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Multi-observable uncertainty bounds via matrix inequalities

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Abstract. We establish new uncertainty bounds for commutators and anti-commutators of observables in both finite- and infinite-dimensional quantum systems. Beginning with a short, elementary proof of Robertson's standard uncertainty principle, we then derive inequalities whose lower bounds remain nonzero, regardless of the number of observables. The main result is a novel trace inequality comparing the block matrices of commutators and anti-commutators of several observables.

1 Introduction

Heisenberg's uncertainty principle [22], one of the cornerstones of quantum mechanics, asserts that the position and momentum of a particle on a line cannot be simultaneously determined with arbitrary accuracy. This places a fundamental limitation on the precision of physical measurements. To make it rigorous, we turn to the Hilbert space formalization of quantum mechanics.

A particle moving on the real line is modeled by the Hilbert space $L^2(\mathbb{R})$. Each unit vector in this space represents a possible quantum state of the particle. If $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$ denotes the state, then the probability of finding the particle in an interval $(a, b) \subset \mathbb{R}$ is given by the *Born rule*:

$$\int_a^b |f(t)|^2 \, dt.$$

The position operator $\hat{x}$ is defined on the domain

$$\mathcal{D}(\hat{x}) = \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} y^2 |f(y)|^2 \, dy < \infty \right\}$$

by

$$\hat{x}(f)(y) = y f(y).$$

Similarly, the momentum operator $\hat{p}$ is defined on

$$\mathcal{D}(\hat{p}) = \{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \}$$

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by

$$\hat{p}(f) = -i\hbar f',$$

where f' denotes the distributional derivative of f, and $\hbar$ is Planck's constant. Both $\hat{x}$ and $\hat{p}$ are unbounded self-adjoint operators, and on the dense subspace of Schwartz functions $\mathcal{S}(\mathbb{R})$, they satisfy the *canonical commutation relation*

$$[\hat{x}, \hat{p}] = i\hbar I.$$

For two self-adjoint operators A and B on a Hilbert space $\mathcal{H}$, and any unit vector $x \in \mathcal{D}(AB - BA)$, the Cauchy–Schwarz inequality yields

$$\begin{aligned} \|Ax\|^2\|Bx\|^2 &\geq |\mathrm{Im}\langle Ax, Bx\rangle|^2 \\ &= \frac{|\langle ABx, x\rangle - \langle BAx, x\rangle|^2}{4} \\ &= \frac{|\langle [A,B]x, x\rangle|^2}{4}. \end{aligned} \tag{1}$$

Note that equality holds in (1) if and only if there exists a real λ such that $Ax = i\lambda Bx$.

Since commutators are unaffected by scalar shifts, one may freely replace A by $A - aI$ and B by $B - bI$ in (1) for any $a, b \in \mathbb{R}$. This observation will lead us to the variance formulation of the uncertainty principle.

Let $f \in \mathcal{S}(\mathbb{R})$ with $\|f\|_2 = 1$. Such functions are smooth, rapidly decaying, and dense in $L^2(\mathbb{R})$. For $a, b \in \mathbb{R}$, one computes

$$\|(\hat{x} - aI)f\|_2^2 = \int_{\mathbb{R}} (y-a)^2 |f(y)|^2 \, dy,$$

the variance of the position observable. On the other hand,

$$\|(\hat{p} - bI)f\|_2^2 = \int_{\mathbb{R}} |-i\hbar f'(y) - bf(y)|^2 \, dy.$$

Passing to the Fourier transform $\hat{f}$ and using Parseval's identity, this expression becomes

$$\|(\hat{p} - bI)f\|_2^2 = 4\pi^2\hbar^2 \int_{\mathbb{R}} (\omega - b)^2 |\hat{f}(\omega)|^2 \, d\omega,$$

which is the variance of the momentum observable.

Substituting into (1) with $A = \hat{x} - aI$ and $B = \hat{p} - bI$, we obtain the Heisenberg–Weyl uncertainty inequality:

$$\int_{\mathbb{R}} (y-a)^2 |f(y)|^2 \, dy \int_{\mathbb{R}} (\omega - b)^2 |\hat{f}(\omega)|^2 \, d\omega \geq \frac{1}{16\pi^2}. \tag{2}$$

Choosing $a = \int_{\mathbb{R}} y|f(y)|^2 dy$ and $b = 2\pi\hbar \int_{\mathbb{R}} \omega |\hat{f}(\omega)|^2 d\omega$ (the expectations of position and momentum, respectively), the terms on the left-hand side become the variances σ_x^2 and σ_p^2. Inequality (2) then reads

$$\sigma_x^2 \, \sigma_p^2 \geq \frac{\hbar^2}{4}.$$

Equality holds precisely for the Gaussians.

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From the perspective of harmonic analysis, (2) expresses a universal limitation:

A function and its Fourier transform cannot both be sharply localized.

This principle underlies many uncertainty relations in analysis (see [17] for an overview). Although we have stated everything for Schwartz functions, the inequality extends to all $f \in \mathcal{D}(\hat{x}) \cap \mathcal{D}(\hat{p})$, albeit with a more involved proof (see [21] for details).

Remark The essence of the uncertainty principle is the non-commutativity of quantum observables, represented by self-adjoint operators on a Hilbert space. This naturally raises broader questions: what uncertainty relations can be derived for arbitrary self-adjoint operators for more than two observables? Also, can we do it for more general states than those given by the unit vectors? The was first addressed by Robertson in 1934, which later led to a whole family of generalized uncertainty relations. It should be noted here that the framework and terminology we are going to use henceforth will be significantly different from Robertson.

Seemingly, the most natural way to discuss this is to use the operator-algebraic approach to quantum mechanics. Let $\mathcal{A}$ be a unital C^*-algebra of *bounded* operators modeling a quantum system and let $\phi : \mathcal{A} \to \mathbb{C}$ be a state. Given a self-adjoint $x \in \mathcal{A}$ representing an observable, its *expectation* and *variance* are given by $\phi(x)$ and $\phi\big(x - \phi(x)\big)^2$, respectively.

Now, let $\{x_j\}_{j=1}^n$ be self-adjoint elements of $\mathcal{A}$. We normalize them so that $\phi(x_j) = 0$ for all $1 \le j \le n$. This will be the assumption throughout the article.

Remark We can do this without loss of generality since our lower bounds will depend only on the commutators $[x_i, x_j]$, and therefore, remain unchanged if the observables are translated by scalars.

The *covariance matrix* of the n-tuple $\tilde{x} = (x_j)_{j=1}^n$ is the $n \times n$ positive matrix given by

$$\mathrm{Cov}_\phi(\tilde{x}) = \big(\mathrm{Re}\ \phi(x_i x_j)\big).$$

This can also be written as

$$\mathrm{Cov}_\phi(\tilde{x}) = M + M^T,$$

where

$$M = \frac{1}{2}\big(\phi(x_i x_j)\big) \tag{3}$$

is a Gram matrix. Note that

$$M - M^T = \frac{1}{2}\big(\phi[x_i, x_j]\big)$$

and

$$M + M^T = \mathrm{Cov}_\phi(\tilde{x}) = \frac{1}{2}\big(\phi\{x_i, x_j\}\big),$$

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where $[x_i, x_j] = x_i x_j - x_j x_i$ is the commutator and $\{x_i, x_j\} = x_i x_j + x_j x_i$ is the anti-commutator of x_i and x_j. In this form, Robertson's standard uncertainty principle (see [35]) can be stated as

$$\det|M - M^T| = \det\left|\left(\phi[x_i, x_j]\right)\right| \leq \det \mathrm{Cov}_\phi(\tilde{x}). \tag{4}$$

Note that for two self-adjoint elements x and y, this reduces to the well-known Schrödinger–Heisenberg uncertainty principle [36]:

$$\phi(x^2)\phi(y^2) - |\mathrm{Re}\ \phi(yx)|^2 \geq \frac{1}{4}|\phi[x, y]|^2.$$

An immediate consequence is the Heisenberg uncertainty principle, now formulated as

$$\phi(x^2)\phi(y^2) \geq \frac{1}{4}|\phi[x, y]|^2.$$

If $\mathcal{A} = M_k(\mathbb{C})$, a state $\phi : M_k(\mathbb{C}) \to \mathbb{C}$ can be written as $\phi(X) = \mathrm{tr}\, AX$ for all $X \in M_k(\mathbb{C})$, for some density matrix A, that is, $A \geq O$ and $\mathrm{tr}\, A = 1$. In this case, for a pair of Hermitian matrices H and K, we can write the Heisenberg uncertainty principle as

$$\mathrm{tr}\, AH^2\ \mathrm{tr}\, AK^2 \geq \frac{|\mathrm{tr}\, A[H, K]|^2}{4}.$$

This is very important in the context of quantum computation and quantum information theory, dealing mainly with finite-dimensional quantum systems (see [40] for more details).

Though inequality (4) generalizes Heisenberg's uncertainty principle for n observables, it has a downside. Note that $i(M - M^T)$ is a real skew-symmetric matrix, and therefore, if n is odd, its determinant vanishes. Thus, for an odd number of observables, inequality (4) fails to capture any quantum phenomenon arising from non-commutativity. This prompted research on uncertainty relations which do not have this drawback. Andai in [1] and Gibilisco et al. in [19] proved the following.

Theorem *Let A be a positive-definite density matrix. Let $\{H_j\}_{j=1}^k$ be Hermitian and let $f : (0, \infty) \to \mathbb{R}$ be a positive operator monotone function satisfying $f(1) = 1$ and $xf(\frac{1}{x}) = f(x)$ for all $x \in (0, \infty)$. Then*

$$\det\left(\frac{f(0)}{2}\gamma_f([A, H_i], [A, H_j])\right) \leq \det\left(\mathrm{Re}\ \mathrm{tr}\, AH_iH_j\right).$$

Here, γ_f denotes the quantum Fisher information metric (see [32, 33]) associated with f. This inequality is also known as the dynamical uncertainty principle and was later generalized to arbitrary covariances by Gibilisco et al. in [18].

The question arises, can we have variants of the *standard* uncertainty principle (4) involving the commutators $[x_i, x_j]$ with a lower bound that does not necessarily vanish depending on the number of observables? Also, can we have a lower bound which is *always* nonzero whenever any two of the observables do not commute, thus capturing any quantum uncertainty arising from non-commutativity? It is to be noted

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that multi-observable uncertainty relations involving the sum of variances have been proved by Chen et al. [13] and Huang [24], but these did not explicitly involve the commutators in the lower bound.

This article is an attempt in that direction. The sum, difference, and geometric mean of positive matrices will all play a key role in the subsequent results.

Lastly, for k Hermitian matrices $\{H_j\}_{j=1}^k$, we discuss the problem of comparing the block matrices $\big([H_i, H_j]\big)$ and $\big(\{H_i, H_j\}\big)$ directly without composing a state entrywise like it is done in (4) and prove an interesting trace inequality. This gives a nonzero lower bound whenever there exist i, j such that $H_iH_j \neq H_jH_i$ and generalizes seamlessly to infinite dimensions. It is to be noted that in this case, we cannot write both sides as the sum and difference of two positive matrices, prompting the use of other techniques.

2 Main results

We first give a short proof of Robertson's standard uncertainty principle (4). For that, we need a lemma.

Lemma 1 *Let A, B be two positive matrices. Then*

$$\det(A-B)^2 \leq \det(A+B)^2.$$

Proof Suffices to prove for positive definite A and B. If A and B commute, the inequality is obvious. Otherwise, we note that

$$\begin{aligned}\det(A-B)^2 &= \det\big(A^{1/2}(I - A^{-1/2}BA^{-1/2})A^{1/2}\big)^2\\ &= \det A^2 \det(I - A^{-1/2}BA^{-1/2})^2\\ &\leq \det A^2 \det(I + A^{-1/2}BA^{-1/2})^2\\ &= \det(A+B)^2.\end{aligned}$$

■

Remark Note that the operator inequality $(A-B)^2 \leq (A+B)^2$ need not be true for positive A and B unless A and B commute.

As an immediate consequence, we have Robertson's uncertainty principle.

Theorem 1 *Let $\{x_i\}_{i=1}^n$ be self-adjoint elements in a C^*-algebra $\mathcal{A}$. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a state. Then,*

$$\det\left(\frac{1}{2}\phi[x_i, x_j]\right)^2 \leq \det \mathrm{Cov}_\phi(\tilde{x})^2.$$

Proof Consider the positive matrix M as given in (3) and apply Lemma 1. ■

Before we state the next lemma, let us have a quick review of the matrix geometric mean, which shall be used subsequently. Given two positive-*definite* matrices A and B, their geometric mean is given by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

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This was introduced by Pusz and Woronowicz in [34] in the context of positive sesquilinear forms, and later studied in detail from the matrix-theoretic perspective by Ando in [2]. Let $\mathbb{P}(n)$ be the set of all $n \times n$ positive-definite matrices equipped with the Riemannian metric

$$\langle H, K\rangle_A = \operatorname{tr}\,(A^{-1}HA^{-1}K) \tag{5}$$

for all $A \in \mathbb{P}(n)$, and Hermitian H and K. It can be shown that for $A, B \in \mathbb{P}(n)$, $A\#B$ is precisely the geodesic midpoint between A and B with respect to the metric (5). In fact, the geodesic joining A and B is given explicitly by

$$t \to A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$.

This geometric viewpoint was developed independently by Bhatia and Holbrook [8] and Moakher [30] and was used to define the geometric mean of k positive-definite matrices for any $k \in \mathbb{N}$.

For positive matrices A and B which are not necessarily invertible, we can define

$$A\#B = \lim_{\varepsilon \to 0+} (A + \varepsilon I)\#(B + \varepsilon I).$$

An important variational characterization is given by

$$A\#B = \max\{X : X^* = X, \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq O\}, \tag{6}$$

as a consequence of which, we get

$$A\#B \leq \frac{A+B}{2}.$$

For details, refer to [2, 4].

The geometric mean is extremely important and finds wide applications in matrix inequalities, quantum information, and even in medical imaging (see [2, 6, 7, 15, 23, 25, 27] for more details).

We now present a couple of lemmas which will be crucial in the later results.

Lemma 2 *Let A, B be positive matrices. Then there exists a unitary U such that*

$$|A - B| \leq (A + B)\#U(A + B)U^*.$$

Proof Note that

$$\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes B,$$

which is positive. Let

$$A - B = U|A - B|$$

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be the polar decomposition of $A - B$. Note that U commutes with $A - B$ and hence, $|A - B|$ since $A - B$ is Hermitian. Observe that

$$\begin{pmatrix} A+B & |A-B| \\ |A-B| & U(A+B)U^* \end{pmatrix} = \begin{pmatrix} I & O \\ O & U \end{pmatrix} \begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix} \begin{pmatrix} I & O \\ O & U^* \end{pmatrix}.$$

Hence, the matrix

$$\begin{pmatrix} A+B & |A-B| \\ |A-B| & U(A+B)U^* \end{pmatrix}$$

is positive and therefore,

$$|A-B| \leq (A+B) \# U(A+B)U^*$$

by (6). ∎

Lemma 3 *Let $A, B \in M_n(\mathbb{C})$ be positive matrices. Then, for any $n \in \mathbb{N}$,*

$$\otimes^n (A \# B) = (\otimes^n A) \# (\otimes^n B).$$

Proof If A and B are positive definite,

$$A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

Using multiplicativity under tensor powers,

$$\otimes^n (A \# B) = (\otimes^n A)^{1/2} [(\otimes^n A)^{-1/2} (\otimes^n B) (\otimes^n A)^{-1/2}]^{1/2} (\otimes^n A)^{1/2}.$$

When A and B are not positive *definite*, we proceed in two steps. First, note by (6) that

$$\begin{pmatrix} A & A\#B \\ A\#B & B \end{pmatrix} \geq O.$$

By Proposition 1.3.2 of [6], a block matrix of the form

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

is positive if and only if there exists a contraction K such that

$$X = A^{1/2} K B^{1/2}.$$

Choose $K \in M_n(\mathbb{C})$ such that $||K|| \leq 1$ and

$$A \# B = A^{1/2} K B^{1/2}.$$

Taking tensor powers,

$$\otimes^n (A \# B) = (\otimes^n A)^{1/2} (\otimes^n K) (\otimes^n B)^{1/2}.$$

Since $|| \otimes^n K|| = ||K||^n \leq 1$, we get

$$\begin{pmatrix} \otimes^n A & \otimes^n (A\#B) \\ \otimes^n (A\#B) & \otimes^n B \end{pmatrix} \geq O.$$

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By (6), this implies

$$\otimes^n(A\#B) \leq (\otimes^n A)\#(\otimes^n B).$$

The final step is to show the reverse bound

$$\otimes^n(A\#B) \geq (\otimes^n A)\#(\otimes^n B).$$

Take $\varepsilon > 0$. Since $A + \varepsilon I$ and $B + \varepsilon I$ are positive *definite*,

$$\otimes^n[(A+\varepsilon I)\#(B+\varepsilon I)] = [\otimes^n(A+\varepsilon I)]\#[\otimes^n(B+\varepsilon I)].$$

It is easy to see by multilinearity that $\otimes^n(A+\varepsilon I) \geq \otimes^n A + \varepsilon^n \otimes^n I$ and $\otimes^n(B+\varepsilon I) \geq \otimes^n B + \varepsilon^n \otimes^n I$. By monotonicity of the geometric mean (see Theorem 4.1.3 in [6]),

$$\begin{aligned}\otimes^n[(A+\varepsilon I)\#(B+\varepsilon I)] &= [\otimes^n(A+\varepsilon I)]\#[\otimes^n(B+\varepsilon I)] \\ &\geq (\otimes^n A + \varepsilon^n \otimes^n I)\#(\otimes^n B + \varepsilon^n \otimes^n I).\end{aligned}$$

The desired inequality is now obtained by taking $\varepsilon \to 0+$. ∎

Remark The comparison argument in the proof was necessary, since the matrix geometric mean is not jointly continuous (see [6]). Hence, we cannot directly say that

$$\lim_{\varepsilon\to 0+}[\otimes^n(A+\varepsilon I)]\#[\otimes^n(B+\varepsilon I)] = (\otimes^n A)\#(\otimes^n B).$$

Tensor powers are highly non-linear, so we had to bound from below with the more tractable $\otimes^n A + \varepsilon^n \otimes^n I$ and $\otimes^n B + \varepsilon^n \otimes^n I$.

Before discussing the applications of Lemmas 2 and 3 to uncertainty bounds, let us quickly recap unitarily invariant norms. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Given any $x, y \in \mathbb{C}^n$, we denote by $x \otimes y^*$ the rank one operator $v \to \langle v, y\rangle x$ for all $v \in \mathbb{C}^n$.

Following [4], a norm $|||.|||$ on $M_n(\mathbb{C})$ is said to be *unitarily invariant* if $|||e_1 \otimes e_1^*||| = 1$ and

$$|||UAV||| = |||A|||$$

for all $A \in M_n(\mathbb{C})$ and $n \times n$ unitaries U and V. We note, as a natural consequence, that for all unit vectors x, y, $|||x \otimes y^*||| = 1$. A natural class of examples is given by the *Schatten p-norms*.

Example 1 Given $X \in M_n(\mathbb{C})$ and $p \geq 1$, define

$$||X||_p = (\operatorname{tr}|X|^p)^{1/p}.$$

This is known as the Schatten p-norm. If $p = 1$, this gives the trace norm $||X||_1 = \operatorname{tr}|X|$. For $p = 2$, it gives the Frobenius norm

$$||X||_2 = (\operatorname{tr} X^* X)^{1/2},$$

while for $p = \infty$, we get the usual operator norm $||.||$.

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It can be shown that any unitarily invariant norm $|||.|||$ satisfies

$$\|X\| \leq |||X||| \leq \|X\|_1$$

for all $X \in M_n(\mathbb{C})$ (see [4]). Thus, the trace norm and operator norm are the two extremes of the class of all unitarily invariant norms. The set of all extreme points of the unit ball $\{X \in M_n(\mathbb{C}) : \|X\|_1 \leq 1\}$ is precisely $\{x \otimes y^* : \|x\| = \|y\| = 1\}$ (see [20]). While for the operator norm, the extreme points of the unit ball are precisely the unitaries (see [6]).

The theory of unitarily invariant norms is very rich. Refer to [4] for a thorough discussion, including a complete characterization and applications to interesting matrix inequalities.

Corollary 1 *Let $\{x_j\}_{j=1}^n$ be self-adjoint elements in a C^*-algebra $\mathcal{A}$ and let $\phi : \mathcal{A} \to \mathbb{C}$ be a state. Then for any unitarily invariant norm $|||.|||$ on $M_n(\mathbb{C})$,*

$$|||\big(\frac{1}{2}\phi[x_i, x_j]\big)||| \leq |||\mathrm{Cov}_\phi(\tilde{x})|||.$$

Proof Consider

$$M = \frac{1}{2}\big(\phi(x_i x_j)\big)$$

as in (3) and recall that

$$M + M^T = \mathrm{Cov}_\phi(\tilde{x}),$$

while

$$M - M^T = \big(\frac{1}{2}\phi[x_i, x_j]\big).$$

By Lemma 2, there exists a unitary U such that

$$\begin{aligned} |M - M^T| &\leq (M + M^T)\#U(M + M^T)U^* \\ &\leq \frac{(M + M^T + U(M + M^T)U^*)}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} |||M - M^T||| &= ||| \, |M - M^T| \, ||| \\ &\leq |||M + M^T||| \\ &= |||\mathrm{Cov}_\phi(\tilde{x})|||. \end{aligned}$$

■

Remark Taking the Frobenius norm $\|.\|_2$ (see Example 1) in Corollary 1, we have

$$\frac{1}{2}\sum_{i<j} |\phi[x_i, x_j]|^2 \leq \|\mathrm{Cov}_\phi(\tilde{x})\|_2^2 \leq \big(\mathrm{tr}\ \mathrm{Cov}_\phi(\tilde{x})\big)^2 = \big[\sum_j \sigma^2(x_j)\big]^2, \tag{7}$$

where $\sigma^2(x_j) = \phi(x_j^2)$, which is the variance of x_j. This gives a non-vanishing lower bound of the sum of variances in terms of the Lie brackets of the observables, leading to an uncertainty relation.

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Given a positive matrix $A = (a_{ij})$, it is well known that

$$\det A \leq \prod_j a_{jj}.$$

Using this, we see that

$$\det|M - M^T| \leq \prod_j \langle |M - M^T| e_j, e_j \rangle,$$

where M is given by (3). The right-hand side is non-vanishing and we would like to have an upper bound for it in terms of $\mathrm{Cov}_\phi(\tilde{x})$. We need the following inequality.

Theorem 2 *Let $A, B \in M_n(\mathbb{C})$ be positive matrices. Then*

$$\Big[\prod_j \langle |A - B| e_j, e_j \rangle\Big]^{2/n} \leq \frac{\mathrm{tr}\, A + \mathrm{tr}\, B}{n} \Big[\prod_j \langle (A + B) e_j, e_j \rangle\Big]^{1/n}.$$

Proof By lLemma 2, there exists a unitary U such that

$$|A - B| \leq (A + B) \# U (A + B) U^*.$$

Taking tensor powers and using Lemma 3,

$$\otimes^n |A - B| \leq [\otimes^n (A + B)] \# [\otimes^n U (A + B) U^*].$$

Let $\{e_j\}_{j=1}^n$ be the standard basis vectors of $\mathbb{C}^n$. Consider the state $\xi : \otimes^n M_n(\mathbb{C}) \to \mathbb{C}$ given by

$$\xi(X) = \langle X(\otimes_{j=1}^n e_j), \otimes_{j=1}^n e_j \rangle.$$

Now,

$$\begin{aligned}
\Pi_j \langle |A - B| e_j, e_j \rangle^2 &= \xi\Big(\otimes^n |A - B|\Big)^2 \\
&\leq \xi\Big(\otimes^n (A + B)\Big)\ \xi\Big(\otimes^n U (A + B) U^*\Big) \\
&\leq \prod_j \langle (A + B) U^* e_j, U^* e_j \rangle\ \prod_j \langle (A + B) e_j, e_j \rangle \\
&\leq \left[\frac{\mathrm{tr}\, (A + B)}{n}\right]^n\ \prod_j \langle (A + B) e_j, e_j \rangle.
\end{aligned}$$

Taking powers of $1/n$, we are done. ∎

Remark The inequality

$$\xi\Big(\otimes^n |A - B|\Big)^2 \leq \xi\Big(\otimes^n (A + B)\Big)\ \xi\Big(\otimes^n U (A + B) U^*\Big)$$

holds because for any state $\phi : M_n(\mathbb{C}) \to \mathbb{C}$, we have $\phi(A \# B) \leq [\phi(A)\phi(B)]^{1/2}$ for any two positive matrices A and B. For details, see [15].

Applying Theorem 2 to the positive matrices M and M^T, we have the following.

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Corollary 2 *Let $\{x_j\}_{j=1}^n$ be self-adjoint elements in a C^*-algebra $\mathcal{A}$ and let $\phi : \mathcal{A} \to \mathbb{C}$ be a state. Let $\sigma(x_j) = \sqrt{\phi(x_j^2)}$ be the standard deviation of x_j. Then*

$$\frac{1}{4}\prod_{k=1}^{n}\langle|(\phi[x_i, x_j])|e_k, e_k\rangle^{2/n} \leq \left[\frac{\sum \sigma^2(x_k)}{n}\right]\left[\prod_k \sigma^2(x_k)\right]^{1/n}. \tag{8}$$

From inequality (8), we also get

$$\frac{1}{2}\prod\langle|(\phi[x_i, x_j])|e_k, e_k\rangle^{1/n} \leq \frac{\sum \sigma^2(x_k)}{n}. \tag{9}$$

This gives a non-vanishing lower bound for the arithmetic mean of the variances and hence, an uncertainty relation. Note that for two self-adjoint elements x_1 and x_2, the matrix

$$\left|\left(\phi[x_i, x_j]\right)\right|$$

is just

$$\begin{pmatrix} |\phi[x_1, x_2]| & 0 \\ 0 & |\phi[x_1, x_2]| \end{pmatrix}$$

and hence, inequality (9) becomes

$$|\phi[x_1, x_2]| \leq \sigma^2(x_1) + \sigma^2(x_2).$$

Note that this is weaker than the Heisenberg uncertainty principle. The importance of inequality (9) lies in the fact that for several self-adjoint elements $\{x_j\}_{j=1}^n$, the matrix

$$|(\phi[x_i, x_j])|$$

is no longer scalar. Hence, unlike the case for two observables, the left-hand side of inequality (9) is bigger than

$$\frac{1}{2}[\det|(\phi[x_1, x_j])|]^{1/n}.$$

Compare this to Robertson's bound in (4) and notice the improvement on the left-hand side. Weakening the right-hand side of (4) with the arithmetic mean of variances is the price being paid for the non-vanishing improvement on the left.

The next inequality gives another nonzero lower bound in terms of the entries $\phi[x_i, x_j]$.

Theorem 3 *Let $\mathcal{A}$ be a C^*-algebra and let $\{x_j\}_{j=1}^n$ be self-adjoint elements of $\mathcal{A}$. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a state. Then*

$$\frac{1}{4}\prod_k\left[\sum_{i\neq k}|\phi[x_i, x_k]|^2\right]^{1/n} \leq \|\mathrm{Cov}_\phi(\tilde{x})\|\left[\frac{\sum \sigma^2(x_k)}{n}\right]^{1/2}\left[\prod_k \sigma(x_k)\right]^{1/n}.$$

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Proof For each k,

$$\begin{aligned}\frac{1}{4}\sum_{i\neq k}|\phi[x_i,x_k]|^2 &= \langle (M-M^T)^2 e_k e_k\rangle \\ &\leq ||M-M^T||\ \langle |M-M^T|e_k,e_k\rangle \\ &\leq ||\mathrm{Cov}_\phi(\tilde{x})||\ \langle |M-M^T|e_k,e_k\rangle,\end{aligned}$$

where the last inequality follows from Corollary 1. Taking geometric mean on both sides and using Corollary 2, we are done. ∎

An important question that arises now is, can we get an uncertainty relation for which the lower bound is always nonzero whenever any two of the observables do not commute? To answer this, we need to compare the block matrices $\left([x_i,x_j]\right)$ and $\left(\{x_i,x_j\}\right)$ without composing a state entry-wise.

We do this by obtaining a trace inequality, which as remarked later, holds for tracial C^*-algebras as well but for the sake of better clarity, we stick to $M_n(\mathbb{C})$, our observables being Hermitian matrices $\{H_j\}_{j=1}^k$.

Note that, in this case, the block matrix $\left(H_jH_i\right)$ is not necessarily positive even though $\left(H_iH_j\right)$ always is. This is demonstrated by the following example.

Example 2. Let $\theta\in\mathbb{R}$ and let

$$H_1=\begin{pmatrix}\frac{\sin\ \theta}{2} & \cos\ \theta\\ \cos\ \theta & \sin\ \theta\end{pmatrix}$$

and

$$H_2=\begin{pmatrix}\cos\ \theta & \sin\ \theta\\ \sin\ \theta & \frac{\cos\ \theta}{2}\end{pmatrix}.$$

Let e_1,e_2 be the standard basis vectors of $\mathbb{C}^2$. Then the matrix

$$\left(\langle H_ie_j,H_je_i\rangle\right)$$

has negative determinant and hence is not positive.

Hence, unlike the previous results, we cannot use the idea of writing both sides as the sum and difference of two positive matrices. However, we can still obtain an inequality with the trace norm (see Example 1) such that the left-hand side vanishes if and only if all the observables commute. For two Hermitian matrices H_1 and H_2, this takes the form

$$\left|\left|\begin{pmatrix}O & [H_1,H_2]\\ [H_2,H_1] & O\end{pmatrix}\right|\right|_1\leq 2(\mathrm{tr}\ H_1^2+\mathrm{tr}\ H_2^2).$$

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This is easy to prove, since,

$$\begin{aligned}\left\|\begin{pmatrix} O & [H_1,H_2] \\ [H_2,H_1] & O \end{pmatrix}\right\|_1 &= 2\ \mathrm{tr}\,\big|[H_1,H_2]\big| \\ &\le 2\big(\|H_1H_2\|_1 + \|H_2H_1\|_1\big) \\ &\le 4\sqrt{\mathrm{tr}\ H_1^2\ \mathrm{tr}\ H_2^2} \\ &\le 2(\mathrm{tr}\ H_1^2 + \mathrm{tr}\ H_2^2).\end{aligned}$$

The next theorem generalizes this to k Hermitian matrices $\{H_j\}_{j=1}^k$.

Theorem 4 *Let $\{H_j\}_{j=1}^k$ be $n \times n$ Hermitian matrices. Then*

$$\big\|\big([H_i,H_j]\big)\big\|_1 \le (k-1)\mathrm{tr}\,\big(\{H_i,H_j\}\big).$$

Proof Let

$$A = \big([H_i,H_j]\big).$$

Note that A is Hermitian and

$$\|A\|_1 = \mathrm{tr}\ (A^2)^{1/2}.$$

Let $\varphi : M_k(M_n(\mathbb{C})) \to M_k(M_n(\mathbb{C}))$ be given by

$$\Phi\begin{pmatrix} A_{11} & \dots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \dots & A_{kk} \end{pmatrix} = \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{kk} \end{pmatrix}$$

for all block matrices $(A_{ij}) \in M_k(M_n(\mathbb{C}))$. This operation is known as the *pinching* and is a central tool in matrix analysis, with numerous applications to matrix inequalities (see [4, 5]). It also constitutes an important example of a *quantum channel* known as a *conditional expectation* (see [12] for details) and plays a significant role in the derivation of quantum entropy inequalities.

Proceeding with our proof, consider the matrix

$$D = \Phi(A^2) = \begin{pmatrix} -\sum_{j\neq 1}[H_1,H_j]^2 & & \\ & \ddots & \\ & & -\sum_{j\neq n}[H_n,H_j]^2 \end{pmatrix}.$$

Note that

$$D = \frac{1}{2\pi}\int_0^{2\pi} U(\theta)A^2U(\theta)^*\,d\theta,$$

where

$$U(\theta) = \begin{pmatrix} I & & & \\ & e^{i\theta}I & & \\ & & \ddots & \\ & & & e^{i(k-1)\theta}I \end{pmatrix}.$$

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By operator concavity of the square root,

$$D^{1/2} \geq \frac{1}{2\pi} \int_0^{2\pi} U(\theta)|A|U(\theta)^* d\theta.$$

Taking traces, we get the inequality

$$\|A\|_1 \leq \sum_i \mathrm{tr}\left(\sum_j \left|[H_i, H_j]\right|^2\right)^{1/2}). \tag{10}$$

Now, for each i, consider the block matrix

$$\tilde{H}_i = \begin{pmatrix} [H_i, H_1] & O & \cdots & O \\ \vdots & & & \\ [H_i, H_k] & O & \cdots & O \end{pmatrix}.$$

Note that inequality (10) now becomes

$$\|A\|_1 \leq \sum_i \|\tilde{H}_i\|_1.$$

Let

$$Y_i = \begin{pmatrix} H_iH_1 & O & \cdots & \cdots & O \\ \vdots & & & & \\ H_iH_{i-1} & & & & \\ & & & & \\ O & & & & \\ & & & & \\ H_iH_{i+1} & & & & \\ \vdots & & & & \\ H_iH_k & O & \cdots & \cdots & O \end{pmatrix}$$

and

$$Z_i = \begin{pmatrix} H_1H_i & O & \cdots & \cdots & O \\ \vdots & & & & \\ H_{i-1}H_i & & & & \\ & & & & \\ O & & & & \\ & & & & \\ H_{i+1}H_i & & & & \\ \vdots & & & & \\ H_kH_i & O & \cdots & \cdots & O \end{pmatrix},$$

where the O in the first column is at the i^{th} position. Note that $\tilde{H}_i = Y_i - Z_i$. Inequality (10) now gives

$$\begin{aligned} \|A\|_1 &\leq \sum_i \|Y_i - Z_i\|_1 \\ &\leq \sum_i \|Y_i\|_1 + \|Z_i\|_1. \end{aligned} \tag{11}$$

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Consider the $k \times k$ block matrices

$$\widehat{H}_i = \begin{pmatrix} H_1 & O & \cdots & \cdots & O \\ \vdots & & & & \\ H_{i-1} & & & & \\ O & & & & \\ H_{i+1} & & & & \\ \vdots & & & & \\ H_k & O & \cdots & \cdots & O \end{pmatrix}, \quad \tilde{Y}_i = \begin{pmatrix} H_i & & & & & & \\ & \ddots & & & & & \\ & & H_i & & & & \\ & & & O & & & \\ & & & & H_i & & \\ & & & & & \ddots & \\ & & & & & & H_i \end{pmatrix}$$

and

$$\tilde{K}_i = \begin{pmatrix} H_i & O & \cdots & \cdots & O \\ \vdots & & & & \\ H_i & & & & \\ O & & & & \\ H_i & & & & \\ \vdots & & & & \\ H_i & O & \cdots & \cdots & O \end{pmatrix}, \quad \widehat{Z}_i = \begin{pmatrix} H_1 & & & & & & \\ & \ddots & & & & & \\ & & H_{i-1} & & & & \\ & & & O & & & \\ & & & & H_{i+1} & & \\ & & & & & \ddots & \\ & & & & & & H_k \end{pmatrix}.$$

Here, O is the i^{th} entry in the first column of $\widehat{H}_i$ and $\tilde{K}_i$, and the i^{th} diagonal entry of $\tilde{Y}_i$ and $\widehat{Z}_i$. Note that $Y_i = \widehat{H}_i \tilde{Y}_i$ and $Z_i = \tilde{K}_i \widehat{Z}_i$. By Cauchy–Schwarz and A.M–G.M,

$$\begin{aligned} ||Y_i||_1 &\leq ||\widehat{H}_i||_2 ||\tilde{Y}_i||_2 \\ &\leq \frac{||\widehat{H}_i||_2^2 + ||\tilde{Y}_i||_2^2}{2} \\ &\leq \frac{(k-1)\ \text{tr}\ H_i^2 + \sum_{j \neq i} \text{tr}\ H_j^2}{2}. \end{aligned} \tag{12}$$

Similarly,

$$||Z_i||_1 \leq \frac{(k-1)\ \text{tr}\ H_i^2 + \sum_{j \neq i} \text{tr}\ H_j^2}{2}. \tag{13}$$

By (11)–(13),

$$\begin{aligned} ||A||_1 &\leq (k-1) \sum_i \text{tr}\ H_i^2 + \sum_i \Big[\sum_j \text{tr}\ H_j^2 - \text{tr}\ H_i^2 \Big] \\ &= 2(k-1) \sum_i \text{tr}\ H_i^2 \\ &= (k-1)\ \text{tr}\ \big(\{H_i, H_j\} \big). \end{aligned}$$

■

It is to be noted that even though we have stuck to Hermitian matrices keeping up with the spirit of the article, a slight variant of Theorem 4 can be obtained for a k-tuple $\{X_j\}_{j=1}^k$ of matrices which are not necessarily Hermitian. In fact, the same proof, with

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very minor modifications, shows that

$$\left\|\left([X_i, X_j]\right)\right\|_1 \leq 2(k-1)\sum_{i=1}^{k}\|X_i\|_2^2. \tag{14}$$

This gives a trace norm bound for the block matrix formed by the commutators $[X_i, X_j]$. Commutator bounds for *two* matrices are well-studied in the literature. The Böttcher–Wenzel inequality (see [3, 10, 11, 14, 29, 37, 38]) states that for two matrices X and Y,

$$\|[X, Y]\|_2 \leq \sqrt{2}\|X\|_2\|Y\|_2. \tag{15}$$

Later, Wenzel and Audenaert [39] used complex interpolation theory to generalize inequality (15) for arbitrary Schatten-p norms. Bhatia and Kittaneh [9] gave unitarily invariant norm bounds on the commutator under the assumption of normality of one of the matrices and showed connections to spectral variation.

Inequality (14), to the best of our knowledge, is the first trace norm bound for the block matrix of commutators $\left([X_i, X_j]\right)$. Note that upper bounds for the trace norm are generally very useful in matrix theory, since it is the largest unitarily invariant norm (see the discussion following Example 1).

Remarks

1. Note that Theorem 4 gives a refinement of Corollary 1 when ϕ is the normalized trace on $M_n(\mathbb{C})$ and $|||.|||$ is the trace norm. In this case, the left-hand side of the inequality in Corollary 1 vanishes, but Theorem 4 gives a non-vanishing lower bound.
2. Theorem 4 *always* gives a nonzero lower bound whenever any two of the observables do not commute.
3. Let $\mathcal{A}$ be a C^*-algebra with a faithful finite trace τ (see [26, 28] for details). Given $x \in \mathcal{A}$, its trace norm is given by

$$\|x\|_1 = \tau(|x|)$$

for all $x \in \mathcal{M}$. We can lift τ to $\tau_k : M_k(\mathcal{A}) \to \mathbb{C}$ via

$$\tau_k(x_{ij}) = \sum_i \tau(x_{ii}).$$

Note that τ_k is positive, faithful, and tracial. Let $\{x_j\}_{j=1}^k$ be elements of $\mathcal{A}$. In this context, inequality (14) reads

$$\left\|\left([x_i, x_j]\right)\right\|_1 \leq 2(k-1)\sum_{i=1}^{k}\tau(x_i^* x_i), \tag{16}$$

where $\|.\|_1$ is the trace norm induced by τ_k on $M_n(\mathcal{A})$. Indeed, the same proof holds, since the only properties of the trace being used there are invariance under unitary conjugation, triangle inequality, and Cauchy–Schwarz inequality, all of which hold in the C^*-algebra setting (see [16, 31]).

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3 Conclusions

In this article, we first obtained several multivariate uncertainty relations for observables in a unital C^*-algebra equipped with a state. The starting point was a determinant inequality for positive matrices, using which we gave a simple proof of Robertson's uncertainty principle [35]. Using the matrix geometric mean, we then obtained some interesting inequalities for a pair of positive matrices A and B. These matrix inequalities were then applied to covariance matrices associated with observables, yielding lower bounds involving their commutators.

The resulting uncertainty relations (see (9), Corollary 1, and Theorem 4) provide nontrivial lower bounds for sums and arithmetic means of variances, regardless of the number of observables. This is an improvement on Robertson's determinant inequality, where the lower bound is zero when the number of observables is odd.

Of particular importance is the trace norm inequality in Theorem 4 comparing the block matrix of commutators with the corresponding block matrix of anti-commutators. As a consequence, one obtains a state-independent lower bound which is nonzero whenever any two observables fail to commute.

One advantage of Theorem 4 is that it extends to tracial C^*-algebras as shown in (16). The standard Böttcher–Wenzel inequality, however, does not. To see this, consider the normalized trace $\tau_n : M_n(\mathbb{C}) \to \mathbb{C}$ given by $\tau_n(A) = \frac{\operatorname{tr} A}{n}$ for all $A \in M_n(\mathbb{C})$. This is a tracial state on $M_n(\mathbb{C})$. We can then modify the Böttcher–Wenzel inequality in (15) as

$$\tau_n|[X, Y]|^2 \leq 2n\, \tau_n(X^*X)\tau_n(Y^*Y).$$

The sharpness of the Böttcher–Wenzel bound (see [11]) and the blow-up of the factor $2n$ imply that we cannot have a bound of the form

$$\tau|[x, y]|^2 \leq C\tau(x^*x)\tau(y^*y)$$

on a tracial C^*-algebra $(\mathcal{A}, \tau)$ for any universal $C > 0$. Our inequality is novel in this sense, since it solves the more general problem of giving a state-independent bound with the block matrix of commutators

$$([x_i, x_j])$$

with a factor depending only on the number of observables, and not on the algebra or the ambient trace.

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References

[1] A. Andai, *Uncertainty principle with quantum Fisher information*. J. Math. Phys. **49**(2008), no. 1, 012106.

[2] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*. Linear Algebra Appl. **26**(1979), 203–241.

[3] K. M. R. Audenaert, *Variance bounds, with an application to norm bounds for commutators*. Linear Algebra Appl. **432**(2010), no. 5, 1126–1143.

[4] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, 169, Springer-Verlag, New York, NY, 1997.

[5] R. Bhatia, *Pinching, trimming, truncating and averaging of matrices*. Amer. Math. Monthly **107**(2000), no. 7, 602–608.

[6] R. Bhatia, *Positive definite matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, New Jersey, 2007.

[7] R. Bhatia, *The Riemannian mean of positive matrices*. In: F. Nielsen and R. Bhatia (eds.), Matrix information geometry, Springer, Berlin, 2012, 35–51.

[8] R. Bhatia and J. Holbrook, *Riemannian geometry and matrix geometric means*. Linear Algebra Appl. **413**(2006), 594–618.

[9] R. Bhatia and F. Kittaneh, *Commutators, pinchings and spectral variation*. Oper. Matrices **2**(2008), no. 1, 143–151.

[10] A. Böttcher and D. Wenzel, *How big can the commutator of two matrices be and how big is it typically?* Linear Algebra Appl. **403**(2005), 216–228.

[11] A. Böttcher and D. Wenzel, *The Frobenius norm and the commutator*. Linear Algebra Appl. **429**(2008), no. 8, 1864–1885.

[12] E. A. Carlen, *Trace inequalities and quantum entropy: An introductory course*. In Entropy and the Quantum, edited by Robert Sims and Daniel Ueltschi, Contemporary Mathematics, vol. 529. 73–140, American Mathematical Society, Providence, RI, 2010.

[13] B. Chen and S.-M. Fei, *Sum uncertainty relations for arbitrary N incompatible observables*. Sci. Rep. **5**(2015), 14238. https://doi.org/10.1038/srep14238

[14] C.-M. Cheng, X.-Q. Jin, and S.-W. Vong, *A survey on the Böttcher-Wenzel conjecture and related problems*. Oper. Matrices **9**(2015), no. 3, 659–673.

[15] M. Congedo, A. Barachant, and R. Bhatia, *Riemannian geometry for EEG-based brain-computer interfaces*. Brain-Comput. Interfaces. **4**(2017), no. 3, 155–174.

[16] R. C. da Silva, *Lecture notes on non-commutative L_p spaces*. Preprint, 2018. https://arxiv.org/abs/1803.02390

[17] G. B. Folland and A. Sitaram, *The uncertainty principle: A mathematical survey*. J. Fourier Anal. Appl. **3**(1997), 207–238.

[18] P. Gibilisco, F. Hiai, and D. Petz, *Quantum covariance, quantum Fisher information and uncertainty relations*. IEEE Trans. Inform. Theory **55**(2009), no. 1, 439–443.

[19] P. Gibilisco, D. Imparato, and T. Isola, *A Robertson-type uncertainty principle and quantum Fisher information*. Linear Algebra Appl. **428**(2008), no. 7, 1706–1724.

[20] R. Grone and M. Marcus, *Isometries of matrix algebras*. J. Algebra **47**(1977), 180–189.

[21] B. C. Hall, *Quantum theory for mathematicians*, Graduate Texts in Mathematics, 267, Springer, New York, NY, 2013.

[22] W. Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik*. Z. Phys. (in German) **43**(1927), no. 3, 172–198.

[23] F. Hiai, *Log majorizations and norm inequalities for exponential operators*. Banach Centre Publ. **38**(1997), 119–181.

[24] Y. Huang, *Variance based uncertainty relations*. Phys. Rev. A **86**(2012), 24101. https://doi.org/10.1103/PhysRevA.86.024101

[25] A. Jenčová, *Geodesic distances on density matrices*. J. Math. Phys. **45**(2004), 1787–1794.

[26] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras. Vol.* II, Academic Press, Orlando, Florida, (1986).

[27] J. D. Lawson and Y. Lim, *The geometric mean, matrices, metrics and more*. Amer. Math. Monthly **108**(2001), 797–812.

[28] B.-R. Li, *Introduction to operator algebras*, World Scientific Publishing, Singapore, 1992.

[29] Z. Lu, *Remarks on the Böttcher-Wenzel inequality*, Linear Algebra Appl. **436**(2012), no. 7, 2531–2535.

[30] M. Moakher, *A differential geometric approach to the geometric mean of symmetric positive definite matrices*. SIAM J. Matrix Anal. Appl. **26**(2005), no. 3, 735–747.

https://doi.org/10.4153/S0008439526102197 Published online by Cambridge University Press

[31] E. Nelson, *Notes on non-commutative integration*. J. Funct. Anal. **15**(1974), 103–116.
[32] D. Petz, *Monotone metrics on matrix spaces*. Linear Algebra Appl. **244**(1996), 81–96.
[33] D. Petz and C. Ghinea, *Introduction to quantum Fisher information*. Quantum Probability and Related Topics (2011), 261–281. doi: 10.1142/9789814338745_0015.
[34] W. Pusz and S. L. Woronowicz, *Functional calculus for sesquilinear forms and the purification map*. Rep. Math. Phys. **8**(1975), 159–170.
[35] H. P. Robertson, *An indeterminacy relation for several observables and its classical interpretation*. Phys. Rev. **46**(1934), 794–801.
[36] E. Schrödinger, *About Heisenberg uncertainty relation (original annotation by A. Angelow and M. C. Batoni)*. Bulg. J. Phys. **26**(2000), nos. 5–6, 193–203.
[37] S.-W . Vong and X.-Q. Jin, *Proof of Böttcher and Wenzel's conjecture*. Oper. Matrices **2**(2008), no. 3, 435–442.
[38] D. Wenzel, *Dominating the commutator*. In: J. A. Ball, V. Bolotnikov, J. W. Helton, L. Rodman, and I. M. Spitkovsky (eds.), Topics in operator theory (1), operators, matrices and analytic functions, Operator Theory: Advances and Applications, 202, Birkhäuser Verlag, Basel, 2010, pp. 579–600.
[39] D. Wenzel and K. M. R. Audenaert, *Impressions of convexity: An illustration for commutator bounds*. Linear Algebra Appl. **433**(2010), nos. 11–12, 1726–1759.
[40] M. M. Wilde, *Quantum information theory*, Cambridge University Press, Cambridge, 2013.

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