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Recurrent surface homeomorphisms

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Abstract

An orientation-preserving, recurrent homeomorphism of the two-sphere which is not the identity is shown to admit exactly two fixed points. A recurrent homeomorphism of a compact surface with negative Euler characteristic is periodic.

1. *Introduction*

A homeomorphism f of a compact metric space (X, d) is recurrent if it admits iterates arbitrarily close to the identity, i.e. if there exists a sequence $n_k \to +\infty$ such that

$$d(f^{n_k}, \mathrm{Id}) \to 0 \quad \text{as} \quad k \to +\infty.$$

This notion is independent of the metric d which defines the topology on X, and is invariant under topological conjugacy.

The set of recurrent homeomorphisms of X includes periodic homeomorphisms and more generally regular homeomorphisms [**1**, **8**], i.e. homeomorphisms such that the sequence of all iterates forms an equicontinuous family:

$$\forall \epsilon > 0, \exists \eta > 0 / d(x, y) \leqslant \eta \Rightarrow d(f^k(x), f^k(y)) \leqslant \epsilon, \quad \forall k \in \mathbf{Z}.$$

In the case of the two-sphere $\mathbf{S}^2$, regular homeomorphisms have been completely classified by Kérékjártó. They are conjugate to the restriction of Euclidean isometries of the ambient 3-space [**1**, **4**, **9**, **10**].

In [**11**], Kérékjártó asked if a non-periodic, orientation-preserving, recurrent homeomorphism of the sphere is always conjugate to an irrational rotation. The answer is now known to be false. In [**6**], Fokkink and Oversteegen gave an example of a periodic-point free, recurrent homeomorphism of the annulus which is not conjugate to an irrational rotation. A similar construction gives an orientation-preserving, recurrent homeomorphism of $\mathbf{S}^2$ with just two fixed points (and no other periodic points) that is not conjugate to an irrational rotation. In that example, the homeomorphism leaves invariant a non-locally connected continuum H with a dense orbit even if it is not minimal: the orbit closure of each point contains a fixed point. Besides the fact that recurrence does not imply regularity, this example shows that the collection of orbit closures may not even form a partition of $\mathbf{S}^2$. However, we will establish here the following result.

Theorem 1·1. *A non-trivial, orientation-preserving and recurrent homeomorphism of the sphere* $\mathbf{S}^2$ *has exactly two fixed points.*

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An immediate corollary of Theorem 1·1 is that an orientation-preserving, recurrent homeomorphism of the closed disc $\mathbf{D}^2$ which is the identity on the boundary is the identity on the whole disc. This is the main ingredient used by Oversteegen and Tymchatyn in [**15**] to show that a recurrent homeomorphism of the Euclidean plane $\mathbf{R}^2$ is periodic. However, we do not think that Theorem 1·1 can be proved by a simple rephrasing of the arguments in [**15**]: one could expect to deduce Theorem 1·1 from the result of Oversteegen and Tymchatyn applying the arguments of Section 4 to the non-compact surface M obtained by removing the fixed points of a recurrent homeomorphism of the sphere. The homeomorphism obtained in such a way is clearly recurrent for the induced, restricted metric on M but not necessarily for a hyperbolic metric on M. Notice that the analogous statement is false in a simpler case: if one removes one or two fixed points of an irrational rotation of the sphere, the induced homeomorphism on M (the plane or the open annulus) is certainly not recurrent for the euclidean metric, otherwise it would be periodic by [**15**]. Moreover, our proof relies on a simple property: the fact that a homeomorphism is recurrent is inherited by the induced homeomorphism on prime ends. Up to our knowledge, this elementary idea has not yet been pointed out in the literature.

Even if he does not seem to have published any proof of it, Kérékjártó announced in [**11**] that a recurrent homeomorphism of a closed, orientable surface of genus $g > 1$ is periodic. This result is a corollary of Theorem 1·1 as we shall see in Section 4. As noted by Kérékjártó in [**12**], this implies in particular that a homeomorphism of a compact surface with negative Euler characteristic cannot be topologically transitive provided it has at least one regular point.

The paper is organized as follows: in Section 2, we review required material from prime ends' theory. For the proofs in this section, we refer the reader to the works of Epstein [**5**] or Mather [**13**]. The proof of Theorem 1·1 is given in Section 3. Section 4 contains several corollaries concerning recurrent homeomorphisms of other surfaces orientable or not.

2. *Prime ends*

Let U be an open, connected subset of the sphere $\mathbf{S}^2$. For a set $A \subset U$ we denote its closure in $\mathbf{S}^2$ by $\mathrm{Cl}\,A$, its closure in U by $\mathrm{Cl}_U A = \mathrm{Cl}\,A \cap U$, its boundary in $\mathbf{S}^2$ by $\mathrm{Bd}\,A$ and its boundary in U by $\mathrm{Bd}_U A = \mathrm{Cl}\,A \cup U \cap \mathrm{Cl}(U \setminus A)$. A topological chain is a sequence $\Omega = (\Omega_i)_{i\in\mathbf{N}}$ of open subsets of U such that:

(1) Ω_i and $\mathrm{Bd}_U \Omega_i$ are non-empty connected sets;
(2) $\mathrm{Cl}_U \Omega_{i+1} \subset \Omega_i$;
(3) $\mathrm{Cl}\,(\mathrm{Bd}_U \Omega_i) \cap \mathrm{Cl}\,(\mathrm{Bd}_U \Omega_j) = \varnothing$ if $i \neq j$;
(4) there exists a point $x \in \mathrm{Cl}\,U$, the principal point of Ω, such that $x \notin \mathrm{Cl}\,(\mathrm{Bd}_U \Omega_i)$ for all i and $\mathrm{Bd}_U \Omega_i \to x$ as $i \to \infty$.

Two chains Ω and Ω' are equivalent, denoted by $\Omega \simeq \Omega'$, if they divide each other, i.e. if for all i there exists j such that $\Omega'_j \subset \Omega_i$ and $\Omega_j \subset \Omega'_i$. The set of all topological chains equivalent to a given chain Ω is called the prime point defined by Ω, and we write $e = [\Omega]$, or $\Omega \in e$. We denote the set of all the prime points by $\hat{U}$.

Let e be a prime point, and let Ω be any topological chain that defines e. The set $Y(e) = \bigcap_{i\in\mathbf{N}} \mathrm{Cl}\,\Omega_i$ depends only on e, and is called the impression of e. The set of principal points of chains $\Omega \in e$ is called the principal set of e, which we denote by $X(e)$.

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It can be shown [**5**, **13**] that $Y(e)$ and $X(e)$ are continua i.e. non-empty, compact, connected sets.

Given $x \in U$, choose any sequence of open discs $(D_i)_{i \in \mathbf{N}}$ such that $\bigcap_{i \in \mathbf{N}} \operatorname{Cl} D_i = \{x\}$ and $\operatorname{Cl} D_{i+1} \subset D_i$. The sequence $(D_i)_{i \in \mathbf{N}}$ is a topological chain, and the prime point $\omega(x)$ which it defines depends only on x. Moreover, we have that $X(\omega(x)) = Y(\omega(x)) = \{x\}$. This allows us to define a one-to-one map $x \mapsto \omega(x)$ from U into $\hat{U}$, and to consider U as a subset of $\hat{U}$. A prime point which is not in $\omega(U)$ is called a prime end, and it is characterized by the fact that $Y(e) \subset \operatorname{Bd} U$ (cf. lemma 2·5 of [**5**]).

A point $x \in \operatorname{Bd} U$ is accessible (from U) provided that there exists a path $\gamma : [0, 1[\to U$ such that $\gamma(t) \to x$ as $t \to 1$. The set of accessible points is dense in $\operatorname{Bd} U$. If $\mathbf{S}^2 \setminus U$ is the union of a finite number of non-degenerate continua, we have the following lemma.

LEMMA 2·1. *For each accessible point $x \in \operatorname{Bd} U$, there exists a prime end $e \in \hat{U} \setminus U$ such that $X(e) = \{x\}$.*

For any open set $V \subset U$, we define

$$\hat{V} = \{e = [\Omega]; \quad \Omega_i \subset V, \quad \text{for some } i\}.$$

With this notation, we have that $\hat{V} \cap \hat{W} = \widehat{V \cap W}$ and $\hat{V} \cup \hat{W} \subset \widehat{V \cup W}$. Therefore, the family $\{\hat{V}; V \subset U\}$ is a basis for a topology τ on $\hat{U}$. It follows from the definition of this topology that:

(1) if $e = [\Omega]$ then $\widehat{\Omega_i}$ is a basis of neighbourhoods of e,

(2) $\omega : U \to \hat{U}$ is a continuous, open map, and $\omega(U)$ is a dense open subset of $\hat{U}$.

Suppose now that we are given a self-homeomorphism f of the pair $(\operatorname{Cl} U, U)$, i.e. such that $f(U) = U$. For each topological chain Ω with principal point x, $f(\Omega) = (f(\Omega_i))_{i \in \mathbf{N}}$ is a topological chain with principal point $f(x)$, and $\Omega' \simeq \Omega$ if and only if $f(\Omega') \simeq f(\Omega)$. Hence f induces a homeomorphism $\hat{f}$ on $\hat{U}$ such that:

(1) for each $V \subset U$, $\hat{f}(\hat{V}) = \widehat{f(V)}$,

(2) for each $e \in \hat{U}$, $X(\hat{f}(e)) = f(X(e))$ and $Y(\hat{f}(e)) = f(Y(e))$.

Clearly, if $x \in U$ is a fixed point of f then $\omega(x)$ is a fixed point for $\hat{f}$. But, in general, nothing can be said about fixed prime ends from the existence of fixed points of f lying on $\operatorname{Bd} U$. Moreover, there are simple examples which show that f may have fixed points on $\operatorname{Bd} U$ although $\hat{f}$ has no fixed point at all. However, if e is a fixed prime end and $X(e) = \{x\}$ then $f(x) = x$ since:

$$\{f(x)\} = X(\hat{f}(e)) = X(e) = \{x\}.$$

The following result is a consequence of Lemma 2·1 and of the density of accessible points in $\operatorname{Bd} U$.

COROLLARY 2·2. *Let $f : (\operatorname{Cl} U, U) \to (\operatorname{Cl} U, U)$ be a homeomorphism. If $\hat{f} | \hat{U} \setminus \omega(U) = \mathrm{Id}$ then $f | \operatorname{Bd} U = \mathrm{Id}$.*

3. *Proof of the main proposition*

Let f be an orientation-preserving homeomorphism of $\mathbf{S}^2$. According to Brouwer's fixed point theorem for the sphere, f has at least one fixed point N. Moreover, if f is recurrent then f induces an orientation-preserving homeomorphism of $\mathbf{R}^2 = \mathbf{S}^2 \setminus \{N\}$

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for which each point is recurrent. According to Brouwer's lemma on translation arcs [**2**, **7**] which asserts that all points of an orientation-preserving, fixed point free homeomorphism of the plane are wandering, f has necessarily a second fixed point S.

To prove Theorem 1·1, it is enough to show that $\mathrm{Fix}f$ is connected provided that f has at least three fixed points. Indeed, if $\mathrm{Fix}f$ is a connected set, each component of $\mathbf{S}^2 \setminus Fixf$ is a simply connected domain of $\mathbf{S}^2$. According to a result of Brown and Kister [**3**], an orientation-preserving homeomorphism of the sphere leaves invariant each component of the complement of its fixed point set. Hence, since every proper, simply connected domain of the sphere is homeomorphic with the plane, the fact that $\mathrm{Fix}f \neq \mathbf{S}^2$ implies the existence of a domain $U \simeq \mathbf{R}^2$ such that $f(U) = U$ and $\mathrm{Fix}f_{|U} = \varnothing$. But each point of U is recurrent under f, this is once again a contradiction with Brouwer's lemma on translation arcs.

LEMMA 3·1. *Let f be a recurrent homeomorphism of $\mathbf{S}^2$ and P and N be two distinct fixed points of f. Let c be a simple closed curve which separates P from N, and let D_P be the disc bounded by c and containing P. There exists a simply-connected continuum K such that:*

(1) $P \in K \subset \mathrm{Cl}\, D_P$;
(2) $K \cap c \neq \varnothing$;
(3) *K is completely invariant*: $f(K) = K$.

Proof. The existence of K is given by the so-called Birkhoff construction. We define inductively a decreasing sequence of Jordan domains as follows:

(1) $D_0 = D_P$;
(2) D_{n+1} = the connected component of P in $f(D_n) \cap D_0$.

That each D_n is a Jordan domain can be established by induction: if $f(D_n) \subset D_0$ or $D_0 \subset f(D_n)$, there is nothing to prove. Otherwise, the two simple closed curves $\mathrm{Bd}\, D_0$ and $f(\mathrm{Bd}\, D_n)$ have more than one common point and D_{n+1} is a Jordan domain (theorem 16·3 of [**14**]).

The set $K = \bigcap_{n \in \mathbf{N}} \mathrm{Cl}\, D_n$ is a simply-connected continuum, contained in $\mathrm{Cl}\, D_P$ and containing P. Moreover $K \subset f(K)$ and since f is recurrent this implies $f(K) = K$.

It remains to show that $K \cap c \neq 0$. If not, we can find an integer $n_0 \geqslant 0$ such that $\mathrm{Cl}\, D_{n_0+1} \cap c = \varnothing$ and so

$$\mathrm{Bd}\, D_{n_0+1} \subset \mathrm{Bd} f(D_{n_0}).$$

Hence $D_{n_0+1} = f(D_{n_0})$ and similarly $D_{n_0+i} = f(D_{n_0+i-1})$ for all $i \geqslant 1$. Therefore,

$$D_{n_0+i} = f^i(D_{n_0}), \quad \forall i \geqslant 1.$$

Since f is recurrent, $\mathrm{Cl} f^i(D_{n_0}) \cap c_{n_0} \neq \varnothing$ for all $i \geqslant 1$ where c_{n_0} is the boundary of D_{n_0} and hence $K \cap c_{n_0} \neq \varnothing$. But c_{n_0} is a subset of $c \cup f(c) \cup \cdots \cup f^{n_0}(c)$ and since $f(K) = K$, this leads to $K \cap c \neq \varnothing$. This gives a contradiction and completes the proof. □

Suppose that $\mathrm{Fix}f$ is not connected. We can write $\mathrm{Fix}f = A \cup B$, where A and B are two disjoint, non-empty closed sets. Let $N \in A$ and $S \in B$. According to 3·1, there is a Jordan curve c, in the complement of $\mathrm{Fix}f = A \cup B$, which separates N from S (see [**14**] or [**16**]). Let D_N (resp. D_S) be the component of $\mathbf{S}^2 \setminus c$ which contains N (resp. S).

If $\mathrm{Fix}f$ has a third fixed point, at least one component of $\mathbf{S}^2 \setminus c$, D_N say, contains two fixed points P and N. Let $J \subset D_N$ be a simple closed curve which separates P and N, and let D'_N be the component of $\mathbf{S}^2 \setminus J$ that contains N. Using the construction of

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Lemma 3·1, first with J and N and then with S and c, we construct two disjoint and non-degenerate continua K_N and K_S such that:

(1) $f(K_N) = K_N$ and $f(K_S) = K_S$,

(2) $\varnothing \neq K^S \cap c \subset \mathrm{Bd}\, K_s \cap c \subset \mathbf{S}^2 \setminus \mathrm{Fix} f$.

The set $U = \mathbf{S}^2 \setminus (K_N \cup K_S)$ is an open topological annulus which contains P and which is invariant under f. According to theorem 6·6 of [**5**], $\hat{U}$ is homeomorphic to a closed annulus an we have the following lemma.

Lemma 3·2 *The homeomorphism $\hat{f}$ induced by f on $\hat{U}$ is recurrent.*

Before giving the proof of this result, we recall an elementary lemma of topology widely used in the theory of prime ends [**13**].

Lemma 3·3. *Let A and B be open subsets of a topological space X. Suppose that X, A, B, $\mathrm{Bd}_X A$ and $\mathrm{Bd}_X B$ are non-empty connected sets, and that $\mathrm{Bd}_X A \cap \mathrm{Bd}_X B \neq \varnothing$. Then exactly one of the following situations holds*:

(1) $X = A \cup B$,

(2) $\mathrm{Cl}_X A \cap \mathrm{Cl}_X B = \varnothing$,

(3) $\mathrm{Cl}_X A \subset B$,

(4) $\mathrm{Cl}_X B \subset A$.

Proof of Lemma 3·2. We fix a metric on the closed annulus $\hat{U}$, and to avoid any confusion with the metric d on U, we denote it by α. Choose a sequence n_k such that $d(f^{n_k}, \mathrm{Id}) \to 0$ on $\mathrm{Cl}\, U$. Let $\epsilon > 0$, $e \in \hat{U}$ and $B_\alpha(e, \epsilon)$ be the ball of radius ϵ around e. For any chain Ω that defines e, we have $\widehat{\Omega_i} \subset B_\alpha(e, \epsilon)$, for large enough i. Hence we can assume that $\widehat{\Omega_1} \subset B_\alpha(e, \epsilon)$. Using recurrence of f on $\mathrm{Cl}\, U$ and applying Lemma 3·3 with $X = U$, $A = \Omega_1$, $B = f^{n_k}(\Omega_2)$, we can find an integer $k(e)$ such that:

$$f^{n_k}(\Omega_2) \subset \Omega_1 \quad \text{for all} \quad k \geqslant k(e).$$

Hence we have found a neighbourhood $\mathscr{V}_e = \hat{\Omega}_2$ of e, and an integer $k(e)$ so that

$$\mathscr{V}_e \quad \text{and} \quad \hat{f}^{n_k}(\mathscr{V}_e) \subset B_\alpha(e, \epsilon), \quad \text{for all} \quad k \geqslant k(e).$$

Since $\hat{U}$ is compact, only a finite number of $\mathscr{V}_e$, say $\mathscr{V}_{e_1}, \ldots, \mathscr{V}_{e_r}$, are necessary to cover $\hat{U}$, and for $k \geqslant \max\{k(e_1), \ldots, k(e_r)\}$, we have that $\alpha(\hat{f}^{n_k}, \mathrm{Id}) \leqslant 2\epsilon$. In other words, $\hat{f}$ is recurrent. □

Hence, $\hat{f}$ is an orientation-preserving, boundary-preserving, recurrent homeomorphism of the closed annulus $\hat{U}$. Since $\hat{f}$ has a fixed point, any lift F of $\hat{f}$ to $\mathbf{R} \times I$ with a fixed point is still recurrent. On each line $\mathbf{R} \times \{i\}$, $(i = 0, 1)$, F is necessarily the identify. Therefore, $\hat{f} | \hat{U} \setminus U$ is the identity and according to Lemma 2·2, $f | \mathrm{Bd}\, U$ is the identity. Since $\mathrm{Bd}\, U$ meets c which lies in $\mathbf{S}^2 \setminus \mathrm{Fix} f$, this is a contradiction. Hence, $\mathrm{Fix} f$ is connected and the proof of 1·1 is now complete.

Corollary 3·4. *An orientation-reversing homeomorphism of the sphere which has a fixed point is an involution.*

Proof. Let f be a recurrent, orientation-reversing homeomorphism of the sphere. Suppose that f has a fixed point x but $f^2 \neq \mathrm{Id}$. According to Theorem 1·1, f^2 has only two fixed points x and y with $f(y) = y$ necessarily. Birkhoff's construction may be applied to obtain two distinct, f-invariant, simply connected continua K_x and K_y. As described in Sections 2 and 3, f induces a recurrent, orientation-reversing

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homeomorphism $\hat{f}$ on the prime end compactification of the open topological annulus $\mathbf{S}^2 \setminus (K_x \cup K_y)$. Since $\hat{f}$ preserves the boundary component of this annulus, it must have a fixed point according to the Lefschetz trace formula and hence $\hat{f}^2$ must be the identity. But this implies that $f^2 = \mathrm{Id}$ on $\mathbf{S}^2 \setminus (K_x \cup K_y)$ which leads to a contradiction with Theorem 1·1.

COROLLARY 3·5. *A non-trivial, recurrent, orientation-preserving homeomorphism of the disc has a unique fixed point. A recurrent, orientation-reversing homeomorphism of the disc is an involution.*

Proof. Taking the double of a recurrent, orientation-preserving homeomorphism of the disc, we obtain at once the first part of 3·5 as a consequence of Theorem 1·1. Let us now consider a recurrent, orientation-reversing homeomorphism f of the disc. As an orientation-reversing homeomorphism of the circle, the restriction of f to the boundary of the disc has a fixed point and hence $f^2 = \mathrm{Id}$ on the boundary of the disc. By previous considerations, this leads to $f^2 = \mathrm{Id}$ and completes the proof.

4. *Recurrent homeomorphisms of surfaces*

Let M^2 be a closed orientable surface of genus $g \geqslant 1$ and $\pi : \tilde{M}^2 \to M^2$ the universal cover of M^2. We can identify $\tilde{M}^2$ either to the euclidean plane $\mathbf{R}^2$ or to the Poincaré disc $\mathbf{D}$ in such a way that M^2 is homeomorphic to the quotient of M^2 by a discrete subgroup Γ of euclidean translations or hyperbolic isometries according to whether $\tilde{M}^2$ is $\mathbf{R}^2$ or $\mathbf{D}$. The metric we shall use on M^2 is the quotient metric defined on $\tilde{M}^2/\Gamma$ by:

$$d(\pi(x), \pi(y)) = \inf_{g,\, h \in \Gamma} \tilde{d}(g.x, h.y),$$

where $\tilde{d}$ is the natural metric on $\tilde{M}^2$.

There is another metric on $\tilde{M}^2$ that we shall use in the following, namely the spherical metric. The Alexandroff compactification of $\tilde{M}^2$, $\tilde{M}^2 \cup \{\infty\}$, is homeomorphic to the sphere $\mathbf{S}^2$. Hence, the standard metric of $\mathbf{S}^2$ induces a metric that we shall call ∂ on $\tilde{M}^2$. These two metrics $\tilde{d}$ and ∂ are not uniformly equivalent on $\tilde{M}^2$ but $\mathrm{Id} : (\tilde{M}^2, \tilde{d}) \to (\tilde{M}^2, \partial)$ is uniformly continuous.

LEMMA 4·1. *Let f be an orientation-preserving recurrent homeomorphism of a closed orientable surface M^2 of genus $g \geqslant 1$. If f has a fixed point and acts trivially on $\Pi_1(M^2)$ then $f = \mathrm{Id}$.*

Proof. A lift of a recurrent homeomorphism of M^2 need not be recurrent in general. However, a lift $\tilde{f}$ which has a fixed point is recurrent for the metric $\tilde{d}$. The fact that $\mathrm{Id} : (\tilde{M}^2, \tilde{d}) \to (\tilde{M}^2, \partial)$ is uniformly continuous is enough to ensure that the extension of $\tilde{f}$ to $\mathbf{S}^2$ (letting $\tilde{f}(\infty) = \infty$) is recurrent for the metric ∂. Moreover if f acts trivially on $\pi_1(M^2)$, $\tilde{f}$ has infinitely many fixed points since it commutes with all covering translations and hence $\tilde{f} = \mathrm{Id}$ according to Theorem 1·1. □

Corollary 4·2. *A recurrent homeomorphism of a compact surface with negative Euler characteristic is periodic.*

Proof. We first note that if the boundary of M^2 s not empty, the natural extension of f to the double DM^2 of M^2 is still recurrent and since $\chi(DM^2) = 2\chi(M^2)$, we are

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reduced to prove Corollary 4·2 for closed surfaces. Moreover, by passing to the orientation covering of M^2 and considering f^2 instead of f is necessary, we may assume that M^2 is orientable with genus $g > 1$ and that f is orientation-preserving. Recall then that every homeomorphism of M^2 which is close enough to the identity must be homotopic to the identity. Hence, since f is recurrent we can find a positive integer n such that f^n is homotopic to the identity. The Lefschetz number of $g = f^n$ is thus $\mathscr{L}(g) = \chi(M^2) < 0$ and g has a fixed point. According to Lemma 4·1, we must have then $g = \mathrm{Id}$ which completes the proof. □

Remark 1. We emphasize on the fact that Corollary 4·2 is false for surfaces with non-negative Euler characteristic. Indeed, the example of [**15**] can be modified to exhibit non-regular, recurrent homeomorphisms on the projective plane, the torus, the Klein bottle, the annulus, the Möbius strip or the disc. However, for surfaces with $\chi(M^2) = 0$, namely the torus, the Klein bottle, the annulus and the Möbius strip, the proof of 4·2 shows that the existence of a periodic point for a recurrent homeomorphism implies that this homeomorphism is periodic itself.

Corollary 4·3. *A non-trivial, recurrent homeomorphism of the projective plane admits a unique fixed point or is an involution.*

Proof. We shall use as model for the projective plane $\mathbf{RP}^2$, the quotient of the sphere $\mathbf{S}^2$ by the involution $\theta: x \mapsto -x$. Let f be a recurrent homeomorphism of the projective plane $\mathbf{RP}^2$. f has two lifts on $\mathbf{S}^2$ which commute with θ. One of them $\tilde{f}_+$ is orientation-preserving and the other $\tilde{f}_- = \theta \circ \tilde{f}_+$ is orientation-reversing. $\tilde{f}_+$ has a fixed point and is therefore recurrent as we have already noticed. Therefore, if f is not trivial, $\tilde{f}_+$ has exactly two fixed points x and $\theta(x)$ which project down to one and only one fixed point of f. Hence, the existence of a second fixed point of f implies that $\tilde{f}_-$ has a fixed point and is therefore recurrent. It must satisfy $\tilde{f}_-^2 = \mathrm{Id}$ according to Corollary 3·4. This shows that f must be an involution and completes the proof. □

Remark 2. Recall that a regular point for a homeomorphism of a compact metric space (X, d) is a point at which the family of all iterates of f, $\{f^n; n \in \mathbf{Z}\}$ is equicontinuous. As noted in the introduction, Corollary 4·2 is the key argument to show that a homeomorphism of a compact surface with negative Euler characteristic which has a regular point cannot be topologically transitive and hence cannot be ergodic with respect to an invariant measure μ such that $\mu(U) > 0$ on open sets U. Indeed, the orbit of a regular point cannot be a dense set, otherwise the homeomorphism is recurrent and hence periodic which gives a contradiction. But the presence of a regular point with a non dense orbit implies existence of proper, non empty invariant open sets.

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