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ESTIMATES OF GREEN'S FUNCTION FOR SECOND-ORDER PARABOLIC EQUATIONS NEAR EDGES

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To Professor V. G. Maz'ya on the occasion of his 75th anniversary

Abstract. We consider the first boundary value problem for a second-order parabolic equation with variable coefficients in the domain $K \times \mathbb{R}^{n-m}$, where K is an m-dimensional cone. The main results of the paper are pointwise estimates of the Green's function.

§1. *Introduction.* This paper deals with the Green's function of the first boundary value problem for a second-order parabolic equation in the domain

$$\mathcal{D} = K \times \mathbb{R}^{n-m}, \quad n \geqslant m \geqslant 2, \tag{1.1}$$

where

$$K = \{x' \in \mathbb{R}^m : \omega = x'/|x'| \in \Omega\}$$

is a cone in $\mathbb{R}^m$ with vertex at the origin. Here Ω is a subdomain of the unit sphere S^{m-1} with boundary $\partial\Omega$ of the class $C^{1,1}$.

The theory of elliptic boundary value problems in domains with angular or conical points and edges is well developed in a large number of publications over the last 50 years. In particular, Maz'ya made important contributions to the development of this theory. Much fewer papers have been published about parabolic problems in piecewise smooth domains. One of the first papers studying the behavior of solutions of the heat equation near conical points was that of Kozlov and Maz'ya [7] in 1987. Initial boundary value problems for general parabolic equations with variable coefficients (depending on the space variable x) in a cone K were studied in [4–6]. In particular, the paper [6] describes the asymptotics of the Green's function for such problems. Concerning parabolic equations in domains with edges, we mention the papers by Solonnikov [14, 15], where the Dirichlet and Neumann problems for the heat equation in a dihedral angle were studied. Here, by means of pointwise estimates for the Green's function, the author proved estimates of the solution in weighted Hölder and L_p Sobolev spaces. Nazarov [13] estimated weighted L_p Sobolev norms for solutions of the Dirichlet and Neumann problems for the heat equation in the domain (1.1), where K is a cone of arbitrary dimension. The asymptotics of solutions of the Dirichlet problem for the heat

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equation near the edge $M = \{0\} \times \mathbb{R}^{n-m}$ of the domain $\mathcal{D}$ is described in our papers [10, 11].

In the present paper, we consider the problem

$$\partial_t u - \mathcal{A}(x, t, \partial_x)u = f \quad \text{in } \mathcal{D} \times \mathbb{R}, \quad u = 0 \text{ on } \partial\mathcal{D} \times \mathbb{R}, \tag{1.2}$$

where

$$\mathcal{A}(x, t, \partial_x) = \sum_{i,j=1}^{n} A_{ij}(x, t)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{n} A_j(x, t)\partial_{x_j} + A(x, t)$$

is a differential operator with variable coefficients depending on both the space variable x and the time t. Throughout the paper, we assume that

$$|A_{ij}(x, t) - \delta_{ij}| \leqslant \epsilon, \tag{1.3}$$

where δ_{ij} denotes the Kronecker symbol and ϵ is a small positive number. Furthermore, we always assume that

$$A_{ij}, A_j, A \in C^{\gamma,\gamma/2}(\overline{\mathcal{D}} \times \mathbb{R}) \tag{1.4}$$

for some $\gamma \in (0, 1)$. Here $C^{\gamma,\gamma/2}(\overline{\mathcal{D}} \times \mathbb{R})$ denotes the set of all continuous and bounded functions on $\overline{\mathcal{D}} \times \mathbb{R}$ with finite half-norm $\| \cdot \|^{\gamma,\gamma/2}$ (see the formula (2.1) below).

The goal of the paper is to prove pointwise estimates for the Green's function $G(x, y, t, s)$ of the problem (1.2). To this end, we derive estimates for the x-derivatives up to order two of solutions of the equation

$$\mathcal{L}u := (\partial_t - \mathcal{A}(x, t, \partial_x))u = 0 \tag{1.5}$$

in a certain subdomain $Q_R^{\mathcal{D}}(x, t)$ of $\mathcal{D}$. Such estimates are given in §2. These estimates are used in §3 (see Theorem 3.2) for the proof of x- and y-derivatives up to order two of the Green's function $G(x, y, t, s)$. Here we suppose that both the coefficients of the operator $\mathcal{A}$ and the coefficients of the adjoint operator of $\mathcal{A}$ satisfy the condition (1.4).

Under stronger assumptions on the coefficients, it is possible to improve the estimates of Theorem 3.2 for x''- and t-derivatives of the Green's function, where $x'' = (x_{m+1}, \dots, x_n)$ is the variable on the edge M. In §4, we prove estimates for x''- and t-derivatives of solutions of the equation (1.5) in the above-mentioned subdomain $Q_R^{\mathcal{D}}(x, t)$. By means of these estimates, we improve and generalize the result of Theorem 3.2 (see Theorem 4.4 at the end of the paper).

§2. *Estimates of solutions of parabolic equations.* In this section, we assume that the coefficients of $\mathcal{L}$ satisfy the conditions (1.3) and (1.4).

2.1. *Local estimates of solutions.* First, we introduce the following notation. For an arbitrary point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we put $x' = (x_1, \dots, x_m)$ and $x'' = (x_{m+1}, \dots, x_n)$. The same notation is used for multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$. If $x = (x', x'') \in \mathcal{D}$, then $r = |x'|$ denotes the distance from the edge and $d(x)$ denotes the distance from the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. Let $x_0 = (x_0', x_0'')$ be an

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arbitrary point in $\mathbb{R}^n$ and let t_0 be an arbitrary real number. Then

$$B_R^n(x_0) = \{x \in \mathbb{R}^n : |x' - x_0'| < R,\ |x'' - x_0''| < R\},$$
$$Q_R(x_0, t_0) = B_R^n(x_0) \times (t_0 - R^2, t_0],$$
$$Q_R^{\mathcal{D}}(x_0, t_0) = \{(x, t) \in Q_R(x_0, t_0) : x \in \mathcal{D}\}.$$

For a cylinder $Q = \Omega \times (t_1, t_2)$ in $\mathbb{R}^n \times \mathbb{R}$, we denote by

$$\partial' Q = (\partial\Omega \times (t_1, t_2)) \cup (\overline{\Omega} \times \{t_1\})$$

its parabolic boundary. Furthermore, we introduce the half-norm

$$\|u\|_Q^{\gamma,\gamma/2} = \sup_{(x,t),(y,t)\in Q} \frac{|u(x,t) - u(y,t)|}{|x - y|^\gamma} + \sup_{(x,t),(x,\tau)\in Q} \frac{|u(x,t) - u(x,\tau)|}{|t - \tau|^{\gamma/2}}. \tag{2.1}$$

By $Du = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$, we denote the gradient of u, while $D^2 u$ denotes the vector of all second-order derivatives $\partial_{x_i}\partial_{x_j} u$. The space $C^{2,1}(Q_R(x,t))$ is defined as the set of all continuous functions u in $Q_R(x,t)$ having continuous derivatives $\partial_t u$, $\partial_{x_i} u$ and $\partial_{x_i}\partial_{x_j} u$.

The next statement can be found (even for more general equations) in [12, Ch. IV].

LEMMA 2.1. *Let R_0 be a fixed positive number. Suppose that $u \in C^{2,1}(Q_R(x,t))$ solves the equation $\mathcal{L}u = f$ in $Q_R(x,t)$ for a certain $R < \min(R_0, d(x))$ and that $f \in L_{p,\infty}(Q_R(x,t))$ for a certain $p > n/2$. Then*

$$|Du(x,t)| \leqslant C\left(R^{1-n/p}\|f\|_{L_{p,\infty}(Q_R(x,t))} + R^{-1}\sup_{Q_R(x,t)} |u|\right),$$

where C may depend on p. If $f \in C^{\gamma,\gamma/2}(Q_R(x,t))$, then the estimate

$$R^\gamma \|D^2 u\|_{Q_{\kappa R}(x,t)}^{\gamma,\gamma/2} + |D^2 u(x,t)| \leqslant C\left(\sup_{Q_R(x,t)} |f| + R^\gamma \|f\|_{Q_R(x,t)}^{\gamma,\gamma/2} + R^{-2}\sup_{Q_R(x,t)} |u|\right) \tag{2.2}$$

holds, where $\kappa \in (0,1)$ and the constant C may depend on R_0.

It is obvious that $d(x)$ depends only on x' and is homogeneous of degree 1. Since $\partial K \setminus \{0\}$ is of the class $C^{1,1}$, the distance $d(x)$ is a $C^{1,1}$ function for $d(x') \leqslant \delta|x'|$ if δ is sufficiently small.

LEMMA 2.2. *Let R_0 be a fixed positive number and let $d(x) < \delta|x'|$, where δ is a positive number less than 1 depending on Ω. Furthermore, let R be a positive number $< \min(R_0, \delta|x'|)$ and let $u \in C^{2,1}(Q_R^{\mathcal{D}}(x,t)) \cap C(\overline{Q_R^{\mathcal{D}}(x,t)})$ be a solution of the problem*

$$\mathcal{L}u = f \quad \text{in } Q_R^{\mathcal{D}}(x,t), \qquad u(y,\tau) = 0 \quad \text{for } (y,\tau) \in \overline{Q_R^{\mathcal{D}}(x,t)}, \quad y \in \partial\mathcal{D}. \tag{2.3}$$

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If $f \in L_{p,\infty}(Q_R(x,t))$ *for a certain* $p > n/2$, *then*

$$|Du(x,t)| \leqslant C\left(R^{1-n/p}\|f\|_{L_{p,\infty}(Q_R^{\mathcal{D}}(x,t))} + R^{-1}\sup_{Q_R^{\mathcal{D}}(x,t)}|u|\right). \tag{2.4}$$

If $f \in C^{\gamma,\gamma/2}(Q_\rho(x,t)) \cap L_p(Q_R^{\mathcal{D}}(x,t))$, $\min(\gamma, 1-\gamma) > (n+2)/p$ *and* $2\rho = \min(R, d(x))$, *then*

$$\rho^\gamma \|D^2u\|^{\gamma,\gamma/2}_{Q_{\kappa\rho}(x,t)} + |D^2u(x,t)| \leqslant C\left(\rho^\gamma\|f\|^{\gamma,\gamma/2}_{Q_\rho(x,t)} + \rho^{-\gamma}R^{\gamma-2}\left(R^{2-(n+2)/p}\|f\|_{L_p(Q_R^{\mathcal{D}}(x,t))} + \sup_{Q_R^{\mathcal{D}}(x,t)}|u|\right)\right), \tag{2.5}$$

where κ *is a positive number less than 1. The constant* C *may depend on* R_0, γ *and* κ.

Proof. Estimate (2.4) can be found in [**12**, Ch. III, §12]. Let us turn to the proof of (2.5). If $R \leqslant c_0 d(x)$, where c_0 is a positive constant, then $R\min(1, c_0^{-1}) \leqslant 2\rho \leqslant R$ and the estimate (2.5) follows directly from (2.2) and from the inequality

$$|f(x,t)| \leqslant \rho^{-(n+2)/p}\|f\|_{L_p(Q_\rho(x,t))} + R^\gamma \|f\|^{\gamma,\gamma/2}_{Q_\rho(x,t)}. \tag{2.6}$$

We assume that $R > c_0 d(x)$, where $c_0 > 1$. Then $\rho = d(x)/2$. By (2.3),

$$(\partial_s - \mathcal{A}(X,s,\partial_X))u(X,s) = f(X,s) \quad \text{for } (X,s) \in Q_R^{\mathcal{D}}(x,t).$$

Let $r = |x'|$. Then the coordinate transformation $y = r^{-1}X$, $\tau = r^{-2}(s-t)$ yields

$$(\partial_\tau - \mathcal{A}_r)v(y,\tau) = F(y,\tau) \quad \text{in } Q_{R/r}^{\mathcal{D}}(x/r, 0), \tag{2.7}$$

where $v(y,\tau) = u(X,s) = u(ry, t+r^2\tau)$, $F(y,\tau) = r^2 f(ry, t+r^2\tau)$ and

$$\mathcal{A}_r = \sum_{i,j=1}^n A_{ij}(ry, t+r^2\tau)\partial_{y_i}\partial_{y_j} + r\sum_{j=1}^n A_j(ry, t+r^2\tau)\partial_{y_j} + r^2 A(ry, t+r^2\tau).$$

With the notation $\tilde{x} = x/r$ and $\tilde{\rho} = \rho/r = d(\tilde{x})/2$, the estimate (2.5) takes the form

$$\tilde{\rho}^\gamma \|D^2v\|^{\gamma,\gamma/2}_{Q_{\kappa\tilde{\rho}}(\tilde{x},0)} + |D^2v(\tilde{x},0)| \leqslant C\left(\tilde{\rho}^\gamma\|F\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\tilde{x},0)} + \tilde{\rho}^{-\gamma}\left(\frac{R}{r}\right)^{\gamma-2}\left(\left(\frac{R}{r}\right)^{2-(n+2)/p}\|F\|_{L_p(Q_{R/r}^{\mathcal{D}}(\tilde{x},0))} + \sup_{Q_{R/r}^{\mathcal{D}}(\tilde{x},0)}|v|\right)\right). \tag{2.8}$$

By our assumptions, both R/r and $d(\tilde{x})$ are less than δ. Near a point $y_0 \in \partial\mathcal{D}$ with $|y_0'| = 1$, the domain can be represented as

$$\{z = (z', z_n) \in \mathbb{R}^n : z_n > \varphi(z')\} \tag{2.9}$$

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in a certain orthogonal system of coordinates z with the center at y_0. Here φ is a $C^{1,1}$ function such that $\varphi(0) = 0$ and $\nabla\phi(0) = 0$. We introduce the function

$$\Phi(\xi) = \int_{\mathbb{R}^{n-1}} h_\mu(\theta)\varphi(\theta\xi_n + \xi')\,d\theta,$$

where $h_\mu(\theta) = \mu^{1-n}h(\theta/\mu)$, $h(\theta) = c\exp((|\theta|^2 - 1)^{-1})$ for $|\theta| \leqslant 1$, $h(\theta) = 0$ for $|\theta| > 1$ and μ is a sufficiently small positive number. The constant c is chosen such that $\|h\|_{L_1} = 1$. Clearly, h and h_μ are non-negative C^∞-functions on $\mathbb{R}^{n-1}$.

We consider the mapping $\mathbb{R}^n_+ \ni \xi \to z \in G$ which is defined by

$$z' = \xi', \qquad z_n = \xi_n + \Phi(\xi). \tag{2.10}$$

If μ is sufficiently small, then

$$|\partial_{\xi_n}\Phi(\xi)| = |\partial_{\xi_n} z_n - 1| \leqslant \tfrac{1}{2}.$$

This implies that the mapping (2.10) is a bi-Lipschitz isomorphism from the half-space $\mathbb{R}^n_+$ onto the set (2.9). Furthermore,

$$|z_n - \xi_n - \varphi(\xi')| = |\Phi(\xi', \xi_n) - \Phi(\xi', 0)| \leqslant \xi_n/2.$$

The last inequality implies that

$$\frac{|\xi|^2}{4\Lambda^2 + 4} \leqslant |z|^2 \leqslant (2\Lambda^2 + 3)|\xi|^2,$$

where Λ is the Lipschitz constant of φ.

The derivatives with respect to the variables z_j satisfy the relations

$$\partial_{z_n} = \frac{1}{1 + \partial_{\xi_n}\Phi}\partial_{\xi_n} \quad \text{and} \quad \partial_{z_j} = \partial_{\xi_j} - \frac{\partial_{\xi_j}\Phi}{1 + \partial_{\xi_n}\Phi}\,\partial_{\xi_n}$$

for $j = 1, \ldots, n-1$. Since $\varphi \in C^{1,1}$, it follows that $\partial^\alpha_\xi \Phi$ is bounded for $|\alpha| \leqslant 2$ and

$$|\partial^\alpha_\xi \Phi(\xi)| \leqslant c\xi_n^{2-|\alpha|}$$

for $|\alpha| \geqslant 2$. Hence, the operator $\mathcal{A}_r$ in the new coordinates ξ takes the form

$$\mathcal{M} = L^{ij}(\xi, \tau)\partial_{\xi_i}\partial_{\xi_j} + L^j(\xi, \tau)\partial_{\xi_j} + L(\xi, \tau)$$

with coefficients L^{ij}, $L \in C^{\gamma, \gamma/2}$. Furthermore, the coefficients L^j are bounded and satisfy the estimate

$$|\partial^\alpha_\xi L^j| \leqslant C\xi_n^{-|\alpha|}$$

for every multi-index α. Let $\mathcal{M}_0$ denote the principal part of the differential operator $\mathcal{M}$. Then (2.7) yields

$$(\partial_\tau - \mathcal{M}_0)v = F_1$$

near the origin in $\mathbb{R}^n_+$, where

$$F_1 = F + (\mathcal{M} - \mathcal{M}_0)v = F + F_0.$$

Furthermore, $v(\xi, \tau) = 0$ for $\xi_n = 0$. Since $\tilde\rho = d(\tilde x)/2 < \delta/2$ and δ is sufficiently small, we may assume that $\xi_n/4 < \tilde\rho < 3\xi_n/4$. Then (2.2)

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implies that

$$\tilde{\rho}^{\gamma}\|D^2v\|^{\gamma,\gamma/2}_{Q_{\kappa\tilde{\rho}}(\xi,0)}+|D^2v(\xi,0)| \\ \leqslant C\left(\tilde{\rho}^{\gamma}\|F_1\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}+\sup_{Q_{\tilde{\rho}}(\xi,0)}|F_1|+\tilde{\rho}^{-2}\sup_{Q_{\tilde{\rho}}(\xi,0)}|v|\right). \tag{2.11}$$

The same estimate holds for the function $V(\xi,\tau)=v(\xi,\tau)-\xi_n(\partial_{\xi_n}v)(\xi_0,\tau_0)$, where (ξ_0,τ_0) is a point in $Q_{\xi_n}(\xi,0)$. Since $V(\xi,\tau)=0$ for $\xi_n=0$, we have

$$\sup_{Q_{\tilde{\rho}}(\xi,0)}|V|\leqslant 2\xi_n\sup_{Q_{\xi_n}(\xi,0)}|\partial_{\xi_n}V|=2\xi_n\sup_{Q_{\xi_n}(\xi,0)}|\partial_{\xi_n}v-(\partial_{\xi_n}v)(\xi_0,\tau_0)|.$$

Hence, the inequality (2.11) for the function V yields

$$\tilde{\rho}^{\gamma}\|D^2v\|^{\gamma,\gamma/2}_{Q_{\kappa\tilde{\rho}}(\xi,0)}+|D^2v(\xi,0)| \\ \leqslant C\left(\tilde{\rho}^{\gamma}\|F_1\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}+\sup_{Q_{\tilde{\rho}}(\xi,0)}|F_1| \right. \\ \left. +\tilde{\rho}^{-2}\xi_n^{1+\sigma}\sup_{(\zeta_1,t_1),(\zeta_2,t_2)\in Q_{\xi_n}(\xi,0)}\frac{|(\partial_{\xi_n}v)(\zeta_1,t_1)-(\partial_{\xi_n}v)(\zeta_2,t_2)|}{|\zeta_1-\zeta_2|^{\sigma}+|t_1-t_2|^{\sigma/2}}\right), \tag{2.12}$$

where σ is an arbitrary real number in $(0,1)$. Using (2.12) together with the inequalities

$$\sup_{Q_{\tilde{\rho}}(\xi,0)}|F_0|\leqslant C\sup_{Q_{\tilde{\rho}}(\xi,0)}(|Dv|+|v|)$$

and

$$\|F_0\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}\leqslant C\left(\|Dv\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}+\xi_n^{-\gamma}\sup_{Q_{\tilde{\rho}}(\xi,0)}|Dv|+\sup_{Q_{\tilde{\rho}}(\xi,0)}|v|\right),$$

we obtain

$$\tilde{\rho}^{\gamma}\|D^2v\|^{\gamma,\gamma/2}_{Q_{\kappa\tilde{\rho}}(\xi,0)}+|D^2u(\xi,0)| \\ \leqslant C\left(\tilde{\rho}^{\gamma}\|F\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}+\sup_{Q_{\tilde{\rho}}(\xi,0)}|F| \right. \\ +\tilde{\rho}^{\sigma-1}\sup_{(\zeta_1,t_1),(\zeta_2,t_2)\in Q_{\xi_n}(\xi,0)}\frac{|(\partial_{\xi_n}v)(\zeta_1,t_1)-(\partial_{\xi_n}v)(\zeta_2,t_2)|}{|\zeta_1-\zeta_2|^{\sigma}+|t_1-t_2|^{\sigma/2}} \\ \left.+\tilde{\rho}^{\gamma}\|Dv\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}+\sup_{Q_{\tilde{\rho}}(\xi,0)}|Dv|+\tilde{\rho}^{\gamma}\sup_{Q_{\tilde{\rho}}(\xi,0)}|v|\right). \tag{2.13}$$

We estimate the terms on the right-hand side of (2.13). By (2.6),

$$\sup_{Q_{\tilde{\rho}}(\xi,0)}|F|\leqslant c(\tilde{\rho}^{\gamma}\|F\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}+\tilde{\rho}^{-(n+2)/p}\|F\|_{L_p(Q_{\tilde{\rho}}(\xi,0))}) \\ \leqslant c\left(\tilde{\rho}^{\gamma}\|F\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}+\tilde{\rho}^{-\gamma}\left(\frac{R}{r}\right)^{\gamma-(n+2)/p}\|F\|_{L_p(Q_{R/r}(\xi,0))}\right),$$

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since $\gamma > (n+2)/p$ and $\tilde{\rho} < R/r$. Let $\tilde{R} = R/r$, $w(\xi,\tau) = v(\tilde{R}\xi, \tilde{R}^2\tau)$ and $H(\xi,\tau) = \tilde{R}^2 F(\tilde{R}\xi, \tilde{R}^2\tau)$. From the equation $(\partial_\tau - \mathcal{M}(\xi,\tau,\partial_\xi))v = F$, it follows that

$$\partial_\tau w - \tilde{R}^2 \mathcal{M}(\tilde{R}\xi, \tilde{R}^2\tau, \tilde{R}^{-1}\partial_\xi) w(\xi,\tau) = H(\xi,\tau).$$

Furthermore, for sufficiently small $\tilde{R} = R/r < \delta$,

$$\begin{aligned}
&\sup_{(\zeta_i,t_i)\in Q_{\xi_n}(\xi,0)} \frac{|(\partial_{\xi_n} v)(\zeta_1,t_1) - (\partial_{\xi_n} v)(\zeta_2,t_2)|}{|\zeta_1-\zeta_2|^\sigma + |t_1-t_2|^{\sigma/2}} \\
&\quad\leqslant \sup_{(\zeta_i,t_i)\in Q^+_{R/r}(\xi,0)} \frac{|(\partial_{\xi_n} v)(\zeta_1,t_1) - (\partial_{\xi_n} v)(\zeta_2,t_2)|}{|\zeta_1-\zeta_2|^\sigma + |t_1-t_2|^{\sigma/2}} \\
&\quad= \tilde{R}^{-1-\sigma} \sup_{(\zeta_i,t_i)\in Q_1(\xi/\tilde{R},0)} \frac{|(\partial_{\xi_n} w)(\zeta_1,t_1) - \partial_{\xi_n} w)(\zeta_2,t_2)|}{|\zeta_1-\zeta_2|^\sigma + |t_1-t_2|^{\sigma/2}},
\end{aligned}$$

where $Q^+_R(x,0)$ is the set of all $(\xi,\tau) \in B_R(x) \times (-R^2, 0]$ such that $\xi_n > 0$. Using the inequalities

$$\|Dw\|_{C^{\sigma,\sigma/2}} = \|Dw\|_{B^{\sigma,\sigma/2}_{\infty,\infty}} \leqslant c\|w\|_{B^{2,1}_{p,\infty}}$$

for $1-\sigma > (n+2)/p$ (see formula (2) in [2, § 18.1] and [2, Theorem 18.4]) and $\|w\|_{B^{2,1}_{p,\infty}} \leqslant c\|w\|_{W^{2,1}_p}$ (see [2, Theorem 18.9]), we get

$$\sup_{(\zeta_i,t_i)\in Q_{\xi_n}(\xi,0)} \frac{|(\partial_{\xi_n} v)(\zeta_1,t_1) - (\partial_{\xi_n} v)(\zeta_2,t_2)|}{|\zeta_1-\zeta_2|^\sigma + |t_1-t_2|^{\sigma/2}} \leqslant c\tilde{R}^{-1-\sigma}\|w\|_{W^{2,1}_p(Q^+_1(\xi/\tilde{R},0))}.$$

Applying [9, Theorem 5.1], we obtain

$$\begin{aligned}
\|w\|_{W^{2,1}_p(Q^+_1(\xi/\tilde{R},0))} &\leqslant c\left(\|H\|_{L_p(Q^+_1(\xi/\tilde{R},0)} + \sup_{Q^+_1(\xi/\tilde{R},0)} |w|\right) \\
&= c\left(\tilde{R}^{2-(n+2)/p}\|F\|_{L_p(Q^+_{R/r}(\xi,0))} + \sup_{Q^+_{R/r}(\xi,0)} |v|\right).
\end{aligned}$$

Thus, for $\sigma = 1-\gamma$, the inequality

$$\begin{aligned}
&\tilde{\rho}^{\sigma-1} \sup_{(\zeta_i,t_i)\in Q_{\xi_n}(\xi,0)} \frac{|(\partial_{\xi_n} v)(\zeta_1,t_1) - (\partial_{\xi_n} v)(\zeta_2,t_2)|}{|\zeta_1-\zeta_2|^\sigma + |t_1-t_2|^{\sigma/2}} \\
&\quad\leqslant c\tilde{\rho}^{-\gamma}\tilde{R}^{\gamma-2}\left(\tilde{R}^{2-(n+2)/p}\|F\|_{L_p(Q^+_{R/r}(\xi,0))} + \sup_{Q^+_{R/r}(\xi,0)} |v|\right) \qquad (2.14)
\end{aligned}$$

holds. Analogously, the terms $\tilde{\rho}^\gamma \|Dv\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)}$ and $\sup|Dv|$ can be estimated by the right-hand side of (2.14) if $1-\gamma > (n+2)/p$. Hence, (2.13) implies

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the estimate

$$\tilde{\rho}^{\gamma}\|D^2v\|^{\gamma,\gamma/2}_{Q_{\kappa\tilde{\rho}}(\xi,0)} + |D^2u(\xi,0)| \leqslant C\Bigg(\tilde{\rho}^{\gamma}\|F\|^{\gamma,\gamma/2}_{Q_{\tilde{\rho}}(\xi,0)} + \tilde{\rho}^{-\gamma}\tilde{R}^{\gamma-2}\Bigg(\tilde{R}^{2-(n+2)/p}\|F\|_{L_p(Q^+_{R/r}(\xi,0))} + \sup_{Q^+_{R/r}(\xi,0)}|v|\Bigg)\Bigg),$$

which is equivalent to (2.8). This proves the proposition. □

As a consequence of the last proposition, we get the estimates

$$|Du(x,t)| \leqslant CR^{-1}\sup_{Q^{\mathcal{D}}_R(x,t)}|u|$$

and

$$|D^2u(x,t)| \leqslant CR^{-2}\left(\frac{d(x')}{|x'|}\right)^{-\varepsilon}\sup_{Q^{\mathcal{D}}_R(x,t)}|u|$$

for $d(x) < \delta|x'|$, $R < \min(R_0, \delta|x'|)$ and for functions $u \in C^{2,1}(Q^{\mathcal{D}}_R(x,t)) \cap C(\overline{Q^{\mathcal{D}}_R(x,t)})$ satisfying the equations

$$\mathcal{L}u = 0 \quad \text{in } Q^{\mathcal{D}}_R(x,t), \quad u(y,\tau) = 0 \quad \text{for } (y,\tau) \in \overline{Q^{\mathcal{D}}_R(x,t)}, \quad y \in \partial\mathcal{D}. \tag{2.15}$$

Here ε is an arbitrarily small positive number.

2.2. *Estimates near the edge.* We construct a barrier function. Let $\Omega_0 = \Omega$, while, for small positive a, the domain Ω_a is defined as the set $\{\omega \in S^{m-1} : \operatorname{dist}(\omega', \Omega) < a\}$. By Δ_ω, we denote the Laplace–Beltrami operator on the unit sphere S^{m-1}. Furthermore, let Λ_a and Φ_a be the first eigenvalue and corresponding positive in Ω_a eigenfunction of the operator $-\Delta_\omega$ on Ω_a with the Dirichlet boundary condition. We normalize the function Φ_a by $\sup_{\Omega_a}\Phi_a = 1$. Then, as is known,

$$\frac{\partial\Phi_0}{\partial\nu} \geqslant \alpha_0 > 0 \quad \text{on } \partial\Omega,$$

where ν is the inner unit normal to $\partial\Omega$. Moreover, by the Hadamard formula (see [3]), we have

$$\Lambda_a = \Lambda_0 - \beta a + o(a), \quad \beta > 0.$$

Furthermore, using a local change of variables, local estimates and the estimate $\|\Phi_a - \Phi_0\|_{L_2(S^{m-1})} \leqslant ca$, we get

$$\frac{\partial\Phi_a}{\partial\nu} \geqslant \alpha_0 - \alpha_1 a \quad \text{on } \partial\Omega$$

with a certain positive constant α_1 independent of a. From this, it follows that

$$\inf_{\Omega}\Phi_a(\omega) \geqslant \alpha_2 a$$

with some positive α_2 independent of a.

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LEMMA 2.3. *Let R_0 be a positive number and let $u \in C^{2,1}(Q_R^{\mathcal{D}}(0,x'',t)) \cap C(\overline{Q_R^{\mathcal{D}}(0,x'',t)})$ be a solution of the equations*

$$\mathcal{L}u = 0 \quad \text{in } Q_R^{\mathcal{D}}(0,x'',t), \quad u(y,t) = 0 \quad \text{for } (y,\tau) \in Q_R(0,x'',t),\ y \in \partial\mathcal{D}. \tag{2.16}$$

Then the estimate

$$|u(x',x'',t)| \leqslant C\left(\frac{|x'|}{R}\right)^{\lambda_0 - c_1\sqrt{\epsilon}} \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u| \quad \text{for } x' \in K, \quad |x'| < R \tag{2.17}$$

holds for $R \leqslant R_0$, where

$$\lambda_0 = \tfrac{1}{2}(2 - m + \sqrt{(m-2)^2 + 4\Lambda_0}) \tag{2.18}$$

is the positive root of the equation $\lambda(\lambda+m-2) = \Lambda_0$ and c_1 is a certain positive constant independent of ϵ. The constant C may depend on ϵ and R_0.

Proof. It suffices to prove (2.17) for small $R_0 \leqslant \epsilon$. We construct a barrier function. Let $\lambda = \lambda_0 - c_1\sqrt{\epsilon}$, let l be the greatest integer less than $\lambda/2$ and let w be the function

$$w(x,t) = \sum_{k=0}^{l}\left(\frac{r}{R}\right)^{\lambda-2k}\Phi_a(\omega)\left(\frac{|x''|^2 - t}{s^2R^2}\right)^k + \left(\frac{|x''|^2 - t}{s^2R^2}\right)^{l+1}, \tag{2.19}$$

where $a = \sqrt{\epsilon}$. Obviously, $w(x,t) \geqslant 0$ for $t \leqslant 0$. We show that w satisfies the inequality $\mathcal{L}w \geqslant 0$ on the set

$$Q_R^{\mathcal{D}}(0,0) = \{(x,t) \in \mathcal{D} \times \mathbb{R} :\ |x'| < R,\ |x''| < R,\ -R^2 < t < 0\}$$

if s and c_1 are sufficiently large. One can easily check that

$$\begin{aligned}(\partial_t - \Delta_{x''})&\left(\frac{|x''|^2 - t}{s^2R^2}\right)^k \\ &= -\frac{k}{s^2R^2}\left(\frac{4(k-1)|x''|^2}{|x''|^2 - t} + 2n - 2m + 1\right)\left(\frac{|x''|^2 - t}{s^2R^2}\right)^{k-1}\end{aligned}$$

and

$$-\Delta_{x'}\left(\frac{r}{R}\right)^{\lambda-2k}\Phi_a(\omega) = \frac{b + 2k(2\lambda - 2k + m - 2)}{R^2}\left(\frac{r}{R}\right)^{\lambda-2k-2}\Phi_a(\omega),$$

where

$$b = \Lambda_a - \lambda(\lambda + m - 2) = \Lambda_a - \Lambda_0 + c_1 a(2\lambda_0 + m - 2) - c_1^2 a^2.$$

Thus,

$$(\partial_t - \Delta_x)w = F_1 - F_2 - F_3,$$

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where

$$F_1 = -\Delta_{x'} w = \sum_{k=0}^{l} \frac{b + 2k(2\lambda - 2k + m - 2)}{R^2} \left(\frac{r}{R}\right)^{\lambda-2k-2} \Phi_a(\omega) \left(\frac{|x''|^2 - t}{s^2 R^2}\right)^k,$$

$$F_2 = -(\partial_t - \Delta_{x''}) \left(\frac{|x''|^2 - t}{s^2 R^2}\right)^{l+1}$$

$$= \frac{l+1}{s^2 R^2} \left(\frac{4l|x''|^2}{|x''|^2 - t} + 2n - 2m + 1\right) \left(\frac{|x''|^2 - t}{s^2 R^2}\right)^l$$

and

$$F_3 = \sum_{k=0}^{l-1} \frac{k+1}{s^2 R^2} \left(\frac{4k|x''|^2}{|x''|^2 - t} + 2n - 2m + 1\right) \left(\frac{r}{R}\right)^{\lambda-2k-2} \Phi_a(\omega) \left(\frac{|x''|^2 - t}{s^2 R^2}\right)^k.$$

Obviously, $F_1 > 0$ and $2(F_2 + F_3) < F_1$ if $r \leqslant R$, $t \leqslant 0$,

$$\frac{k+1}{s^2}(4k + 2n - 2m + 1) \leqslant \frac{b + 2k(2\lambda - 2k + m - 2)}{2} \quad \text{for } k = 0, 1, \ldots, l-1 \tag{2.20}$$

and

$$\frac{l+1}{s^2}(4l + 2n - 2m + 1) \leqslant \frac{b + 2l(2\lambda - 2l + m - 2)}{2} \alpha_2 a. \tag{2.21}$$

Since $b = -\beta a + c_1 a(2\lambda_0 + m - 2) + o(a)$ and $2\lambda_0 + m - 2 > 1$ (see [**8**, Sections 2.1 and 2.2]), we can choose c_1 such that $b > c_1 a/2$. Moreover, $2\lambda - 2l + m - 2 > 0$. Consequently, the number $s \geqslant 1$ can be chosen such that (2.20) and (2.21) are satisfied. Furthermore, using (1.3), we get the estimate

$$|(\Delta - \mathcal{A})w(x,t)| \leqslant \frac{ca^2}{R^2} \sum_{k=0}^{l} \left(\frac{r}{R}\right)^{\lambda-2k-2} \left(\frac{|x''|^2 - t}{s^2 R^2}\right)^k$$

for $(x,t) \in \mathcal{Q}_R^{\mathcal{D}}(0,0)$, $R < \epsilon = a^2$, where c depends only on the coefficients A_j and A of $\mathcal{A}$. Consequently, $2|(\Delta - \mathcal{A})w(x,t)| \leqslant F_1$ if $2ca^2 \leqslant b \inf_\Omega \Phi_a$. The last inequality is satisfied, in particular, under the following condition on the number c_1:

$$2b > c_1 a \quad \text{and} \quad \alpha_2 c_1 \geqslant 4c. \tag{2.22}$$

Thus, it is shown that w satisfies the inequality $\partial_t w - \mathcal{A}(x,t,\partial_x)w \geqslant 0$ on $\mathcal{Q}_R^{\mathcal{D}}(0,0)$ if s and c_1 satisfy (2.20)–(2.22). Furthermore, $w \geqslant 0$ for $t < 0$ and

$$w(x,t) \geqslant c_2 a^{l+1} \quad \text{for all } (x,t) \in \overline{\mathcal{Q}_R^{\mathcal{D}}(0,0)}$$
$$\text{such that } |x'| = R \text{ or } |x''| = R \text{ or } t = -R^2.$$

Here c_2 is a certain positive constant. Now let $u \in C^{2,1}(\mathcal{Q}_R^{\mathcal{D}}(0,0) \cap C(\overline{\mathcal{Q}_R^{\mathcal{D}}(0,0)})$ be a solution of the equations

$$\partial_t u - \mathcal{A}(x,t,\partial_x)u = 0 \quad \text{in } \mathcal{Q}_R^{\mathcal{D}}(0,0), \quad u(x,t) = 0 \quad \text{for } x \in \partial\mathcal{D},$$

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where R satisfies (2.22). We assume that the number q is an upper bound for the coefficient $A(x,t)$ of the operator $\mathcal{A}$ and define

$$v_{1,2}(x,t) = \mathrm{e}^{-qt}\left(u(x,t) \pm \frac{M}{c_2 a^{l+1}} w(x,t)\right), \quad \text{where } M = \sup_{Q_R^{\mathcal{D}}(0,0)} |u|.$$

Then $v_1(x,t) \geqslant 0 \geqslant v_2(x,t)$ for $(x,t) \in \partial' Q_R^{\mathcal{D}}(0,0)$, where $\partial' Q_R^{\mathcal{D}}(0,0)$ denotes the parabolic boundary of $Q_R^{\mathcal{D}}(0,0)$, i.e. that part of the boundary of $Q_R^{\mathcal{D}}(0,0)$ on which $x \in \partial\mathcal{D}$, or $|x'| = R$, or $|x''| = R$ or $t = -R^2$. Furthermore,

$$\partial_t v_1 - (\mathcal{A} - q)v_1 \geqslant 0 \geqslant \partial_t v_2 - (\mathcal{A} - q)v_2 \quad \text{in } Q_R^{\mathcal{D}}(0,0).$$

Thus, by the maximum principle, $v_1 \geqslant 0 \geqslant v_2$ in $Q_R^{\mathcal{D}}(0,0)$. This means that

$$|u(x,t)| \leqslant \frac{M}{c_2 a^{l+1}} w(x,t) \quad \text{for } (x,t) \in Q_R^{\mathcal{D}}(0,0).$$

In particular, it follows that

$$|u(x',0,0)| \leqslant \frac{M}{c_2 a^{l+1}} w(x',0,0) \leqslant \frac{1}{c_2 a^{l+1}} \left(\frac{r}{R}\right)^{\lambda_0 - c_1 a} M \quad \text{for } x' \in K, \quad r < R.$$

Analogously, any solution $u \in C^{2,1}(Q_R^{\mathcal{D}}(0,x'',t)) \cap C(\overline{Q_R^{\mathcal{D}}(0,x'',t)})$ of (2.16) satisfies the estimate

$$|u(x',x'',t)| \leqslant \frac{1}{c_2 a^{l+1}} \left(\frac{r}{R}\right)^{\lambda_0 - c_1 a} \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u| \quad \text{for } x' \in K, \quad r < R$$

if $R \leqslant R_0$ and R_0 is defined by (2.22). This implies (2.17) for $R \leqslant R_0$, where R_0 is an arbitrary constant and the constant C depends on R_0 and ϵ. The lemma is proved. □

In the sequel, λ_0 and c_1 are the same numbers as in the last lemma, i.e. the number λ_0 is given by (2.18) and c_1 is a positive constant independent of ϵ which guarantees that the inequalities (2.22) are valid. Using Lemma 2.3 together with Lemmas 2.1 and 2.2, we can also estimate the derivatives of u in a neighborhood of the edge.

LEMMA 2.4. *Suppose that $0 < \kappa < 1$ and $R < R_0$, where R_0 is a fixed positive number, and that $u \in C^{2,1}(Q_R^{\mathcal{D}}(0,x'',t)) \cap C(\overline{Q_R^{\mathcal{D}}(0,x'',t)})$ satisfies (2.16). Furthermore, let $\lambda \leqslant \lambda_0 - c_1\sqrt{\epsilon}$. Then*

$$|\partial_x^\alpha u(x,t)| \leqslant C R^{-|\alpha|} \left(\frac{|x'|}{R}\right)^{\lambda - |\alpha|} \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u| \tag{2.23}$$

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for $|\alpha| \leqslant 1$ *and* $|x'| < \kappa R$. *Furthermore,*

$$|D^2 u(x,t)| \leqslant C R^{-2} \left(\frac{|x'|}{R}\right)^{\lambda-2} \left(\frac{d(x)}{|x'|}\right)^{-\varepsilon} \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u| \tag{2.24}$$

for $|x'| < \kappa R$, *where* ε *is an arbitrary positive number. The constant* C *may depend on* λ, ε, R_0 *and* ϵ.

Proof. If $|\alpha| = 0$, then (2.23) follows from (2.17). Let $|\alpha| = 1$. From Lemmas 2.1 and 2.2, it follows that

$$|\partial_x^\alpha u(x,t)| \leqslant \frac{C}{|x'|} \sup_{\Pi_s} |u(z,\tau)|,$$

where $s \in (0,1)$ and Π_s denotes the set of all (z,τ) such that $z \in \mathcal{D}$, $s|x'| < |z'| < s^{-1}|x'|$, $|z'' - x''| \leqslant s^{-1}|x'|$ and $t - s^{-2}|x'|^2 < \tau < t$. (Note that $\Pi_s \supset Q_R(x,t)$ for $R = (1-s)|x'|$.) Estimating the right-hand side of the last inequality by means of (2.17), we get

$$|Du(x,t)| \leqslant \frac{C}{|x'|} \left(\frac{|x'|}{R}\right)^{\lambda} \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u|$$

for $|x'| < \kappa R$. This proves (2.23) for $|\alpha| = 1$.

Now let $|\alpha| = 2$. Using (2.2) and (2.5), we obtain

$$|D^2 u(x,t)| \leqslant C d(x)^{-\gamma} |x'|^{\gamma-2} \sup_{\Pi_s} |u|$$

with arbitrary positive γ. Combining the last estimate with (2.23), we obtain (2.24). □

§3. *Estimates of Green's function.* In this section, we assume that the coefficients of $\mathcal{A}$ satisfy the conditions (1.3) and

$$\partial_x^\alpha A_{ij} \in C^{\gamma,\gamma/2}(\overline{\mathcal{D}} \times \mathbb{R}) \quad \text{for } |\alpha| \leqslant 2 \quad \text{and} \quad A, A_j, \partial_{x_i} A_j \in C^{\gamma,\gamma/2}(\overline{\mathcal{D}} \times \mathbb{R}) \tag{3.1}$$

with some $\gamma \in (0,1)$.

Let $\Gamma(x,y,t,s)$ be the Green's function of the equation $\mathcal{L}u = f$ in the whole space (we suppose that the coefficients of $\mathcal{A}$ are extended in a suitable way to $\mathbb{R}^n \times \mathbb{R}$). Then the solution of this equation is given by the formula

$$u(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} \Gamma(x,y,t,s) f(y,s)\, dy\, ds.$$

According to [**1**], the function Γ admits the estimate

$$\Gamma(x,y,t,s) \leqslant \frac{C}{(t-s)^{n/2}} \exp\left(-\frac{\sigma|x-y|^2}{t-s}\right), \tag{3.2}$$

where σ is a positive constant.

Departing from Γ, we can construct the Green's function for the Dirichlet problem in $\mathcal{D} \times \mathbb{R}$. For all $(y,s) \in \mathcal{D} \times \mathbb{R}$, we denote by $G(x,y,t,s)$ the solution

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of the Dirichlet problem

$$\mathcal{L}u = \delta(x-y)\delta(t-s) \quad \text{in } \mathcal{D}\times\mathbb{R}, \quad u|_{x\in\partial\mathcal{D}} = 0,\ u|_{t<s} = 0.$$

By the maximum principle, the Green's function G is positive and satisfies the inequality

$$G(x,y,t,s) \leqslant \Gamma(x,y,t,s) \quad \text{for all } x,y\in\mathcal{D} \text{ and } s<t. \tag{3.3}$$

Similarly, one can construct the Green's function $\widehat{G}(x,y,t,s)$ for the operator

$$\widehat{\mathcal{L}} = \partial_t - \mathcal{A}_*,$$

where

$$\mathcal{A}_*(x,t,\partial_x)u = \sum_{i,j=1}^{n} \partial_{x_i}\partial_{x_j}(A_{ij}(x,-t)u(x,t)) - \sum_{j=1}^{n} \partial_{x_j}(A_j(x,-t)u(x,t)) + A(x,-t)u.$$

The function $u(x,t) = \widehat{G}(x,y,t,s)$ satisfies the equations

$$\widehat{\mathcal{L}}u = \delta(x-y)\delta(t-s) \quad \text{in } \mathcal{D}\times\mathbb{R}, \quad u|_{x\in\partial\mathcal{D}} = 0,\ u|_{t<s} = 0.$$

By changing of variables $t\to -t$ and $s\to -s$, we conclude that the function $v(x,t) = \widehat{G}(x,y,-t,-s)$ solves the problem

$$\mathcal{L}^*v = \delta(x-y)\delta(t-s) \quad \text{in } \mathcal{D}\times\mathbb{R}, \quad v|_{x\in\partial\mathcal{D}} = 0,\ v|_{t>s} = 0,$$

where $\mathcal{L}^* = -\partial_t - \mathcal{A}^*(x,t,\partial_x)$ and $\mathcal{A}^*$ is the formally adjoint operator to $\mathcal{A}$.

Using Green's formula for the functions $u(x,t) = G(x,y,t,s_1)$ and $v(x,t) = \widehat{G}(x,\eta,-t,-s_2)$ with $s_1 < t < s_2$, we get

$$v(y,s_1) = \int_{\mathbb{R}}\int_{\mathcal{D}} \mathcal{L}u\cdot v\,dx\,dt = \int_{\mathbb{R}}\int_{\mathcal{D}} u\cdot\mathcal{L}^*v\,dx\,dt = u(\eta,s_2),$$

which means that

$$G(y,x,-s,-t) = \widehat{G}(x,y,t,s). \tag{3.4}$$

In the following, we will use the notation

$$\mathcal{R}_x = \frac{|x'|}{|x'|+\sqrt{t-s}}, \qquad \mathcal{R}_y = \frac{|y'|}{|y'|+\sqrt{t-s}}.$$

Furthermore, let λ_0 and c_1 be the same numbers as in Lemma 2.3.

LEMMA 3.1. *Let T be a positive constant and let $\lambda \leqslant \lambda_0 - c_1\sqrt{\epsilon}$. Then*

$$|G(x,y,t,s)| \leqslant C(t-s)^{-n/2}\mathcal{R}_x^{\lambda}\mathcal{R}_y^{\lambda}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right) \tag{3.5}$$

for $x,y\in\mathcal{D}$ and $0<t-s<T$. Here σ_1 is a positive constant. The constant C may depend on ϵ and T.

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Proof. First, we prove the inequality

$$G(x, y, t, s) \leqslant C\mathcal{R}_x^{\lambda}(t-s)^{-n/2}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right). \tag{3.6}$$

If $|x'|^2 \geqslant (t-s)/4$, then $1/3 \leqslant \mathcal{R}_x \leqslant 1$ and (3.6) follows directly from (3.3) and (3.2).

Suppose that $|x'|^2 < (t-s)/4$. Applying (2.23) with $R = \sqrt{(t-s)/2}$ and $\alpha = 0$ to the function $u(x, t) = G(x, y, t, s)$, we obtain

$$G(x, y, t, s) \leqslant c\left(\frac{|x'|}{\sqrt{t-s}}\right)^{\lambda} \sup_{(z,\tau)\in Q_R^{\mathcal{D}}(0,x'',t)} G(z, y, \tau, s). \tag{3.7}$$

For $|x'|^2 < (t-s)/4$ and $(z, \tau) \in Q_R^{\mathcal{D}}(0, x'', t)$, we have

$$(t-s)/2 < \tau - s < t - s \quad \text{and} \quad 2|z-y|^2 \geqslant |x-y|^2 - 4(t-s). \tag{3.8}$$

Consequently, by (3.3) and (3.2), the inequality

$$G(z, y, \tau, s) \leqslant c(t-s)^{-n/2}\exp\left(-\frac{\sigma|x-y|^2}{2(t-s)}\right)$$

holds for $(z, \tau) \in Q_R^{\mathcal{D}}(0, x'', t)$. This together with (3.7) implies (3.6).

Analogously, the estimate

$$\widehat{G}(x, y, t, s) \leqslant c\mathcal{R}_x^{\lambda}\,(t-s)^{-n/2}\exp\left(-\frac{\sigma|x-y|^2}{2(t-s)}\right)$$

holds. Using (3.4), we conclude that

$$G(x, y, t, s) \leqslant c\mathcal{R}_y^{\lambda}\,(t-s)^{-n/2}\exp\left(-\frac{\sigma|x-y|^2}{2(t-\tau)}\right).$$

In the case $|x'|^2 \geqslant (t-s)/4$, the last inequality is equivalent to (3.5). If $|x'|^2 < (t-s)/4$, we use again the inequality (3.7) and combine this with the estimate

$$G(z, y, \tau, s) \leqslant c\mathcal{R}_y^{\lambda}\,(t-s)^{-n/2}\exp\left(-\frac{\sigma|x-y|^2}{2(t-\tau)}\right)$$

for $(z, \tau) \in Q_R^{\mathcal{D}}(0, x'', t)$. This leads to the inequality (3.5). □

Next, we estimate the derivatives of Green's function.

THEOREM 3.2. *Suppose that the coefficients of the operator $\mathcal{A}$ satisfy the conditions (1.3) and (3.1). Furthermore, let T be a positive number and let $\lambda < \lambda_0 - c_1\sqrt{\epsilon}$. Then the estimates*

$$|\partial_x^{\alpha}\partial_y^{\beta}G(x, y, t, s)| \leqslant C(t-s)^{-(n+|\alpha|+|\beta|)/2}\mathcal{R}_x^{\lambda-|\alpha|}\mathcal{R}_y^{\lambda-|\beta|} \times\left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_\alpha}\left(\frac{d(y)}{|y'|}\right)^{-\varepsilon_\beta}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right) \tag{3.9}$$

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and

$$|\partial_t \partial_y^\beta G(x, y, t, s)| \leqslant C(t-s)^{-(n+|\beta|+2)/2}\mathcal{R}_x^{\lambda-2}\mathcal{R}_y^{\lambda-|\beta|} \times\left(\frac{d(x)}{|x'|}\right)^{-\varepsilon}\left(\frac{d(y)}{|y'|}\right)^{-\varepsilon_\beta}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right) \quad (3.10)$$

hold for $x, y \in \mathcal{D}$, $0 < t-s < T$, $|\alpha| \leqslant 2$ *and* $|\beta| \leqslant 2$. *Here* σ_1 *is a positive constant,* $\varepsilon_\alpha = 0$ *for* $|\alpha| \leqslant 1$, *while* $\varepsilon_\alpha = \varepsilon$ *is an arbitrarily small positive number if* $|\alpha| = 2$. *The constant* C *may depend on* T, λ *and* ε.

Proof. We start with the case $|\beta| = 0$ in (3.9). If in addition $|\alpha| = 0$, then the estimate (3.9) follows from Lemma 3.1. Let $1 \leqslant |\alpha| \leqslant 2$. Then we consider three cases.

Case 1: $|x'|^2 \leqslant (t-s)/4$. Using Lemma 2.4 with $R = \sqrt{(t-s)/2}$, we obtain

$$|\partial_x^\alpha G(x, y, t, s)| \leqslant C(t-s)^{-|\alpha|/2}\mathcal{R}_x^{\lambda-|\alpha|}\left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_\alpha}\sup_{(z,\tau)\in Q_R^{\mathcal{D}}(0,x'',t)} G(z, y, \tau, s).$$

By Lemma 3.1 and by (3.8), we have

$$G(z, y, \tau, s) \leqslant C(t-s)^{-n/2}\mathcal{R}_y^{\lambda}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right) \quad (3.11)$$

for $(z, \tau) \in Q_R^{\mathcal{D}}(0, x'', t)$. Combining the last two inequalities, we get (3.9).

Case 2: $|x'|^2 \geqslant (t-s)/4$ *and* $d(x) \geqslant \delta|x'|$, *where* $\delta \in (0, 1)$. Note that $1/3 \leqslant \mathcal{R}_x \leqslant 1$ for $|x'|^2 \geqslant (t-s)/4$. We put $R = \min(\delta|x'|/2, \sqrt{(t-s)/2})$. Applying Lemma 2.1 to the function $u(x, t) = G(x, y, t, s)$, we obtain

$$|\partial_x^\alpha G(x, y, t, s)| \leqslant CR^{-|\alpha|}\sup_{(z,\tau)\in Q_R(x,t)} G(z, y, \tau, s). \quad (3.12)$$

Since $(t-s)/2 < \tau - s < t-s$ and $2|z-y|^2 \geqslant |x-y|^2 - 4R^2 \geqslant |x-y|^2 - 2(t-s)$ for $(z, \tau) \in Q_R(x, t)$, Lemma 3.1 yields

$$\sup_{(z,\tau)\in Q_R(x,t)} G(z, y, \tau, s) \leqslant C(t-s)^{-n/2}\mathcal{R}_y^{\lambda}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right).$$

The last estimate together with the obvious inequality $4R \geqslant \delta\sqrt{t-s}$ and (3.12) prove (3.9) for $|\beta| = 0$ in this case.

Case 3: $|x'|^2 \geqslant (t-s)/4$ *and* $d(x) \leqslant \delta|x'|$. Here we may assume that δ is sufficiently small. If $|\alpha| = 1$, then (2.4) yields

$$|\partial_x^\alpha G(x, y, t, s)| \leqslant \frac{C}{R}\sup_{(z,\tau)\in Q_R^{\mathcal{D}}(x,t)} |G(z, y, \tau, s)|, \quad (3.13)$$

where $R = \sqrt{(t-s)/2}$. Using the inequalities $(t-s)/2 \leqslant \tau - s \leqslant t-s$, $2|z-y|^2 \geqslant |x-y|^2 - 4R^2$ for $(z, \tau) \in Q_R^{\mathcal{D}}(x, t)$ and Lemma 3.1, we obtain

$$\sup_{(z,\tau)\in Q_R^{\mathcal{D}}(x,t)} |G(z, y, \tau, s)| \leqslant C(t-s)^{-n/2}\mathcal{R}_y^{\lambda}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right), \quad (3.14)$$

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which together with (3.13) leads to (3.9) for $|\alpha| = 1$ and $|\beta| = 0$. If $|\alpha| = 2$, then (2.5) yields

$$|\partial_x^\alpha G(x, y, t, s)| \leqslant cR^{-2}\left(\frac{R}{\rho}\right)^{\varepsilon} \sup_{(z,\tau)\in Q_R^{\mathcal{D}}(x,t)} |G(z, y, \tau, s)|, \tag{3.15}$$

where $R = \min(\delta|x'|, \sqrt{(t-s)/2})$ and $\rho = \frac{1}{2}\min(R, d(x))$. Obviously, $2R \geqslant \delta\sqrt{t-s}$ and $R/\rho \leqslant 2\delta|x'|/d(x)$. Therefore, the inequalities (3.14) and (3.15) imply (3.9) for $|\alpha| = 2$ and $|\beta| = 0$.

Thus, the inequality (3.9) is proved for $|\beta| = 0$ and $|\alpha| \leqslant 2$. Analogously, we get

$$\begin{aligned}&|\partial_x^\alpha \widehat{G}(x, y, t, s)|\\ &\quad\leqslant C(t-s)^{-(n+|\alpha|)/2}\mathcal{R}_x^{\lambda-|\alpha|}\mathcal{R}_y^{\lambda}\left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_\alpha}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right),\end{aligned}$$

which together with (3.4) implies that

$$\begin{aligned}&|\partial_y^\beta G(x, y, t, s)|\\ &\quad\leqslant C(t-s)^{-(n+|\beta|)/2}\mathcal{R}_x^{\lambda}\mathcal{R}_y^{\lambda-|\beta|}\left(\frac{d(y)}{|y'|}\right)^{-\varepsilon_\beta}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right).\end{aligned} \tag{3.16}$$

This means that (3.9) is valid for $\alpha = 0$ and $|\beta| \leqslant 2$. In the general case $|\alpha| \leqslant 2$ and $|\beta| \leqslant 2$, we repeat the above proof (when $\beta = 0$), but, instead of Lemma 3.1, we use the estimate (3.16).

Finally, the inequality (3.10) follows from (3.9), since the derivative with respect to t can be expressed by the second derivatives with respect to x. □

§4. *Estimates of x''- and t-derivatives.* The goal of this section is to improve the estimates in Theorem 3.2 for x''- and t-derivatives of Green's function. For this, we have to prove estimates for the x''- and t-derivatives of solutions of the equations (2.15) and (2.16), generalizing the results of Lemmas 2.2 and 2.4. This requires some additional conditions on the coefficients of the operator $\mathcal{A}$.

4.1. *Local estimates for x''- and t-derivatives of solutions.* First, we consider solutions of the equations $\mathcal{L}u = 0$ in $Q_R^{\mathcal{D}}(x, t)$, where $R < \delta|x'|$, $0 < \delta < 1$.

LEMMA 4.1. *Suppose that the coefficients of $\mathcal{A}$ satisfy the conditions (1.3), (3.1) and*

$$\sum_{i,j=1}^{n} |\partial_{x''}^\alpha \partial_t^k A_{ij}| + \sum_{j=1}^{n} |\partial_{x''}^\alpha \partial_t^k A_j| + |\partial_{x''}^\alpha \partial_t^k A| \leqslant C \quad \text{for } |\alpha| \leqslant l_1, \quad k \leqslant l_2. \tag{4.1}$$

Furthermore, let R_0 be a fixed positive number and let $d(x) < \delta|x'|$, where δ is a positive number less than 1 depending on Ω. If $u \in C^{2,1}(Q_R^{\mathcal{D}}(x, t)) \cap$

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$C(\overline{Q_R^{\mathcal{D}}(x,t)})$ solves (2.15) for a certain $R < \min(R_0, \delta|x'|)$, then

$$|D^j \partial_{x''}^{\alpha} \partial_t^k u(x,t)| \leqslant C R^{-j-|\alpha|-2k} \left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_j} \sup_{Q_R^{\mathcal{D}}(x,t)} |u| \quad \text{for } j \leqslant 2, \quad |\alpha| \leqslant l_1,\ k \leqslant l_2, \tag{4.2}$$

where $\varepsilon_j = 0$ for $j \leqslant 1$ and ε_j is an arbitrarily small positive number if $j = 2$. The constant C may depend on R_0 and ε_j.

Proof. We prove the lemma by induction in $l = l_1 + l_2$. Moreover, we show that

$$|D \partial_{x''}^{\alpha} \partial_t^k u(x,t)| \leqslant c R^{-1} \sum_{i=0}^{k} \sum_{|\gamma| \leqslant |\alpha|} \sup_{Q_R^{\mathcal{D}}(x,t)} |\partial_{x''}^{\gamma} \partial_t^i u| \tag{4.3}$$

and

$$\rho^{\varepsilon} \|D^2 \partial_{x''}^{\alpha} \partial_t^k u\|_{Q_{\kappa\rho}(x,t)}^{\varepsilon, \varepsilon/2} + |D^2 \partial_{x''}^{\alpha} \partial_t^k u(x,t)| \leqslant C \rho^{-\varepsilon} R^{\varepsilon - 2} \sum_{i=0}^{k} \sum_{|\gamma| \leqslant |\alpha|} \sup_{Q_R^{\mathcal{D}}(x,t)} |\partial_{x''}^{\gamma} \partial_t^i u| \tag{4.4}$$

for $2\rho = \min(R, d(x))$, $|\alpha| \leqslant l_1$ and $k \leqslant l_2$. Note that $\rho^{-1} R \leqslant c d(x)^{-1} |x'|$ under the conditions of the lemma.

For $l_1 = l_2 = 0$, the estimates (4.2)–(4.4) follow immediately from Lemma 2.2. Suppose that $|\alpha| = l_1$, $k = l_2$, $|\alpha| + k = l \geqslant 1$ and that the estimates (4.2)–(4.4) are proved for $|\alpha| + k \leqslant l - 1$. Since

$$\partial_{x''}^{\alpha} \partial_t^k u = \partial_{x''}^{\alpha} \partial_t^{k-1} \mathcal{A}(x, t, \partial_x) u \quad \text{in } Q_R^{\mathcal{D}}(x,t)$$

for $k \geqslant 1$, it follows from the induction hypothesis that

$$|\partial_{x''}^{\alpha} \partial_t^k u(x,t)| \leqslant c R^{-2k-|\alpha|} \left(\frac{d(x)}{|x'|}\right)^{-\varepsilon} \sup_{Q_R^{\mathcal{D}}(x,t)} |u| \tag{4.5}$$

(in the case $k = 0$, this estimate is even valid with $\varepsilon = 0$ by the induction hypothesis). Let $0 < \kappa < \kappa' < 1$ and let $\psi = \psi(y,s)$ be a cut-off function on $\mathcal{D} \times (-\infty, t]$ depending on the parameters x'' and t such that $\psi(y,s) = 1$ for $(y,s) \in Q_{\kappa R}^{\mathcal{D}}(x,t)$, $\psi(y,s) = 0$ for $(y,s) \notin Q_{\kappa' R}^{\mathcal{D}}(0, x'', t)$ and

$$|\partial_s^j \partial_y^{\beta} \psi(y,s)| \leqslant C_{j,\beta} R^{-2j-|\beta|}.$$

Then

$$(\partial_s - \mathcal{A}(y,s,\partial_y)) \psi(y,s) \partial_{y''}^{\alpha} \partial_s^k u(y,s) = \psi [\partial_{y''}^{\alpha} \partial_s^k, \mathcal{A}(y,s,\partial_y)] u + [\partial_s - \mathcal{A}(y,s,\partial_y), \psi] \partial_{y''}^{\alpha} \partial_s^k u = f(y,s)$$

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and $\psi\partial^{\alpha}_{y''}\partial^{k}_{s}u = 0$ on $\partial\mathcal{D}\times(-\infty, t]$. Here $[A, B] = AB - BA$ denotes the commutator of the operators A and B. Thus,

$$\partial^{\alpha}_{x''}\partial^{k}_{t}u(x,t) = \int_{Q^{\mathcal{D}}_{\kappa' R}(x,t)} G(x, y, t, s) f(y, s)\, dy\, ds.$$

The function f admits the decomposition $f = f_1 + f_2$, where

$$f_1(y,s) = -\sum_{i,j}(A_{ij}(y,s) + A_{ji}(y,s))(\partial_{y_i}\psi(y,s))\partial_{y_j}\partial^{\alpha}_{y''}\partial^{k}_{s}u(y,s),$$

while $f_2(y, s)$ contains only the derivatives $\partial^{\alpha}_{y''}\partial^{k}_{s}u$ and $\partial^{\gamma}_{y}\partial^{\delta}_{y''}\partial^{j}_{s}u$, where $|\gamma| \leqslant 2$ and $j + |\delta| \leqslant l - 1$. Applying the induction hypothesis, (4.5) and the estimate for the derivatives of ψ, we get

$$|f_2(y,s)| \leqslant cR^{-2k-|\alpha|-2}\left(\frac{d(y)}{|y'|}\right)^{-\varepsilon} \sup_{Q^{\mathcal{D}}_{R}(x,t)} |u| \tag{4.6}$$

for $(y, s) \in Q_{\kappa' R}(x, t)$. In the integral

$$\int_{Q^{\mathcal{D}}_{\kappa' R}(x,t)} G(x, y, t, s) f_1(y, s)\, dy\, ds,$$

we can integrate by parts with respect to y. Using the estimate for $\partial^{\alpha}_{y} G(x, y, t, s)$ in Theorem 3.2 and the estimate (4.5), we obtain

$$\left|\int_{Q^{\mathcal{D}}_{\kappa' R}(x,t)} G(x, y, t, s) f_1(y, s)\, dy\, ds\right| \leqslant cR^{-2k-|\alpha|-1} \sup_{Q^{\mathcal{D}}_{R}(x,t)} |u|$$
$$\times \int_{Q^{\mathcal{D}}_{\kappa' R}(x,t)} (t-s)^{-(n+1)/2}\mathcal{R}^{\lambda-1}_{y}\left(\frac{d(y)}{|y'|}\right)^{-\varepsilon}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right) dy\, ds.$$

The same estimate holds for the integral of $G(x, y, t, s) f_2(y, s)$ by means of (4.6). Substituting $x' = RX'$, $y' = Ry'$, $t - s = R^2\tau$ and using the equality

$$\int (t-s)^{(m-n)/2}\exp\left(-\frac{\sigma_1|x''-y''|^2}{t-s}\right) dy'' = C < \infty, \tag{4.7}$$

we get

$$|\partial^{\alpha}_{x''}\partial^{k}_{t}u(x,t)| \leqslant cR^{-2k-|\alpha|} \sup_{Q^{\mathcal{D}}_{R}(x,t)} |u|$$
$$\times \int_0^{\kappa'}\int_K \left(\frac{|Y'|}{|Y'|+\sqrt{\tau}}\right)^{\lambda-1}\left(\frac{d(Y')}{|Y'|}\right)^{-\varepsilon}\exp\left(-\frac{\sigma_1|X'-Y'|^2}{\tau}\right)\frac{dY'\,d\tau}{\tau^{(m+1)/2}}.$$

The integral on the right-hand is equal to

$$\int_0^{\kappa'}\int_K \tau^{-1/2}\left(\frac{|z'|}{|z'|+1}\right)^{\lambda-1}\left(\frac{d(z')}{|z'|}\right)^{-\varepsilon}\exp(-\sigma_1|z'-\tau^{-1/2}X'|^2)\, dz'\, d\tau,$$

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where $z' = \tau^{-1/2}Y'$. We may assume that $0 < \lambda < 1$. Estimating the inner integral by means of the inequalities $|z'|^{\lambda-1}(|z'|+1)^{1-\lambda} \leqslant c(1+|z'|^{\lambda-1})$ and

$$\int_K |z'|^\delta \left(\frac{d(z')}{|z'|}\right)^{-\varepsilon} \exp(-\sigma_1|q'-z'|^2)\,dz' \leqslant C \quad \text{if } -m < \delta \leqslant 0,$$

we obtain

$$|\partial_{x''}^\alpha \partial_t^k u(x,t)| \leqslant cR^{-2k-|\alpha|} \sup_{Q_R^{\mathcal{D}}(x,t)} |u|. \tag{4.8}$$

We consider the x-derivatives of $\partial_{x''}^\alpha \partial_t^k u$. By (2.3),

$$\mathcal{L}\partial_{x''}^\alpha \partial_t^k u = [\partial_{x''}^\alpha \partial_t^k, \mathcal{A}]u = F \quad \text{in } Q_R^{\mathcal{D}}(x,t),\ \partial_{x''}^\alpha \partial_t^k u = 0 \quad \text{on } \partial Q_R^{\mathcal{D}}(x,t) \cap \partial\mathcal{D}.$$

Using (2.4), we obtain

$$|D\partial_{x''}^\alpha \partial_t^k u(x,t)| \leqslant C\left(R^{1-n/p}\,\|F\|_{L_{p,\infty}(Q_{\kappa R}^{\mathcal{D}}(x,t))} + R^{-1} \sup_{Q_{\kappa R}^{\mathcal{D}}(x,t)} |\partial_{x''}^\alpha \partial_t^k u|\right)$$

for $p > n/2$. Obviously, F depends only on the derivatives $D^j \partial_{x''}^\gamma \partial_t^i u$ with $j \leqslant 2$ and $i + |\gamma| \leqslant l-1$. Thus, by the induction hypothesis,

$$\begin{aligned} |F(y,s)| &\leqslant cR^{-2}\left(\frac{d(y')}{|y'|}\right)^{-\varepsilon} \sum_{i+|\gamma|\leqslant l-1} \sup_{Q_{\kappa' R}^{\mathcal{D}}(y,s)} |\partial_{x''}^\gamma \partial_t^i u| \\ &\leqslant cR^{-2}\left(\frac{d(y')}{|y'|}\right)^{-\varepsilon} \sum_{i+|\gamma|\leqslant l-1} \sup_{Q_R^{\mathcal{D}}(x,t)} |\partial_{x''}^\gamma \partial_t^i u| \end{aligned}$$

if $(y,s) \in Q_{\kappa R}^{\mathcal{D}}(x,t)$ and κ' is sufficiently small. Consequently,

$$\|F\|_{L_{p,\infty}(Q_{\kappa R}^{\mathcal{D}}(x,t))} \leqslant cR^{-2+n/p} \sum_{i+|\gamma|\leqslant l-1} \sup_{Q_R^{\mathcal{D}}(x,t)} |\partial_{x''}^\gamma \partial_t^i u|.$$

This implies (4.3). Furthermore, by (2.5),

$$\begin{aligned} &\rho^\varepsilon \|D^2 \partial_{x''}^\alpha \partial_t^k u\|_{Q_{\kappa\rho}(x,t)}^{\varepsilon,\varepsilon/2} + |D^2 \partial_{x''}^\alpha \partial_t^k u(x,t)| \\ &\leqslant C\Big(\rho^\varepsilon \|F\|_{Q_\rho(x,t)}^{\varepsilon,\varepsilon/2} + \rho^{-\varepsilon} R^{\varepsilon-2}(R^{2-(n+2)/p}\|F\|_{L_p(Q_{\kappa R}^{\mathcal{D}}(x,t))} \\ &\quad + \sup_{Q_{\kappa R}^{\mathcal{D}}(x,t)} |\partial_{x''}^\alpha \partial_t^k u|)\Big), \end{aligned}$$

where $p > (n+2)/\varepsilon$. We can again estimate the norms of F on the right-hand side by means of the induction hypothesis and obtain (4.4). Combining (4.3) and (4.4) with (4.8), we get (4.2). □

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4.2. *Estimates of x''- and t-derivatives near the edge.* In this subsection, we estimate the x''- and t-derivatives of solutions of the equation $\mathcal{L}u = 0$ in $Q_R^{\mathcal{D}}(0, x'', t)$. For this, we need the following lemma.

LEMMA 4.2. *Let I denote the integral*

$$\int_{Q_R^{\mathcal{D}}(0,x'',t)} \frac{\mathcal{R}_x^{\lambda-\alpha}\mathcal{R}_y^{\lambda-\beta}}{(t-s)^{(n+k)/2}} \left(\frac{|y'|}{R}\right)^{\lambda-\gamma} \left(\frac{d(y)}{|y'|}\right)^{-\varepsilon} \exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right) dy\, ds,$$

where $k = 0$ or $k = 1$. If $0 \leqslant \varepsilon < 1$, $\lambda - \gamma > -m$ and $2\lambda - \beta - \gamma > -m$, then

$$I \leqslant cR^{2-k}|X'|^{\lambda+2-k-\gamma} J,$$

where $X' = R^{-1}x'$, $J = 1$ for $\gamma+k-2 > \alpha$, $J = \log(1+|X'|^{-1})$ for $\gamma+k-2 = \alpha$ and $J = |X'|^{\gamma+k-2-\alpha}$ for $\gamma + k - 2 < \alpha$.

Proof. By (4.7), we have

$$I \leqslant C \int_{t-R^2}^{t} \int_{y'\in K,\ |y'|<R} (t-s)^{-(m+k)/2} \mathcal{R}_x^{\lambda-\alpha}\mathcal{R}_y^{\lambda-\beta} \left(\frac{|y'|}{R}\right)^{\lambda-\gamma} \left(\frac{d(y)}{|y'|}\right)^{-\varepsilon} \times \exp\left(-\frac{\sigma_1|x'-y'|^2}{t-s}\right) dy'\, ds.$$

Substituting $t - s = R^2\tau$, $x' = RX'$ and $y' = RY'$, we get

$$I \leqslant CR^{2-k} \int_0^1 \int_{Y'\in K, |Y'|<1} \tau^{-(m+k)/2} \left(\frac{|X'|}{|X'|+\sqrt{\tau}}\right)^{\lambda-\alpha} \left(\frac{|Y'|}{|Y'|+\sqrt{\tau}}\right)^{\lambda-\beta} \times |Y'|^{\lambda-\gamma} \left(\frac{d(Y')}{|Y'|}\right)^{-\varepsilon} \exp\left(-\frac{\sigma_1|X'-Y'|^2}{\tau}\right) dY'\, d\tau.$$

The substitution $z' = \tau^{-1/2}Y'$ yields

$$I \leqslant CR^{2-k} \int_0^1 \int_{z'\in K, |z'|<\tau^{-1/2}} \tau^{(\lambda-\gamma-k)/2} \left(\frac{|X'|}{|X'|+\sqrt{\tau}}\right)^{\lambda-\alpha} \left(\frac{|z'|}{|z'|+1}\right)^{\lambda-\beta} \times |z'|^{\lambda-\gamma} \left(\frac{d(z')}{|z'|}\right)^{-\varepsilon} \exp(-\sigma_1|\tau^{-1/2}X' - z'|^2)\, dz'\, d\tau.$$

Estimating the inner integral by means of the inequalities

$$\left(\frac{|z'|}{|z'|+1}\right)^{\lambda-\beta} \leqslant c(1 + |z'|^{\min(0,\lambda-\beta)})$$

and

$$\int_K |z'|^{\delta} \left(\frac{d(z')}{|z'|}\right)^{-\varepsilon} \exp(-\sigma_1|q'-z'|^2)\, dz' \leqslant C(1+|q'|)^{\delta} \quad \text{if } \delta > -m,$$

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we obtain

$$I \leqslant cR^{2-k}\int_0^1 \tau^{(\lambda-\gamma-k)/2}\left(\frac{|X'|}{|X'|+\sqrt{\tau}}\right)^{\lambda-\alpha}\left(1+\frac{|X'|}{\sqrt{\tau}}\right)^{\lambda-\gamma} d\tau$$
$$= cR^{2-k}|X'|^{\lambda-\alpha}\int_0^1 \tau^{-k/2}(|X'|+\sqrt{\tau})^{\alpha-\gamma}\, d\tau$$
$$= 2cR^{2-k}|X'|^{\lambda+2-k-\gamma}\int_0^{|X'|^{-1}} s^{1-k}(1+s)^{\alpha-\gamma}\, ds.$$

This implies the desired inequality. □

In the next lemma, we generalize the assertions of Lemma 2.4.

LEMMA 4.3. *Suppose that the coefficients of $\mathcal{A}$ satisfy the conditions (1.3), (3.1) and (4.1). Furthermore, let $0 < \kappa < 1$, $\lambda < \lambda_0 - c_1\sqrt{\epsilon}$ and $R < R_0$, where R_0 is a fixed positive number. If $u \in C^{2,1}(Q_R^{\mathcal{D}}(0,x'',t)) \cap C(\overline{Q_R^{\mathcal{D}}(0,x'',t)})$ satisfies the equations (2.16), then*

$$|D^j \partial_{x''}^{\alpha}\partial_t^k u(x,t)| \leqslant CR^{-j-|\alpha|-2k}\left(\frac{|x'|}{R}\right)^{\lambda-j}\left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_j}\sup_{Q_R^{\mathcal{D}}(0,x'',t)}|u| \quad (4.9)$$

for $j \leqslant 2$, $|\alpha| \leqslant l_1$, $k \leqslant l_2$ and $|x'| < \kappa R$. Here $\varepsilon_j = 0$ for $j \leqslant 1$, while ε_j is an arbitrarily small positive number if $j = 2$. The constant C may depend on R_0 and ε_j.

Proof. We prove the lemma by induction in $l = l_1 + l_2$. For $l_1 = l_2 = 0$, we may refer to Lemma 2.4. Suppose that $|\alpha| + k = l \geqslant 1$ and that the estimate (4.9) is proved for $|\alpha| + k \leqslant l - 1$. Since $\partial_{x''}^{\alpha}\partial_t^k u = \partial_{x''}^{\alpha}\partial_t^{k-1}\mathcal{A}(x,t,\partial_x)u$ for $k \geqslant 1$, it follows from the induction hypothesis that

$$|\partial_{x''}^{\alpha}\partial_t^k u(x,t)| \leqslant cR^{-2k-|\alpha|}\left(\frac{|x'|}{R}\right)^{\lambda-2}\left(\frac{d(x)}{|x'|}\right)^{-\varepsilon}\sup_{Q_R^{\mathcal{D}}(0,x'',t)}|u| \quad (4.10)$$

for $|x'| < \kappa R$, where ε is an arbitrarily small positive number. In the case $k = 0$, even a sharper estimate is valid by the induction hypothesis. We improve the estimate (4.10) in two steps. Let $0 < \kappa < \kappa' < 1$ and let $\psi = \psi(y,s)$ be a cut-off function on $\mathcal{D} \times \mathbb{R}$ depending on the parameters x'' and t such that $\psi(y,s) = 1$ for $(y,s) \in Q_{\kappa R}^{\mathcal{D}}(0,x'',t)$, $\psi(y,s) = 0$ for $(y,s) \notin Q_{\kappa' R}^{\mathcal{D}}(0,x'',t)$ and

$$|\partial_s^k \partial_y^{\beta}\psi(y,s)| \leqslant C_{k,\beta}R^{-2k-|\beta|}.$$

Then

$$(\partial_s - \mathcal{A}(y,s,\partial_y))\psi(y,s)\partial_{y''}^{\alpha}\partial_s^k u(y,s)$$
$$= \psi[\partial_{y''}^{\alpha}\partial_s^k, \mathcal{A}(y,s,\partial_y)]u + [\partial_s - \mathcal{A}(y,s,\partial_y), \psi]\partial_{y''}^{\alpha}\partial_s^k u = f(y,s).$$

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Thus,

$$\partial_{x''}^{\alpha}\partial_t^k u(x,t) = \int_{Q^{\mathcal{D}}_{\kappa' R}(0,x'',t)} G(x,y,t,s) f(y,s)\, dy\, ds$$

for $|x'| < \kappa R$. The function f admits the decomposition $f = f_1 + f_2$, where

$$f_1(y,s) = -\sum_{i,j}(A_{ij}(y,s) + A_{ji}(y,s))(\partial_{y_i}\psi(y,s))\partial_{y_j}\partial_{y''}^{\alpha}\partial_s^k u(y,s),$$

while $f_2(y,s)$ contains only the derivatives $\partial_{y''}^{\alpha}\partial_s^k u$ and $\partial_y^{\gamma}\partial_{y''}^{\delta}\partial_s^j u$, where $|\gamma| \leqslant 2$ and $j + |\delta| \leqslant l - 1$. Applying the induction hypothesis and (4.10), we get the estimate

$$|f_2(y,s)| \leqslant cR^{-2k-|\alpha|-2}\left(\frac{|y'|}{R}\right)^{\lambda-2}\left(\frac{d(y)}{|y'|}\right)^{-\varepsilon}\sup_{Q^{\mathcal{D}}_R(0,x'',t)}|u|$$

for $(y,s) \in Q_{\kappa' R}(0,x'',t)$. This estimate together with (3.5) and Lemma 4.2 implies that

$$\left|\int_{Q_{\kappa' R}(0,x'',t)} G(x,y,t,s) f_2(y,s)\, dy\, ds\right| \leqslant cR^{-2k-|\alpha|}|X'|^{\lambda}\log(1+|X'|^{-1})\sup_{Q^{\mathcal{D}}_R(0,x'',t)}|u|$$

for $|x'| < \kappa R$, where $X' = R^{-1}x'$. Furthermore, integration by parts yields

$$\int_{Q_{\kappa' R}(0,x'',t)} G(x,y,t,s) f_1(y,s)\, dy\, ds = \sum_{i,j=1}^{n}\int_{Q_{\kappa' R}(0,x'',t)}(\partial_{y''}^{\alpha}\partial_s^k u(y,s))\partial_{y_j}(G(x,y,t,s)(A_{ij}+A_{ji})\partial_{y_i}\psi)\, dy\, ds.$$

By Theorem 3.2,

$$|\partial_{y_j}(G(x,y,t,s)(A_{ij}+A_{ji})\partial_{y_i}\psi)| \leqslant cR^{-1}(t-s)^{-(n+1)/2}\mathcal{R}_x^{\lambda}\mathcal{R}_y^{\lambda-1}\exp\left(-\frac{\sigma_1|x-y|^2}{t-s}\right).$$

This together with (4.10) and Lemma 4.2 yields

$$\left|\int_{Q_{\kappa' R}(0,x'',t)} G(x,y,t,s) f_1(y,s)\, dy\, ds\right| \leqslant cR^{-2k-|\alpha|}|X'|^{\lambda-1}\sup_{Q^{\mathcal{D}}_R(0,x'',t)}|u|$$

for $|x'| < \kappa R$. Here we used the fact that $2\lambda > 3 - m$. (It follows directly from the definition of λ_0 that $\lambda_0 > 0$; moreover, it is well known that $\lambda_0 > 1/2$ if $m = 2$.) Thus, we obtain

$$|\partial_{x''}^{\alpha}\partial_t^k u(x,t)| \leqslant cR^{-2k-|\alpha|}\left(\frac{|x'|}{R}\right)^{\lambda-1}\sup_{Q^{\mathcal{D}}_R(0,x'',t)}|u| \tag{4.11}$$

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for $|x'| < \kappa R$. Using (4.11) instead of (4.10), we get the estimate

$$\left|\int_{Q_{\kappa' R}(0,x'',t)} G(x, y, t, s) f_1(y, s)\, dy\, ds\right| \leqslant cR^{-2k-|\alpha|}|X'|^{\lambda} \log(1 + |X'|^{-1}) \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u|,$$

which proves (4.9) for $j = 0$.

We prove (4.9) for $j = 1$. As in the proof of Lemma 2.4, we denote by Π_s the set of all (y, τ) such that $z \in \mathcal{D}$, $s|x'| < |y'| < s^{-1}|x'|$, $|y'' - x''| \leqslant s^{-1}|x'|$ and $t - s^{-2}|x'|^2 < \tau < t$. Here s is a positive number less than 1. Suppose that $|x'| < \kappa R$. By (4.3) (with $R = (1 - s)|x'|$), we have

$$|D\partial_{x''}^{\alpha}\partial_t^k u(x, t)| \leqslant c|x'|^{-1} \sum_{i=0}^{k} \sum_{|\gamma|\leqslant|\alpha|} \sup_{(y,\tau)\in\Pi_s} |\partial_{y''}^{\gamma}\partial_{\tau}^{i} u(y, \tau)|.$$

If $1 - s$ is sufficiently small, then the estimate (4.9) with $j = 0$ yields

$$|\partial_{y''}^{\gamma}\partial_t^i u(y, \tau)| \leqslant CR^{-i-|\gamma|}\left(\frac{|y'|}{R}\right)^{\lambda} \sup_{Q_{\kappa' R}^{\mathcal{D}}(0,y'',\tau)} |u| \leqslant CR^{-i-|\gamma|}\left(\frac{|x'|}{R}\right)^{\lambda} \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u|$$

for $(y, \tau) \in \Pi_s$ and, consequently,

$$|D\partial_{x''}^{\alpha}\partial_t^k u(x, t)| \leqslant CR^{-2k-|\alpha|-1}\left(\frac{|x'|}{R}\right)^{\lambda-1} \sup_{Q_R^{\mathcal{D}}(0,x'',t)} |u|.$$

Analogously, (4.9) holds by means of (4.4) for $j = 2$. The proof is complete. □

4.3. *Estimates for x''- and t-derivatives of Green's function.* We assume that the conditions of the last lemma are satisfied for the coefficients of both the operators $\mathcal{A}$ and $\mathcal{A}_*$. Then, analogously to Theorem 3.2, we can prove the following estimates.

THEOREM 4.4. *Suppose that the coefficients of the operator $\mathcal{A}$ satisfy the conditions (1.3), (3.1) and*

$$\sum_{i,j=1}^{n} \sum_{|\gamma|\leqslant 2} |\partial_x^{\gamma}\partial_{x''}^{\alpha''}\partial_t^k A_{ij}| + \sum_{j=1}^{n} \sum_{|\gamma|\leqslant 1} |\partial_x^{\gamma}\partial_{x''}^{\alpha''}\partial_t^k A_j| + |\partial_{x''}^{\alpha''}\partial_t^k A| \leqslant C$$
$$\text{for } |\alpha''| \leqslant l_1,\ k \leqslant l_2.$$

Furthermore, let T be a positive number and let $\lambda < \lambda_0 - c_1\sqrt{\epsilon}$. Then the following estimate is satisfied for $x, y \in \mathcal{D}$, $0 < t - s < T$, $|\alpha'| \leqslant 2$, $|\beta'| \leqslant 2$,

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$|\alpha|, |\beta| \leqslant l_1 + 2$ *and* $j, k \leqslant l_2$*:*

$$|\partial_x^\alpha \partial_y^\beta \partial_t^j \partial_s^k G(x, y, t, s)| \leqslant C(t-s)^{-(n+|\alpha|+|\beta|+2j+2k)/2} \mathcal{R}_x^{\lambda-|\alpha'|} \mathcal{R}_y^{\lambda-|\beta'|} \times \left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_{\alpha'}} \left(\frac{d(y)}{|y'|}\right)^{-\varepsilon_{\beta'}} \exp\left(-\frac{\sigma_1 |x-y|^2}{t-s}\right). \tag{4.12}$$

Here σ_1 *is a positive constant,* $\varepsilon_{\alpha'} = 0$ *for* $|\alpha'| \leqslant 1$*, while* $\varepsilon_{\alpha'}$ *is an arbitrarily small positive number if* $|\alpha'| = 2$*. The constant* C *may depend on* T*,* λ*,* $\varepsilon_{\alpha'}$ *and* $\varepsilon_{\beta'}$*.*

Proof. As in the proof of Theorem 3.2, we start with the case $|\beta| = k = 0$. If $|x'|^2 \leqslant (t-s)/4$, then it follows from Lemma 4.3 with $R = \sqrt{(t-s)/2}$ that

$$|\partial_x^\alpha \partial_t^j G(x, y, t, s)| \leqslant C(t-s)^{-j-|\alpha|/2} \mathcal{R}_x^{\lambda-|\alpha'|} \left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_{\alpha'}} \sup_{(z,\tau)\in Q_R^{\mathcal{D}}(0,x'',t)} |G(z, y, \tau, s)|$$

for $|\alpha'| \leqslant 2$, $|\alpha''| \leqslant l_1$ and $j \leqslant l_2$. If $|x'|^2 > (t-s)/4$, then $1/3 < \mathcal{R}_x < 1$ and Lemma 4.1 with $R = \delta\sqrt{(t-s)/4}$ yields

$$|\partial_x^\alpha \partial_t^j G(x, y, t, s)| \leqslant C(t-s)^{-j-|\alpha|/2} \left(\frac{d(x)}{|x'|}\right)^{-\varepsilon_{\alpha'}} \sup_{(z,\tau)\in Q_R^{\mathcal{D}}(x,t)} |G(z, y, \tau, s)|.$$

Using the estimates (3.11) and (3.14), respectively, we obtain the estimate (4.12) with $|\beta| = k = 0$. Analogously, we obtain the estimate

$$|\partial_y^\beta \partial_s^k G(x, y, t, s)| \leqslant C(t-s)^{-(n+|\beta|+2k)/2} \mathcal{R}_x^{\lambda} \mathcal{R}_y^{\lambda-|\beta'|} \left(\frac{d(y)}{|y'|}\right)^{-\varepsilon_{\beta'}} \exp\left(-\frac{\sigma_1 |x-y|^2}{t-s}\right). \tag{4.13}$$

Repeating the considerations of the first part of the proof with $\partial_y^\beta \partial_s^k G(x, y, t, s)$ instead of $G(x, y, t, s)$ and using (4.13), we obtain the estimate (4.12) for arbitrary α, β, j, k satisfying the conditions of the theorem. □

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