)|)\,\mathrm{d}x$. Besides, the above inequality shows that $\int_{-\infty}^{+\infty} \Theta_{V_u}(\{f > t\}, U) \le \Theta_{V_u}(f, U) < +\infty$, that is, $t \mapsto \Theta_{V_u}(\{f > t\}, U) \in L^1(\mathbb{R})$. To finish, the case of nondirectional variation easily follows from the integral geometric formula of Proposition 1 and Fubini's theorem. □

Remark 6. Since the level sets $\{f > t\}$, $t \in \mathbb{R}$, of a stationary RF f are stationary random sets, when dealing with stationary RFs, Proposition 6 gives a similar result for variation intensities, that is,

$$\theta_{V_u}(f) = \int_{-\infty}^{+\infty} \theta_{V_u}(\{f > t\})\,\mathrm{d}t \quad \text{and} \quad \theta_V(f) = \int_{-\infty}^{+\infty} \theta_V(\{f > t\})\,\mathrm{d}t. \tag{13}$$

6. Illustration: variation intensities of some classical random field models

In this section we illustrate the different results of Sections 4 and 5 by computing the variation intensities of various RFs and random set models, namely Gaussian random fields, Gaussian excursion sets, Poisson shot noise of random sets, Boolean models, colored dead leaves RFs, and colored tessellations. Most of the obtained formulas remain valid in the degenerate cases, and, thus, provide a characterization for the finiteness of the variation intensities of the considered RF or random set models.

6.1. Gaussian random fields and Gaussian excursion sets

A jointly measurable RF f_{G} is a *stationary Gaussian RF* with mean $\mu \in \mathbb{R}$ and covariance function $C\colon \mathbb{R}^d \to \mathbb{R}$ if, for all $p \in \mathbb{N}$, $x_1, \dots, x_p \in \mathbb{R}^d$, and $w_1, \dots, w_p \in \mathbb{R}$, the RV $\sum_{i=1}^p w_i f_{\mathrm{G}}(x_i)$ is normal with mean $\sum_{i=1}^p w_i \mu$ and variance $\sum_{i,j=1}^p w_i w_j C(x_j - x_i)$. We mention that the measurability assumption is not restrictive since, in what follows, we will consider only stationary Gaussian RFs whose covariance function is regular at the origin, and as soon as the covariance function C is continuous in 0, the RF is continuous in probability and, thus, it has a measurable version [13, p. 171]. For all $t \in \mathbb{R}$, we consider the random set $\{f_{\mathrm{G}} > t\}$, called the Gaussian excursion set, and we denote by ν_t its variogram, that is, the function defined for all $y \in \mathbb{R}^d$ by $\nu_t(y) = \mathbb{P}(f_{\mathrm{G}}(y) > t, f_{\mathrm{G}}(0) \le t)$. As shown in [18, Proposition 16.1.1],

$$\nu_t(y) = \frac{1}{\pi} \int_0^{\arcsin(\sqrt{(C(0)-C(y)}/2C(0))} \exp\left(-\frac{t^2}{2C(0)}(1 + \tan^2 s)\right) \mathrm{d}s. \tag{14}$$

From this expression, Lantuéjoul asserted that the excursion sets of f_{G} have finite 'specific perimeter' if and only if $C(0) - C(y)$ is proportional to $|y|^2$ [18, p. 207]. Our next proposition completes this observation in computing the expression of the specific variations of the Gaussian excursion sets $\{f_{\mathrm{G}} > t\}$, as well as the variation intensity of the Gaussian RF f_{G}.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Proposition 7. (i) *Variation intensity of a stationary Gaussian RF. Let f_G be a stationary Gaussian RF with mean μ and covariance C, and let $u \in S^{d-1}$.*

- *(Variation intensity.) It holds that f_G has finite-directional variation intensity $\theta_{V_u}(f_G)$ in the direction u if and only if the one-dimensional restriction of the covariance $C_u : r \mapsto C(ru)$ is twice differentiable at 0, and in this case, noting that $C_u''(0)$ is the second derivative in 0 of C_u, $\theta_{V_u}(f_G) = (-2C_u''(0)/\pi)^{1/2}$. Consequently, f_G has finite variation intensity $\theta_V(f_G)$ if and only if, for all $u \in S^{d-1}$, the one-dimensional restrictions of the covariance $C_u : r \mapsto C(ru)$ are twice differentiable at 0, and in this case, $\theta_V(f_G) = (1/2\omega_{d-1})\int_{S^{d-1}} (-2C_u''(0)/\pi)^{1/2}\mathcal{H}^{d-1}(\mathrm{d}u)$.*
- *(Variation intensity of excursion sets.) For all $t \in \mathbb{R}$, $\{f_G > t\}$ has finite specific directional variation in the direction u if and only if the one-dimensional restriction C_u of the covariance is twice differentiable at 0, and, in this case,*

$$\theta_{V_u}(\{f_G > t\}) = \sqrt{\frac{-2C_u''(0)}{\pi}}\frac{1}{\sqrt{2\pi C(0)}}\exp\left(-\frac{t^2}{2C(0)}\right) = \theta_{V_u}(f_G)p_{f_G}(t),$$

where $p_{f_G}(t) = (2\pi C(0))^{-1/2}\exp(-t^2/2C(0))$ is the first-order density of the RF f_G. Consequently, $\{f_G > t\}$ has finite specific variation in the direction u if and only if, for all $u \in S^{d-1}$, the one-dimensional restrictions $C_u : r \mapsto C(ru)$ are twice differentiable at 0, and, in this case, $\theta_V(\{f_G > t\}) = \theta_V(f_G)p_{f_G}(t)$.

Proof. This is a straightforward application of Theorem 3. For all $u \in S^{d-1}$ and $r \in \mathbb{R}$, $f_G(ru) - f_G(0)$ follows a normal distribution with mean 0 and variance $2(C(0) - C(ru))$, and, thus, $\mathbb{E}(|f_G(ru) - f_G(0)|) = (2/\sqrt{\pi})\sqrt{C(0) - C(ru)}$. Hence, $\lim_{r\to 0}\mathbb{E}(|f_G(0; ru)|)$ exists if and only if $(C(0) - C(ru))/r^2$ admits a limit in 0, that is, if and only if $C_u : r \mapsto C(ru)$ is twice differentiable at 0 with $C_u'(0) = 0$. But, since C_u is even, if it is differentiable at 0 then necessarily $C_u'(0) = 0$. As for the expression of $\theta_{V_u}(f_G)$, note that $\lim_{r\to 0}(C(0) - C(ru))/r^2 = -C_u''(0)/2$; hence, by Theorem 3 $\theta_{V_u}(f_G) = \lim_{r\to 0}\mathbb{E}(|f_G(0; ru)|) = \sqrt{-2C_u''(0)/\pi}$. The case of nondirectional variation follows by (9).

We now consider excursion sets. By (12), $\theta_{V_u}(\{f_G > t\}) = 2\lim_{r\to 0} v_t(ru)/|r|$. The function $h : x \mapsto (1/\pi)\int_0^{\arcsin(x)} \exp(-(t^2/2C(0))(1 + \tan^2 s))\,\mathrm{d}s$ is $\mathcal{C}^1$ at $x = 0$ and

$$h'(0) = \arcsin'(0)\frac{1}{\pi}\exp\left(-\frac{t^2}{2C(0)}(1 + \tan^2(\arcsin(0)))\right) = \frac{1}{\pi}\exp\left(-\frac{t^2}{2C(0)}\right).$$

Hence, thanks to (14), we deduce that

$$\begin{aligned}\theta_{V_u}(\{f_G > t\}) &= 2\lim_{r\to 0}\frac{v_t(ru)}{|r|}\\ &= \frac{2}{\pi}\exp\left(-\frac{t^2}{2C(0)}\right)\lim_{r\to 0}\sqrt{\frac{C(0) - C(ru)}{2C(0)r^2}}\\ &= \frac{2}{\pi}\exp\left(-\frac{t^2}{2C(0)}\right)\sqrt{\frac{-C_u''(0)}{4C(0)}}.\end{aligned}$$

□

Remark 7. (*Regularity of a Gaussian RF.*) It is worth noting that the necessary and sufficient condition for a stationary Gaussian RF to be of locally bounded variation implies a stronger

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

regularity on the sample paths than just being of locally bounded variation. First, the differentiability at the origin of the covariance implies that there exist $\rho > 0$ and $K > 0$ such that, for all $x \in B(0,\rho)$, $C(0) - C(x) \leq K|x|$, and, thus, the sample paths of a stationary Gaussian RF f_{G} are Hölder continuous for all exponents $s < \frac{1}{2}$ (see [6, Proposition 5.2] which summarizes results from [1]). In addition, according to [24], if the covariance function C is twice differentiable at the origin in every direction, then the sample paths of a stationary Gaussian RF f_{G} are a.s. in the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d) \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$. Hence, by Proposition 7, $\theta_V(f_{\mathrm{G}}) < +\infty$ implies that $f_{\mathrm{G}} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ a.s., and, consequently, its variation $|Df_{\mathrm{G}}|$ is absolutely continuous with respect to the Lebesgue measure. Hence, if $f_{\mathrm{G}} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. and has a finite variation intensity, then it is a 'smooth' function among the functions of locally bounded variation since its variation measure Df_{G} has neither a jump part nor a Cantor part (see [2, Section 3.9] for more details on the decomposition of the variation measure Df of functions of bounded variation). This shows that the space of functions of locally bounded variation is not of interest for a stationary Gaussian RF.

Remark 8. (*Regularity of Gaussian excursion sets.*) For all $t \in \mathbb{R}$, we have

$$\theta_{V_u}(\{f_{\mathrm{G}} > t\}) = \theta_{V_u}(f_{\mathrm{G}})p_{f_{\mathrm{G}}}(t) \quad \text{and} \quad \theta_V(\{f_{\mathrm{G}} > t\}) = \theta_V(f_{\mathrm{G}})p_{f_{\mathrm{G}}}(t),$$

and we can check that the coarea formulas of (13) hold. These formulas show that the regularity of a stationary Gaussian RF corresponds to that of its level sets: a stationary Gaussian RF has finite variation intensity if and only if all its level sets have finite variation intensities if and only if at least one of its level sets has finite variation intensity. We also note that $\theta_V(\{f_{\mathrm{G}} > t\}) = \theta_V(f_{\mathrm{G}})p_{f_{\mathrm{G}}}(t)$ is in accordance with the 'Rice formula for the expectation of the geometric measure of a level set' derived by Azaïs and Wschebor [4, Theorem 6.8]. Indeed, under some smoothness assumption on the paths of f_{G}, thanks to the stationarity of f_{G}, we have [4, Theorem 6.8] $\mathbb{E}(\mathcal{H}^{d-1}(\{y,\ f_{\mathrm{G}}(y) = t\} \cap (0,1)^d)) = \mathbb{E}(\|\nabla f(0)\| | f(0) = t)p_{f_{\mathrm{G}}}(t)$. But since, for a stationary Gaussian RF, $\nabla f(0)$ and $f(0)$ are independent, we can obtain

$$\mathbb{E}(\mathcal{H}^{d-1}(\{y,\ f_{\mathrm{G}}(y) = t\} \cap (0,1)^d)) = \mathbb{E}(\|\nabla f(0)\|)p_{f_{\mathrm{G}}}(t) = \theta_V(f_{\mathrm{G}})p_{f_{\mathrm{G}}}(t) = \theta_V(\{f_{\mathrm{G}} > t\}).$$

6.2. Germ–grain models

In this section we compute the variation intensities of several germ–grain models under very broad assumptions for the grain distribution. A germ–grain model defines an RF by combining a collection of colored random sets given as a marked Poisson process according to an interaction principle (addition, supremum, occultation, etc.). The random sets are called the *grains*, while the random set locations are called the *germs* [18], [29]. Although most of the models evoked here could be defined using RAMS to define the grains (see also the recent paper of Rataj regarding random sets of finite perimeter [22]), we prefer to use the classical framework of RACS to simplify the presentation.

6.2.1. *Poisson shot noise.* We first consider the Poisson shot noise model that is obtained in summing random functions placed on the points of a Poisson process. More formally, the *Poisson shot noise* associated with the independently marked Poisson process $\Pi = \{(x_j, \kappa_j)\} \subset \mathbb{R}^d \times K$ and the *impulse function* $h\colon \mathbb{R}^d \times K \to \mathbb{R}$ is the RF f_{SN} defined by

$$f_{\mathrm{SN}}(x) = \sum_{(x_j,\kappa_j)\in\Pi} h(x - x_j, \kappa_j),$$

where $\Pi = \{(x_j, \kappa_j)\} \subset \mathbb{R}^d \times K$ is an independently marked Poisson process having intensity measure $\lambda\mathcal{L}^d \otimes P_\kappa$, $\lambda > 0$, and P_κ is the probability distribution of the marks (see, e.g. [18]).

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

The impulse function $h\colon \mathbb{R}^d \times K \to \mathbb{R}$ is supposed to be $\mathcal{L}^d \otimes P_\kappa$-integrable, which ensures that f_{SN} has finite expectation by Campbell's theorem (see, e.g. [17]). We first show that if the impulse function h has a finite mean total variation then f_{SN} has locally bounded variation and its variation intensity is finite.

Proposition 8. (Bounds on the variation intensities of Poisson shot noises.) *Let $u \in S^{d-1}$, and suppose that $h(\cdot,\kappa) \in \mathrm{BV}_u(\mathbb{R}^d)$ P_κ-a.s. with $\mathbb{E}(|D_u h(\cdot,\kappa)|(\mathbb{R}^d)) < +\infty$. Then the shot noise f_{SN} has locally bounded directional variation in the direction u and $\theta_{V_u}(f_{\mathrm{SN}}) \le \lambda\mathbb{E}(|D_u h(\cdot,\kappa)|(\mathbb{R}^d))$. Consequently, if $h(\cdot,\kappa) \in \mathrm{BV}(\mathbb{R}^d)$, P_κ-a.s. and $\mathbb{E}(|Dh(\cdot,\kappa)|(\mathbb{R}^d)) < +\infty$ then f_{SN} has locally bounded variation and $\theta_V(f_{\mathrm{SN}}) \le \lambda\mathbb{E}(|Dh(\cdot,\kappa)|(\mathbb{R}^d))$.*

Proof. Let $u \in S^{d-1}$ and $r \in \mathbb{R}$. Using the triangular inequality and Campbell's theorem (see, e.g. [17]),

$$\begin{aligned}\mathbb{E}(|f_{\mathrm{SN}}(ru) - f_{\mathrm{SN}}(0)|) &\le \mathbb{E}\Bigg(\sum_{(x_j,\kappa_j)\in\Pi} |h(ru - x_j,\kappa_j) - h(-x_j,\kappa_j)|\Bigg)\\ &\le \lambda \int_{\mathbb{R}^d\times K} |h(ru - x,\kappa) - h(-x,\kappa)|\,\mathrm{d}x\,P_\kappa(\mathrm{d}\kappa).\end{aligned}$$

By Lemma 1 with $U = \mathbb{R}^d$, P_κ-a.s. $\int_{\mathbb{R}^d} |h(ru - x,\kappa) - h(-x,\kappa)|\,\mathrm{d}x \le |D_u h(\cdot,\kappa)|(\mathbb{R}^d)|r|$. Hence, $\mathbb{E}(|f_{\mathrm{SN}}(ru) - f_{\mathrm{SN}}(0)|) \le \lambda\mathbb{E}(|D_u h(\cdot,\kappa)|(\mathbb{R}^d))|r|$, that is, f_{SN} is directionally Lipschitz in mean. By Proposition 4, we conclude that f_{SN} is a.s. in $\mathrm{BV}_{u,\mathrm{loc}}(\mathbb{R}^d)$ and $\theta_{V_u}(f_{\mathrm{SN}}) \le \lambda\mathbb{E}(|D_u h(\cdot,\kappa)|(\mathbb{R}^d))$. The upper bound on $\theta_V(f_{\mathrm{SN}})$ is obtained by integrating over all directions $u \in S^{d-1}$. □

Proposition 8 simply establishes that the mean variation of the sum is lower than the sum of the mean variations. It only gives an upper bound on the variation intensity of the Poisson shot noise. Biermé and Desolneux [5] recently studied the mean perimeter of excursion sets of Poisson shot noise models within the framework of functions of bounded variation with more specific techniques. Under the hypothesis that the functions $h(\cdot,\kappa)$ have no Cantor part, they obtained an expression for the mean variation intensity of Poisson shot noises [5, Theorem 3] which is the sum of two terms: one term for the approximate differential of f_{SN} [2] and another term from the jump part of f_{SN}. This second term is explicit and results from the observation that the jump part of f_{SN} is the sum of the jump parts of the derivatives $Dh(\cdot - x_j,\kappa_j)$. Consequently, when the functions h are indicator functions of sets of finite perimeter, the upper bound of Proposition 8 is reached, and, thus, Proposition 8 cannot be improved in general. The following proposition sums up this observation.

Proposition 9. (Variation of shot noises of random indicator functions.) *Consider a shot noise of the form*

$$f_{\mathrm{SN}}(x) = \sum_{(x_j,a_j,X_j)\in\Pi} a_j \mathbf{1}(x \in x_j + X_j),$$

where the Poisson process Π_λ has intensity measure $\lambda\mathcal{L}^d \otimes P_X \otimes P_a$, $\lambda \ge 0$, where P_X is a probability distribution over the set $\mathcal{F}$ of closed subsets of $\mathbb{R}^d$ and $P_a \in L^1(\Omega)$. Suppose that the RACS $X \sim P_X$ has finite mean Lebesgue measure and finite perimeter. Then f_{SN} has a.s. bounded variation and

$$\theta_V(f) = \lambda\mathbb{E}(|a|)\mathbb{E}(\mathrm{Per}(X)) \quad \textit{and} \quad \theta_{V_u}(f) = \lambda\mathbb{E}(|a|)\mathbb{E}(V_u(X)), \qquad u \in S^{d-1}.$$

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Proof. The equation $\theta_V(f) = \lambda\mathbb{E}(|a|)\mathbb{E}(\mathrm{Per}(X))$ is given in [5, Theorem 3]. By Proposition 8, this implies that $\theta_{V_u}(f) = \lambda\mathbb{E}(|a|)\mathbb{E}(V_u(X))$ for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$. We conclude that it holds for all u using a continuity argument (see Remark 1). □

6.2.2. *Boolean models.* We now turn to the computation of the variation intensities of Boolean models. Recall that the homogeneous *Boolean random set* with intensity λ and grain distribution P_X is the stationary RACS Z_B defined by $Z_\mathrm{B} = \bigcup_{j\in\mathbb{N}} x_j + X_j$, where $\{(x_j, X_j)\}$ is an independently marked stationary Poisson process in the space $\mathbb{R}^d \times \mathcal{F}$ having intensity measure $\lambda\mathcal{L}^d \otimes P_X$, $\lambda \geq 0$ (see, e.g. [18], [25], and [29]). Starting from the well-known expression of the variogram of the Boolean set Z_B, it was shown in [10] using (12) that

$$\begin{aligned} \theta_{V_u}(Z_\mathrm{B}) &= \lambda\mathbb{E}(V_u(X))\exp(-\lambda\mathbb{E}(\mathcal{L}^d(X))), \qquad u \in S^{d-1}, \\ \theta_V(Z_\mathrm{B}) &= \lambda\mathbb{E}(\mathrm{Per}(X))\exp(-\lambda\mathbb{E}(\mathcal{L}^d(X))). \end{aligned} \tag{15}$$

As discussed in [10], we recall that (15) is valid for *any* grain distribution P_X and that it generalizes known results for Boolean models with convex grains [25, p. 386]. Similar generalizations involving intensity of surface measures deriving from Steiner formula have recently been established [14], [31], under some technical hypotheses on the RACS X. The above formulas are similar but not identical since the outer Minkowski content of a set differs from its (variational) perimeter [30].

Several RF models can be considered as generalizations of Boolean random sets [27]. Here we consider a simple example called random Boolean islands. Let $\Phi = \{(x_j, X_j, a_j)\}$ be an independently marked Poisson process taking values in $\mathbb{R}^d \times \mathcal{F} \times [0, +\infty)$ and having intensity measure $\lambda\mathcal{L}^d \otimes P_X \otimes P_a$, $\lambda \geq 0$. We define the *Boolean random field* f_B associated to this process by $f_\mathrm{B}(y) = \sup(\{0\} \cup \{a_j,\ y \in x_j + X_j\})$. We note that, with this model, the colored random sets are superimposed according to a hierarchy: the lighter sets are placed above the darker sets. Note that if $a_j = 1$ a.s. then f_B is the indicator function of the Boolean random set Z_B. More generally, the upper level sets of f_B are Boolean random sets. Indeed, for all $t \geq 0$, $\{y,\ f_\mathrm{B}(y) > t\}$ is the Boolean model associated with the Poisson process $\sum_\Phi \mathbf{1}(a_j > t)\delta_{x_j, X_j}$. Relying on this observation, we obtain an expression for the variation intensities of Boolean RFs using the coarea formula for variation intensities.

Proposition 10. (Variation intensities of Boolean random fields.) *Let f_B be the Boolean RF with Poisson intensity λ, grain distribution P_X, and gray-level distribution P_a. Let X denote a RACS with distribution P_X, and let a denote an RV with distribution P_a. Then,*

$$\begin{aligned} \theta_{V_u}(f_\mathrm{B}) &= \lambda\mathbb{E}(V_u(A))\int_0^{+\infty} P_a(\{a > t\})\exp(-\lambda\mathbb{E}(\mathcal{L}^d(X))P_a(\{a > t\}))\,\mathrm{d}t, \qquad u \in S^{d-1}, \\ \theta_V(f_\mathrm{B}) &= \lambda\mathbb{E}(\mathrm{Per}(A))\int_0^{+\infty} P_a(\{a > t\})\exp(-\lambda\mathbb{E}(\mathcal{L}^d(X))P_a(\{a > t\}))\,\mathrm{d}t. \end{aligned}$$

Proof. Note that $\{y,\ f_\mathrm{B}(y) > t\}$ is the Boolean model associated with the Poisson process $\sum_\Phi \mathbf{1}(a_j > t)\delta_{x_j, X_j}$. This Poisson process has grain distribution P_X and intensity $\lambda P_a(\{a > t\})$. Hence, $\theta_{V_u}(\{f_\mathrm{B} > t\}) = \lambda\mathbb{E}(V_u(A))P_a(\{a > t\})\exp(-\lambda\mathbb{E}(\mathcal{L}^d(X))P_a(\{a > t\}))$. The expressions for $\theta_{V_u}(f_\mathrm{B})$ and $\theta_V(f_\mathrm{B})$ are obtained using the coarea formula for variation intensities (see (13)). □

6.2.3. *Colored dead leaves model.* The dead leaves model [7], [9], [16], also initially introduced by Matheron, is a germ–grain model where the interaction rule is occultation, that is, where

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

the grains $x_j + X_j$ hide each other. For this germ–grain model, the grains are chronologically ordered by a time $t_j \in (-\infty, 0)$, called *falling time*. The leaves are the points of the Poisson process $\Phi = \{(t_j, x_j, X_j, a_j)\} \subset (-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R}$ with intensity measure $\mathcal{L}^1 \otimes \mathcal{L}_d \otimes P_X \otimes P_a$. For each leaf (t_j, x_j, X_j, a_j), the random set $x_j + X_j$ associated with the random color a_j is partially or totally hidden by its subsequent leaves, that is, the leaves which fall after $t = t_j$. In the end, at time $t = 0$ the only remaining part of $x_j + X_j$ is the *visible part* V_j, that is, the set

$$V_j = (x_j + X_j) \setminus \left(\bigcup_{(t_k, x_k, X_k, a_k) \in \Phi,\, t_k > t_j} x_k + X_k \right).$$

(Our definition of the *visible parts* V_j is slightly different from that given in [7]. This is because we do not enforce the condition that the visible parts be closed sets.) As soon as $\mathbb{E}(\mathcal{L}^d(X)) > 0$, the whole Euclidean space $\mathbb{R}^d$ is covered by the random sets $x_j + X_j$, and, consequently, each point $y \in \mathbb{R}^d$ belongs to a unique visible part. The *colored dead leaves RF* f_{CDL} is the RF defined by assigning to each $y \in \mathbb{R}^d$ the color a_j of the unique visible part V_j such that $y \in V_j$. More formally, f_{CDL} is defined by

$$f_{\mathrm{CDL}}(y) = \sum_{(t_j, x_j, X_j, a_j) \in \Phi} a_j \mathbf{1}(y \in V_j),$$

but note that, for each point y, the above sum has only one nonnull term. Even though occultation between objects is also observable with Boolean RFs, colored dead leaves RFs vary from this first model: here the ordering of the objects is not related to their gray level, and the whole domain is by construction fully covered by objects.

In the next proposition we give the variation intensities of this RF model. Relying on Theorem 3, it consists in computing $\mathbb{E}(|f_{\mathrm{CDL}}(ru) - f_{\mathrm{CDL}}(0)|)$. To this end, we consider the set of leaves that cover (or hit) the two points 0 and ru (or both). We consider the restriction $\Phi^{\{0,ru\}}$ of the leaves of Φ which hit the set $\{0, ru\}$, that is,

$$\Phi^{\{0,ru\}} = \{(t_j, x_j, X_j, a_j) \in \Phi,\ x_j + X_j \cap \{0, ru\} \neq \varnothing\}.$$

According to [11, Proposition 5], $\Phi^{\{0,ru\}}$ is an independently marked Poisson process with ground process $\{t_j,\ (t_j, x_j, X_j, a_j) \in \Phi^{\{0,ru\}}\}$ of intensity $2\gamma_X(0) - \gamma_X(ru)$ (where γ_X denotes the mean geometric covariogram of X) and marks (x_j, X_j, a_j). The marks a_j are independent and identically distributed (i.i.d.) with distribution P_a, and are independent of (x_j, X_j). As for the distribution of the marks (x_j, X_j), we are only interested in the probabilities

$$\mathbb{P}(\{0, ru\} \subset x_j + X_j) = \frac{\mathcal{L}^d \otimes P_X(\{(x, X),\ \{0, ru\} \subset x + X\})}{\mathcal{L}^d \otimes P_X(\{(x, X),\ \{0, ru\} \cap x + X \neq \varnothing\})} = \frac{\gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)},$$

and, by symmetry and complementarity,

$$\begin{aligned}\mathbb{P}(0 \in x_j + X_j \text{ and } ru \notin x_j + X_j) &= \mathbb{P}(ru \in x_j + X_j \text{ and } 0 \notin x_j + X_j) \\ &= \frac{\gamma_X(0) - \gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)}.\end{aligned}$$

In addition, if we denote by (t_0, x_0, X_0, a_0) the last leaf of $\Phi^{\{0,ru\}}$, that is, the leaf such that $t_0 = \sup\{t_j,\ (t_j, x_j, X_j, a_j) \in \Phi^{\{0,ru\}}\}$, then (x_0, X_0, a_0) has the same distribution as any mark (x_j, X_j, a_j) of $\Phi^{\{0,ru\}}$, the shifted Poisson process $\Phi_{t_0} = \{(t - t_0, x, X, a),\ (t, x, X, a) \in \Phi \text{ and } t < t_0\}$ has the same distribution as Φ, and (x_0, X_0, a_0) and Φ_{t_0} are independent.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

Proposition 11. (Variation intensities of the colored dead leaves RF.) *Suppose that* $0 < \mathbb{E}(\mathcal{L}^d(X)) < +\infty$ *and that* $a \in L^1(\Omega)$. *Let* a_1 *and* a_2 *be two independent RVs with distribution* P_a. *Then, for all* $u \in S^{d-1}$,

$$\theta_{V_u}(f_{\mathrm{CDL}}) = \mathbb{E}(|a_1 - a_2|)\frac{\mathbb{E}(V_u(X))}{\mathbb{E}(\mathcal{L}^d(X))}, \quad \textit{and} \quad \theta_V(f_{\mathrm{CDL}}) = \mathbb{E}(|a_1 - a_2|)\frac{\mathbb{E}(\mathrm{Per}(X))}{\mathbb{E}(\mathcal{L}^d(X))}.$$

Proof. We first compute the expectation $\mathbb{E}(|f_{\mathrm{CDL}}(ru) - f_{\mathrm{CDL}}(0)|)$ for $r \in \mathbb{R}$ and $u \in S^{d-1}$. If the points ru and 0 are in the same visible part V_j, then $f_{\mathrm{CDL}}(ru) = f_{\mathrm{CDL}}(0)$. If ru and 0 are in different visible parts, then both $f_{\mathrm{CDL}}(ru)$ and $f_{\mathrm{CDL}}(0)$ have distribution P_a and they are independent. Hence,

$$\mathbb{E}(|f_{\mathrm{CDL}}(ru) - f_{\mathrm{CDL}}(0)|) = \mathbb{E}(|a_1 - a_2|)\mathbb{P}(\{ru \text{ and } 0 \text{ belong to different visible parts}\}).$$

Now, ru and 0 belong to different visible parts if the last leaf covering either ru or 0 does not cover both points. Noting that $x_0 + X_0$ is the last leaf of $\Phi^{\{0,ru\}}$, we have

$$\begin{aligned}
&\mathbb{P}(\{ru \text{ and } 0 \text{ belong to different visible parts}\}) \\
&\quad = \mathbb{P}(0 \in x_0 + X_0 \text{ and } ru \notin x_0 + X_0) + \mathbb{P}(ru \in x_0 + X_0 \text{ and } 0 \notin x_0 + X_0) \\
&\quad = 2\frac{\gamma_X(0) - \gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)}.
\end{aligned}$$

(Up to our slightly different definition of the visible parts, this probability is also given by [7, Equation (12)].) Hence, by Theorem 3 and (11),

$$\begin{aligned}
\theta_{V_u}(f_{\mathrm{CDL}}) &= \lim_{r\to 0} \mathbb{E}(|f_{\mathrm{CDL}}(0; ru)|) \\
&= \mathbb{E}(|a_1 - a_2|) \lim_{r\to 0} 2\frac{\gamma_X(0) - \gamma_X(ru)}{|r|}\frac{1}{2\gamma_X(0) - \gamma_X(ru)} \\
&= \mathbb{E}(|a_1 - a_2|)\frac{\mathbb{E}(V_u(X))}{\mathbb{E}(\mathcal{L}^d(X))}.
\end{aligned}$$

Integrating over all directions gives the expression for $\theta_V(f_{\mathrm{CDL}})$. □

Remark 9. The expression of the variation intensity

$$\theta_V(f_{\mathrm{CDL}}) = \mathbb{E}(|a_1 - a_2|)\frac{(\mathbb{E}(\mathrm{Per}(X))}{\mathbb{E}(\mathcal{L}^d(X)))}$$

is in accordance with our expectation. Indeed, $\mathbb{E}(|a_1 - a_2|)$ is the mean contrast between two distinct visible parts, whereas the ratio $\mathbb{E}(\mathrm{Per}(X))/\mathbb{E}(\mathcal{L}^d(X))$ is known to be the mean length of the cell boundary per unit area when the RACS are random polygons [9]. We also mention that the variation intensity of the transparent dead leaves RF defined in [11] can be computed using the same technique and that it takes the same form with a different contrast term depending on the degree of transparency.

6.2.4. *Colored tessellations.* A colored tessellation is the RF obtained by assigning a random color to each cell of a random partition of the plane. The interaction principle which is at work for colored tessellations is arguably juxtaposition.

A (random) tessellation is a random partition $\bigcup_j C_j = \mathbb{R}^d$ of the Euclidean space $\mathbb{R}^d$, the sets C_j being called *cells* of the tessellation. Even though random tessellations have been

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

widely studied, there lacks a general acknowledged definition. This is principally because most studied tessellation models involve only convex cells. Nevertheless, tessellations can constitute nonconvex (and even nonconnected) cells, such as the tessellation corresponding to the dead leaves model defined in [7]. Following [7] and [28], we consider a quite general definition. A *(random) tessellation* is a point process $T = \sum_j \delta_{C_j}$ taking values in the set $\mathcal{K}'$ of nonempty compact sets and which satisfies the following additional properties:

- for all compact sets K, the number of sets C_i intersecting K is finite;
- for all $j \neq k$, $\mathrm{int}C_j \cap \mathrm{int}C_k \neq \varnothing$;
- $\bigcup_j C_k = \mathbb{R}^d$;
- for all j, $\mathcal{L}^d(\partial C_j) = 0$.

With these conditions, almost every point $x \in \mathbb{R}^d$ belongs to a unique cell C_j. We will only consider *stationary* tessellations, that is, tessellations such that, for all $x \in \mathbb{R}^d$, $\sum_j \delta_{x+C_j} \overset{\mathrm{D}}{=} \sum_j \delta_{C_j}$, where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution. Thanks to the stationarity, for these tessellations, every point $x \in \mathbb{R}^d$ a.s. belongs to a unique cell C_j.

Given a stationary tessellation $T = \sum_j \delta_{C_j}$, we define a stationary RF f_T by associating a random intensity $a_j \in \mathbb{R}$ to each cell C_j. The real RVs a_j are i.i.d. with common distribution P_a. More formally, the *colored tessellation* T_{c} associated to the tessellation T and with color distribution P_a is the independently marked point process $T_{\mathrm{c}} = \sum_j \delta_{(C_j, a_j)}$, where the marks a_j have common distribution P_a. If μ denotes the intensity measure of the point process $\sum_j \delta_{C_j}$, then $T_{\mathrm{c}} = \sum_j \delta_{(C_j, a_j)}$ has intensity measure $\mu \otimes P_a$. Its associated RF f_T is defined as follows: $f_T(x) = a_j$, where a_j is the color of the a.s. unique cell C_j containing x. Note that f_T can also be defined as a sum over the marked point process $f_T(x) = \sum_j a_j \mathbf{1}(x \in C_j)$.

Before establishing the expression of the variation intensities of f_T, we need to introduce the fundamental notion of a typical cell. First, we interpret a stationary tessellation as a point process in $\mathbb{R}^d$ marked with random sets by introducing a centroid map. Recall that $\mathcal{K}'$ denotes the set of nonempty compact sets of $\mathbb{R}^d$. A *centroid map* is a measurable application $z\colon \mathcal{K}' \mapsto \mathbb{R}^d$ such that $z(x + C) = x + z(C)$. Second, given a centroid map z, any stationary tessellation $T = \sum_j \delta_{C_j}$ is decomposed into the stationary marked point process $\sum_j \delta_{(z(C_j), C_j - z(C_j))}$. According to [25, Section 4.1], we deduce that, for any stationary tessellation T, there exists a distribution Q over $\mathcal{K}'_0 = \{K \in \mathcal{K}',\ z(K) = 0\}$ such that, for all measurable functions $f\colon \mathcal{K}' \mapsto \mathbb{R}+$,

$$\mathbb{E}\left(\sum_j f(C_j)\right) = \frac{1}{\mathbb{E}(\mathcal{L}^d(\mathcal{C}))} \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} f(x + K)\, \mathrm{d}x\, Q(\mathrm{d}K), \tag{16}$$

where, by definition, Q is the distribution of the *typical cell* of T denoted by $\mathcal{C} \sim Q$. The key result needed to compute the variation intensity of randomly colored stationary tessellations is the following proposition, which is stated without proof in [18]. The proof reproduced below was personally communicated to the author by Professor Pierre Calka.

Proposition 12. (Stationary tessellations and mean covariogram.) *Let $T = \sum_j \delta_{C_j}$ be a stationary tessellation, and let $\gamma_{\mathcal{C}}\colon h \mapsto \mathbb{E}(\mathcal{C} \cap (h + \mathcal{C}))$ be the mean covariogram of its typical cell $\mathcal{C}$. Then, for all $h \in \mathbb{R}^d$, $\mathbb{P}(\{0 \text{ and } h \text{ belong to the same cell}\}) = \gamma_{\mathcal{C}}(h)/\gamma_{\mathcal{C}}(0)$.*

Proof. Define $\rho(h) = \mathbb{P}(\{0 \text{ and } h \text{ belong to the same cell}\})$. First, by stationarity, $\rho(h) = \mathbb{P}(\{-h \text{ and } 0 \text{ belong to the same cell}\})$. Second, note that, for all $C \in \mathcal{K}$, $\{0, -h\} \subset C$ if and

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

only if $0 \in C \cap (h + C)$. Hence, applying (16) with $K \mapsto \mathbf{1}(0 \in K \cap (h + K))$, we obtain

$$\begin{aligned}\rho(h) &= \mathbb{E}\Big(\sum_j \mathbf{1}(0 \in C_j \cap (h + C_j))\Big) \\ &= \frac{1}{\mathbb{E}(\mathcal{L}^d(\mathcal{C}))} \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}(0 \in (x + K) \cap (h + x + K)) \, \mathrm{d}x \, Q(\mathrm{d}K) \\ &= \frac{1}{\gamma_{\mathcal{C}}(0)} \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}(-x \in K \cap (h + K)) \, \mathrm{d}x \, Q(\mathrm{d}K) \\ &= \frac{1}{\gamma_{\mathcal{C}}(0)} \int_{\mathcal{K}_0} \mathcal{L}^d(K \cap (h + K)) Q(\mathrm{d}K) \\ &= \frac{\gamma_{\mathcal{C}}(h)}{\gamma_{\mathcal{C}}(0)}.\end{aligned}$$

□

Proposition 13. (Variation intensities of colored tessellations.) *Let $T_{\mathrm{c}} = \sum_j \delta_{(C_j, a_j)}$ be a randomly colored stationary tessellation, let f_T be its associated stationary RF, and denote by $\mathcal{C}$ the typical cell of T. Let a_1 and a_2 be i.i.d. RVs with distribution P_a. For all $u \in S^{d-1}$, $\theta_{V_u}(f_T) = \mathbb{E}(|a_1 - a_2|)\frac{1}{2}(\mathbb{E}(V_u(\mathcal{C}))/\mathbb{E}(\mathcal{L}^d(\mathcal{C})))$, and $\theta_V(f_T) = \mathbb{E}(|a_1 - a_2|)\frac{1}{2}(\mathbb{E}(\mathrm{Per}(\mathcal{C}))/\mathbb{E}(\mathcal{L}^d(\mathcal{C})))$.*

Proof. We compute $\mathbb{E}(|f_T(ru) - f_T(0)|)$. We have $|f_T(ru) - f_T(0)| = |a_k - a_j|$ if $0 \in C_j$ and $ru \in C_k$ with $j \neq k$, and 0 if 0 and ru belong to the same cell. By Proposition 12, and since, for $j \neq k$, a_j and a_k are independent,

$$\begin{aligned}\mathbb{E}(|f_T(ru) - f_T(0)|) &= \mathbb{E}(|a_1 - a_2|)\mathbb{P}(\{0 \text{ and } ru \text{ are in different cells}\}) \\ &= \mathbb{E}(|a_1 - a_2|)\frac{\gamma_{\mathcal{C}}(0) - \gamma_{\mathcal{C}}(ru)}{\gamma_{\mathcal{C}}(0)}.\end{aligned}$$

By (11) applied with $X = \mathcal{C}$ and Theorem 3, $\theta_{V_u}(f_T) = \mathbb{E}(|a_1 - a_2|)\frac{1}{2}(\mathbb{E}(V_u(\mathcal{C}))/\mathbb{E}(\mathcal{L}^d(\mathcal{C})))$. Integrating over all directions we obtain the expression for $\theta_V(f_T)$. □

Observe that $\theta_V(f_T) = \mathbb{E}(|a_1 - a_2|)\frac{1}{2}(\mathbb{E}(\mathrm{Per}(\mathcal{C}))/\mathbb{E}(\mathcal{L}^d(\mathcal{C})))$ is in accordance with our expectation. Indeed, $\mathbb{E}(|a_1 - a_2|)$ is the mean contrast between two adjacent cells, whereas $\frac{1}{2}(\mathbb{E}(\mathrm{Per}(\mathcal{C}))/\mathbb{E}(\mathcal{L}^d(\mathcal{C})))$ is known to be the mean length of the tessellations boundary per unit area [25, Section 10.1].

Acknowledgements

The author is grateful to Professor Pierre Calka for his help in establishing the proof of Proposition 12, to Professor Yann Gousseau for his various comments that contributed to improving a preliminary version of this paper, as well as to the anonymous reviewers for their relevant suggestions.

References

[1] Adler, R. J. (1981). *The Geometry of Random Fields*. John Wiley, Chichester.
[2] Ambrosio, L., Fusco, N. and Pallara, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press.
[3] Aubert, G. and Kornprobst, P. (2006). *Mathematical Problems in Image Processing* (Appl. Math. Sci. **147**), 2nd edn. Springer, New York.
[4] Azaïs, J.-M. and Wschebor, M. (2009). *Level Sets and Extrema of Random Processes and Fields*. John Wiley, Hoboken, NJ.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press

[5] Biermé, H. and Desolneux, A. (2016). On the perimeter of excursion sets of shot noise random fields. *Ann. Prob.* **44,** 521–543.
[6] Biermé, H., Meerschaert, M. M. and Scheffler, H.-P. (2007). Operator scaling stable random fields. *Stoch. Process. Appl.* **117,** 312–332.
[7] Bordenave, C., Gousseau, Y. and Roueff, F. (2006). The dead leaves model: a general tessellation modeling occlusion. *Adv. Appl. Prob.* **38,** 31–46.
[8] Chlebík, M. (1997). On variation of sets. Preprint 44, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig.
[9] Cowan, R. and Tsang, A. K. L. (1994). The falling-leaves mosaic and its equilibrium properties. *Adv. Appl. Prob.* **26,** 54–62.
[10] Galerne, B. (2011). Computation of the perimeter of measurable sets via their covariogram. Applications to random sets. *Image Anal. Stereol.* **30,** 39–51.
[11] Galerne, B. and Gousseau, Y. (2012). The transparent dead leaves model. *Adv. Appl. Prob.* **44,** 1–20.
[12] Galerne, B. and Lachièze-Rey, R. (2015). Random measurable sets and covariogram realizability problems. *Adv. Appl. Prob.* **47,** 611–639.
[13] Gikhman, I. I. and Skorokhod, A. V. (1974). *The Theory of Stochastic Processes. I*. Springer, Berlin.
[14] Hug, D., Last, G. and Weil, W. (2004). A local Steiner-type formula for general closed sets and applications. *Math. Z.* **246,** 237–272.
[15] Ibragimov, I. A. (1995). Remarks on variations of random fields. *J. Math. Sci.* **75,** 1931–1934.
[16] Jeulin, D. (1997). Dead leaves models: from space tessellation to random functions. In *Proceedings of the International Symposium on Advances in Theory and Applications of Random Sets*, World Scientific, River Edge, NJ, pp. 137–156.
[17] Kingman, J. F. C. (1993). *Poisson Processes* (Oxford Studies Prob. **3**). Oxford University Press.
[18] Lantuéjoul, C. (2002). *Geostatistical Simulation: Models and Algorithms*. Springer, Berlin.
[19] Matheron, G. (1975). *Random Sets and Integral Geometry*. John Wiley, New York.
[20] Meyer, Y. (2001). *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations* (Univ. Lecture Ser. **22**). American Mathematical Society, Providence, RI.
[21] Molchanov, I. (2005). *Theory of Random Sets*. Springer, London.
[22] Rataj, J. (2015). Random sets of finite perimeter. *Math. Nachr.* **288,** 1047–1056.
[23] Rudin, L. I., Osher, S. and Fatemi, E. (1992). Nonlinear total variation based noise removal algorithms. *Physica D* **60,** 259–268.
[24] Scheuerer, M. (2010). Regularity of the sample paths of a general second order random field. *Stoch. Process. Appl.* **120,** 1879–1897.
[25] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.
[26] Serra, J. (1982). *Image Analysis and Mathematical Morphology*. Academic Press, London.
[27] Serra, J. (ed.) (1988). *Image Analysis and Mathematical Morphology*, Vol. 2, *Theoretical Advances*. Academic Press, London.
[28] Stoyan, D. (1986). On generalized planar random tessellations. *Math. Nachr.* **128,** 215–219.
[29] Stoyan, D., Kendall, W. S. and Mecke, J. (1995). *Stochastic Geometry and Its Applications*, 2nd edn. John Wiley, Chichester.
[30] Villa, E. (2009). On the outer Minkowski content of sets. *Ann. Mat. Pura Appl. (4)* **188,** 619–630.
[31] Villa, E. (2010). Mean densities and spherical contact distribution function of inhomogeneous Boolean models. *Stoch. Anal. Appl.* **28,** 480–504.

https://doi.org/10.1017/apr.2016.60 Published online by Cambridge University Press