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TWO MOMENT PROBLEMS FOR BOUNDED FUNCTIONS

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1. Let (a, b) denote an *unbounded* interval. Given the numbers $s_0, \ldots, s_{n-1}$, consider the problem of finding what conditions they must satisfy in order that there shall be a function $f(x)$, $0 \leqslant f \leqslant 1$, such that

$$s_k = \int_a^b x^k f(x)\,dx \quad (k = 0, \ldots, n-1).$$

This problem, with minor differences, has been solved in the case where (a, b) is $(-\infty, \infty)$ by Achyèser and Krein (*Communications de la Société Mathématique de Kharkoff*, Série 4, 12 (1935), 13–35). If we write

$$\exp\left(\sum_0^{n-1} \frac{s_k}{z^{k+1}}\right) = 1 + \sum_0^{n-1} \frac{t_k}{z^{k+1}} + \ldots, \tag{1}$$

then the result of Achyèser and Krein (loc. cit. Satz 1, p. 19 and Bemerkung, p. 21), when adapted to our slightly different formulation, can be expressed by saying that it is necessary and sufficient in order that the equations

$$s_k = \int_{-\infty}^{\infty} x^k f(x)\,dx \quad (k = 0, \ldots, n-1) \tag{2}$$

shall be satisfied, that the quadratic form

$$\sum_{i,k=0}^{m} t_{i+k}\, x_i x_k$$

shall be positive definite or semi-definite, where $2m$ is the greatest even integer which does not exceed $n-1$.

Now it is a fact, known, perhaps, or conjectured by students of this subject, that the required property of the quadratic form is a necessary and sufficient condition in order that there shall be a function $\sigma(x)$, bounded and non-decreasing in $(-\infty, \infty)$, such that

$$t_k = \int_{-\infty}^{\infty} x^k\,d\sigma \quad (k = 0, \ldots, n-1). \tag{3}$$

It therefore seems desirable to give a proof of the equivalence of (2) and (3) which brings out the fundamental relation between the functions f and σ. This relation is expressed by the following theorem.

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THEOREM I. (i) *If $f(x)$ is integrable in $(-\infty,\infty)$ and $0\leqslant f\leqslant 1$, then there is a function $\sigma(x)$, bounded and non-decreasing in $(-\infty,\infty)$, such that*

$$\exp\left(\int_{-\infty}^{\infty}\frac{f(x)}{\zeta-x}dx\right)=1+\int_{-\infty}^{\infty}\frac{d\sigma}{\zeta-x}, \tag{4}$$

where $\zeta=\xi+i\eta$ and $\eta>0$.

(ii) *If $f(x)=0$ throughout an interval, then $\sigma(x)$ is constant in that interval.*

(iii) *If there is a function $\sigma(x)$, bounded and non-decreasing in $(-\infty,\infty)$, then there is a function $f(x)$, integrable in $(-\infty,\infty)$, such that $0\leqslant f\leqslant 1$, and which satisfies* (4).

(iv) *If $\sigma(x)$ is constant for $x>0$, then $f(x)=0$ for $x>0$.*

As regards the origin of the theorem, we may remark that Achyèser and Krein (loc. cit. p. 19) use the function

$$\exp\left(\int_{-N}^{N}\frac{f(x)}{\zeta-x}dx\right),$$

where $0\leqslant f\leqslant 1$, and show that it is of the form

$$1+\int_{-N}^{N}\frac{d\sigma}{\zeta-x}.$$

Thus (i) and (ii) form a generalization of their result.

We shall use theorem I to prove the equivalence of (2) and (3) when $n-1$ is even. For the case $n-1$ odd, see §13.

THEOREM II. *Let $n-1$ be even. Given the numbers $s_0,\ldots,s_{n-1}$, define $t_0,\ldots,t_{n-1}$ by* (1). *A necessary and sufficient condition that there shall be a function $f(x)$, where $0\leqslant f\leqslant 1$, which satisfies* (2) *is that there shall be a non-decreasing function $\sigma(x)$ which satisfies* (3).

We shall also use theorem I to solve the analogous problem for a semi-infinite interval, a case which is not considered by Achyèser and Krein. In view of theorem I (iv), it is convenient to choose the interval as $(-\infty,0)$. We shall prove

THEOREM III. *Given the numbers $s_0,\ldots,s_{n-1}$, define $t_0,\ldots,t_{n-1}$ by* (1). *A necessary and sufficient condition that there shall be a function $f(x)$, where $0\leqslant f\leqslant 1$, which satisfies*

$$s_k=\int_{-\infty}^{0}x^k f(x)\,dx\quad(k=0,\ldots,n-1), \tag{2'}$$

is that there shall be a non-decreasing function $\sigma(x)$ which satisfies

$$t_k=\int_{-\infty}^{0}x^k\,d\sigma\quad(k=0,\ldots,n-1). \tag{3'}$$

We add a remark on notation. In what follows, when an upper limit to an integral is omitted, then the upper limit is ∞. When the lower limit is omitted, then the lower limit is $-\infty$.

2. We require a lemma on harmonic functions which are of constant sign in a half-plane.

LEMMA 1. *If $J(\xi,\eta)$ is a harmonic function such that $J\geqslant 0$ if $\eta>0$, then there is a number $d\geqslant 0$ and a function $G(x)$, bounded and non-decreasing in $(-\infty,\infty)$, such that*

$$J=d\eta+\int\frac{\eta(1+x^2)\,dG}{\eta^2+(x-\xi)^2}. \tag{5}$$

For a proof, see M. H. Stone, *Linear Transformations in Hilbert Space* (1932), p. 574.

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3. *Proof of* (i). If $\eta > 0$ and $0 \leqslant f \leqslant 1$, then

$$0 \leqslant \frac{\eta f(x)}{\eta^2 + (x-\xi)^2} \leqslant \frac{\eta}{\eta^2 + (x-\xi)^2},$$

and

$$0 \leqslant \int \frac{\eta f(x)}{\eta^2 + (x-\xi)^2}\, dx \leqslant \pi.$$

But

$$\mathbf{I} \frac{1}{\zeta - x} = -\frac{\eta}{\eta^2 + (x-\xi)^2}. \tag{6}$$

Hence, writing

$$\gamma = \alpha - i\beta = \int \frac{f(x)}{\zeta - x}\, dx,$$

we have $0 \leqslant \beta \leqslant \pi$. If now

$$e^{\gamma} = e^{\alpha}(\cos\beta - i\sin\beta) = K - iJ,$$

then $J = J(\xi, \eta) \geqslant 0$. But e^{γ} is regular in $\eta > 0$. Hence J is harmonic in that domain. By lemma 1, we can write (5). Put $\xi = 0$. Then

$$J(0, \eta) = d\eta + \int \frac{\eta(1+x^2)}{\eta^2 + x^2} dG \tag{7}$$

$$= d\eta + o(\eta)$$

as $\eta \to \infty$. But when $\xi = 0$,

$$\gamma = \int \frac{f(x)}{i\eta - x}\, dx = O(\eta^{-1}),$$

and so

$$K - iJ = e^{\gamma} = 1 + O(\eta^{-1}).$$

Thus $J = O(\eta^{-1})$. It follows that $d = 0$ in (7), and that the integral is $O(\eta^{-1})$. Then

$$\int \frac{\eta^2(1+x^2)}{\eta^2 + x^2} dG = O(1).$$

This is true if the limits are $-\eta, \eta$. In this range, $\eta^2/(\eta^2 + x^2)$ is between 1 and $\frac{1}{2}$. Hence

$$\int (1+x^2)\, dG = O(1).$$

We can therefore write

$$\sigma(x) = \int^x (1+u^2)\, dG(u),$$

and then $\sigma(x)$ will be bounded and non-decreasing in $(-\infty, \infty)$. Then the second member of (4) is regular in $\eta > 0$, and (5), with $d = 0$, becomes

$$-J = \mathbf{I} \int \frac{d\sigma}{\zeta - x}.$$

Thus

$$e^{\gamma} = \int \frac{d\sigma}{\zeta - x} + c,$$

where c is a real constant. On putting $\xi = 0$ and letting $\eta \to \infty$, we find that $c = 1$.

4. *Proof of* (ii). Equating imaginary parts in (4), and using the notation of § 3, we have

$$e^{\alpha}\sin\beta = J = \int \frac{\eta}{\eta^2 + (x-\xi)^2} d\sigma.$$

Further,

$$\beta = \int \frac{\eta}{\eta^2 + (x-\xi)^2} f(x)\, dx.$$

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If $f(x) = 0$ throughout the open interval Δ, then $\beta \to 0$ as $\eta \to 0$ for every ξ of that interval. Hence $J \to 0$, since $\alpha = O(1)$ as $\eta \to 0$. But

$$J \geqslant \int_{\xi-\eta}^{\xi+\eta} \frac{\eta}{\eta^2+(x-\xi)^2} d\sigma \geqslant \frac{1}{2\eta} \int_{\xi-\eta}^{\xi+\eta} d\sigma \geqslant 0.$$

Hence, the symmetric derivative of $\sigma(x)$ is 0 at every point of Δ. Thus σ is constant in Δ.

5. In order to prove (iii), we require a lemma on harmonic functions which are bounded in a half-plane.

LEMMA 2. *If $J(\xi, \eta)$ is a harmonic function such that $0 \leqslant J \leqslant \pi$ in $\eta > 0$, then there is a function $f(x)$ defined in $(-\infty, \infty)$, satisfying $0 \leqslant f \leqslant 1$, and such that*

$$J = \int \frac{\eta}{\eta^2+(x-\xi)^2} f(x)\, dx. \tag{8}$$

By lemma 1 we have (5). Here, the integral is $o(\eta)$ as $\eta \to \infty$, and, by hypothesis, the same is true of J. Hence, $d = 0$. Then

$$\pi \geqslant J \geqslant \frac{1}{2\eta} \int_{\xi-\eta}^{\xi+\eta} (1+x^2)\, dG \geqslant 0.$$

Thus, if

$$s(x) = \int_0^x (1+u^2)\, dG(u),$$

then the non-decreasing function $s(x)$ has a bounded symmetric derivative. Hence $s(x)$ is the integral of a bounded non-negative function $f(x)$. We thus have (8). Since the second member tends to $\pi f(\xi)$ as $\eta \to 0$ for almost all ξ, it follows that $f \leqslant 1$.

6. *Proof of* (iii). If $\sigma(x)$ is constant, then $f(x) = 0$. Excluding this case, the second member of (4) is regular in $\eta > 0$, and its imaginary part,

$$-\int \frac{\eta\, d\sigma}{\eta^2+(x-\xi)^2},$$

is negative. Hence we can write

$$1 + \int \frac{d\sigma}{\zeta - x} = Ae^{i\theta}, \tag{9}$$

where $A > 0$ and $-\pi < \theta < 0$. Then

$$\mathrm{I} \log Ae^{i\theta} = \theta,$$

where here, and subsequently, the principal value of the logarithm is intended. But $\log Ae^{i\theta}$ is regular in $\eta > 0$. Thus θ is harmonic in that domain. By lemma 2, we can write

$$\theta = \theta(\xi, \eta) = -\int \frac{\eta f(x)}{\eta^2+(x-\xi)^2} dx, \tag{10}$$

where $0 \leqslant f \leqslant 1$.

When $\xi = 0$,

$$Ae^{i\theta} = 1 + \int \frac{d\sigma}{i\eta - x} = 1 + O(\eta^{-1})$$

as $\eta \to \infty$. Hence, taking logs and equating imaginary parts, $\theta = O(\eta^{-1})$. By (10),

$$\int \frac{\eta f(x)}{\eta^2+x^2} dx = O(\eta^{-1}).$$

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This is true if the limits are $-\eta, \eta$. It then follows that

$$\int_{-\eta}^{\eta} f(x)\,dx = O(1),$$

i.e. $f(x)$ is integrable in $(-\infty, \infty)$.

We can now write $$\mathbf{I} \log A e^{i\theta} = \mathbf{I} \int \frac{f(x)}{\zeta - x}\,dx,$$

where the integral is a regular function of ζ in the domain $\eta > 0$. By (9),

$$\log\left(1 + \int \frac{d\sigma}{\zeta - x}\right) = \int \frac{f(x)}{\zeta - x}\,dx + c,$$

where c is a real constant. Put $\xi = 0$ and let $\eta \to \infty$. We find $c = 0$.

7. *Proof of* (iv). For a fixed $\xi > 0$ and variable $x < 0$, $|\zeta - x| > \xi - x > \xi$. Hence

$$1 + \int \frac{d\sigma}{\zeta - x} \to 1 + \int^{0} \frac{d\sigma}{\xi - x} = B(\xi),$$

where $B > 0$. Now if $w \to w_0$ and $w_0 > 0$, then $\log w \to \log w_0$, where the logs denote principal values (a result which is not true if $w_0 < 0$). Hence

$$\int \frac{f(x)}{\zeta - x}\,dx \to \log B(\xi).$$

This implies that the imaginary part of the first member, viz.

$$-\int \frac{\eta f(x)}{\eta^2 + (x - \xi)^2}\,dx,$$

tends to 0 if $\xi > 0$. Since, however, it tends to $-\pi f(\xi)$ for almost all ξ, it follows that $f(x) = 0$ if $x > 0$.

8. To prove theorem II, we require two lemmas.

LEMMA 3. *If $f(x)$ is defined in $(-\infty, \infty)$ and $f \geqslant 0$, and there are real numbers s_k such that*

$$\int \frac{f(x)}{i\eta - x}\,dx = \sum_{0}^{n-1} \frac{s_k}{(i\eta)^{k+1}} + o(\eta^{-n})$$

as $\eta \to \infty$, then $$s_k = \int x^k f(x)\,dx \quad (k = 0, \ldots, n-1).$$

LEMMA 4. *If $\sigma(x)$ is non-decreasing in $(-\infty, \infty)$ and there are real numbers t_k such that*

$$\int \frac{d\sigma}{i\eta - x} = \sum_{0}^{n-1} \frac{t_k}{(i\eta)^{k+1}} + o(\eta^{-n})$$

as $\eta \to \infty$, then $$t_k = \int x^k d\sigma \quad (k = 0, \ldots, n-1).$$

If n is odd, then in lemma 3 s_{n-1} is the integral of a positive function. Hence all the integrals s_k are absolutely convergent. If n is even then s_{n-2} is the integral of a positive function and the integrals s_k are absolutely convergent if $k \leqslant n-2$, while the equation for $k = n-1$ means

$$s_{n-1} = \lim_{\eta \to \infty} \int_{-\eta}^{\eta} x^{n-1} f(x)\,dx.$$

A similar remark applies to lemma 4.

The proofs of the lemmas are almost identical. We shall consider only the first.

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9. If $n = 1$, the hypothesis is

$$\int \frac{f(x)}{i\eta - x} dx = \frac{s_0}{i\eta} + o(\eta^{-1}).$$

Equating imaginary parts, we find

$$\int \frac{\eta^2}{\eta^2 + x^2} f(x)\, dx = s_0 + o(1). \tag{11}$$

This implies $$\int_{-\eta}^{\eta} \frac{\eta^2}{\eta^2 + x^2} f(x)\, dx = O(1),$$

and so $$\int_{-\eta}^{\eta} f(x)\, dx = O(1).$$

Thus the non-negative function $f(x)$ is integrable in $(-\infty, \infty)$. We can therefore take the limit as $\eta \to \infty$ in (11), and obtain the desired result

$$s_0 = \int f(x)\, dx. \tag{12}$$

If $n = 2$, the hypothesis is

$$\int \frac{f(x)}{i\eta - x} dx = \frac{s_0}{i\eta} + \frac{s_1}{(i\eta)^2} + o(\eta^{-2}).$$

By the case $n = 1$, we have (12). Hence

$$\int \frac{xf(x)}{i\eta - x} dx = \frac{s_1}{i\eta} + o(\eta^{-1}).$$

Equating real and imaginary parts, we obtain

$$\int \frac{\eta^2}{\eta^2 + x^2} xf(x)\, dx = s_1 + o(1), \tag{13}$$

$$\int \frac{x^2}{\eta^2 + x^2} f(x)\, dx = o(\eta^{-1}). \tag{14}$$

Now $$\int_{\eta} \frac{\eta^2}{\eta^2 + x^2} xf(x)\, dx = O\left[\eta \int_{\eta} \frac{x^2}{\eta^2 + x^2} f(x)\, dx\right] = o(1)$$

by (14). Similarly, $$\int^{-\eta} \frac{\eta^2}{\eta^2 + x^2} xf(x)\, dx = o(1).$$

Hence, (13) implies $$\int_{-\eta}^{\eta} \frac{\eta^2 x}{\eta^2 + x^2} f(x)\, dx = s_1 + o(1). \tag{15}$$

In (14), we can introduce the limits $-\eta$, η. Hence

$$\int_{-\eta}^{\eta} \frac{x^3}{\eta^2 + x^2} f(x)\, dx = o(1).$$

Adding this to (15), $$\int_{-\eta}^{\eta} xf(x)\, dx = s_1 + o(1),$$

and so $$s_1 = \int xf(x)\, dx.$$

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10. We can complete the proof by induction. If $n = 2k+1$, the hypothesis is

$$\int \frac{f(x)}{i\eta - x} dx = \sum_0^{2k} \frac{s_j}{(i\eta)^{j+1}} + o(\eta^{-(2k+1)}).$$

By the lemma for $n = 2k$, we can write

$$s_j = \int x^j f(x)\, dx \tag{16}$$

for $j = 0, \ldots, 2k-1$. Then

$$\int f(x) \left(\frac{1}{i\eta - x} - \sum_0^{2k-1} \frac{x^j}{(i\eta)^{j+1}} \right) dx = \frac{s_{2k}}{(i\eta)^{2k+1}} + o(\eta^{-(2k+1)}),$$

i.e.

$$\int \frac{x^{2k} f(x)}{i\eta - x} dx = \frac{s_{2k}}{i\eta} + o(\eta^{-1}).$$

Here, $x^{2k} f \geqslant 0$. By the lemma with $n = 1$, (16) holds for $j = 2k$.

If $n = 2k+2$, the lemma for $n = 2k$ allows us to write (16) for $j = 0, \ldots, 2k-1$. The hypothesis and the preceding argument show that

$$\int \frac{x^{2k} f(x)}{i\eta - x} dx = \frac{s_{2k}}{i\eta} + \frac{s_{2k+1}}{(i\eta)^2} + o(\eta^{-2}).$$

By the lemma with $n = 2$, (16) holds for $j = 2k, 2k+1$.

11. We now turn to the proof of theorem II, and first show that (2) implies (3). We remind the reader that since $n-1$ is even, the integral for s_{n-1} is absolutely convergent. Hence

$$\int \frac{f(x)}{i\eta - x} dx$$

can be written as

$$\int f(x) \left(\sum_0^{n-1} \frac{x^j}{(i\eta)^{j+1}} + \frac{x^n}{(i\eta)^n (i\eta - x)} \right) dx = \sum_0^{n-1} \frac{s_j}{(i\eta)^{j+1}} + \frac{1}{(i\eta)^n} \int \frac{x^n f(x)}{i\eta - x} dx.$$

The last integral is $o(1)$ as $\eta \to \infty$. Hence

$$\int \frac{f(x)}{i\eta - x} dx = \sum_0^{n-1} \frac{s_j}{(i\eta)^{j+1}} + o(\eta^{-n}),$$

and

$$\exp\left(\int \frac{f(x)}{i\eta - x} dx \right) = 1 + \sum_0^{n-1} \frac{t_j}{(i\eta)^{j+1}} + o(\eta^{-n})$$

by (1).

By theorem I (i) (in which we choose $\xi = 0$), the first member of the preceding equation is

$$1 + \int \frac{d\sigma}{i\eta - x}.$$

By lemma 4, (3) follows.

To prove that (3) implies (2), we use theorem I (iii) and lemma 3.

12. We now prove theorem III, and here $n-1$ is not restricted to be even. Given (2′), we can use the argument of 11, observing that, since $f(x) = 0$ if $x > 0$, the integral for s_{n-1} is necessarily absolutely convergent. Then (2′) implies (3), and by theorem I (ii), this is (3′). Similarly for the proof that (3′) implies (2′).

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13. In theorems II and III, given a function $f(x)$ which satisfies (2) or (2′), there is a function $\sigma(x)$ which satisfies (3) or (3′) respectively and which is connected with $f(x)$ by (4); and conversely. The enunciation of theorem II is correct when $n-1$ is odd, but in this case, given $f(x)$ satisfying (2), though we can prove that there is a function $\sigma(x)$ satisfying (3), we do not prove that $\sigma(x)$ is connected with $f(x)$ by (4). And a similar remark applies to the converse. We shall treat this case in a subsequent note.

14. My attention has been drawn to the fact that lemma 4 is a refinement of a theorem of Hamburger and R. Nevanlinna. Cf. Hamburger, *Math. Ann.* 81 (1920), Satz 9, 268–74; R. Nevanlinna, *Ann. Acad. Sci. Fennicae*, 18 (1922), no. 5, 45–8.

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THE COMPLETE SYSTEM OF THE BINARY (3, 1) FORM

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1. The only double binary forms for which the complete systems of invariants and covariants are known appear to be the (1, 1), (2, 1) and (2, 2) forms, for which the complete systems were determined sixty years ago by Peano*. In the present paper we determine the complete system of the binary (3, 1) form and establish its irreducibility. The system proves to contain twenty forms, and is thus scarcely more complicated than that of the (2, 2) form which includes eighteen concomitants, and much simpler than that of the (2, 1, 1) form derived by Gilham†. It is hoped in a subsequent note to interpret the system geometrically.

2. Let
$$f \equiv A_x^3 \equiv a_x^3 \alpha_y \tag{1}$$
be a (3, 1) form in the two sets of binary variables (x_0, x_1), (y_0, y_1) which undergo independent linear transformations. It is convenient to use small roman letters to indicate symbolic factors involving x and small greek letters to refer to symbolic factors involving y, and to employ capital letters whenever we wish to consider a form with reference to the variable x only. Equivalent symbols will be represented by accents. If f_1 and f_2 are two double binary forms the expression $(f_1, f_2)^{\lambda,\mu}$ will denote their transvectant of indices λ, μ, the former index referring to the variables (x_0, x_1) and the latter to the variables (y_0, y_1).

* Peano, *Giorn. matematico*, 20 (1882), 79–100. For further references, including simultaneous systems, see Turnbull, *The Theory of Determinants, Matrices, and Invariants* (Blackie, 1928), p. 243.

† Gilham, *Proc. London Math. Soc.* (2), 32 (1931), 259–72 and 38 (1935), 271–2.

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