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Generalised Bohr compactification and model-theoretic connected components

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Abstract

For a group G first order definable in a structure M, we continue the study of the "definable topological dynamics" of G (from [**9**] for example). The special case when *all* subsets of G are definable in the given structure M is simply the usual topological dynamics of the discrete group G; in particular, in this case, the words "externally definable" and "definable" can be removed in the results described below.

Here we consider the mutual interactions of three notions or objects: a certain model-theoretic invariant $G^*/(G^*)^{000}_M$ of G, which appears to be "new" in the classical discrete case and of which we give a direct description in the paper; the [externally definable] generalised Bohr compactification of G; [externally definable] strong amenability. Among other things, we essentially prove: (i) the "new" invariant $G^*/(G^*)^{000}_M$ lies in between the externally definable generalised Bohr compactification and the definable Bohr compactification, and these all coincide when G is definably strongly amenable and all types in $S_G(M)$ are definable; (ii) the kernel of the surjective homomorphism from $G^*/(G^*)^{000}_M$ to the definable Bohr compactification has naturally the structure of the quotient of a compact (Hausdorff) group by a dense normal subgroup; (iii) when $\mathrm{Th}(M)$ is NIP, then G is [externally] definably amenable iff it is externally definably strongly amenable.

In the situation when all types in $S_G(M)$ are definable, one can just work with the definable (instead of externally definable) objects in the above results.

0. *Introduction*

The introduction of methods and ideas from topological dynamics, such as minimal flows and Ellis groups, into the model-theoretic study of definable groups was initiated by Newelski in [**17**] in the context of the action of a definable group on its space of types. Among

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the existing model-theoretic invariants of a definable group G are certain quotients of a saturated elementary extension G^* of G by various "connected components", which can be also seen as "quotients" of the type space, and an important aspect of Newelski's approach [**17**, **18**] was to try to relate some of these invariants to dynamical invariants such as the Ellis group. In this paper, we try to go deeper in this direction, relating these model-theoretic invariants to the so-called generalised Bohr compactification from [**11**]. As mentioned in the abstract, our model-theoretic set-up subsumes the case when G is simply a discrete group. Our results go both ways. On the one hand, in Theorem 0·1, we show that the most general model-theoretic invariant $G^*/(G^*)^{000}_M$ is a group lying in between the externally definable generalised Bohr compactification and the definable Bohr compactification. In the classical discrete case, $G^*/(G^*)^{000}_M$ appears to be a new invariant of G, which we will give an intrinsic description of as a universal "quasihomomorphic" image of βG (see Proposition 1·20). On the other hand, the externally definable generalised Bohr compactification gives a new tool for describing $G^*/(G^*)^{000}_M$. We should remark that $G^*/(G^*)^{000}_M$ does NOT have the structure of a compact (Hausdorff) group, or even topological group in any meaningful sense, so a lot of previous work has been aimed at trying to understand it as a mathematical object. And this is one of the main aims and outcomes of the current paper (see Theorem 0·2). Using [**7**], one can also apply our results to get many new examples of discrete groups whose Bohr compactification is trivial whereas the generalised Bohr compactification is non-trivial.

Throughout the paper, we use various notions, theorems and tricks from classical topological dynamics (mostly from [**11**]) adapted to our "definable" context. But we would like to emphasise the proofs of our main results are essentially new.

In the rest of this introduction, we will recall some terminology, give background and formulate our main results. More technical definitions will be given in the next section.

Let G be a group $\varnothing$-definable in a first order structure M (i.e. both the group itself and the graph of the group operation are $\varnothing$-definable sets, that is the sets of realisations in M of some formulas without parameters). By $S_G(M)$ we denote the space of complete types over M containing the formula defining G; equivalently, this is the space of ultrafilters of definable (with parameters from M) subsets of G, equipped with the Stone topology. By $S_{G,\mathrm{ext}}(M)$ we denote the space of all externally definable complete types over M containing G, that is the space of ultrafilters in the Boolean algebra of all externally definable subsets of G (i.e. subsets which are intersections with G of sets definable in arbitrary elementary extensions of M).The advantage of $S_{G,\mathrm{ext}}(M)$ is that it allows us to develop topological dynamics without any additional assumptions on M (or on the theory of M); in order to work smoothly with $S_G(M)$, one would have to assume that all types in $S_G(M)$ are definable (i.e. for every $p \in S_G(M)$ and formula $\varphi(x; y)$ the set of all $m \in M$ such that $\varphi(x, m) \in p$ is definable). Note that if all types in $S_G(M)$ are definable, then $S_G(M)$ coincides with $S_{G,\mathrm{ext}}(M)$. In the case when M is the group G expanded by predicates for all subsets of G, all types in $S_G(M)$ are definable, so $S_G(M)$ coincides with $S_{G,\mathrm{ext}}(M)$, and moreover, this is just βG – the Stone–Čech compactification of G, that is the space of ultrafilters in the Boolean algebra of all subsets of G.

Now, G acts by translations as groups of homeomorphisms of the compact spaces $S_{G,\mathrm{ext}}(M)$, $S_G(M)$ and βG, turning them into G-flows (which turn out to be universal in certain categories, see Section 1). Let us focus on $S_{G,\mathrm{ext}}(M)$. The rest of this paragraph is a collection of classical results of Ellis adapted to our context. There is a certain semigroup structure on $S_{G,\mathrm{ext}}(M)$ called the Ellis semigroup; the semigroup operation will be denoted by $*$. Let $\mathcal{M}$ be a minimal G-subflow of $S_{G,\mathrm{ext}}(M)$. Then $\mathcal{M}$ is the disjoint union of sets of

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the form $u * \mathcal{M}$ with u ranging over all idempotents in $\mathcal{M}$. Moreover, each $u * \mathcal{M}$ is a group (with $*$ as the group operation), and the isomorphism type of all these groups is the same and does not depend on the choice of $\mathcal{M}$. This isomorphism type (or just the group $u * \mathcal{M}$ for any idempotent u) is called the Ellis group of the flow $S_{G,\mathrm{ext}}(M)$.

Let $\mathfrak{C} \succ M$ be a monster model extending M (namely, a model which is κ-saturated (i.e. every type over a set of parameters of cardinality less than κ has a realization in this model) and κ-strongly homogeneous (i.e. every partial elementary map between sets of cardinality less than κ extends to an automorphism of the whole model) for a big enough cardinal κ). By G^* we denote the interpretation of G in $\mathfrak{C}$ (i.e. the set of realisations in $\mathfrak{C}$ of the formula defining G in M). Let A be any small set of parameters from $\mathfrak{C}$ (small means of cardinality less than the degree of saturation of $\mathfrak{C}$). Recall that an A-type-definable set is the set of realisations of a type over A (i.e. it is a (possibly) infinite intersection of A-definable sets), and an A-invariant set is a set invariant under all automorphisms of the monster model fixing A pointwise. By a bounded cardinal we will mean a cardinal less than the degree of saturation of $\mathfrak{C}$. The following connected components and their quotients play a fundamental role in the study of groups from the model-theoretic perspective:

(i) the intersection of all A-definable subgroups of G^* of finite index, denoted by G^{*0}_A;
(ii) the smallest A-type-definable subgroup of G^* of bounded index, denoted by G^{*00}_A;
(iii) the smallest A-invariant subgroup of G^* of bounded index, denoted by G^{*000}_A.

It is clear that $G^{*000}_A \leqslant G^{*00}_A \leqslant G^{*0}_A \leqslant G^*$, and it is easy to show that all these groups are normal in G^*. Sometimes these connected components do not depend on the choice of A, e.g. in the so-called NIP theories. In such situations, we skip the parameter set A.

The significance of the above components in model theory was discussed in various papers (e.g. see [**6**, **8**, **12**]). In particular, they play a fundamental role in the study of stable, simple and NIP groups, and they are precisely related to strong types in different senses [**8**]. Note also that all quotients such as G^*/G^{*0}_A, G^*/G^{*00}_A, G^*/G^{*000}_A or G^{*00}_A/G^{*00}_A are certain invariants of the group G (in the sense that they do not depend on the choice of the monster model) and it is desirable to understand their algebraic, topological and possibly some other structure. The components G^{*0}_A and G^{*00}_A and the quotients G^*/G^{*0}_A and G^*/G^{*00}_A are well understood. Namely, these quotients are equipped with the logic topology (in which closed sets are the sets with type-definable preimages in G^*). Then G^*/G^{*0}_A is a profinite group (in fact, G^*/G^{*0}_M is the definable profinite completion of G) and G^*/G^{*00}_A is a compact Hausdorff group (in fact, G^*/G^{*00}_M is the definable Bohr compactification of G which was proved in [**9**]). (If one takes as M the group G equipped with predicates for all subsets, then G^*/G^{*0}_M is the profinite completion of G and G^*/G^{*00}_M is the Bohr compactification of G.) This becomes particularly interesting in o-minimal structures due to Pillay's conjecture which describes G^*/G^{*00}_A as a compact Lie group of an appropriate dimension, and so associates with the group G a classical mathematical object G^*/G^{*00}_A (see [**19**]). However, G^{*000}_A and the quotients G^*/G^{*000}_A and G^{*00}_A/G^{*000}_A are much more mysterious objects. Until quite recently, it was not known whether always $G^{*00}_A = G^{*000}_A$. The first counter-example, which is the universal cover of $\mathrm{SL}_2(\mathbb{R})$, was found in [**4**], and then whole families of counter-examples were found in [**7**]. Recall that the logic topology on G^*/G^{*000}_A is not necessarily Hausdorff, and on G^{*00}_A/G^{*000}_A it is trivial, so it is rather useless. Thus, one of the main goals here is to understand the structure or to learn how to measure the complexity of G^{*00}_A/G^{*000}_A. Progress in this direction is done in this paper (see Theorem 0·2 and Corollaries 0·4 and 0·5 below).

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Now, we are going to relate the Ellis group $u * \mathcal{M}$ to connected components and to formulate the main results of this paper.

Newelski [**17**] found a natural epimorphism, say θ, from $u * \mathcal{M}$ to G^*/G^{*00}_M, and he conjectured that it is an isomorphism. In [**10**], the authors found a counter-example (namely, the group $G = \mathrm{SL}_2(\mathbb{R})$ defined in $M = (\mathbb{R}, +, \cdot)$, where $u * \mathcal{M} \cong \mathbb{Z}_2$ and G^*/G^{*00}_M is trivial, so these groups are not isomorphic), and more examples of this kind were found in [**13**]. We note in Section 3 that there is also a natural epimorphism f from $u * \mathcal{M}$ to G^*/G^{*000}_M such that f composed with the natural epimorphism π from G^*/G^{*000}_M to G^*/G^{*00}_M coincides with θ. This immediately shows that any example where $G^{*000}_M \neq G^{*00}_M$ is a counter-example to Newelski's conjecture (at least in the strong form predicting that θ is an isomorphism; a weaker form, saying that there is an abstract isomorphism, is also false by the example from [**10**] or by any example from [**7**] with $G^{*00}_M = G^* \neq G^{*000}_M$). Precise definitions of f and θ are given in Section 3.

Our goal is to understand better (or to refine) the following sequence of epimorphisms:

$$u * \mathcal{M} \xrightarrow{f} G^*/G^{*000}_M \xrightarrow{\pi} G^*/G^{*00}_M \,. \tag{0·1}$$

In the context of classical topological dynamics, Glasner (see [**11**, chapter VIII]) introduces the generalised Bohr compactification which is meaningful in some situations in which the Bohr compactification is trivial. Analogously, we define an externally definable generalised Bohr compactification of G. As in the classical situation, one can introduce a certain topology, called τ-topology, on $u * \mathcal{M}$ so that if $H(u * \mathcal{M})$ is the intersection of the τ-closures of τ-neighbourhoods of u, then $u * \mathcal{M}/H(u * \mathcal{M})$ (equipped with the quotient topology induced by τ-topology) becomes a compact Hausdorff group which turns out to be exactly the externally definable generalised Bohr compactification of G. However, the proof of this fact (given in Section 2) in our externally definable context is somewhat different than in the classical case (although it uses various tricks and computations from Glasner's book).

Our first main theorem yields a refinement of the sequence of epimorphisms (0·1).

THEOREM 0·1. *Let G be an arbitrary group $\varnothing$-definable in an arbitrary structure M, and let $A \subseteq M$ be any set of parameters. Let $f : u * \mathcal{M} \to G^*/G^{*000}_A$ be the natural epimorphism (defined in Section 3), and suppose that $u * \mathcal{M}$ is equipped with the τ-topology and $u * \mathcal{M}/H(u * \mathcal{M})$ – with the induced quotient topology. Then*:

(i) *f is continuous*;
(ii) *$H(u * \mathcal{M}) \leqslant \ker(f) := f^{-1}(eG^{*000}_A)$*;
(iii) *the formula $p * H(u * \mathcal{M}) \mapsto f(p)$ yields a well-defined continuous epimorphism $\bar{f}$ from $u * \mathcal{M}/H(u * \mathcal{M})$ to G^*/G^{*000}_A.*

In particular, we get the following sequence of continuous epimorphisms:

$$u * \mathcal{M} \twoheadrightarrow u * \mathcal{M}/H(u * \mathcal{M}) \xrightarrow{\bar{f}} G^*/G^{*000}_A \xrightarrow{\pi} G^*/G^{*00}_A \,, \tag{0·2}$$

where $u * \mathcal{M}/H(u * \mathcal{M})$ is the externally definable generalised Bohr compactification of G.

Using Theorem 0·1, we obtain the following property of G^{*00}_A/G^{*000}_A.

THEOREM 0·2. *For an arbitrary group G which is $\varnothing$-definable in an arbitrary model M and for any set of parameters A the group G^{*00}_A/G^{*000}_A is isomorphic to the quotient of a compact Hausdorff group by a dense subgroup. More precisely, assume $A \subseteq M$, and for $Y := \ker(\bar{f})$ let $\mathrm{cl}_\tau(Y)$ be its closure inside $u * \mathcal{M}/H(u * \mathcal{M})$. Then $\bar{f}$ restricted to $\mathrm{cl}_\tau(Y)$*

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induces an isomorphism between $\mathrm{cl}_\tau(Y)/Y$ *(the quotient of a compact group by a dense subgroup) and* G^{*00}_A/G^{*000}_A.

The first part of the above theorem has been known to be true in some special situations, e.g. for groups definable in o-minimal expansions of real closed fields [**5**] or for some group extensions [**7**, **16**]. However, there have been no methods so far that could work in general. The goal here is to understand the invariant G^{*00}_A/G^{*000}_A (as was mentioned before, the logic topology on this group is trivial so useless). Theorem 0·2 is interesting in its own right, but it is also possible that it will help us to deduce something new about the Borel cardinality of G^{*00}_A/G^{*000}_A (see [**14**, **15**, **16**] for definitions) which in a sense measures the complexity of this quotient in descriptive set theoretic terms. It may also be profitable to view objects such as G^{*00}_A/G^{*000}_A from the point of view of noncommutative geometry, and in fact, Theorem 0.2 gives a representation as a "typical" bad quotient of noncommutative geometry.

We easily get the following corollary of Theorem 0·2.

COROLLARY 0·3. *Let* $A \subseteq M$. *Then*:

(i) *if the epimorphism* $\bar{f} : u * \mathcal{M}/H(u * \mathcal{M}) \to G^*/G^{*000}_A$ *is an isomorphism, then* $G^{*000}_A = G^{*00}_A$.

(ii) *if the epimorphism* $f : u * \mathcal{M} \to G^*/G^{*000}_A$ *is an isomorphism, then* $H(u * \mathcal{M})$ *is trivial and* $G^{*000}_A = G^{*00}_A$.

So far we have been working in full generality. In the next result, we need to assume that all types in $S_G(M)$ are definable (still covering the situation when predicates for all subsets of G are added to the language, so still extending the classical discrete topological dynamics). There is a classical notion of a strongly amenable group (see Definition 1·15) and it is claimed in [**11**, chapter IX, corollary 4·3] that for such groups the generalized Bohr compactification is isomorphic to the Bohr compactification. We reprove this theorem in our more general, definable context, showing that the desired isomorphism is exactly the composition of the epimorphism $\bar{f} : u * \mathcal{M}/H(u * \mathcal{M}) \to G^*/G^{*000}_M$ from Theorem 0·1 with the natural epimorphism $\pi : G^*/G^{*000}_M \to G^*/G^{*00}_M$. Thus, we get the following:

COROLLARY 0·4. *Assume all types in* $S_G(M)$ *are definable and suppose* G *is definably strongly amenable. Then the natural epimorphism* $\pi \circ \bar{f} : u * \mathcal{M}/H(u * \mathcal{M}) \to G^*/G^{*00}_M$ *is an isomorphism, so* $G^{*000}_M = G^{*00}_M$.

Now, any nilpotent group G is strongly amenable as a discrete group, so definably strongly amenable for any structure on G. Hence:

COROLLARY 0·5. *Assume all types in* $S_G(M)$ *are definable and suppose* G *is nilpotent. Then the natural epimorphism* $\pi \circ \bar{f} : u * \mathcal{M}/H(u * \mathcal{M}) \to G^*/G^{*00}_M$ *is an isomorphism, so* $G^{*000}_M = G^{*00}_M$.

Since this applies to the situation when predicates for all subsets of G are in the language, Corollary 0·5 yields a partial answer to [**7**, question 4·28] asking for which groups G expanded by predicates for all subsets one has $G^{*000}_G = G^{*00}_G$. Recall that by [**7**, corollary 4·27] (which was obtained by completely different methods), we know that if G is a free group in at least 2 free generators, expanded by predicates for all subsets, then $G^{*000}_G \neq G^{*00}_G$ (and similarly for surface groups of genus at least 2). However, it had not been clear whether this can happen even for abelian groups. Corollary 0·5 shows that that this is impossible in the class of all nilpotent groups. Jakub Gismatullin has told us that he has a different proof of

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the fact that $G^{*000}_M = G^{*00}_M$ in the particular case of abelian groups but in a somewhat more general context (weakening the definability of types assumption).

In all the results above, we used Theorem 0·1 to understand better connected components and their quotients. There is also an application of Theorem 0·1 which goes the other way around: using some knowledge on connected components together with this theorem, we can produce many examples of discrete groups whose generalized Bohr compactification differs from the Bohr compactification. Namely, working in the situation when predicates for everything are in the language and $G^{*00}_M = G \neq G^{*000}_M$, using Theorem 0·1 together with the fact that G^*/G^{*00}_M is the Bohr compactification of G, we conclude that the Bohr compactification of G is trivial whereas the generalised Bohr compactification is non-trivial. Paper [**7**] provides many concrete examples where this can be applied, for examples various extensions of symplectic groups $\mathrm{Sp}_{2n}(k)$ over ordered fields k.

We also study the relationship between strong amenability and amenability. In topological dynamics, [**11**, chapter III, theorem 3·1] shows that strong amenability implies amenability. Since there are solvable groups which are not strongly amenable, we see that the converse is not true. However, we prove that the counterparts of these notions in a definable context are equivalent for NIP theories.

THEOREM 0·6. *If G is a group definable in a NIP theory, then G is externally definably strongly amenable if and only if it is [externally] definably amenable.*

We already mentioned that G^*/G^{*00}_M is the definable Bohr compactification of G. In the last section, we give a characterisation of the externally definable Bohr compactification of G as a quotient of some subgroup of G by a certain component.

1. *Preliminaries*

In this section, we are going to recall some definitions from topological dynamics, mainly in a modified form to fit the definable or externally definable context. We also make some basic observations. Our main reference for classical topological dynamics is [**11**], but [**1**] may also be helpful.

1·1. *A few basic facts from topological dynamics*

Recall that a G-flow is a pair (G, X), where G is a group acting on a compact Hausdorff space by homeomorphisms. We always consider discrete flows, i.e. with no topology on G (or, if one prefers, with the discrete topology on G).

Definition 1·1. The Ellis semigroup of the flow (G, X), denoted by $E(X)$, is the closure of the collection of functions $\{\pi_g : g \in G\}$ (where $\pi_g : X \to X$ is given by $\pi_g(x) = gx$) in the space X^X equipped with the product topology, with composition as semigroup operation.

This semigroup operation is continuous in the left coordinate, $E(X)$ is also a G-flow, and minimal subflows of $E(X)$ are exactly minimal left ideals with respect to the semigroup structure on $E(X)$. We have the following fundamental fact proved by Ellis.

Fact 1·2. Let $\mathcal{M}$ be a minimal ideal in $E(X)$, and let $J(\mathcal{M})$ be the set of all idempotents in $\mathcal{M}$. Then the following items hold:

(i) for any $p \in \mathcal{M}$, $E(X)p = \mathcal{M}p = \mathcal{M}$;
(ii) $\mathcal{M}$ is the disjoint union of sets $u\mathcal{M}$ with u ranging over $J(\mathcal{M})$;

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(iii) for each $u \in J(\mathcal{M})$, $u\mathcal{M}$ is a group with the neutral element u, where the group operation is the restriction of the semigroup operation on $E(X)$;
(iv) all the groups $u\mathcal{M}$ (for $u \in J(\mathcal{M})$) are isomorphic, even when we vary the minimal ideal $\mathcal{M}$

The isomorphism type of the groups $u\mathcal{M}$ (or just any of these groups) from the above fact is called the Ellis group of the flow X.

Definition 1·3. A G-ambit is a G-flow (G, X, x_0) with a distinguished point $x_0 \in X$ such that the orbit Gx_0 is dense.

With the natural notion of flow homomorphism we have the following fact.

Fact 1·4. $(G, \beta G, e)$ is the unique up to isomorphism universal G-ambit.

Using this fact, one gets an "action" of βG on any G-flow (G, X), namely for $x \in X$ there is a unique flow homomorphism $h_x : (G, \beta G, e) \to (G, X, x)$, and for $p \in \beta G$ we define $px = h_x(p)$. More explicitly, this action is given by $px = \lim g_i x$ for any net (g_i) of elements of G converging to p in βG. In particular, βG acts on itself, and denoting this action by $*$, one has $(p * q)x = p(qx)$ for all $p, q \in \beta G$ and $x \in X$. In particular, $*$ is a semigroup operation on βG which is continuous on the left and whose restriction to $G \times G$ is the original group operation on G. One easily checks that $(\beta G, *) \cong E(\beta G)$ (by sending $p \in \beta G$ to the function $(x \mapsto px) \in E(\beta G)$). In particular, Fact 1·2 applies to $(\beta G, *)$ in place of $E(\beta G)$. In a moment, we will recall a very explicit model-theoretic description of $*$.

1·2. *Definable and externally definable context*

We fix a group G which is $\varnothing$-definable in a first order structure M and a model N which is an $|M|^+$-saturated elementary extension of M (saturation means that every type over less than $|M|^+$ parameters from N has a realisation in N). We also fix a monster model $\mathfrak{C} \succ N$.

By $S_{G,M}(N)$ we denote the space of all types in $S_G(N)$ finitely satisfiable in M (which means that every formula in each such a type has a realization in M). Note that $S_{G,M}(N)$ is a closed subset of $S_G(N)$. Then, $S_{G,ext}(M)$ is naturally homeomorphic to $S_{G,M}(N)$, and we identify these spaces. By $S_{G,M}(N)(\mathfrak{C})$ we will denote the set of realizations in $\mathfrak{C}$ of all types in $S_{G,M}(N)$ (this is an N-type-definable set). Now, we recall the definition of a definable function from [**9**].

Definition 1·5. Let C be a compact space. A map $f : G \to C$ is [externally] definable if for all disjoint, closed subsets C_1 and C_2 of C the preimages $f^{-1}[C_1]$ and $f^{-1}[C_2]$ can be separated by an [externally] definable subset of G.

For functions defined on the monster model we take another definition, but the lemma below shows that both definitions are compatible.

Definition 1·6. Let C be a compact space.
(i) A function $f : G^* \to C$ is M-definable if for every closed $C_1 \subseteq C$ the preimage $f^{-1}[C_1]$ is type-definable over M.
(ii) A function $f : S_{G,M}(N)(\mathfrak{C}) \to C$ is externally definable if for every closed $C_1 \subseteq C$ the preimage $f^{-1}[C_1]$ is type-definable over N.

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Remark 1·7. A function f is externally definable according to Definition 1·6 iff there is a continuous function $h : S_{G,M}(N) \to C$ such that $f = h \circ r$, where $r : S_{G,M}(N)(\mathfrak{C}) \to S_{G,M}(N)$ is the obvious map.

LEMMA 1·8.

(1) *Definable context*:
   (i) *if $f : G \to C$ is definable, then it extends uniquely to an M-definable function $f^* : G^* \to C$. Moreover, f^* is given by the formula $f^*(a) = \bigcap_{\varphi \in \mathrm{tp}(a/M)} \mathrm{cl}(f[\varphi(M)])$;*
   (ii) *conversely, if $f^* : G^* \to C$ is an M-definable function, then $f^*|_G : G \to C$ is definable*;

(2) *Externally definable context*:
   (i) *if $f : G \to C$ is externally definable, then it extends uniquely to an externally definable function $f^* : S_{G,M}(N)(\mathfrak{C}) \to C$. Moreover, f^* is given by the formula $f^*(a) = \bigcap_{\varphi \in \mathrm{tp}(a/N)} \mathrm{cl}(f[\varphi(M)])$;*
   (ii) *conversely, if $f^* : S_{G,M}(N)(\mathfrak{C}) \to C$ is an externally definable function, then $f^*|_G : G \to C$ is also externally definable.*

Proof. (1) is exactly [**9**, lemma 3·2], and (2) can be proved in the same fashion.

The definable part of the next definition comes from [**9**, definition 3·5].

Definition 1·9.

(i) An [externally] definable G-flow is a G-flow (G, X) such that for every $x \in X$ the map $f_x : G \to X$ defined by $f_x(g) = gx$ is [externally] definable.
(ii) An [externally] definable G-ambit is an [externally] definable G-flow (G, X, x_0) with a distinguished point x_0 such that the orbit Gx_0 is dense in X.

The proofs of [**9**, lemma 3·7(i) and proposition 3·8] can be adapted also to the externally definable context, so we have

Fact 1·10.

(i) If all types in $S_G(M)$ are definable, then $(G, S_G(M), \mathrm{tp}(e/M))$ is the unique up to isomorphism universal definable G-ambit.
(ii) $(G, S_{G,M}(N), \mathrm{tp}(e/N))$ is the unique up to isomorphism universal externally definable G-ambit.

As in the case of βG in the previous subsection, by universality, we get:

(i) assume all types in $S_G(M)$ are definable. Then $S_G(M)$ "acts" on every definable G-flow (G, X) via $px = \lim g_i x$ for any net (g_i) of elements of G converging to p in $S_G(M)$. In particular, it acts on itself, yielding a semigroup operation $*$ on $S_G(M)$ which is continuous on the left. Then $(S_G(M), *) \cong E(S_G(M))$. In particular, Fact 1·2 applies to the semigroup $(S_G(M), *)$. The reason why we say that $S_G(M)$ "acts" on (G, X) is that $(p * q)x = p(qx)$ for any $p, q \in S_G(M)$ and $x \in X$.
(ii) $(G, S_{G,M}(N), \mathrm{tp}(e/N))$ "acts" on every definable G-flow (G, X) via $px = \lim g_i x$ for any net (g_i) of elements of G converging to p in $S_{G,M}(N)$. In particular, it acts on itself, yielding a semigroup operation $*$ on $S_{G,M}(N)$ which is continuous on the left. Then $(S_{G,M}(N), *) \cong E(S_{G,M}(N))$. In particular, Fact 1·2 applies to the semigroup $(S_{G,M}(N), *)$. We still have $(p * q)x = p(qx)$ for any $p, q \in S_{G,M}(N)$ and $x \in X$.

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Throughout this paper, when we refer to particular results in [**11**], we often mean (without saying it) the obvious counterparts of the statements from [**11**] in our [externally] definable context.

The following explicit description of $*$ was established in [**18**]. This description applies, in particular, to the universal G-ambit $(\beta G, *)$ by taking as M the group G equipped with predicates for all subsets of G.

Fact 1·11.

(i) Assume all types in $S_G(M)$ are definable. Take any $p, q \in S_G(M)$. Then $p * q := \mathrm{tp}(ab/M)$, where $\mathrm{tp}(b/M) = q$ and $\mathrm{tp}(a/M, b)$ is the unique (by the definability of all types in $S_G(M)$) coheir of p over M, b, that is the unique type $\bar{p} \in S_G(M, b)$ extending p and finitely satisfiable in M.

(ii) Take any $p, q \in S_{G,M}(N)$. Then $p * q := \mathrm{tp}(ab/N)$, where $\mathrm{tp}(b/N) = q$ and $\mathrm{tp}(a/N, b)$ is the unique extension of p to a complete type over N, b which is finitely satisfiable in M.

The following easy remark is fundamental in many proofs in this paper.

Remark 1·12. A product of [externally] definable G-flows is an [externally] definable G-flow.

Proof. Let X_i, $i \in I$, be [externally] definable G-flows. Let $X = \prod_i X_i$, which is naturally a G-flow. We will show that it is [externally] definable. Consider any disjoint closed subsets D_1 and D_2 of X and $x = (x_i)_{i \in I} \in X$. We need to show that the sets $D_1' := \{g \in G : gx \in D_1\}$ and $D_2' := \{g \in G : gx \in D_2\}$ can be separated by an [externally] definable subset of G. Wlog we can assume that D_1 and D_2 are subbasic closed sets of the form $D_1 = A_1 \times \cdots \times A_n \times \prod_{i \neq i_1, \ldots, i_n} X_i$ and $D_2 = B_1 \times \cdots \times B_n \times \prod_{i \neq i_1, \ldots, i_n} X_i$ for some closed subsets $A_1, B_1 \subseteq X_{i_1}, \ldots, A_n, B_n \subseteq X_{i_n}$.

The assumption that $D_1 \cap D_2 = \varnothing$ implies that $A_j \cap B_j = \varnothing$ for some $j \in \{1, \ldots, n\}$. By the [external] definability of the G-flow X_{i_j}, we get that $D_1'' := \{g \in G : gx_{i_j} \in A_j\}$ and $D_2'' := \{g \in G : gx_{i_j} \in B_j\}$ can be separated by an [externally] definable subset of G. Since clearly $D_1' \subseteq D_1''$ and $D_2' \subseteq D_2''$, the proof is completed.

The following definition is classical.

Definition 1·13. Two points x and y of a G-flow (G, X) are said to be proximal if there exists a net (g_i) in G such that $\lim g_i x = \lim g_i y$. A flow (G, X) is proximal if any two points in it are proximal.

It is proved in [**11**, chapter II, proposition 4·2] that the unique minimal subflow of the product of all (up to isomorphism) minimal proximal G-flows is the unique up to isomorphism universal minimal proximal G-flow. Having Remark 1·12, Glasner's proof extends to the [externally] definable context.

COROLLARY 1·14. *There exists a unique up to isomorphism universal minimal [externally] definable proximal G-flow, which will be denoted by (G, π_{def}) [respectively, $(G, \pi_{\mathrm{extdef}})$].*

Recall the definition of amenability and strong amenability in our context.

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Definition 1·15.

(i) G is said to be [externally] definably amenable if there is a left invariant, finitely additive probability measure on [externally] definable subsets of G.

(ii) G is said to be [externally] definably strongly amenable if the only minimal [externally] definable proximal G-flow is the trivial one. Equivalently, if the universal minimal [externally] definable proximal G-flow is trivial.

In classical topological dynamics, it is known that each strongly amenable group is amenable, but the converse is not true. It is also known that all nilpotent groups are strongly amenable (so also [externally] definably strongly amenable).

Recall that a group compactification of a given (discrete) group G is a homomorphism from G to a compact Hausdorff group K with dense image (or just this compact group K). (For convenience we will write "compactification" instead of "group compactification".) There is always a unique up to isomorphism universal compactification of G, and it is called the Bohr compactification of G.

By an [externally] definable compactification of G we mean an [externally] definable homomorphism as above.

Remark 1·16. There is a unique universal [externally] definable compactification of G, which will be called the [externally] definable Bohr compactification of G.

Proof. As in the classical case, let K' be the product of all (up to isomorphism) [externally] definable compactifications of G, and let K be the closure of the image of the diagonal homomorphism from G (i.e. $K = \mathrm{cl}(\{(g, g, \dots) : g \in G\})$). Clearly K is a compactification of G. [External] definability of this compactification follows from Remark 1·12. The facts that it is universal and so unique up to isomorphism are obvious.

In [**9**, proposition 3·4], the authors gave the following model-theoretic realization of the definable Bohr compactification.

Fact 1·17. Let G be a group definable in M. Then G^*/G^{*00}_M is the definable Bohr compactification of G.

Almost all the theory recalled and further developed in this paper works in the externally definable case without many differences in comparison with the definable case. However, the last fact is an exception, and the last section of the paper is devoted to a model-theoretic description of the externally definable Bohr compactification of G. We will need there the following lemma.

LEMMA 1·18. *Let $f : G \to C$ be an externally definable homomorphism from G to a compact group C. Let $f^* : S_{G,M}(N)(\mathfrak{C}) \to C$ be the externally definable map obtained in Lemma* 1·8*, and let $h : S_{G,M}(N) \to C$ be the map provided by Remark* 1·7*.*

(i) *For any $a_1, \dots, a_n \in S_{G,M}(N)(\mathfrak{C})$ such that $a_1 \cdot \ldots \cdot a_n \in S_{G,M}(N)(\mathfrak{C})$ one has $f^*(a_1 \cdot \ldots \cdot a_n) = f^*(a_1) \cdot \ldots \cdot f^*(a_n)$.*

(ii) *For any $a \in S_{G,M}(N)(\mathfrak{C})$, one has $a^{-1} \in S_{G,M}(N)(\mathfrak{C})$ and $f^*(a^{-1}) = f^*(a)^{-1}$.*

(iii) *The map h is a semigroup homomorphism.*

Proof. For (i) and (ii) it is enough to apply the argument similar to the one from the proof of [**9**, proposition 3·4]. In fact, (ii) follows easily from (i).

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(iii) Consider any $p, q \in S_{G,M}(N)$. Take $b \models q$ and a such that $\mathrm{tp}(a/N, b)$ is the unique extension of p to a complete type over N, b which is finitely satisfiable in M. Then $p * q = \mathrm{tp}(ab/N) \in S_{G,M}(N)$. Therefore, by (ii), $h(p * q) = f^*(ab) = f^*(a)f^*(b) = h(p)h(q)$.

One can also think of Fact 1·17 as a description of G^*/G^{*00}_M as a universal object in a certain category. Now, we will find such a description for G^*/G^{*000}_M. First, we recall a basic fact on connected components and we fix some notation.

Fact 1·19. Consider any $A \subseteq M$. Then G^{*000}_A consists of all products of finitely many elements of the form $b^{-1}a$, where (a, b) extends to an infinite A-indiscernible sequence, and G^{*000}_M consists of all products of finitely many elements of the form $b^{-1}a$, where $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$. In particular, if $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, then $b^{-1}a \in G^{*000}_M \subseteq G^{*000}_A$.

Let X and Y be definable subsets of G. Then $X \cdot Y$ will denote the set of products $x \cdot y$, $x \in X$, $y \in Y$, viewed first as a definable subset of G, and then as a clopen subset of $S_G(M)$. For $p, q \in S_G(M)$, we define $p \cdot q$ as the intersection of all clopen sets $X \cdot Y$ for $X \in p$, $Y \in q$, so a closed subset of $S_G(M)$. Equivalently, $p \cdot q = \{\mathrm{tp}(a \cdot b/M) : \mathrm{tp}(a/M) = p \text{ and } \mathrm{tp}(b/M) = q\}$.

PROPOSITION 1·20. *Consider surjective maps $f : S_G(M) \to H$, where H is a group, and for all $p, q \in S_G(M)$, $f(p) \cdot f(q)$ is equal to the common value of $f(r)$ for all $r \in p \cdot q$. Then there is a (unique) universal such map, and it is precisely the map onto G^*/G^{*000}_M given by $\mathrm{tp}(a/M) \mapsto aG^{*000}_M$.*

When H is a compact Hausdorff group and the surjective map $f : S_G(M) \to H$ is continuous and $f|_G$ is a homomorphism, then the above condition is automatically satisfied.

Proof. Uniqueness will be clear once we show that the surjective map $f_0 : S_G(M) \to G^*/G^{*000}_M$ given by $f_0(\mathrm{tp}(a/M)) = aG^{*000}_M$ is universal.

First, we check that f_0 satisfies the requirement. The fact that f_0 is well-defined (i.e. does not depend on the choice of a realizing a given type) follows from the observation that whenever two elements have the same type over M, then they lie in the same coset modulo G^{*000}_M (see Fact 1·19). Now, take $p, q \in S_G(M)$ and $a \models p, b \models q$. Then $f_0(\mathrm{tp}(ab/M)) = f_0(\mathrm{tp}(a/M)) \cdot f_0(\mathrm{tp}(b/M))$, which follows immediately from the definition of f_0.

Now, consider any $f : S_G(M) \to H$ as in the proposition. Since $f(\mathrm{tp}(e_G/M)) = f(\mathrm{tp}(e_G/M) \cdot \mathrm{tp}(e_G/M)) = f(\mathrm{tp}(e_G/M))^2$, we see that $f(\mathrm{tp}(e_G/M)) = e_H$. So, if for some $a, b \in G^*$ we have $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, then $f(\mathrm{tp}(b^{-1}a/M)) = f(\mathrm{tp}(b^{-1}/M)) \cdot f(\mathrm{tp}(a/M)) = f(\mathrm{tp}(b^{-1}b/M)) = e_H$; in particular, $f(p^{-1}) = f(p)^{-1}$ for all $p \in S_G(M)$, where $p^{-1} = \mathrm{tp}(a^{-1}/M)$ for any $a \models p$.

In order to finish the proof, it is enough to show that $h : G^*/G^{*000}_M \to H$ given by $h(aG^{*000}_M) = f(\mathrm{tp}(a/M))$ is a well-defined homomorphism. So, take $a, b \in G^*$ such that $aG^{*000}_M = bG^{*000}_M$. By Fact 1·19, this means that $b^{-1}a = \beta_1^{-1}\alpha_1 \cdot \ldots \cdot \beta_n^{-1}\alpha_n$ for some $\alpha_i, \beta_i \in G^*$ such that $\mathrm{tp}(\alpha_i/M) = \mathrm{tp}(\beta_i/M)$ for all i. Therefore, by the last paragraph and the property of f, it follows that $f(\mathrm{tp}(a/M)) = f(\mathrm{tp}(b/M))$, so we have proved that h is well-defined. The fact that it is a homomorphism follows from the property of f.

The additional part of the proposition is an easy exercise using continuity.

Now, we recall the notion of a group extension from [**11**, chapter VIII].

Definition 1·21. Let $\chi : (G, Y) \to (G, X)$ be a homomorphism of minimal flows. We say that χ is a group extension if there is a compact Hausdorff group K acting faithfully and continuously on the right on Y in such a way that:

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(i) for every $y \in Y$, $\chi^{-1}(\chi(y)) = yK$, and
(ii) for all $y \in Y$, $g \in G$ and $k \in K$, $(gy)k = g(yk)$.

In the above situation, we also say that (G, Y, K) is a group extension of (G, X).

In particular, each element of K in the above definition is an automorphism of the flow (G, Y). In fact, it is easy to see that K is the group of all automorphisms of (G, Y) preserving the fibers of χ.

We talk about [externally] definable group extension when the flows (G, X) and (G, Y) are [externally] definable.

Remark 1·22. Let (G, X) be a minimal [externally] definable G-flow. Then there is a unique up to isomorphism universal (minimal) [externally] definable group extension (G, Y, K) of (G, X).

Proof. We elaborate slightly on the proof from [**11**, page 102], where the author deals with certain pointed G-flows. Consider the family $\chi_i : (G, Y_i, K_i) \to (G, X)$, $i \in I$, of all up to isomorphisms [externally] definable group extensions of (G, X).

Fix any point $x_0 \in X$ and choose an idempotent $u \in \beta G$ in some minimal left ideal of βG with $ux_0 = x_0$ (it exists by [**11**, chapter I, proposition 3·1(2)]). For any $i \in I$ choose $y_i \in \chi_i^{-1}(x_0)$. We have $\chi_i(uy_i) = u\chi_i(y_i) = ux_0 = x_0$, and so replacing y_i by uy_i, we can assume that $uy_i = y_i$ for all i.

Let $Z = \prod_i Y_i$ and $z_0 \in Z$ be the sequence (y_i). Finally, put $Y = \mathrm{cl}(Gz_0)$. Since u is an idempotent and clearly $uz_0 = z_0$, by [**11**, chapter I, proposition 3·1], we conclude that (G, Y) is minimal. By Remark 1·12, this flow is [externally] definable. Take $K' = \prod_i K_i$ and $K = \{k \in K' : z_0k \in Y\}$. Now, it is easy to check that (G, Y, K) is a universal [externally] definable group extension of (G, X).

The proof of uniqueness [**11**, p. 102] is not quite complete, because one should work in the category of non-pointed flows. Consider two universal [externally] definable group extensions $f_1 : (G, Y_1, K_1) \to (G, X)$ and $f_2 : (G, Y_2, K_2) \to (G, X)$. Choose any point $y_1 \in Y_1$ and let $x_1 = f_1(y_1)$. By the universality of (G, Y_1, K_1) and (G, Y_2, K_2), there are homomorphisms $\varphi : (G, Y_1) \to (G, Y_2)$ and $\psi : (G, Y_2) \to (G, Y_1)$ satisfying $f_2 \circ \varphi = f_1$ and $f_1 \circ \psi = f_2$. Then $\psi(\varphi(y_1)) \in f_1^{-1}(x_1) = y_1K_1$, so, by the minimality of the flow (G, Y_1), we get that the endomorphism $\psi \circ \varphi$ coincides with some $k_1 \in K_1$, so it is an automorphism. Similarly $\varphi \circ \psi$ is an automorphism. Thus, φ is an isomorphism.

Now, we recall one of the central notions for this paper (see [**11**, chapter VIII]).

Definition 1·23. The universal [externally] definable group extension $(G, \pi^{\#}_{\mathrm{def}}, \Sigma_{\mathrm{def}}(G))$ [respectively, $(G, \pi^{\#}_{\mathrm{extdef}}, \Sigma_{\mathrm{extdef}}(G))$] of the universal minimal [externally] definable proximal G-flow (G, π_{def}) [respectively, $(G, \pi_{\mathrm{extdef}})$] will be called the universal [externally] definable compactification flow of G, and $\Sigma_{\mathrm{def}}(G)$ [respectively, $\Sigma_{\mathrm{extdef}}(G)$]– the generalized [externally] definable Bohr compactification of G.

Recall that a minimal G-flow (G, X) is said to be regular if for any $x, y \in X$ there is an automorphism f of X such that $f(x)$ and y are proximal. A regular G-flow (G, X) is called a compactification flow of G if the group of all automorphisms of (G, X) is a compact Hausdorff topological group (with the pointwise convergence topology), and this compact group of automorphisms is called a generalized compactification of G (note that it is not necessarily a compactification of G). It is proved in [**11**, chapter VIII, proposition 3·1] that (G, X) is a compactification flow of G if and only if it is a group extension of a proximal flow, and it

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follows from the proof that the group K associated with this group extension is exactly the group of all automorphisms of (G, X). In particular, $\Sigma_{\mathrm{def}}(G)$ [respectively, $\Sigma_{\mathrm{extdef}}(G)$] is the compact group of all automorphisms of $(G, \pi^{\#}_{\mathrm{def}})$ [respectively, of $(G, \pi^{\#}_{\mathrm{extdef}})$].

In this paper, we will usually work in $(S_{G,\mathrm{ext}}(M), *)$. So $\mathcal{M}$ will be a minimal left ideal in this semigroup and $u \in \mathcal{M}$ – an idempotent. Then $u\mathcal{M}$ is the Ellis group of the flow $(G, S_{G,\mathrm{ext}}(M))$. Now, we adapt the notation from [**11**, chapter I·4]. For any externally definable G-flow (G, X) and a point $x_0 \in X$ such that $ux_0 = x_0$, we define the Galois (or Ellis) group of (X, x_0) as

$$\mathcal{G}(X, x_0) = \{\alpha \in u\mathcal{M} : \alpha x_0 = x_0\}.$$

Whenever there is a homomorphism f between externally definable pointed flows (X, x_0) and (Y, y_0) (where $ux_0 = x_0$ and $uy_0 = y_0$), then $\mathcal{G}(X, x_0) \leqslant \mathcal{G}(Y, y_0)$. Recall that a G-flow homomorphism is called proximal if any two points in each fiber are proximal. Then, h is proximal if and only if $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$.

We finish with a short discussion on equicontinuity. The essential notion here is uniformity. For the definition of uniformity and a concise exposition of basic material on this topic the reader is referred to the second appendix in [**1**].

Definition 1·24. A G-flow (G, X) is said to be equicontinuous if for any index α from the unique uniformity on X there is an index β such that $g\beta \in \alpha$ for all $g \in G$.

Without giving the definition of uniformity in general, let us only say that in a compact Hausdorff space there is a unique uniformity generating the underlying topology, and it consists of all neighbourhoods of the diagonal. In a compact Hausdorff group K, this uniformity has a basis consisting of the sets $\{(a, b) \in K \times K : (\exists U \in \mathcal{U})(a^{-1}b \in U)\}$, where $\mathcal{U}$ consists of all neighbourhoods of the neutral element in K.

2. *Generalised Bohr compactification as a quotient of the Ellis group*

As before, G is a group $\varnothing$-definable in a model M of an arbitrary theory, $N \succ M$ is $|M|^+$-saturated, and $\mathfrak{C} \succ N$ is a monster model. We identify $S_{G,\mathrm{ext}}(M)$ with $S_{G,M}(N)$. Recall that the semigroup operation in $S_{G,\mathrm{ext}}(M)$ is denoted by $*$, but quite often we will just skip it. Moreover, $\mathcal{M}$ is a minimal subflow of $(G, S_{G,ext}(M))$ and $u \in \mathcal{M}$ is an idempotent. Thus, $u\mathcal{M}$ is the Ellis group of $S_{G,ext}(M)$.

In this section, we will generalize [**11**, chapter IX, theorem 4·2] on presenting the generalized Bohr compactification as a quotient of the Ellis group in our externally definable [or definable, assuming that all types in $S_G(M)$ are definable] context. The general approach via the so called τ-topology is the same as in [**11**, chapter IX]. However, there are some difficulties and the proof of the main theorem is different (although it uses various tricks and computations from [**11**, chapter IX]). The reason is that we do not know whether the G-flow $2^{S_{G,\mathrm{ext}}(M)}$ of all non-empty closed subsets of $S_{G,\mathrm{ext}}(M)$ is externally definable, and so we do not have a continuous on the left "action" of the universal externally definable G-ambit $(G, S_{G,\mathrm{ext}}(M), e)$ on this flow, and in consequence, we do not have all the nice properties of the circle operation considered in [**11**, chapter IX]. We will be rather precise about what goes through and what does not.

Although we do not have the natural "action" of $S_{G,\mathrm{ext}}(M)$ on $2^{S_{G,\mathrm{ext}}(M)}$, we can take the statement in point (1) of [**11**, chapter IX, lemma 1·1] as the definition of $\circ$.

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Definition 2·1. For $A \subseteq S_{G,\mathrm{ext}}(M)$ and $p \in S_{G,\mathrm{ext}}(M)$, $p \circ A$ is defined as the set of all points $x \in S_{G,\mathrm{ext}}(M)$ for which there exist nets (x_i) in A and (g_i) in G such that $\lim g_i = p$ and $\lim g_i x_i = x$.

It is easy to check that $p \circ A$ is closed, $pA \subseteq p \circ A$ and $p \circ (q \circ A) \subseteq (pq) \circ A$ (but we do not know whether $p \circ (q \circ A) = (pq) \circ A$; in topological dynamics, it follows from the existence of the action of βG on $2^{\beta G}$). Let us check for example that $p \circ (q \circ A) \subseteq (pq) \circ A$.

Take $x \in p \circ (q \circ A)$. Then, there are nets (g_i) and (x_i) of elements of G and $q \circ A$, respectively, such that $\lim g_i = p$ and $\lim g_i x_i = x$. Next, for every i there are nets $(h_{i,j})$ and $(y_{i,j})$ from G and A, respectively, satisfying $\lim_j h_{i,j} = q$ and $\lim_j h_{i,j} y_{i,j} = x_i$. Take any neighborhood U of pq and V of x. There is an index $i(U, V)$ such that $g_{i(U,V)} q \in U$ and $g_{i(U,V)} x_{i(U,V)} \in V$. So, there is an index $j(U, V)$ such that $g_{i(U,V)} h_{i(U,V),j(U,V)} \in U$ and $g_{i(U,V)} h_{i(U,V),j(U,V)} y_{i(U,V),j(U,V)} \in V$. All of this implies that $\lim_{U,V} g_{i(U,V)} h_{i(U,V),j(U,V)} = pq$ and $\lim_{U,V} (g_{i(U,V)} h_{i(U,V),j(U,V)}) y_{i(U,V),j(U,V)} = x$.

Definition 2·2. For $A \subseteq u\mathcal{M}$, define $\mathrm{cl}_\tau(A) = (u \circ A) \bigcap u\mathcal{M}$.

Now, the proofs of [**11**, chapter IX, 1·2-1·12 (except 1·12(2))] go through (with some slight modifications) in our context. So we have all these results at our disposal. We also have a "definable" variant of [**11**, theorem 2·1] from there. However, a problem appears in [**11**, chapter IX·3], as for a τ-closed subgroup F of $u\mathcal{M}$ we do not have the canonical externally definable G-flow $\mathcal{A}(F) := \{p \circ F : p \in \mathcal{M}\} \subseteq 2^{\mathcal{M}}$ whose Galois group at $u \circ F$ is F (because we do not know whether $2^{\mathcal{M}}$ is an externally definable G-flow). Instead of this, we will use some other argument (based on [**11**, chapter IX, proposition 4·1] and some tricks from [**11**, chapter IX]). Let us note that [**11**, lemma 3·4, chapter IX] also goes through in our context.

We will not repeat here all the necessary results from [**11**]. Let us only recall a few basic facts. Namely, cl_τ is a closure operator on subsets of $u\mathcal{M}$, and it induces the so-called τ-topology on $u\mathcal{M}$. This topology is compact and T_1, and multiplication is continuous on each coordinate separately.

Definition 2·3. $H(u\mathcal{M})$ is the intersection of the sets $\mathrm{cl}_\tau(V)$ with V ranging over all τ-neighborhoods of u in the group $u\mathcal{M}$.

Then $H(u\mathcal{M})$ is a τ-closed, normal subgroup of $u\mathcal{M}$, $u\mathcal{M}/H(u\mathcal{M})$ is a compact Hausdorff group, and for any τ-closed subgroup K of $u\mathcal{M}$, $u\mathcal{M}/K$ is a Hausdorff space if and only if $K \supseteq H(u\mathcal{M})$.

Our goal in this section is to show that $u\mathcal{M}/H(u\mathcal{M})$ is the generalized externally definable Bohr compactification of G.

LEMMA 2·4. *Define* $\sim$ *on* $\mathcal{M}$ *by*:

$$p \sim q \iff p \in q \circ u\mathcal{M} \ \wedge \ q \in p \circ u\mathcal{M} \ \wedge \ up^{-1}q \in H(u\mathcal{M}).$$

Then $\sim$ *is a closed,* G*-invariant equivalence relation on* $\mathcal{M}$*. So* $(G, \mathcal{M}/\sim)$ *is a minimal externally definable* G*-flow. Moreover,* $\mathcal{G}(\mathcal{M}/\sim, u/\sim) = H(u\mathcal{M})$.

Note that whenever we compute the inverse p^{-1} of an element $p \in \mathcal{M}$, we do this inside the group $v\mathcal{M}$, where v is the unique idempotent such that $p \in v\mathcal{M}$.

Proof. The fact that $\sim$ is a G-invariant equivalence relation uses the argument from the first paragraph of the proof of [**11**, chapter IX, proposition 4·1]. The fact that $\sim$ is closed is

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an exercise on limits of nets, and it also uses the argument from the second paragraph of that proof. External definability of $(G, \mathcal{M}/\sim)$ follows from the external definability of $(G, \mathcal{M})$. Let us show now that $\mathcal{G}(\mathcal{M}/\sim, u/\sim) = H(u\mathcal{M})$.

Take $a \in u\mathcal{M}$ such that $a(u/\sim) = u/\sim$, i.e., $au/\sim = u/\sim$. Then $au \sim u$, so $u(u^{-1}(au)) \in H(u\mathcal{M})$, so $a \in H(u\mathcal{M})$. To prove the opposite inclusion, consider any $a \in H(u\mathcal{M})$. Then $uu^{-1}(au) = a \in H(u\mathcal{M})$. Moreover, $a \in u\mathcal{M} = u(u\mathcal{M}) \subseteq u \circ u\mathcal{M}$ and $u \in a(u\mathcal{M}) \subseteq a \circ u\mathcal{M}$. All of this means that $u \sim au$, so $a(u/\sim) = u/\sim$, which completes the proof.

THEOREM 2·5. *$\mathcal{M}/\sim$ is the universal externally definable compactification flow of G, and the compact group $u\mathcal{M}/H(u\mathcal{M})$ is the generalised externally definable Bohr compactification of G.*

Proof. In this proof, instead of $H(u\mathcal{M})$ we will write H. We start from the following claim which is crucial.

Claim. The compact group $u\mathcal{M}/H(u\mathcal{M})$ acts faithfully, jointly continuously and by automorphisms on the right on the flow $\mathcal{M}/\sim$ in the following way

$$[p(u/\sim)] \bullet (aH) = p(a(u/\sim)) = pa/\sim .$$

Proof of the claim. First, we check that $\bullet$ is well-defined, i.e., that it does not depend on the choice of a in the coset aH and that it does not depend on the choice of p. The first statement follows from Lemma 2·4, more precisely, from the fact that H fixes $u/\sim$. For the second statement, consider any $p, q \in \mathcal{M}$ such that $p(u/\sim) = q(u/\sim)$. We need to show that $pa/\sim = qa/\sim$, i.e., $pa \sim qa$. Since $p \sim q$, we have that $up^{-1}q \in H$. Thus, $u(pa)^{-1}(qa) = a^{-1}p^{-1}qa = a^{-1}(up^{-1}q)a \in H$ (as H is a normal subgroup of $u\mathcal{M}$ and $a \in u\mathcal{M}$). It remains to check that $pa \in qa \circ u\mathcal{M}$ (the symmetric condition follows in the same way). Since $p \sim q$, there are $g_i \in G$ and $x_i \in u\mathcal{M}$ such that $\lim g_i = q$ and $\lim g_i x_i = p$. Then, $pa = \lim(g_i a)(a^{-1}x_i a)$ and $\lim g_i a = qa$. Thus, one can find nets (g'_j) in G and (y_j) in $u\mathcal{M}$ such that $\lim g'_j = qa$ and $\lim g'_j(a^{-1}y_j a) = pa$, and hence $pa \in qa \circ u\mathcal{M}$.

Now, we check that for any $aH \in u\mathcal{M}/H$, $\bullet(aH)$ is an automorphism of the flow $\mathcal{M}/\sim$. G-invariance is clear: $[g(p(u/\sim))] \bullet (aH) = gpa/\sim = g([p(u/\sim)] \bullet (aH))$ for any $g \in G$ and $p \in \mathcal{M}$. To show continuity, consider any net (p_i) in $\mathcal{M}$ such that $\lim p_i(u/\sim) = p(u/\sim)$ and suppose for a contradiction that $\lim p_i a/\sim \neq pa/\sim$. There exists a subnet (q_j) of (p_i) such that $\lim q_j a/\sim$ exists and is different from $pa/\sim$. There is a subnet (s_k) of (q_j) such that $\lim s_k$ exists in $\mathcal{M}$, and so $\lim s_k a \notin pa/\sim$. But (s_k) is a subnet of (p_i), so $\lim s_k \in p/\sim$, hence, by the previous paragraph, $\lim s_k a \in pa/\sim$, a contradiction.

The fact that $\bullet$ is an action of $u\mathcal{M}/H(u\mathcal{M})$ on $\mathcal{M}/\sim$ is clear from the definition. It is faithful, because if $\bullet(aH)$ is a trivial automorphism, then $a \sim u$, so $a = uu^{-1}a \in H$.

We have already checked that the action $\bullet$ is continuous in the first coordinate. In order to see that it is continuous as a 2-variable function, by the joint continuity theorem (see [**1**, chapter 4, theorem 7]), it remains to show that it is continuous in the second coordinate. For this, it is enough to check that the function $f : u\mathcal{M} \to (\mathcal{M}/\sim)^{\mathcal{M}/\sim}$ given by the formula $f(a)(p/\sim) = pa/\sim$ is continuous. So, consider any net (a_i) in $u\mathcal{M}$ such that $\tau\text{-}\lim a_i = a \in u\mathcal{M}$. Our goal is to show that $\lim pa_i/\sim = pa/\sim$ for any $p \in \mathcal{M}$. We will be done if we prove that for any subnet (b_j) of (a_i) for which (pb_j) converges in the usual topology on $\mathcal{M}$ to some r one has $r \sim pa$. Using [**11**, chapter IX, lemmas 1·4 and 1·5], the assumption that $\lim pb_j = r$ implies that $\tau\text{-}\lim upb_j = ur$, so, by [**11**, chapter IX,

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proposition 1·7], $\tau\text{-lim}\, ur^{-1}pb_j = \tau\text{-lim}(ur^{-1})(upb_j) = ur^{-1}ur = ur^{-1}r = u$. On the other hand, $\tau\text{-lim}\, ur^{-1}pb_j = ur^{-1}pa$. Therefore, since $u\mathcal{M}/H$ is Hausdorff, we conclude that $ur^{-1}paH = uH = H$, so

$$ur^{-1}pa \in H. \tag{2·1}$$

Since $b_j, a \in u\mathcal{M}$, we easily get

$$r = \lim pb_j \in p \circ u\mathcal{M} = pa \circ u\mathcal{M}. \tag{2·2}$$

Consider any open neighbourhoods U of r and V of pa. Choose $j(U, V)$ such that $pb_{j(U,V)} \in U$ (it exists as $\lim pb_j = r$). Choose $g_{j(U,V)} \in U \cap G$ so close to $pb_{j(U,V)}$ that $g_{j(U,V)}b_{j(U,V)}^{-1}a \in V$ (it exists as $pb_{j(U,V)}b_{j(U,V)}^{-1}a = pa$ and multiplication in $\mathcal{M}$ is continuous in the first coordinate). Then $\lim_{U,V} g_{j(U,V)} = r$ and $\lim_{U,V} g_{j(U,V)}b_{j(U,V)}^{-1}a = pa$, so

$$pa \in r \circ u\mathcal{M}. \tag{2·3}$$

By (2·1), (2·2) and (2·3), we have $r \sim pa$, which was our goal, and the proof of the claim is completed.

Let $(G, \pi_{\text{extdef}}, x)$ be the universal minimal externally definable proximal G-flow (where $ux = x$), and let $\varphi\colon (\mathcal{M}, u) \to (\pi_{\text{extdef}}, x)$ be the epimorphism given by $\varphi(p) = px$ (recall that $px := \lim g_i x$ for an arbitrary net (g_i) in G converging to p).

We claim that φ factors through the quotient map $i : (\mathcal{M}, u) \to (\mathcal{M}/\sim, u/\sim)$. (Indeed, if $p \sim q$, then $p \in q \circ u\mathcal{M}$, so $p = \lim g_i x_i$ for some nets (x_i) in $u\mathcal{M}$ and (g_i) in G satisfying $\lim g_i = q$, and so $px = \lim g_i x_i x = \lim g_i x = qx$ (we use here the fact that $\mathcal{G}(\pi_{\text{extdef}}, x) = u\mathcal{M}$ as (G, π_{extdef}) is proximal).) So, let $\bar{\varphi}\colon \mathcal{M}/\sim \to \pi_{\text{extdef}}$ be the map such that $\bar{\varphi}i = \varphi$ (i.e. $\bar{\varphi}(p/\sim) = px$).

Define

$$\mathcal{N} := (\mathcal{M}/\sim)/(u\mathcal{M}/H),$$

the space of orbits on $\mathcal{M}/\sim$ under the action $\bullet$ of $u\mathcal{M}/H$. Note that by the claim, $\mathcal{N}$ is a minimal externally definable G-flow.

Directly from the definition of $\mathcal{N}$ (and the claim), we see that $(\mathcal{M}/\sim, u\mathcal{M}/H)$ is a group extension of $\mathcal{N}$. We will prove that:

(i) $\mathcal{N} \cong \pi_{\text{extdef}}$;

(ii) $\bar{\varphi}\colon (\mathcal{M}/\sim, u\mathcal{M}/H, u/\sim) \to (\pi_{\text{extdef}}, x)$ is the universal externally definable group extension of π_{extdef}.

From this it follows that $\mathcal{M}/\sim$ is the universal externally definable compactification flow of G and that $u\mathcal{M}/H$ is the generalized externally definable Bohr compactification of G.

Note that $\bar{\varphi}$ factors through the quotient map $j : \mathcal{M}/\sim \to \mathcal{N}$. (Indeed, if $q \sim pa$ for some $a \in u\mathcal{M}$, then $qx = pax = px$). So, let $\bar{\bar{\varphi}}\colon \mathcal{N} \to \pi_{\text{extdef}}$ be such that $\bar{\bar{\varphi}}j = \bar{\varphi}$.

Since $\mathcal{G}(\mathcal{N}, j(i(u))) = u\mathcal{M} = \mathcal{G}(\pi_{\text{extdef}}, x)$, by [**11**, chapter I, proposition 4·1(2)], we get that $\bar{\bar{\varphi}}$ is a proximal epimorphism, so, as (G, π_{extdef}) is a proximal flow, $\mathcal{N}$ is also proximal. Since minimal proximal flows do not have endomorphisms different from the identity (see [**11**, chapter II, lemma 4·1]) and π_{extdef} is the universal externally definable minimal proximal flow, we conclude that $\bar{\bar{\varphi}}$ is an isomorphism, so (i) is proved.

Since $j\colon (\mathcal{M}/\sim, u\mathcal{M}/H) \to \mathcal{N}$ is an externally definable group extension and $\bar{\bar{\varphi}}$ is an isomorphism satisfying $\bar{\bar{\varphi}}j = \bar{\varphi}$, we get that $\bar{\varphi}\colon (\mathcal{M}/\sim, u\mathcal{M}/H, u/\sim) \to (\pi_{\text{extdef}}, x)$ is an externally definable group extension. Finally, the universality of this extension follows from

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the obvious counterpart of [**11**, chapter IX, theorem 2·1(3)], the fact that group extensions are distal (i.e. any two points in each fiber are either equal or not proximal) and the equality $\mathcal{G}(\mathcal{M}/\sim, u/\sim) = H$.

3. *Generalised Bohr compactification and connected components*

The aim of this section is to prove Theorem 0·1. But first we need to define the epimorphisms θ and f discussed before this theorem.

We take the notation from Section 2. So we will be working in the general externally definable context. Recall that this applies to the definable case (assuming that all types in $S_G(M)$ are definable), and, in particular, to the case when predicates for all subsets of G are in the language (so to classical topological dynamics). Theorem 0·1 is new even in this last situation.

Take any set $A \subseteq M$. Newelski [**17**] proved that the function $\theta : u\mathcal{M} \to G^*/G^{*00}_A$ given by $f(\mathrm{tp}(a/N)) = aG^{*00}_A$ is a well-defined group epimorphism. One can easily refine this in the following way.

PROPOSITION 3·1. *The function $f : u\mathcal{M} \to G^*/G^{*000}_A$ given by $f(\mathrm{tp}(a/N)) = aG^{*000}_A$ is a well-defined group epimorphism. Moreover, the composition of f with the natural map from G^*/G^{*000}_A to G^*/G^{*00}_A is exactly θ.*

Proof. Consider $\hat{f} : S_{G,M}(N) \to G^*/G^{*000}_A$ given by the same formula as f. First note that $\hat{f}$ is well-defined, because if $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, then $b^{-1}a \in G^{*000}_A$.

Now, we check that $\hat{f}$ is a semigroup homomorphism. Take any $p, q \in S_{G,M}(N)$. By Fact 1·11, $p * q = \mathrm{tp}(ab/N)$, where $\mathrm{tp}(b/N) = q$ and $\mathrm{tp}(a/N, b)$ is the unique extension of p to a complete type over N, b which is finitely satisfiable in M. So, $\hat{f}(p) = aG^{*000}_A$, $\hat{f}(q) = bG^{*000}_A$ and $\hat{f}(p * q) = abG^{*000}_A$.

It remains to prove that f is onto. Since each type in $S_G(M)$ can be extended to a type in $S_{G,M}(N)$, we easily get that $\hat{f}$ is onto. We already know that $\hat{f}$ is a homomorphism. Thus, as $\mathcal{M} = S_{G,M}(N) * u$, we conclude that $\hat{f}|_{\mathcal{M}}$ is onto, and so $f = \hat{f}|_{u*\mathcal{M}}$ is also onto.

We have explained what the function f in Theorem 0·1 is, and now we are ready to prove this theorem.

Proof of Theorem 0·1. (1) Let $\bar{D} \subseteq G^*/G^{*000}_A$ be closed, i.e., $D := \{g \in G^* : gG^{*000}_A \in \bar{D}\}$ is type-definable. The goal is to show that $f^{-1}[\bar{D}]$ is a τ-closed subset of $u\mathcal{M}$.

Consider any $p \in \mathrm{cl}_\tau(f^{-1}[\bar{D}])$. By the definition of the τ-topology, there are $g_i \in G$ and $p_i \in f^{-1}[\bar{D}]$ such that $\lim_i g_i = u$ and $\lim_i g_i p_i = p$ (these limits are in the usual topology on $S_G(N)$). As $f^{-1}[\bar{D}] = \{\mathrm{tp}(a/N) : a \in D\} \bigcap u\mathcal{M}$ and $g_i p_i = \mathrm{tp}(g_i a_i/N)$ for $a_i \models p_i$, by compactness, we get that there are $a \models u$ and $b \in D$ such that $ab \models p$. Then, since $f(u) = eG^{*000}_A$, we have $a \in G^{*000}_A$. As $b \in D$ and D is closed under multiplication by elements of G^{*000}_A, we get $ab \in D$, so $p = \mathrm{tp}(ab/N)$ belongs to $f^{-1}[\bar{D}]$.

(2) By Fact 1·19, we know that $G^{*000}_A = \bigcup_{n\in\omega} F_n$, where F_n is the A-type-definable set consisting of products of n (equivalently, at most n) elements of the form $b^{-1}a$ where (a, b) extends to an A-indiscernible sequence. Put $\widetilde{F}_n := \{\mathrm{tp}(a/N) : a \in F_n\}$, a closed subset of $S_G(N)$. Since $f(u) = eG^{*000}_A$, we can find $n \in \omega$ such that $u \in \widetilde{F}_n$.

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Let π be the partial type over A defining F_{2n} and closed under conjunction. Consider any $\varphi(x) \in \pi$. Let

$$V := [\neg\varphi(x)] \bigcap u\mathcal{M},$$

where $[\neg\varphi(x)]$ is the closed subset of $S_G(N)$ consisting of types containing $\neg\varphi(x)$.

Claim.

(i) $u \notin \mathrm{cl}_\tau(V)$.

(ii) $\mathrm{cl}_\tau(u\mathcal{M} \setminus \mathrm{cl}_\tau(V)) \subseteq \mathrm{cl}_\tau(u\mathcal{M} \setminus V) \subseteq \widetilde{F}^{\varphi}_{3n}$, where $\widetilde{F}^{\varphi}_{3n} := \{\mathrm{tp}(ab/N) : a \in F_n \wedge b \models \varphi(x)\}$.

Proof of the claim. (i) Suppose for a contradiction that $u \in \mathrm{cl}_\tau(V)$. Arguing as in (1), we get that there are $a \models u$ and $b \models \neg\varphi(x)$ such that $ab \models u$. Then $a \in F_n$ and $ab \in F_n$, so $b \in F_{2n}$, and hence $b \models \varphi(x)$, a contradiction.
(ii) We need to check that $\mathrm{cl}_\tau(u\mathcal{M} \setminus V) \subseteq \widetilde{F}^{\varphi}_{3n}$. Consider any $p \in \mathrm{cl}_\tau(u\mathcal{M} \setminus V)$. As before, there are $a \models u$ and $b \models \varphi(x)$ such that $ab \models p$. Then $a \in F_n$, so $\mathrm{tp}(ab/N) \in \widetilde{F}^{\varphi}_{3n}$.

Notice that $\bigcap_{\varphi(x)\in\pi} \widetilde{F}^{\varphi}_{3n} = \widetilde{F}_{3n}$. So, by the claim,

$$\begin{aligned} H(u\mathcal{M}) &= \bigcap\{\mathrm{cl}_\tau(U) : U \ \tau\text{-neighbourhood of } u\} \subseteq \textstyle\bigcap_{\varphi(x)\in\pi} \widetilde{F}^{\varphi}_{3n} \bigcap u\mathcal{M} = \widetilde{F}_{3n} \bigcap u\mathcal{M} \\ &\subseteq f^{-1}(eG^{*000}_{M}), \end{aligned}$$

which completes the proof of (2).

(3) follows from (1) and (2).

The fact that $u\mathcal{M}/H(u\mathcal{M})$ is the generalised externally definable Bohr compactification of G was proved in Theorem 2·5.

Note that from Theorem 0·1 it follows immediately that the epimorphism θ from $u\mathcal{M}$ to G^*/G^{*00}_{A} is also continuous.

4. *The proof of Theorem* 0·2

This section is devoted to the proof of Theorem 0·2, which will consist of a series of lemmas.

Proof of Theorem 0·2. Enlarging M if necessary, we can assume that $A \subseteq M$, and we aim at proving the second (more precise) statement of the theorem. We take the notation as in Section 2.

The situation is as follows. By Theorem 0·1, we have the natural continuous epimorphism $\bar{f} : u\mathcal{M}/H(u\mathcal{M}) \to G^*/G^{*000}_{A}$ given by $\bar{f}(pH(u\mathcal{M})) = f(p)$, where $f : u\mathcal{M} \to G^*/G^{*000}_{A}$ is defined by $f(\mathrm{tp}(a/N)) = aG^{*000}_{A}$. Next, $Y := \ker(\bar{f}) = \bar{f}^{-1}(eG^{*000}_{A})$, and $\mathrm{cl}_\tau(Y)$ is the closure of Y computed inside the compact topological group $u\mathcal{M}/H(u\mathcal{M})$. Thus, $\bar{f}$ induces an isomorphism between $\mathrm{cl}_\tau(Y)/Y$ and $\bar{f}[\mathrm{cl}_\tau(Y)]$. To prove the theorem, we need to show that $\bar{f}[\mathrm{cl}_\tau(Y)] = G^{*00}_{A}/G^{*000}_{A}$. Of course, $\bar{f}[\mathrm{cl}_\tau(Y)] \leqslant G^{*00}_{A}/G^{*000}_{A}$, because by Theorem 0·1, the preimage by $\bar{f}$ of the closed subgroup G^{*00}_{A}/G^{*000}_{A} is closed, so it contains $\mathrm{cl}_\tau(Y)$. The rest of the proof is devoted to the proof of the opposite inclusion.

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Let P_u be the preimage of Y by the quotient map π from $u\mathcal{M}$ to $u\mathcal{M}/H(u\mathcal{M})$, and let $S = \mathrm{cl}_\tau(P_u)$. Then $\pi[S] = \mathrm{cl}_\tau(Y)$, as π is continuous and closed. Hence, $\bar{f}[\mathrm{cl}_\tau(Y)] = f[S]$. Since a well-known result (see e.g. [**8**, proposition 3·5]) says that G^{*00}_A/G^{*000}_A is the closure of the neutral element in G^*/G^{*000}_A, we conclude that in order to finish the proof, we need to show that $f[S]$ is closed in G^*/G^{*000}_A.

All the sets Y, S and P_u are defined in terms of the idempotent $u \in \mathcal{M}$ chosen at the beginning. For any other idempotent $v \in \mathcal{M}$ we have the analogous sets, in particular, we have the set $P_v \subseteq v\mathcal{M}$.

Let $\hat{f} : S_{G,M}(N) \to G^*/G^{*000}_A$ be the semigroup homomorphism extending f given by $\hat{f}(\mathrm{tp}(a/N)) = aG^{*000}_A$ (see the proof of Proposition 3·1). By J we will denote the set of all idempotents in $\mathcal{M}$. By $S_{G^{*000}_A,M}(N)$ we will, of course, denote the subset of $S_{G,M}(N)$ consisting of types of elements from G^{*000}_A.

Below, sometimes we use $*$ to denote the semigroup operation in $S_{G,M}(N)$, but sometimes we do not use any symbol in such a situation.

LEMMA 4·1. *For any idempotent $v \in J$ one has $P_v = \ker(\hat{f}) \bigcap v\mathcal{M} = vS_{G^{*000}_A,M}(N)v$, and for any other idempotent $w \in J$ we have $P_v = vP_w$. Also, P_v is a subgroup of $v\mathcal{M}$ for $v \in J$.*

Proof. The first equality is obvious from the definition of P_v. The rest follows easily from the fact that $\hat{f}$ is a semigroup homomorphism mapping both J and $S_{G^{*000}_A,M}(N)$ to the neutral element of G^*/G^{*000}_A.

Let us check for example that $P_v = vP_w$. Since $\hat{f}[vP_w] = \hat{f}(v)\hat{f}[P_w] = \{eG^{*000}_A\}$, we get $P_v \supseteq vP_w$. For the opposite inclusion, take any $p \in P_v$. Then $p = vq$ for some $q \in \mathcal{M}$. Since $\hat{f}$ is a homomorphism and $v, w, p \in \ker(\hat{f})$, we get that $q, wq \in \ker \hat{f}$, so $wq \in P_w$ and $p = v(wq) \in vP_w$.

Remark 4·2. For any $D \subseteq G^*/G^{*000}_A$ one has that D is closed iff $\hat{f}^{-1}[D]$ is closed.

Proof. Since $A \subseteq M$, D is closed iff $E := \{\mathrm{tp}(a/M) : aG^{*000}_A \in D\}$ is closed in $S_G(M)$. The restriction map $\alpha : S_{G,M}(N) \to S_G(M)$ is continuous and closed. Moreover, $\hat{f}^{-1}[D] = \alpha^{-1}[E]$ and $\alpha[\hat{f}^{-1}[D]] = E$. Thus, D is closed iff E is closed iff $\hat{f}^{-1}[D]$ is closed.

Remark 4·3. For any $D \subseteq G^*/G^{*000}_A$ one has that D is closed iff $\hat{f}^{-1}[D] \bigcap \mathcal{M}$ is closed in $\mathcal{M}$. In particular, in our context, in order to prove that $f[S]$ is closed, it is enough to show that $\hat{f}^{-1}[f[S]] \bigcap \mathcal{M}$ is closed in $\mathcal{M}$.

Proof. Note that $\hat{f}^{-1}[D] \bigcap \mathcal{M} = \hat{f}^{-1}[D] * u$. Moreover, $*$ is continuous in the left coordinate and $S_{G,M}(N)$ is compact Hausdorff. Thus, $\hat{f}^{-1}[D] \bigcap \mathcal{M}$ is closed iff $\hat{f}^{-1}[D] * u$ is closed iff $\hat{f}^{-1}[D]$ is closed (in $S_{G,M}(N)$) iff D is closed (where the last equivalence follows from the previous remark).

So, in order to finish the proof of the theorem, it remains to show that $\hat{f}^{-1}[f[S]] \bigcap \mathcal{M}$ is closed in $\mathcal{M}$.

LEMMA 4·4. $\hat{f}^{-1}[f[S]] \bigcap \mathcal{M} = J * (u \circ P_u) * S_{G^{*000}_A,M}(N) * u = J * (u \circ P_u)$.

Proof. The second equality follows easily from the continuity of $*$ on the left and from the observation that $P_u * S_{G^{*000}_A,M}(N) * u = P_u$ which is an immediate consequence of Lemma 4·1.

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So, we concentrate on the first equality. The inclusion ($\supseteq$) follows from the observation that $f[S] = \hat{f}[u \circ P_u]$ (being an obvious consequence of the fact that $S = u(u \circ P_u)$ noted in [**11**, chapter IX, lemma 1·3]) and from the fact that $\hat{f}$ is a semigroup homomorphism which maps all idempotents and all elements from $S_{G^{*000}_A,M}(N)$ to the neutral element in G^*/G^{*000}_A.

($\subseteq$) Take any $q \in \hat{f}^{-1}[f[S]] \cap \mathcal{M}$. Then $\hat{f}(q) = f(s)$ for some $s \in S = u(u \circ P_u)$. There is $v \in J$ such that $q \in v\mathcal{M}$. Then $\hat{f}(q) = \hat{f}(vs)$. So $\hat{f}((vs)^{-1}q) = eG^{*000}_A$, i.e., $(vs)^{-1}q \in S_{G^{*000}_A,M}(N) \cap \mathcal{M} = S_{G^{*000}_A,M}(N) * u$. Therefore, $vq \in vs * S_{G^{*000}_A,M}(N) * u$, but $vq = q$, so $q \in vs * S_{G^{*000}_A,M}(M) * u \subseteq J * (u \circ P_u) * S_{G^{*000}_A,M}(N) * u$.

LEMMA 4·5. *Let $v \in J$.*

(i) *$v \circ P_u = v \circ P_v$ and $u \circ P_v = u \circ P_u$.*

(ii) *$v(u \circ P_u) \subseteq v \circ P_v$ and $u(v \circ P_v) \subseteq u \circ P_u$.*

(iii) *$v(u \circ P_u) = v(v \circ P_v)$ and $u(v \circ P_v) = u(u \circ P_u)$.*

Proof. In all three points, it is clearly enough to show only one of the two symmetric statements.

Point (i) is a consequence of the following two sequences of inclusions which follow from Lemma 4·1 and basic properties of $\circ$.

$$v \circ P_v = v \circ (vP_u) \subseteq v \circ (v \circ P_u) \subseteq (vv) \circ P_u = v \circ P_u$$

and

$$v \circ P_u = v \circ (uvP_u) = v \circ (uP_v) \subseteq v \circ (u \circ P_v) \subseteq (vu) \circ P_v = v \circ P_v.$$

(ii) By (i), we have

$$v(u \circ P_u) \subseteq v \circ (u \circ P_u) \subseteq (vu) \circ P_u = v \circ P_u = v \circ P_v.$$

(iii) By (ii), we get $v(u \circ P_u) = v(v(u \circ P_u)) \subseteq v(v \circ P_v)$ and similarly $u(v \circ P_v) \subseteq u(u \circ P_u)$. It follows that $u(u \circ P_u) = u(v(u \circ P_u)) \subseteq uv(v \circ P_v) \subseteq u(u \circ P_u)$. So the second inclusion must be the equality, i.e., $u(v \circ P_v) = u(u \circ P_u)$.

The next lemma is a corollary of the above considerations.

LEMMA 4·6. $\hat{f}^{-1}[f[S]] \cap \mathcal{M} = \bigcup_{v \in J} v \circ P_v = \bigcup_{v \in J} v \circ P_u$.

Proof. The second equality follows from Lemma 4·5(i). For the inclusion ($\subseteq$) in the first equality, note that by Lemmas 4·4 and 4·5(ii),

$$\hat{f}^{-1}[f[S]] \cap \mathcal{M} = \bigcup_{v \in J} v(u \circ P_u) \subseteq \bigcup_{v \in J} v \circ P_v.$$

For ($\supseteq$) notice that by Lemma 4·5(iii), we have $\hat{f}[v \circ P_v] = \hat{f}[v(v \circ P_v)] = \hat{f}[v(u \circ P_u)] = \hat{f}[u \circ P_u] = f[S]$, and so $v \circ P_v \subseteq \hat{f}^{-1}[f[S]] \cap \mathcal{M}$.

LEMMA 4·7. $\mathrm{cl}(J) \subseteq \ker(\hat{f}) \cap \mathcal{M}$. *Equivalently,* $\mathrm{cl}(J) \subseteq \bigcup_{v \in J} P_v = \bigcup_{v \in J} vP_u$.

Proof. The second part is equivalent to the first one, because $\bigcup_{v \in J} v\mathcal{M} = \mathcal{M}$ and $P_v = \ker(\hat{f}) \cap v\mathcal{M}$ for $v \in J$. To show the first part, consider any $p \in \mathrm{cl}(J)$.

For an arbitrary $v \in J$, we have that $v * v = v$. Thus, by Fact 1·11, for any $b \models v$ and a realizing the unique extension of v to a complete type over N, b finitely satisfiable in M,

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we have that $ab \models v$. So, denoting by $F(x)$ the type over M saying that $x = yz^{-1}$ for some $y \equiv_M z$, we get that $a = abb^{-1} \models F$. So, for any $v \in J$ and $c \models v$ we have that $c \models F$.

Since in every open neighbourhood of p there is an element from J, we have that for every $\varphi(x) \in p$ there is $c \models \varphi(x) \wedge F(x)$. By compactness, there is $c \models p(x) \wedge F(x)$. So $c \in G^{*000}_A$ and $p \in \hat{f}^{-1}(eG^{*000}_A) \bigcap \mathcal{M}$, and we are done.

LEMMA 4·8. *$\bigcup_{v \in J} v \circ P_u$ is closed.*

Proof. Consider any $q \in \mathrm{cl}(\bigcup_{v \in J} v \circ P_u)$. Then there are nets (v_i) in J and (x_i) in $\mathcal{M}$ such that $x_i \in v_i \circ P_u$ and (x_i) converges to q. We can find subnets (w_j) of (v_i) and (y_j) of (x_i) for which (w_j) converges to some $r \in \mathrm{cl}(J)$ (and still $y_j \in w_j \circ P_u$). Thus, we can find nets (g_k) in G and (p_k) in P_u such that (g_k) converges to r and $(g_k p_k)$ converges to q. So we have proved that $q \in r \circ P_u$.

By Lemma 4·7, we get that $r = wp$ for some $w \in J$ and $p \in P_u$. Since P_u is a subgroup of $u\mathcal{M}$, we also have that $p^{-1}P_u = P_u$. Thus, we conclude that

$$q \in r \circ P_u = r \circ (p^{-1}P_u) \subseteq r \circ (p^{-1} \circ P_u) \subseteq (rp^{-1}) \circ P_u = w \circ P_u \subseteq \bigcup_{v \in J} v \circ P_u,$$

which completes the proof of the lemma.

By Lemmas 4·6 and 4·8 together with the conclusion right after Remark 4·3, the proof of Theorem 0·2 has been completed.

Note that if one is not interested in proving that G^{*00}_A/G^{*000}_A is the quotient of a compact Hausdorff group by a dense subgroup but just by any subgroup, then such a weaker statement can be easily deduced from Theorem 0·1. Namely, by this theorem, $Z := \bar{f}^{-1}[G^{*00}_A/G^{*000}_A]$ is closed (so compact Hausdorff) in $u\mathcal{M}/H(u\mathcal{M})$, and clearly $\bar{f}$ induces an isomorphism between Z/Y and G^{*00}_A/G^{*000}_A, where $Y = \ker(\bar{f})$. In a similar fashion, we get

COROLLARY 4·9. *For an arbitrary group G which is $\varnothing$-definable in an arbitrary model M and for any set of parameters A the group G^*/G^{*000}_A is isomorphic to a quotient of a compact Hausdorff group. In the above notation, it will be isomorphic to the quotient of $u\mathcal{M}/H(u\mathcal{M})$ by $\ker(\bar{f})$.*

We finish this section proving Corollary 0·3.

Proof of Corollary 0·3. (ii) follows from (i) using the fact that $\bar{f}$ is naturally induced by f. To see (i), notice that $Y := \ker(\bar{f})$ is a singleton, so it is closed, so $\mathrm{cl}_\tau(Y) = Y$, and, by Theorem 0·2, $\{eG^{*000}_A\} = \bar{f}[Y] = \bar{f}[\mathrm{cl}_\tau(Y)] = G^{*00}_A/G^{*000}_A$, which implies that $G^{*000}_A = G^{*00}_A$.

5. *Strongly amenable case*

We will prove Corollary 0·4, and then we will give some discussion around the proof.

Proof of Corollary 0·4. Recall that we assume that all types in $S_G(M)$ are definable, so $S_{G,\mathrm{ext}}(M) = S_G(M)$, and so $\mathcal{M}$ is a minimal left ideal in $S_G(M)$ and $u \in \mathcal{M}$ is an idempotent. Let $\pi \circ \bar{f} : u\mathcal{M}/H(u\mathcal{M}) \to G^*/G^{*00}_M$ be denoted by ζ. It is an epimorphism and the goal is to prove that it is an isomorphism.

It is enough to show that there is a definable (in the sense of Definition 1·5) homomorphism $\eta : G \to u\mathcal{M}/H(u\mathcal{M})$ with dense image and such that $(\zeta \circ \eta)(g) = gG^{*00}_M$

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for all $g \in G$. Indeed, since G^*/G^{*00}_M is the definable Bohr compactification of G, universality yields a continuous homomorphism $h : G^*/G^{*00}_M \to u\mathcal{M}/H(u\mathcal{M})$ such that $h(gG^{*00}_M) = \eta(g)$ for all $g \in G$; then $(\zeta \circ h)(gG^{*00}_M) = gG^{*00}_M$, so $\zeta \circ h = \mathrm{id}$, so ζ is injective (as h is onto by the denseness of $\mathrm{Im}(\eta)$), so we are done.

Let (G, X, K, x_0) be the universal definable compactification flow of G (where $ux_0 = x_0$). So, by definable strong amenability, it is the universal group extension of the trivial G-flow. By Theorem 2·5 and its proof, we know that $K = u\mathcal{M}/H(u\mathcal{M})$, and it acts on the right on X by $(px_0)k = p(kx_0)$ for $p \in \mathcal{M}$. Since (G, X, K) is a group extension of the trivial G-flow, we have that $X = x_0K$, and for any $g \in G$, $x \in X$ and $k \in K$ one has $g(xk) = (gx)k$. Then K is the group of all automorphisms of the G-flow X. Also, the function $k \mapsto x_0k$ is a homeomorphism between K and X.

One easily gets that (G, X) is equicontinuous. Thus, by [**11**, chapter I, theorem 3·3], the Ellis semigroup $E(X)$ is a compact topological group consisting of homeomorphisms. By the definition of $E(X)$, we also have a natural homomorphism $\chi : G \to E(X)$ (namely, $\chi(g)$ is the translation by g) with dense image. Since $E(X)$ is a subflow of the product G-flow X^X, by Remark 1·12, we easily get that χ is definable.

Define a function $\alpha : E(X) \to X^X$ by

$$(px_0)\alpha(\beta) = p(\beta x_0)$$

for $p \in E(X)$. To see that α is well-defined, notice that whenever $px_0 = qx_0$, then $p(x_0k) = (px_0)k = (qx_0)k = q(x_0k)$ for all $k \in K$, and hence $p = q$. This computation also shows that α is injective.

Claim. α is a topological isomorphism from $E(X)$ onto K.

Proof of the claim. First, let us check that for every $\beta \in E(X)$, $\alpha(\beta)$ is an automorphism of (G, X), and so $\alpha(\beta) \in K$. Of course, for $g \in G$ and $p \in E(X)$ we have $(gpx_0)\alpha(\beta) = gp(\beta x_0) = g(p\beta x_0) = g((px_0)\alpha(\beta))$. Now, we show continuity of $\alpha(\beta)$. Assume $\lim_i p_i x_0 = px_0$. Then $\lim_i (p_i x_0)k = (px_0)k$, and so $\lim_i p_i(x_0k) = p(x_0k)$, for all $k \in K$. But $x_0K = X$, so $\lim_i p_i(\beta x_0) = p(\beta x_0)$, i.e. $\lim_i (p_i x_0)\alpha(\beta) = (px_0)\alpha(\beta)$. Thus, $\alpha(\beta)$ is continuous.

The fact that $\alpha : E(X) \to K$ is a group monomorphism is obvious. To see that α is onto, consider any $k \in K$. Then, since (G, X) is a minimal flow, there is $\beta \in E(X)$ such that $x_0\alpha(\beta) = \beta x_0 = x_0k$, so $\alpha(\beta) = k$.

It remains to check that α is continuous. For this, we need to show that for any $p \in E(X)$ and U an open subset of X, the collection B of all $\beta \in E(X)$ satisfying $(px_0)\alpha(\beta) \in U$ is open in $E(X)$. But since $E(X)$ is group, we get that $\beta \in B$ iff $p\beta x_0 \in U$ iff $\beta x_0 \in p^{-1}U$. As $E(X)$ consists of homeomorphisms of X, $p^{-1}U$ is open in X, and so B is open in $E(X)$.

Now, define $\eta : G \to u\mathcal{M}/H(u\mathcal{M})$ as $\alpha \circ \chi$. By the above observations that α is a topological isomorphism and χ is a definable homomorphism with dense image, we conclude that η is a definable (group) homomorphism with dense image. To finish the proof of the corollary, we need to show that $(\zeta \circ \eta)(g) = gG^{*00}_M$. For this, it is enough to see that $\eta(g) = (ugu)H(u\mathcal{M})$ (because then, taking $b \models u$ and a realizing the unique coheir of u over M, b, we have $ugu = \mathrm{tp}(agb/M)$ and $a, b \in G^{*00}_M$, so $\zeta(ugu H(u\mathcal{M})) = agbG^{*00}_M = gG^{*00}_M$).

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Since $ux_0 = x_0$, we have that $u(x_0k) = (ux_0)k = x_0k$ for all $k \in K$, and so u acts trivially on X. Thus, $x_0\alpha(\chi(g)) = \chi(g)x_0 = gx_0 = ugux_0 = x_0(uguH(u\mathcal{M}))$. So $\eta(g) = \alpha(\chi(g)) = uguH(u\mathcal{M})$.

Note that we obtained an explicit formula for the function $\eta : G \to u\mathcal{M}/H(u\mathcal{M})$ in the proof above, namely $\eta(g) = uguH(u\mathcal{M})$. As was easily checked, even without using strong amenability, this function satisfies $(\zeta \circ \eta)(g) = gG^{*00}_M$. The whole difficulty was to get that η is a definable homomorphism with dense image, and for that we used strong amenability. It is worth to emphasise that in general η will not be a homomorphism with dense image. To see that, it is enough to take any group G, with predicates for all subsets added to the language, such that $G^{*00}_M \neq G^{*000}_M$ (e.g. as G one can take the free group in at least 2 generators [**7**]). Then clearly ζ is not injective, but if η was a homomorphism with dense image, then the second paragraph of the above proof would imply that ζ is injective, a contradiction.

Since all nilpotent groups are strongly amenable, Corollary 0·5 follows immediately from Corollary 0·4. The solvable case remains open. The following general and related question looks interesting.

Question 5·1. (i) Let G be a group definable in a structure M. Suppose G is definably amenable (in the sense of Definition 1·15). Is it the case that $G^{*00}_M = G^{*000}_M$?
(ii) The special case of (i) where all subsets of G are definable in M: Namely, if the discrete group G is amenable, is it the case that the Bohr compactification of G coincides with the "new" invariant G^*/G^{*000}_M?

It would be natural to make a definability of types assumption in (i), but the question makes sense without it, and a positive answer was given in [**12**] when T has NIP. But the NIP case can also be deduced from Corollary 0·4 and Theorem 0·6 (proved in the next section), bearing in mind the invariance of the relevant connected components when passing to M^{ext} which was proved in [**2**] (see also the beginning of the next section for a short discussion on M^{ext}).

6. *Strong amenability and amenability under NIP*

The main point of this section is to give a proof of Theorem 0·6, but we also take the opportunity to recall some issues on definable measures.

See earlier papers such as [**12**] for the definition of the property NIP, but if it helps the reader, it means that any definable family of definable sets has finite Vapnik-Chervonenkis dimension.

With any first order structure M, we can associate the structure M^{ext} with the same universe as M and predicates for all externally definable subsets of M. Recall that Shelah proved that if M is a model of a NIP theory, then $\text{Th}(M^{ext})$ is also NIP, it has quantifier elimination, and all complete types over M^{ext} in this theory are definable. Therefore, if G is a group definable in a model M of a NIP theory, then externally definable strong amenability of G is equivalent to definable strong amenability of G treated as a group definable in M^{ext}. On the other hand, by [**2**, theorem 3·17], we know that in a NIP theory definable amenability is preserved under passing to M^{ext} and coincides with externally definable amenability. That is why, replacing M by M^{ext}, we can assume that all types in $S_G(M)$ are definable and we can rewrite Theorem 0·6 in the following equivalent form.

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THEOREM 6·1. *Let G be a group definable in a model M of NIP theory. Assume that all types in $S_G(M)$ are definable. Then G is definably strongly amenable if and only if it is definably amenable.*

From now on, let G be a group definable in a first order structure M, and assume that all types in $S_G(M)$ are definable. Let $T = \mathrm{Th}(M)$.

First, we will prove the implication ($\leftarrow$) by showing the following:

PROPOSITION 6·2. *Assume that T is NIP and G is definably amenable. Suppose P is a definable minimal proximal G-flow. Then P is trivial, i.e., it is a single point.*

Then, after a discussion on the (compact) space $\mathcal{M}(S_G(M))$ of Borel probability measures on $S_G(M)$ and on definability of measures, we will obtain the implication ($\rightarrow$) by showing the following two facts.

PROPOSITION 6·3. *Suppose T is NIP. Then $\mathcal{M}(S_G(M))$ is a definable G-flow.*

PROPOSITION 6·4. *Suppose that the natural action of G on $\mathcal{M}(S_G(M))$ is definable (i.e. $\mathcal{M}(S_G(M))$ is a definable G-flow). Assume G is definably strongly amenable. Then G is definably amenable.*

Proof of Proposition 6·2. We will use recent work of Chernikov and Simon [**3**], and we also refer to Glasner's book [**11**].

First we recall the Baire-generic compact domination theorem from [**3**]. Since G is definably amenable, there is a strongly f-generic global type $q \in S_G(\mathfrak{C})$ (according to the terminology from [**3**]) which was proved in [**12**]. Let C be the closure of the orbit G^*q in $S_G(\mathfrak{C})$, and let $\pi : S_G(\mathfrak{C}) \to G^*/G^{*00}$ be the obvious continuous map. By $\pi|_C : C \to G^*/G^{*00}$ we denote the restriction of π to C. For a clopen subset X of $S_G(\mathfrak{C})$ define

$$E_X := \{h \in G^*/G^{*00} : (\pi|_C)^{-1}(h) \bigcap X \neq \varnothing \text{ and } (\pi|_C)^{-1}(h) \bigcap X^c \neq \varnothing\}.$$

[**3**, theorem 5·2] tells us that E_X is closed and has empty interior. Now, we want to deduce a counterpart of this working in our non-saturated model M. Let $r : S_G(\mathfrak{C}) \to S_G(M)$ be the restriction map. Then $r[C]$ is a G-subflow of the flow $(G, S_G(M))$. Choose a minimal subflow $\mathcal{M}$ of $S_G(M)$ which is contained in $r[C]$. Let now $\theta : \mathcal{M} \to G^*/G^{*00}$ be the canonical, continuous, surjective homomorphism (it was defined on $u\mathcal{M}$ in Section 3, but the same definition works on $\mathcal{M}$). Then $\theta \circ (r|_{r^{-1}(\mathcal{M})}) = \pi|_{r^{-1}(\mathcal{M})}$. Thus, by the fact that E_X has empty interior for any clopen $X \subseteq S_G(\mathfrak{C})$, we easily get the following claim. (Closedness follows from the continuity of θ.)

Claim 1. For every clopen subset X of $\mathcal{M}$ (i.e. given by a formula over M) the set $\{g \in G^*/G^{*00} : \theta^{-1}(g)$ meets both X and $X^c\}$ is closed and has empty interior.

Let $x_0 \in P$. By Fact 1·10 and the comments below it, we know that $S_G(M)$ acts on P in the sense that $ex = x$ and $(p * q)x = p(qx)$ for all $x \in X$ and $p, q \in S_G(M)$. Let $\phi : S_G(M) \to P$ be defined by $\phi(p) = px_0$. It is a continuous function, and $\phi|_{\mathcal{M}}$ is onto P. Let us write ϕ' for the restriction of ϕ to $\mathcal{M}$.

Suppose for a contradiction that P is non-trivial.

Claim 2. We can write $\mathcal{M}$ as the union of finitely many clopen subsets $V_1, \ldots, V_n$, such that for each i, $\phi'(V_i)$ is a proper subset of P.

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Proof of Claim 2. Easy using that P is non-trivial, ϕ' is continuous and surjective, and $\mathcal{M}$ is a Stone space. Namely, first write P as the union of finitely many proper open subsets $U_1, \ldots, U_k$. Let V_i' be $\phi'^{-1}(U_i)$. Then the V_i' form a covering of $\mathcal{M}$. Each V_i is open so a union of clopen subsets $V_{i,j}'$. By compactness, a finite union of some of the $V_{i,j}'$ cover $\mathcal{M}$ and this is the required covering.

Applying Claim 1 to all the V_i from Claim 2, we obtain $g \in G^*/G^{*00}$ and i (without loss $i = 0$) such that $\theta^{-1}(g)$ is contained in V_0. Let $U_0 = \phi'(V_0)$. So U_0 is a proper subset of P. Let $p \in \theta^{-1}(g)$, and let $x = \phi'(p) = px_0$. Now, let $y \in P \setminus U_0$. By [**11**, proposition 3·2(2), chapter I], there is some idempotent $v \in S_G(M)$ (even in a minimal ideal of $S_G(M)$) such that $vx = y$. As v is an idempotent, $\hat{\theta}(v) = eG^{*00}$ (where $\hat{\theta}$ is the obvious extension of θ to the function from $S_G(M)$ to G^*/G^{*00}). Also, since $\mathcal{M}$ is a left ideal, $vp \in \mathcal{M}$. Therefore, $vp \in \theta^{-1}(g) \subseteq V_0$. So $\phi'(vp) \in U_0$. But $\phi'(vp) = (vp)x_0 = v(px_0) = vx = y \notin U_0$, a contradiction. So the proof has been completed.

Let us discuss now the topology on the space of measures $\mathcal{M}(S_G(M))$. A regular probability measure on $S_G(M)$ is the "same thing" as a finitely additive probability measure on the Boolean algebra $B_G(M)$ of clopens (i.e. a Keisler measure on G over M). The collection of such finitely additive measures on $B_G(M)$ is a closed subset of $[0, 1]^{B_G(M)}$ equipped with the product topology, yielding the structure of a compact Hausdorff space on $\mathcal{M}(S_G(M))$. On the other hand, $\mathcal{M}(S_G(M))$ can be identified (Riesz representation theorem) with a compact convex subset of the dual space to the space of continuous functions from $S_G(M)$ to $\mathbb{R}$ equipped with the weak $*$-topology. One can check that the two topologies coincide.

We will deduce below Proposition 6·3 from [**2**, theorem 2·7] which proves definability of measures assuming definability of types, in the NIP environment.

As in the case of groups (see Section 1·2), there are on the face of it two definitions of a definable map from a definable set to a compact space; the first, in a saturated environment, and the second in a not necessarily saturated environment. The definitions are set up to be compatible and we repeat the explanation.

Recall that M denotes an arbitrary model, and $\mathfrak{C}$ a monster model. For any set X definable in M, by X^* we denote its interpretation in $\mathfrak{C}$.

Definition 6·5. Let X by any set definable in M, and let C be a compact Hausdorff space.

(i) A map $f : X^* \to C$ is said to be definable over M if the premiage of any closed subset of C is type-definable over M. (So this precisely means that f is induced by a continuous map $\bar{f} : S_X(M) \to C$.)

(ii) A map $f : X \to C$ is said to be definable if for any two closed disjoint subsets C_1, C_2 of C, $f^{-1}[C_1]$ and $f^{-1}[C_2]$ are separated by a definable (with parameters from M of course) subset of X.

Fact 6·6. The map taking f to $f|_X$ sets up a bijection between M-definable maps from X^* to C and definable maps from X to C.

As a prequel we consider definable types. For $\phi(x, y)$ an L-formula (or even L_M-formula), let Y_ϕ be the sort of the y variable. Let $p(x) \in S_x(M)$. For p to be definable means precisely that for each $\phi(x, y)$, the map $f_{p,\phi}$ from $Y_\phi(M)$ to $\{0, 1\}$ such that $f_{p,\phi}(b) = 1$ iff $\phi(x, b) \in p$, is definable. A special case of Fact 6·6 is that if $p \in S_x(M)$ is definable, then p has a unique extension $p' \in S_x(\mathfrak{C})$ which is definable over M.

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Now for measures. Let X be an M-definable set, and μ a Keisler measure on X over M, namely $\mu \in \mathcal{M}(S_X(M))$ according to earlier identifications. To say that μ is definable needs to be understood in terms of Definition 6·5(ii), as we are in a not necessarily saturated environment. So μ is *definable* if for all $\phi(x, y)$, the map $f_{\mu,\phi} : Y_\phi(M) \to [0, 1]$ taking b to $\mu(\phi(x, b))$ is definable in the sense of Definition 6·5(ii). And Fact 6·6 implies that such a definable μ has a unique extension to a global Keisler measure μ' on X which is definable over M in the sense of Definition 6·5(i): namely for each ϕ the map from Y^*_ϕ to $[0, 1]$ taking b to $\mu'(\phi(x, b))$ is definable over M in the sense of Definition 6·5(i).

Now for definable actions. We are in the context of a group G definable in M. As was recalled in Definition 1·9, an action of G on a compact space C by homeomorphisms is said to be definable if for each $x \in C$ the map $f_x : G \to C$ defined by $f_x(g) = gx$ is definable in the sense of Definition 6·5(ii). We will be interested in proving that the action of G on the compact space $\mathcal{M}(S_G(M))$ is definable.

Proof of Proposition 6·3. [**2**, theorem 2·7] shows (using the VC-theorem) that if μ is a Keisler measure on G over M, then μ is definable in the sense of Definition 6·5(ii). We want to deduce from this that the action of G on $\mathcal{M}(S_G(M))$ is definable. Fix $\mu \in \mathcal{M}(S_G(M))$, and we have to prove that the map, which we will call f_μ, taking $g \in G$ to $g\mu$ is definable. So we fix disjoint closed subsets D_1 and D_2 of $\mathcal{M}(S_G(M))$ and we have to show that $f_\mu^{-1}[D_1]$ and $f_\mu^{-1}[D_2]$ are separated by a definable subset of G. By the description above of the topology on $\mathcal{M}(S_G(M))$, we may assume that D_1 is of the form $\{\rho \in \mathcal{M}(S_G(M)) : \rho(X_1) \in C_1$ and $\rho(X_2) \in C_2$ and $\ldots$ and $\rho(X_k) \in C_k\}$, where $X_1, \ldots, X_k$ are definable (over M) subsets of G and $C_1, \ldots, C_k$ are closed subsets of $[0, 1]$. And similarly for D_2.

Now let μ' be the unique extension of μ to a global Keisler measure which is definable over M (given by Fact 6·6 as discussed above). For $g \in G^*$, let $g\mu'|_M$ denote the restriction of the global measure $g\mu'$ to M. So $g\mu'|_M \in \mathcal{M}(S_G(M))$. Let $Y_1 = \{g \in G^* : g\mu'|_M \in D_1\}$, which is precisely $\{g \in G^* : \mu'(g^{-1}X_1) \in C_1$ and $\ldots$ and $\mu'(g^{-1}X_k) \in C_k\}$, which is type-definable over M, by definability of μ'. Similarly $Y_2 = \{g \in G^* : g\mu'|_M \in D_2\}$ is type-definable over M. As Y_1 and Y_2 are disjoint, we obtain by compactness an M-definable subset Y of G such that Y^* separates them. Then clearly Y separates $f_\mu^{-1}[D_1]$ and $f_\mu^{-1}[D_2]$ as required.

Proof of Proposition 6·4. We follow the lines of the proof that strong amenability implies amenability (see [**11**, chapter III, theorem 3·1]). Recall that definable strong amenability of G means that there is no nontrivial definable minimal proximal G-flow. We are assuming definable strong amenability of G and want to deduce definable amenability, namely the existence of a G-invariant Keisler measure on G over M. By assumption, the natural action of G on $\mathcal{M}(S_G(M))$ by homeomorphisms is definable. By [**11**, theorem 2·3, chapter III], there is a minimal G-subflow X of $\mathcal{M}(S_G(M))$ which is strongly proximal. Strong proximality means that the action of G on $\mathcal{M}(X)$ is proximal. Now, X is a definable G-flow (as a subflow of the definable G-flow $\mathcal{M}(S_G(M))$). One can show that the action of G on $\mathcal{M}(X)$ is also definable, but we will not need it. Now, X embeds as a G-flow, homeomorphically into $\mathcal{M}(X)$: map $x \in X$ to the measure concentrating on x. Hence, X is a proximal G-flow. As X is minimal and G is definably strongly amenable, X is a singleton $\{x\}$, and so x is a G-invariant measure on $S_G(M)$, yielding definable amenability of G.

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7. *A characterisation of the externally definable Bohr compactification*

Our goal here is to give a description (in the spirit of Fact 1·17) of the externally definable Bohr compactification of a given group as a certain quotient by a connected component. However, in order to do that, we will have to work in a certain subgroup (which is invariant but not necessarily type-definable) of the given group. We will also get appropriate counterparts of Theorem 0·1 and Corollary 0·4.

For the rest of this section, let $H = \langle S_{G,M}(N)(\mathfrak{C})\rangle \leqslant G^*$. So H is an N-invariant subgroup. (This group depends on the choice of N, so it should be denoted by something like H_N, but we denote it by H for simplicity.) Let H_N^{000} be the smallest N-invariant subgroup of H of bounded index. As usual, one easily shows that H_N^{000} exists and is a normal subgroup of H, and one gets an absolute bound on the index $[H : H_N^{000}]$ (using Erdős-Rado theorem). The proof of the next proposition is written in such way that it shows this, too.

PROPOSITION 7·1. $H_N^{000} = \langle a^{-1}b : \mathrm{tp}(a/N) = \mathrm{tp}(b/N) \in S_{G,M}(N)\rangle$.

Proof. Let H_0 be the right-hand side of this equality.

($\subseteq$) First, we show that H_0 is normal in H. For this, it is enough to check that for any a, a', b such that $\mathrm{tp}(a/N) = \mathrm{tp}(a'/N) \in S_{G,M}(N)$ and $\mathrm{tp}(b/N) \in S_{G,M}(N)$ one has that $b^{-1}(a^{-1}a')b \in H_0$. We can find b' realizing the unique extension of $\mathrm{tp}(b/N)$ to a complete type over N, a, a' finitely satisfiable in M. Then $b'^{-1}b \in H_0$, so it is enough to show that $b'^{-1}(a^{-1}a')b' \in H_0$. By the fact that $\mathrm{tp}(b'/N, a, a')$ is finitely satisfiable in M and $\mathrm{tp}(a/N) = \mathrm{tp}(a'/N)$, we easily get that $\mathrm{tp}(a/N, b') = \mathrm{tp}(a'/N, b')$. Hence, $\mathrm{tp}(b'^{-1}ab'/N) = \mathrm{tp}(b'^{-1}a'b'/N)$. Using the fact that $\mathrm{tp}(b'/N, a)$ and $\mathrm{tp}(a/N)$ are finitely satisfiable in M, we easily get that $\mathrm{tp}(b'^{-1}ab'/N) \in S_{G,M}(N)$. Thus, $b'^{-1}(a^{-1}a')b' = (b'^{-1}ab')^{-1}(b'^{-1}a'b') \in H_0$.

By the normality of H_0, one easily gets that $[H : H_0] \leqslant |S_{G,M}(N)|\cdot\aleph_0$. Since clearly H_0 is N-invariant, we conclude that H_0 is an N-invariant normal subgroup of H of bounded index.

($\supseteq$) Let L be any N-invariant subgroup of H of bounded index. Consider any a, b such that $\mathrm{tp}(a/N) = \mathrm{tp}(b/N) \in S_{G,M}(N)$. Then there is $c \models \mathrm{tp}(a/N)$ such that both pairs (a, c) and (b, c) can be extended to arbitrarily long infinite sequences indiscernible over N. We will show that $a^{-1}c \in L$ and $c^{-1}b \in L$, which implies $a^{-1}b \in L$, finishing the proof.

Let $(a_i)_i$ be an N-indiscernible sequence extending (a, c) of length greater that $[H : L]$. Note that all a_i's are in H. If $a^{-1}c \notin L$, then $a_i^{-1}a_j \notin L$ for all $i < j$, which contradicts the choice of the length of the sequence $(a_i)_i$.

Remark 7·2. The map $\pi : S_{G,M}(N)(\mathfrak{C}) \to H/H_N^{000}$ given by $\pi(a) = aH_N^{000}$ is onto.

Proof. It is enough to show that whenever $\mathrm{tp}(a/N), \mathrm{tp}(b/N) \in S_{G,M}(N)$, one has $abH_N^{000} \cap S_{G,M}(N)(\mathfrak{C}) \neq \varnothing$. Take a' such that $\mathrm{tp}(a'/N, b)$ is the unique extension of $\mathrm{tp}(a/N)$ to a complete type over N, b which is finitely satisfiable in M. Then $\mathrm{tp}(a'b/N) \in S_{G,M}(N)$, and, by Proposition 7·1, $(a'b)^{-1}ab = b^{-1}a'^{-1}ab \in b^{-1}H_N^{000}b = H_N^{000}$, so $a'b \in abH_N^{000}$.

Now, we introduce the logic topology on H/H_N^{000}.

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Definition 7·3. We say that $D \subseteq H/H_N^{000}$ is closed if $\pi^{-1}[D]$ is a type-definable subset of $S_{G,M}(N)(\mathfrak{C})$ (equivalently, if it is type-definable over N), where π is the map defined in Remark 7·2.

Let $H_N^{00} \leqslant H$ be the preimage of the closure of the neutral element of H/H_N^{000} by the quotient map from H to H/H_N^{000}.

PROPOSITION 7·4. *H_N^{00} is a normal subgroup of H.*

Proof. First we show that for any $a \in H$ the function $f_a : H/H_N^{000} \to H/H_N^{000}$ given by $f_a(xH_N^{000}) = axH_N^{000}$ is continuous (i.e. the group operation in H/H_N^{000} is continuous on the right). By Remark 7·2, we can choose $a' \in aH_N^{000}$ such that $\mathrm{tp}(a'/N) \in S_{G,M}(N)$. Consider any closed set $D \subseteq H/H_N^{000}$. Then

$$\pi^{-1}[f_a^{-1}[D]] = \{b \in S_{G,M}(N)(\mathfrak{C}) : (\exists b' \equiv_N b)(a'b' \in \pi^{-1}[D])\}$$

is type-definable, so $f_a^{-1}[D]$ is closed.

Since $f_{a^{-1}}$ is the inverse of f_a, we get that f_a is a homeomorphism. Similarly, one gets that inversion and conjugation by any element are homeomorphisms of H/H_N^{000}.

So, in order to finish the proof, it is enough to show that H_N^{00} is closed under multiplication. Since for any $a \in H$, f_a is a homeomorphism, we get that $\mathrm{cl}(aH_N^{000}) = a\,\mathrm{cl}(eH_N^{000})$. Now, consider any $a, b \in H_N^{00}$. We have $abH_N^{000} \in a\,\mathrm{cl}(eH_N^{000}) = \mathrm{cl}(aH_N^{000}) \subseteq \mathrm{cl}(eH_N^{000}) = H_N^{00}/H_N^{000}$, so $ab \in H_N^{00}$.

Let $\bar{\pi} : S_{G,M}(N)(\mathfrak{C}) \to H/H_N^{00}$ be the map given by $\bar{\pi}(a) = aH_N^{00}$. By Remark 7·2, $\bar{\pi}$ is onto. We introduce the logic topology on H/H_N^{00} as in Definition 7·3.

PROPOSITION 7·5.

(i) *H/H_N^{00} is a compact Hausdorff topological group.*

(ii) *H/H_N^{000} is a compact topological group (which is not necessarily Hausdorff).*

Proof. (i) Compactness of H/H_N^{00} is clear from the definition of the logic topology and the fact that $\bar{\pi}$ is onto.

Now we will show that H/H_N^{00} is Hausdorff. Consider any $aH_N^{00} \in H/H_N^{00}$ and choose $a' \in aH_N^{00} \cap S_{G,M}(N)(\mathfrak{C})$. One easily gets

$$\bar{\pi}^{-1}(aH_N^{00}) = \left\{b \in S_{G,M}(N)(\mathfrak{C}) : (\exists b' \equiv_N b)(a'^{-1}b' \in \pi^{-1}[H_N^{00}/H_N^{000}])\right\}.$$

$\pi^{-1}[H_N^{00}/H_N^{000}]$ is an N-type-definable set, so it is defined by a collection of formulas $\theta_i(x)$, $i \in I$, over N closed under conjunction. Choose also a collection of formulas $\varphi_j(x, y)$, $j \in J$, over N which defines the condition $x \equiv_N y$ and which is closed under conjunction. For any $i \in I$ and $j \in J$ consider

$$O_{a',i,j} := \left\{cH_N^{00} : \bar{\pi}^{-1}(cH_N^{00}) \subseteq \{b \in S_{G,M}(N)(\mathfrak{C}) : (\exists b')(\varphi_j(b', b) \wedge \theta_i(a'^{-1}b'))\}\right\}.$$

Clearly $aH_N^{00} \in O_{a',i,j}$, and one can check that $O_{a',i,j}$ is open in the logic topology on H/H_N^{00}.

Consider any $a_1H_N^{00} \neq a_2H_N^{00}$ in H/H_N^{00}. Take $a_1' \in a_1H_N^{00} \cap S_{G,M}(N)(\mathfrak{C})$ and $a_2' \in a_2H_N^{00} \cap S_{G,M}(N)(\mathfrak{C})$. We claim that there are $i \in I$ and $j \in J$ such that $O_{a_1',i,j} \cap O_{a_2',i,j} = \varnothing$, which of course witnesses that H/H_N^{00} is Hausdorff. If it is not the case, then, by compactness, there are $b', b'' \in S_{G,M}(N)(\mathfrak{C})$ such that $b' \equiv_N b''$ and $a_1'^{-1}b', a_2'^{-1}b'' \in \pi^{-1}[H_N^{00}/H_N^{000}]$, so $a_1'^{-1}b'b''^{-1}a_2' \in H_N^{00}$, so $a_1'H_N^{00} = a_2'H_N^{00}$, a contradiction.

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The fact that group inversion in H/H_N^{00} is continuous is very easy. Let us check that multiplication is continuous. The map $\bar{\pi}$ factors through the natural map $r : S_{G,M}(N)(\mathfrak{C}) \to S_{G,M}(N)$, i.e., there is a continuous map $p : S_{G,M}(N) \to H/H_N^{00}$ such that $\bar{\pi} = p \circ r$. Then $\bar{\pi} \times \bar{\pi} = (p \times p) \circ (r \times r)$. Denote the group operation in H/H_N^{00} by $\cdot$.

Take any closed subset $D \subseteq H/H_N^{00}$, and put $S := (\cdot \circ (\bar{\pi} \times \bar{\pi}))^{-1}[D]$. Then

$$S = \left\{(a, b) \in S_{G,M}(N)(\mathfrak{C}) \times S_{G,M}(N)(\mathfrak{C}) : (\exists a' \equiv_N a)(\exists b' \equiv_N b)(a'b' \in \bar{\pi}^{-1}[D])\right\},$$

which is of course type-definable. Thus, $(r \times r)[S]$ is closed, and so $((p \times p) \circ (r \times r))[S]$ is closed, too (as $S_{G,M}(N)$ is compact, H/H_N^{00} is Hausdorff and $p \times p$ is continuous). But the last set is exactly $\cdot^{-1}[D]$, so we are done.

(ii) is left as an exercise.

Remark 7·6.
(i) $\pi[G]$ is a dense subgroup of H/H_N^{000}.
(ii) $\bar{\pi}[G]$ is a dense subgroup of H/H_N^{00}.

Proof. It is enough to show (i). Take the induced map $h : S_{G,M}(N) \to H/H_N^{000}$. Consider any open subset U of H/H_N^{000}. Then, there is a formula $\varphi(x)$ with parameters from N such that $\varphi(G) \neq \emptyset$ and $[\varphi(x)] \bigcap S_{G,M}(N) \subseteq h^{-1}[U]$. Take $g \in \varphi(G)$. Then $gH_N^{000} \in U$.

By Proposition 7·5(i), Remark 7·6(ii), the definition of the logic topology on H/H_N^{00} and by point 2)(ii) of Lemma 1·8, we get

COROLLARY 7·7. *H/H_N^{00} is an externally definable compactification of G.*

Now, we can prove the promised description of the externally definable Bohr compactification of G.

PROPOSITION 7·8. *H/H_N^{00} is the externally definable Bohr compactification of G.*

Proof. Let C be an externally definable compactification of G; so there is an externally definable homomorphism $f : G \to C$, and Lemma 1·8 yields the externally definable map $f^* : S_{G,M}(N)(\mathfrak{C}) \to C$ given by $f^*(a) = \bigcap_{\varphi \in \mathrm{tp}(a/N)} \mathrm{cl}(f[\varphi(M)])$.

Claim. If $a, b \in S_{G,M}(N)(\mathfrak{C})$ and $aH_N^{000} = bH_N^{000}$, then $f^*(a) = f^*(b)$.

Proof of the claim. By Remark 7·1, there are some $a_1, \ldots, a_n, a_1', \ldots, a_n' \in H$ satisfying $\mathrm{tp}(a_i/N) = \mathrm{tp}(a_i'/N) \in S_{G,M}(N)$ and such that $a = b(a_1^{-1}a_1') \cdot \ldots \cdot (a_n^{-1}a_n')$. Since a, b and all a_i's and a_i''s are in the domain of f^*, Lemma 1·18 gives us $f^*(a) = f^*(b) \prod_i f^*(a_i^{-1}) f^*(a_i') = f^*(b) \prod_i f^*(a_i)^{-1} f^*(a_i) = f^*(b)$, which completes the proof of the claim.

By the claim and Remark 7·2, we get a well-defined map $\Phi : H/H_N^{000} \to C$ given by

$$\Phi(aH_N^{000}) = f^*(a')$$

for any $a' \in aH_N^{000} \bigcap S_{G,M}(N)(\mathfrak{C})$.

Let us check that Φ is a homomorphism. Consider any $a, b \in H$. Take $a', b' \in S_{G,M}(N)(\mathfrak{C})$ in the cosets aH_N^{000}, bH_N^{000}, respectively, so that $\mathrm{tp}(a'/N, b')$ is finitely satisfiable in M. Then $a'b' \in abH_N^{000} \bigcap S_{G,M}(N)(\mathfrak{C})$. By Lemma 1·18, we get $\Phi(aH_N^{000} \cdot bH_N^{000}) = \Phi(a'b'H_N^{000}) = f^*(a'b') = f^*(a')f^*(b') = \Phi(aH_N^{000})\Phi(bH_N^{000})$.

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The fact that Φ is continuous is clear, as for a closed subset D of C one has $\pi^{-1}[\Phi^{-1}[D]] = f^{*-1}[D]$ is a type-definable subset of $S_{G,M}(N)(\mathfrak{C})$.

Thus, $\ker(\Phi)$ is a closed subgroup of H/H_N^{000}, so it contains H_N^{00}/H_N^{000}. This yields a natural continuous homomorphism from H/H_N^{00} to $(H/H_N^{000})/\ker(\Phi)$ commuting with the natural homomorphisms from G. Since H/H_N^{000} is compact and C is compact Hausdorff, we have that $(H/H_N^{000})/\ker(\Phi)$ is topologically isomorphic to C via the natural isomorphism also commuting with the homomorphisms from G (note that for $g \in G$, one has $\Phi(gH_N^{000}) = f^*(g) = f(g)$). Thus, we have found a continuous homomorphism from H/H_N^{00} to C commuting with the homomorphisms from G.

The proof of Theorem 0·1 goes through, and we get the following result.

PROPOSITION 7·9. *Let $\mathcal{M}$ be a minimal ideal in $S_{G,ext}(M) = S_{G,M}(N)$ and let $u \in \mathcal{M}$ be an idempotent. Then, the function $f : u\mathcal{M} \to H/H_N^{000}$ given by $\mathrm{tp}(a/N) \mapsto aH_N^{000}$ is a well-defined epimorphism. Equip $u\mathcal{M}$ with the τ-topology. Then*:

(i) *f is continuous*;

(ii) *$H(u\mathcal{M}) \leqslant f^{-1}[H_N^{000}]$*;

(iii) *the formula $\mathrm{tp}(a/N)H(u\mathcal{M}) \mapsto aH_N^{000}$ yields a well-defined continuous epimorphism from $u\mathcal{M}/H(u\mathcal{M})$ to H/H_N^{000}.*

Thus, as in the case of Corollary 0·4, we get

COROLLARY 7·10. *Suppose G is externally definably strongly amenable (i.e., the only minimal externally definable proximal flow is the trivial one). Then the natural epimorphism $\zeta : u\mathcal{M}/H(u\mathcal{M}) \to H/H_N^{00}$ is an isomorphism, so $H_N^{000} = H_N^{00}$.*

One can also check that the proof of Theorem 0·2 goes through in our current context, yielding the obvious counterpart of this theorem.

All of this leads to various natural questions, some of which could probably be answered negatively by giving suitable counter-examples. We discuss only some of them.

Since H/H_N^{00} is the externally definable Bohr compactification of G, it does not depend on the choice of N. Is H/H_N^{000} also independent of the choice of N (recall that H depends on N)? We think that this should be true. If $N_2 \succ N_1 \succ M$ are $|M|^+$-saturated, then there is a continuous epimorphism from the quotient obtained for N_2 to the one obtained for N_1, but it is not clear whether it is injective.

There is also a question whether the externally definable Bohr compactification coincides with the definable Bohr compactification. Probably not. It would be also interesting to describe the externally definable Bohr compactification in terms of connected components of G^* (not of H). For example, G^*/G_M^{*00} as well as $\mathrm{cl}(G/G_N^{*00})$ (where $N \succ M$ is $|M|^+$-saturated) are externally definable compactifications. Is any of them the externally definable Bohr compactification? Is the natural continuous epimorphism from $\mathrm{cl}(G/G_N^{*00})$ to G^*/G_M^{*00} an isomorphism? Is $\mathrm{cl}(G/G_N^{*00})$ independent of the choice of N? There is a natural continuous epimorphism from H/H_N^{00} to $\mathrm{cl}(G/G_N^{*00})$ and also to G^*/G_M^{*00}. Is any of them an isomorphism? They are isomorphisms when all types in $S_G(M)$ are definable, as then the externally definable Bohr compactification coincides with the definable Bohr compactification.

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