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Upper l_p-estimates in vector sequence spaces, with some applications

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1. *Introduction*

In [**11**], Partington proved that if λ is a Banach sequence space with a monotone basis having the Banach–Saks property, and (X_n) is a sequence of Banach spaces each having the Banach–Saks property, then the vector sequence space $\sum_\lambda X_n$ has this same property. In addition, Partington gave an example showing that if λ and each X_n have the weak Banach–Saks property, then $\sum_\lambda X_n$ need not have the weak Banach–Saks property.

In this paper we shall show that if λ and X have the p-Banach–Saks property (or its weak version) of Johnson [**7**], or the S_p-property of Knaust and Odell [**10**], or they admit upper l_p-estimates in the stronger sense of [**3**], then the vector sequence space $\lambda(X)$ has that same property. The case of a sequence of Banach spaces will also be considered. Applications of these results are as follows.

(i) If X and λ have the hereditary Dunford–Pettis property, then the vector-valued sequence space $\lambda(X)$ has the same property. This includes the cases $\lambda = c_0, l_1$ of Cembranos [**4**, **5**].

(ii) We obtain new examples of Λ-spaces, in the sense of Carne, Cole and Gamelin [**2**]. For instance, we shall show that if X denotes the dual of any of the following spaces, Tsirelson's original space, or any of Schachermayer's spaces, then, for any $1 < p < \infty$, $l_p(X)$ is a Λ-space.

Throughout the paper, p^* denotes the conjugate number of p for $1 \leqslant p < \infty$; if $p = 1, l_{p^*}$ plays the role of c_0. We shall use λ to denote a Banach sequence space having a monotone Schauder basis (e_k), i.e. $\|\sum_k a_k e_k\| \leqslant \|\sum_k b_k e_k\|$ when $|a_k| \leqslant |b_k|$ for every k. If (X_n) is a sequence of Banach spaces, the vector sequence space $\sum_\lambda X_n$ is the space of sequences (x_n), with $x_n \in X_n$ for each n, such that $\sum_n \|x_n\| e_n$ is in λ. The norm of $\sum_\lambda X_n$ is defined by $\|(x_n)\| = \|\sum_n \|x_n\| e_n\|_\lambda$. If $X_n = X$ for all n, then we simply write $\lambda(X)$.

A sequence (x_n) in a Banach space X is said to be a Banach–Saks sequence if it has norm convergent arithmetic means, i.e. if the sequence $(x_1 + \ldots + x_n)/n$ converges. A Banach space X is said to have the Banach–Saks property when each bounded sequence admits a Banach–Saks sub-sequence.

The definition of the (weak) p-Banach–Saks property is due to Johnson [**7**]. A sequence (x_n) in a Banach space X is said to be p-Banach–Saks, for $1 < p < \infty$, if for some constant C and all $n \in \mathbb{N}$, $\|\sum_{k=1}^n x_k\| \leqslant C.n^{1/p}$. We shall say that a sequence (x_n) is p-Banach–Saks convergent to $x \in X$ if the sequence $(x_n - x)$ is p-Banach–Saks. A Banach space X is said to have the p-Banach–Saks property, for $1 < p < \infty$, if a constant $C > 0$ exists such that every sequence in the unit ball admits a p-

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Banach–Saks convergent sub-sequence satisfying the above inequality for that C. It is said to have the weak p-Banach–Saks property if the above property of extraction of sub-sequences holds for weakly null sequences.

Another strengthening of the Banach–Saks property is the following: a sequence (x_n) in a Banach space X is said to be weakly p-summable $(1 \leqslant p < \infty)$ if there is a constant $C > 0$ such that $\sup_n \|\sum_{k=1}^n \alpha_k x_k\| \leqslant C . \|(\alpha_n)\|_{p^*}$ for any sequence $(\alpha_n) \in l_{p^*}$. We shall denote by $w_p\{(x_n)\}$ the infimum of those constants C. We shall say that the sequence (x_n) is weakly p-convergent to $x \in X$ if the sequence $(x_n - x)$ is weakly p-summable. We say that a Banach space X has property W_p, or that $X \in W_p$, if every bounded sequence admits a weakly-p-convergent sub-sequence. The weak version of W_p is the property S_{p^*} defined in [**10**]: every weakly null sequence has a sub-sequence satisfying a C-upper l_p-estimate for some $C > 0$. Therefore properties W_p and S_{p^*} are equivalent in reflexive spaces.

It is obvious that W_p implies p^*-Banach–Saks, and that p^*-Banach–Saks implies Banach–Saks. We have proved in [**3**] that if X has the p-Banach–Saks property then, for every $r > p^*$, it has the W_r property (see Remark 2 below).

According to Knaust and Odell [**10**], given a Banach space X in W_p, there is a constant, which we shall denote by $c_p(X)$, such that any normalized weakly null sequence (x_n) admits a weakly p-summable sub-sequence $(x_{n(k)})$ satisfying $w_p\{(x_{n(k)})\} \leqslant c_p(X)$.

Our main result in its simplest form is

THEOREM 1. *Let* $1 < p < +\infty$. *If* $X, \lambda \in W_p$, *then* $\lambda(X) \in W_p$.

2. *Proof of Theorem* 1

Step 1. Let $1 \leqslant s \leqslant +\infty$. If X, Y are Banach spaces belonging to W_p, then Knaust and Odell's uniform constant clearly satisfies $c_p(\sum_s X \times Y) \leqslant (c_p(X)^s + c_p(Y)^s)^{1/s}$. We shall show that $c_p(X \times Y)$ is equal to $\max\{c_p(X), c_p(Y)\}$ whenever $X \times Y$ is endowed with a norm induced by a monotone norm F in $\mathbb{R}^2$, in the sense that

(i) $\|(x, y)\| = F(\|x\|, \|y\|)$, and

(ii) $F(a_1, a_2) \leqslant F(b_1, b_2)$ if $|a_i| \leqslant |b_i|$ for $i = 1, 2$.

To verify this, let us take a normalized weakly null sequence (f_n, g_n) in $X \times Y$ and define

$$r_1 = \liminf \|f_n\|.$$

Without loss of generality one can assume $r_1 > 0$. We shall consider a sub-sequence $(f_k)_{k \in J}$ such that $\|f_k\| \to r_1$. Similarly, define

$$r_2 = \liminf\{\|g_n\| : n \in J\}$$

where it is again assumed that $r_2 > 0$. Extract a sub-sequence $(f_{n(k)}, g_{n(k)})$ such that

$$\|f_{n(k)}\| - 2^{-k} r_1 \leqslant r_1 \leqslant \|f_{n(k)}\| + 2^{-k} \quad \text{and} \quad \|g_{n(k)}\| - 2^{-k} r_2 \leqslant r_2 \leqslant \|g_{n(k)}\| + 2^{-k}.$$

Notice that

$$F(r_1, r_2) \leqslant F(\|f_{n(k)}\| + 2^{-k}, \|g_{n(k)}\| + 2^{-k}) \leqslant F(\|f_{n(k)}\|, \|g_{n(k)}\|) + F(2^{-k}, 2^{-k})$$

which implies $F(r_1, r_2) \leqslant 1$.

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Thus, for any sequence (ξ_k) in the unit ball of l_{p^*},

$$\begin{aligned}\left\|\sum_k \xi_k (f_{n(k)}, g_{n(k)})\right\| &= \left\|\left(\sum_k \xi_k f_{n(k)}, \sum_k \xi_k g_{n(k)}\right)\right\| = F\left(\left\|\sum_k \xi_k f_{n(k)}\right\|, \left\|\sum_k \xi_k g_{n(k)}\right\|\right) \\ &\leqslant F((1+2^{-k})\, r_1\, c_p(X), (1+2^{-k})\, r_2\, c_p(Y)) \\ &\leqslant (1+2^{-k})\, F(r_1, r_2) \max\{c_p(X), c_p(Y)\} \\ &\leqslant (1+2^{-k}) \max\{c_p(X), c_p(Y)\},\end{aligned}$$

from which our assertion follows.

Step 2. Let (f_n) be a normalized weakly null sequence in $\lambda(X)$. Since X has property W_p, by a standard diagonalization procedure one can obtain a sub-sequence such that, for all $k \in \mathbb{N}$, the sequence $\{f_n(k)\}_n$ is weakly-p-summable in X.

Applying the idea of Step 1, and then diagonalizing, one can obtain another sub-sequence, which we still denote (f_n), such that, if Δ is a finite subset of $\mathbb{N}$ with $m = \min(\Delta)$, then

$$\left\| \sum_{k \leqslant m} \left\| \sum_{n \in \Delta} \xi_n f_n(k) \right\| e_k \right\|_\lambda \leqslant c_p(X)$$

for any sequence (ξ_n) in the unit ball of l_{p^*}.

Notice that if the sequence (f_n) is replaced by the sequence (g_n) defined as

$$\begin{aligned} g_n(k) &= f_n(k) \quad \text{for} \quad k \leqslant n, \\ g_n(k) &= 0 \qquad \text{for} \quad k > n, \end{aligned}$$

then, if $M = \max(\Delta)$, $\left\| \sum_{k \leqslant M} \left\| \sum_{n \in \Delta} \xi_n g_n(k) \right\| e_k \right\|_\lambda \leqslant c_p(X)$,

whence it follows that $\left\| \sum_{k \leqslant n} \left\| \sum_n \xi_n f_n(k) \right\| e_k \right\|_\lambda \leqslant c_p(X)$.

Step 3. Let $\epsilon > 0$. First determine inductively a sub-sequence (f_N). Assume that the term f_N has been chosen, and find an index $F(N)$ such that

$$\left\| \sum_{k=F(N)}^{\infty} \|f_N(k)\|\, e_k \right\|_\lambda \leqslant \epsilon . 2^{-N},$$

and set $f_{N+1} = f_{F(N)}$.

Pick norm one elements $x^*(k) \in X^*$, such that

$$\langle x^*(k), f_N(k) \rangle = \|f_N(k)\|, \quad F(N-1) \leqslant k < F(N),$$

and form the operator $\phi : \lambda(X) \to \lambda$, defined by $\phi(f) = (\langle x^*(k), f(k) \rangle)_{k \in \mathbb{N}} \in \lambda$.

The image of (f_N) is weakly null, and thus (f_N) admits a sub-sequence, which we shall call (f_n), whose image is weakly-p-summable. This sub-sequence can be chosen such that $w_p\{(\phi(f_n))\} \leqslant c_p(\lambda)$.

We shall now show that (f_n) is weakly-p-summable in $\lambda(X)$:

$$\begin{aligned}\left\| \sum_{n=1}^{\infty} \xi_n f_n \right\|_{\lambda(X)} &= \left\| \sum_k \left\| \sum_{n=1}^{\infty} \xi_n f_n(k) \right\| e_k \right\|_\lambda \\ &\leqslant \left\| \sum_{k \leqslant n} \left\| \sum_{n=1}^{\infty} \xi_n f_n(k) \right\| e_k \right\|_\lambda + \left\| \sum_{k > n} \left\| \sum_{n=1}^{\infty} \xi_n f_n(k) \right\| e_k \right\|_\lambda\end{aligned}$$

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$$\leqslant c_p(X)+\left\|\sum_{m=0}^{\infty}\sum_{F(m)\leqslant k<F(m+1)}\left\|\xi_{m+1}f_{m+1}(k)\right\|e_k\right\|_\lambda$$

$$+\left\|\sum_{m=0}^{\infty}\sum_{F(m)\leqslant k<F(m+1)}\left\|\sum_{n=1}^{m}\xi_n f_n(k)\right\|e_k\right\|_\lambda$$

$$\leqslant c_p(X)+\left\|\sum_{m=0}^{\infty}\sum_{F(m)\leqslant k<F(m+1)}\left\|\xi_{m+1}f_{m+1}(k)\right\|e_k\right\|_\lambda+\epsilon$$

$$=c_p(X)+\left\|\sum_{m=0}^{\infty}\sum_{F(m)\leqslant k<F(m+1)}|\xi_{m+1}||\langle x^*(k),f_{m+1}(k)\rangle|\,e_k\right\|_\lambda+\epsilon$$

$$\leqslant c_p(X)+\left\|\sum_k|\langle x^*(k),\sum_n\xi_n f_n(k)\rangle|\,e_k\right\|_\lambda$$

$$+\left\|\sum_{m=0}^{\infty}\sum_{F(m)\leqslant k<F(m+1)}\left|\left\langle x^*(k),\sum_{n=m+2}^{\infty}\xi_n f_n(k)\right\rangle\right|e_k\right\|_\lambda+2\epsilon$$

$$\leqslant c_p(X)+\left\|\sum_{n=1}^{\infty}\xi_n\phi(f_n)\right\|_\lambda+c_p(X)+2\epsilon\leqslant 2c_p(X)+c_p(\lambda)+2\epsilon.$$ ∎

Comment. We obtained $c_p(\lambda(X))\leqslant 2c_p(X)+c_p(\lambda)$. A little more care with the calculations shows that $c_p(\lambda(X))\leqslant c_p(X)+c_p(\lambda)$.

Remark 1. Recall that if $1<s<+\infty$ then $l_s\in W_{s^*}$. Therefore if $\lambda=l_s$ we should take $p\geqslant s^*$.

Remark 2. The corresponding result for the p-Banach–Saks property is a little worse, and it relies on one of the main results of [3]: if any bounded sequence admits a p^*-Banach–Saks convergent sub-sequence, then any bounded sequence admits a sub-sequence which is weakly-r-convergent for all $r>p^*$. This result should first be used to prove that X and λ are in W_r for all $r>p^*$. Then continue with Steps 1, 2, 3.

Remark 3. Theorem 1 remains true for an arbitrary sequence (X_n) of Banach spaces if and only if the sequence of their uniform constants $c_r(X_n)$ is bounded for all $r>p^*$: if $c_r(X_n)\to+\infty$, then $\Sigma_p X_n$ cannot be in W_r, since in that case

$$c_r(\Sigma_p X_n)\geqslant c_r(X_n).$$

Remark 4. With the limitations noted in Remark 3, if we only assume that λ and each X_n have the S_p property of Knaust (or the original p-Banach–Saks property of Johnson, or the formally weaker version of this in [6]), then $\Sigma_\lambda X_n$ has that same property.

Remark 5. The proof of Partington in [11] cannot be adapted to cover this case: if (g_n) is a sequence p-Banach–Saks convergent to g, then the arithmetic means of g_n converge to g, but not the pth-arithmetic means. This prevents the utilization of the sequence of norms $(\|g_n(k)\|)$.

Remark 6. Theorem 1 for $L_p(X)$ is false. Following [1] it is not difficult to see that space E_2 of Schachermayer has property W_2, while $L_2(E_2)$ does not even have the Banach–Saks property.

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3. *Applications*

The hereditary Dunford–Pettis property. A Banach space X is said to have the Dunford–Pettis property if weakly compact operators defined on X are compact, or, equivalently, it for arbitrary weakly null sequences (x_n) and (x_n^*) in X and X^* respectively, $\lim \langle x_n^*, x_n \rangle = 0$. A Banach space X is said to have the hereditary Dunford–Pettis property if each closed subspace of X has the Dunford–Pettis property.

No infinite-dimensional Banach space can have property W_1. Let us consider the non-reflexive version: we shall say that a Banach space X has property weak W_1 if every weakly null sequence admits a weakly-1-summable sub-sequence. It is not difficult to prove that property weak W_1 is equivalent to the hereditary Dunford–Pettis property and, therefore, to property S of Knaust and Odell [**9**].

Using the main result of [**9**], i.e. the existence of a uniform constant for Banach spaces having the property weak W_1, one can reproduce the proof of Theorem 1 to obtain that if X and λ have property weak W_1, then $\lambda(X)$ has property weak W_1.

THEOREM 2. *If X and λ have the hereditary Dunford–Pettis property, then $\lambda(X)$ has the hereditary Dunford–Pettis property.*

This includes the cases $\lambda = c_0, l_1$ of Cembranos [**4**, **5**].

Λ-spaces. Let X be a (real or complex) Banach space. For $m = 1, 2, \ldots$, a map P is said to be a continuous m-homogeneous polynomial if it has the form

$$P(x) = A(x, x, \ldots, x),$$

where A is a continuous m-linear scalar valued form on X^m. A continuous polynomial is a finite sum $P = P_0 + P_1 + \ldots + P_n$, where P_0 is constant and each P_m is a continuous m-homogeneous polynomial. A sequence $(x_n) \subseteq X$ is said to be weak-polynomial convergent to x if $P(x_n) \to P(x)$ for all continuous polynomials on X. In [**2**], Carne, Cole, and Gamelin defined a Λ-space as a Banach space such that every sequence weak-polynomial convergent to zero is norm null, and conjectured that, for $1 < p < \infty$, L_p-spaces are Λ-spaces, proving themselves the conjecture for $p \geqslant 2$. In [**8**], Jaramillo and Prieto proved, using results of [**3**], that super-reflexive spaces are Λ-spaces. Usually, it is not easy to prove that a Banach space is a Λ-space. For this reason it is worth mentioning that the proof of Jaramillo and Prieto works without change for Banach spaces X such that, for some $p, X^* \in W_p$. This observation, together with Theorem 1, allows one to prove that several interesting Banach spaces are Λ-spaces. For instance, if T denotes Tsirelson's original space, and E_q denotes Schachermayer's spaces [**1**, **12**], $1 < q < +\infty$, then, for every $1 < p < \infty$, it follows that $l_p(T^*)$, and $l_p(E_q)^*)$ are Λ-spaces. It is only necessary to verify the following more or less well-known points: that $T \in W_p$ for any $p > 1$, that, as we have already observed, $E_p \in W_p$; and, which is very easy to see, that $X \in W_p$ if and only if $X^{**} \in W_p$. Moreover, if X is a Banach space in W_p, and λ a Banach sequence space having a monotone Schauder basis in W_q, then $\lambda(X)^*$ is a Λ-space.

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