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EXPONENTIAL STABILITY OF TIMOSHENKO BEAM SYSTEM WITH DELAY TERMS IN BOUNDARY FEEDBACKS*

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Abstract. In this paper, the stability of a Timoshenko beam with time delays in the boundary input is studied. The system is fixed at the left end, and at the other end there are feedback controllers, in which time delays exist. We prove that this closed loop system is well-posed. By the complete spectral analysis, we show that there is a sequence of eigenvectors and generalized eigenvectors of the system operator that forms a Riesz basis for the state Hilbert space. Hence the system satisfies the spectrum determined growth condition. Then we conclude the exponential stability of the system under certain conditions. Finally, we give some simulations to support our results.

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1. Introduction

Time delay is a universal phenomenon existing in almost every engineering practices, such as electrical and mechanical engineering, biology and so on (see [8,14,15]). Normally, the presence of delays makes the systems less productive, less optimal and less stable (see [2,10]). However, sometimes it also can play a positive role in the performance of the systems (see [4,16]).

Due to the effect of time delays, it is necessary to take the delays into account when we discuss the control problem of the systems. For elastic systems, such as wave equations and beam equations, there have been some nice results on control problem with time delays. For example, Liang *et al.* in [5] introduced the modified Smith predictor to the boundary control of wave equation and Euler-Bernoulli beam equation with a delayed boundary measurement. Xu *et al.* in [21,24] considered the Riesz basis property, exact controllability and stability of the string systems with time-delayed boundary feedback control. Mörgul in [9] designed a class of dynamical controllers to robustly stabilize the wave equations against small time delays in the feedback loop. Nicaise and Pignotti in [11] get the exponential stability of the wave equation with boundary or internal distributed delay.

However, at present, there is few result about the stability of Timoshenko beam with time-delayed feedback control. Timoshenko beam model constitutes a weakly coupled system. This weakly coupled property causes that it is usually difficult to consider the control problem for this kind of system. Our aim in this paper is

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to discuss the stability of a Timoshenko beam with time delays in the boundary control. What we are interested in is that, under the time-delayed feedback controls, whether the system is still exponentially stable, since it can get the exponential stability without time delays (see [19]).

Let us recall the Timoshenko beam model:

$$\begin{cases} \rho w_{tt}(x,t) - k(w_{xx} - \varphi_x)(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ I_\rho \varphi_{tt}(x,t) - EI\varphi_{xx}(x,t) - k(w_x - \varphi)(x,t) = 0, \quad x \in (0,1),\ t > 0, \end{cases} \tag{1.1}$$

where ρ, I_ρ, EI, k are mass density, moment of mass inertia, rigidity coefficient and shear modulus of elastic beam, respectively. For more precise physical meanings of them, see Timoshenko's book [17]. In the sequel, we shall use the abbreviations $w_t = \frac{\partial w}{\partial t}$ and $w_x = \frac{\partial w}{\partial x}$.

The beam is clamped at the left end, *i.e.*,

$$w(0,t) = \varphi(0,t) = 0, \quad t > 0. \tag{1.2}$$

The external shearing force and bending moment in which there exist time delays are applied at the right end,

$$\begin{cases} k(w_x - \varphi)(1,t) = \mu_1 v_1(t) + (1-\mu_1)v_1(t-\tau_1),\ t > 0, \\ EI\varphi_x(1,t) = \mu_2 v_2(t) + (1-\mu_2)v_2(t-\tau_2),\ t > 0, \end{cases} \tag{1.3}$$

where $\mu_i \in (0,1),\ i = 1,2.$ $v_i(t-\tau_i) := \widetilde{f}_i(t-\tau_i),\ t \in (0,\tau_i)$, and $\widetilde{f}_i(\theta),\ i = 1,2$ are the given functions.

The control design (1.3) has been used in [12,24], in which the authors considered 1-d wave systems with time delays. When $\mu_i = 1,\ i = 1,2$, Xu and Feng [19] and Kim and Renardy [3] have used the feedback control laws

$$\begin{cases} v_1(t) = -\alpha_1 w_t(1,t),\ t > 0,\ \alpha_1 > 0, \\ v_2(t) = -\alpha_2 \varphi_t(1,t),\ t > 0,\ \alpha_2 > 0 \end{cases} \tag{1.4}$$

to exponentially stabilize the system (1.1)–(1.3). However, if $\mu_i \neq 1$, the stability of the system (1.1)–(1.3) is unknown. In this paper, we shall discuss this case. Under certain conditions, we prove that the system is still exponentially stable when $\mu_i \neq 1$.

In addition, the initial conditions are given as follows:

$$\begin{cases} w(x,0) = \widetilde{w}_0(x), \quad w_t(x,0) = \widetilde{w}_1(x), \\ \varphi(x,0) = \widetilde{\varphi}_0(x), \quad \varphi_t(x,0) = \widetilde{\varphi}_1(x). \end{cases} \tag{1.5}$$

Thus, under the feedback control laws (1.4), we get the following closed loop system:

$$\begin{cases} \rho w_{tt}(x,t) - k(w_{xx} - \varphi_x)(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ I_\rho \varphi_{tt}(x,t) - EI\varphi_{xx}(x,t) - k(w_x - \varphi)(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ w(0,t) = \varphi(0,t) = 0,\ t > 0, \\ k(w_x - \varphi)(1,t) = -\alpha_1\mu_1 w_t(1,t) - \alpha_1(1-\mu_1)w_t(1,t-\tau_1),\ t > 0, \\ EI\varphi_x(1,t) = -\alpha_2\mu_2\varphi_t(1,t) - \alpha_2(1-\mu_2)\varphi_t(1,t-\tau_2),\ t > 0, \\ w(x,0) = \widetilde{w}_0(x), \quad w_t(x,0) = \widetilde{w}_1(x), \quad \varphi(x,0) = \widetilde{\varphi}_0(x), \quad \varphi_t(x,0) = \widetilde{\varphi}_1(x), \\ w_t(1,t-\tau_1) = -\alpha_1^{-1}\widetilde{f}_1(t-\tau_1),\ \varphi_t(1,t-\tau_2) = -\alpha_2^{-1}\widetilde{f}_2(t-\tau_2), \quad t \in (0,\tau_i),\ i = 1,2. \end{cases} \tag{1.6}$$

Set

$$p(x,t) := w_t(1,t-x\tau_1), \quad q(x,t) := \varphi_t(1,t-x\tau_2).$$

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Then the system (1.6) is equivalent to

$$\begin{cases} \rho w_{tt}(x,t) - k(w_{xx} - \varphi_x)(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ I_\rho \varphi_{tt}(x,t) - EI\varphi_{xx}(x,t) - k(w_x - \varphi)(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ \tau_1 p_t(x,t) + p_x(x,t) = 0, \quad \tau_2 q_t(x,t) + q_x(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ w(0,t) = \varphi(0,t) = 0, \quad t > 0, \\ p(0,t) = w_t(1,t), \quad q(0,t) = \varphi_t(1,t), \quad t > 0, \\ k(w_x - \varphi)(1,t) = -\alpha_1\mu_1 w_t(1,t) - \alpha_1(1-\mu_1)p(1,t), \quad t > 0, \\ EI\varphi_x(1,t) = -\alpha_2\mu_2\varphi_t(1,t) - \alpha_2(1-\mu_2)q(1,t), \quad t > 0, \\ w(x,0) = \widetilde{w}_0(x), \quad w_t(x,0) = \widetilde{w}_1(x), \quad \varphi(x,0) = \widetilde{\varphi}_0(x), \quad \varphi_t(x,0) = \widetilde{\varphi}_1(x), \\ p(x,0) = -\alpha_1^{-1}\widetilde{f}_1(-x\tau_1),\ q(x,0) = -\alpha_2^{-1}\widetilde{f}_2(-x\tau_2), \quad x \in (0,1). \end{cases} \tag{1.7}$$

Note that this system is too complex to construct a Lyapunov function or a multiplier to analyze its stability. Herein we mainly employ the Riesz basis approach. For a vibrating system, the Riesz basis generation is the most profound result. It forms a basis not only for the expansion of the solution in terms of the (generalized) eigenvectors of the system but also for the spectrum determined growth condition. Since the spectral analysis is a key to develop Riesz basis property, we shall have the complete spectral analysis for the system (1.7). Based on the distribution of the spectrum of this system, we prove that there exists a sequence of (generalized) eigenvectors of the system (1.7) that forms a Riesz basis for the state space. Hence, the spectrum determined growth condition holds, *i.e.*, the growth order of the system is determined *via* its spectral bound. Therefore, we conclude the exponential stability by showing that the imaginary axis is not an asymptote of the spectrum of the system operator.

The remaining part of this paper is organized as follows. In Section 2, the system (1.7) is formulated in a Hilbert space setting and the well-posedness of the system is proved. Section 3 is devoted to the spectral analysis of the system. In Section 4, the completeness as well as the Riesz basis property of the (generalized) eigenvectors of the system operator is presented. The exponential stability of the system is discussed in Section 5. Finally, in Section 6, some simulations are given to support our results.

2. Well-posedness of the system

In this section, we shall study the well-posedness of the closed loop system (1.7). To this end, we formulate this system in an appropriate Hilbert space setting.

Set

$$V^k(0,1) := \{f \in H^k(0,1) | f(0) = 0\},$$

where $H^k(0,1)$ is the usual Sobolev space of order k.

Let the state space be

$$\mathcal{H} := V^1(0,1) \times L^2(0,1) \times L^2(0,1) \times V^1(0,1) \times L^2(0,1) \times L^2(0,1) \tag{2.1}$$

endowed with an inner product: for $W_i = (w_i, z_i, p_i, \varphi_i, \psi_i, q_i) \in \mathcal{H},\ i = 1, 2$

$$\begin{aligned} (W_1, W_2)_{\mathcal{H}} \quad := \quad & \int_0^1 k(w_{1,x} - \varphi_1)\overline{(w_{2,x} - \varphi_2)}\mathrm{d}x + \int_0^1 EI\varphi_{1,x}\overline{\varphi_{2,x}}\mathrm{d}x + \int_0^1 \rho z_1 \overline{z_2}\mathrm{d}x \\ & + \int_0^1 I_\rho \psi_1 \overline{\psi}_2 \mathrm{d}x + \int_0^1 p_1 \overline{p_2}\mathrm{d}x + \int_0^1 q_1 \overline{q_2}\mathrm{d}x \end{aligned} \tag{2.2}$$

where k, ρ, I_ρ, EI are all positive parameters in the system (1.7). A direct verification shows that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Hilbert space.

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We define the operator $\mathcal{A}$ in $\mathcal{H}$ as follows

$$\mathcal{A}(w,z,p,\varphi,\psi,q)^\tau = \begin{pmatrix} z \\ \rho^{-1}k(w_{xx}-\varphi_x) \\ -\tau_1^{-1}p_x \\ \psi \\ I_\rho^{-1}EI\varphi_{xx}+I_\rho^{-1}k(w_x-\varphi) \\ -\tau_2^{-1}q_x \end{pmatrix} \tag{2.3}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (w,z,p,\varphi,\psi,q)^\tau \in V^2(0,1)\times V^1(0,1)\times H^1(0,1)\times V^2(0,1)\times V^1(0,1)\times H^1(0,1) \;\middle|\; \begin{array}{l} k(w_x(1)-\varphi(1)) = -\alpha_1\mu_1 z(1) - \alpha_1(1-\mu_1)p(1),\ z(1)=p(0) \\ EI\varphi_x(1) = -\alpha_2\mu_2\psi(1) - \alpha_2(1-\mu_2)q(1),\ \psi(1)=q(0) \end{array} \right\}.$$

Consequently, the problem (1.7) can be rewritten as an evolutionary equation in $\mathcal{H}$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}W(t) = \mathcal{A}W(t), \quad t>0, \\ W(0) = W_0, \end{cases} \tag{2.4}$$

where $W(t) = (w(x,t), \dot{w}(x,t), p(x,t), \varphi(x,t), \dot{\varphi}(x,t), q(x,t))^\tau$ and

$$W_0 = (\widetilde{w}_0, \widetilde{w}_1, -\alpha_1^{-1}\widetilde{f}_1, \widetilde{\varphi}_0, \widetilde{\varphi}_1, -\alpha_2^{-1}\widetilde{f}_2)^\tau.$$

We have the following result:

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then for any $\alpha_i, \mu_i \in \mathbb{R}_+$, $i=1,2$, $0\in\rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$.*

Proof. Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Let $\mu_i,\ \alpha_i \in \mathbb{R}_+,\ i=1,2$ be given. For any $F=(f_1,f_2,f_3,f_4,f_5,f_6)\in\mathcal{H}$, we consider the solvability of equation $\mathcal{A}W=F$, where $W=(w,z,p,\varphi,\psi,q)^\tau\in\mathcal{D}(\mathcal{A})$, *i.e.*,

$$\begin{cases} z(x) = f_1(x), \quad \rho^{-1}k(w_{xx}-\varphi_x)(x) = f_2(x), \\ \tau_1^{-1}p_x(x) = -f_3(x), \quad \psi(x) = f_4(x), \\ I_\rho^{-1}EI\varphi_{xx}(x) + I_\rho^{-1}k(w_x-\varphi)(x) = f_5(x), \quad \tau_2^{-1}q_x(x) = -f_6(x) \end{cases} \tag{2.5}$$

with the boundary conditions

$$\begin{cases} w(0)=\varphi(0)=0, \quad p(0)=z(1), \quad q(0)=\psi(1), \\ k(w_x(1)-\varphi(1)) = -\alpha_1\mu_1 z(1) - \alpha_1(1-\mu_1)p(1), \\ EI\varphi_x(1) = -\alpha_2\mu_2\psi(1) - \alpha_2(1-\mu_2)q(1). \end{cases} \tag{2.6}$$

Solving the ordinary differential equations (2.5)–(2.6) yields

$$\begin{cases} p(x) = f_1(1) - \tau_1\int_0^x f_3(s)\mathrm{d}s, \quad q(x) = f_4(1) - \tau_2\int_0^x f_6(s)\mathrm{d}s, \\ \varphi(x) = -EI^{-1}\alpha_2 f_4(1)x + EI^{-1}\alpha_2(1-\mu_2)\tau_2 x\int_0^1 f_6(s)\mathrm{d}s - EI^{-1}\int_0^x\int_s^1 (I_\rho f_5(r) - G_1(r))\mathrm{d}r\mathrm{d}s, \\ w(x) = -EI^{-1}\alpha_2 f_4(1)\frac{x^2}{2} + EI^{-1}\alpha_2(1-\mu_2)\tau_2\frac{x^2}{2}\int_0^1 f_6(s)\mathrm{d}s - EI^{-1}\int_0^x\int_0^s\int_r^1 (I_\rho f_5(\xi) - G_1(\xi))\mathrm{d}\xi\mathrm{d}r\mathrm{d}s \\ \qquad\quad + k^{-1}\int_0^x G_1(s)\mathrm{d}s, \end{cases} \tag{2.7}$$

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where

$$G_1(x) := \left[-\alpha_1 f_1(1) + \alpha_1\tau_1(1-\mu_1)\int_0^1 f_3(x)\mathrm{d}x\right] - \int_x^1 \rho f_2(s)\mathrm{d}s.$$

For these $p(x), q(x), \varphi(x), w(x)$, which are given by (2.7), we see that $W = (w, f_1, p, \varphi, f_4, q) \in \mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ and $\mathcal{A}(w, f_1, p, \varphi, f_4, q) = F$. Hence $\mathcal{A}^{-1}$ exists. Thus, the Sobolev's Embedding Theorem implies that $\mathcal{A}^{-1}$ is a compact operator on $\mathcal{H}$. □

As a direct consequence of Lemma 2.1, we have the following result.

Corollary 2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the spectrum of $\mathcal{A}$ only consists of isolated eigenvalues of finite multiplicity.*

Suppose that

$$\mu_i \geq \frac{1}{2}, \quad i = 1, 2. \tag{2.8}$$

Under this condition, we shall show that $\mathcal{A}$ is dissipative in $\mathcal{H}$. To this end, we choose the positive real constants $\eta_i,\ i = 1, 2$ such that

$$\tau_i(1-\mu_i)\alpha_i \leq \eta_i \leq \tau_i(3\mu_i - 1)\alpha_i,\ i = 1, 2. \tag{2.9}$$

These constants $\eta_i,\ i = 1, 2$ exist due to the condition (2.8). Then we introduce a new inner product in $\mathcal{H}$: for $W_j = (w_j, z_j, p_j, \varphi_j, \psi_j, q_j)^\tau \in \mathcal{H},\ j = 1, 2,$

$$\begin{aligned}(W_1, W_2)_1 \quad = \quad & \int_0^1 k(w_{1,x} - \varphi_1)\overline{(w_{2,x} - \varphi_2)}\mathrm{d}x + \int_0^1 EI\varphi_{1,x}\overline{\varphi_{2,x}}\mathrm{d}x + \int_0^1 \rho z_1\overline{z_2}\mathrm{d}x \\ & + \int_0^1 I_\rho\psi_1\overline{\psi}_2\mathrm{d}x + \int_0^1 \eta_1 p_1\overline{p_2}\mathrm{d}x + \int_0^1 \eta_2 q_1\overline{q_2}\mathrm{d}x.\end{aligned} \tag{2.10}$$

It is easy to verify that $(W_1, W_2)_1$ is equivalent to the inner product $(W_1, W_2)_{\mathcal{H}}$. Then we have the following result:

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and the condition (2.8) be fulfilled. Then $\mathcal{A}$ is dissipative in $(\mathcal{H},\ \|\cdot\|_1)$.*

Proof. For any real $W = (w, z, p, \varphi, \psi, q)^\tau \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned}(\mathcal{A}W, W)_1 \quad = \quad & k(w_x - \varphi)z|_0^1 + EI\varphi_x\psi|_0^1 - \frac{\tau_1^{-1}\eta_1}{2}p^2|_0^1 - \frac{\tau_2^{-1}\eta_2}{2}q^2|_0^1 \\ = \quad & -\left(\alpha_1\mu_1 - \frac{\eta_1}{2\tau_1}\right)(p(0))^2 - \left(\alpha_2\mu_2 - \frac{\eta_2}{2\tau_2}\right)(q(0))^2 - \frac{\eta_1}{2\tau_1}(p(1))^2 - \frac{\eta_2}{2\tau_2}(q(1))^2 \\ & -\alpha_1(1-\mu_1)p(0)p(1) - \alpha_2(1-\mu_2)q(0)q(1) \\ \leq \quad & -\left[\alpha_1\mu_1 - \frac{\eta_1}{2\tau_1} - \frac{\alpha_1(1-\mu_1)}{2}\right](p(0))^2 - \left[\alpha_2\mu_2 - \frac{\eta_2}{2\tau_2} - \frac{\alpha_2(1-\mu_2)}{2}\right](q(0))^2 \\ & -\left[\frac{\eta_1}{2\tau_1} - \frac{\alpha_1(1-\mu_1)}{2}\right](p(1))^2 - \left[\frac{\eta_2}{2\tau_2} - \frac{\alpha_2(1-\mu_2)}{2}\right](q(1))^2\end{aligned} \tag{2.11}$$

with

$$-\alpha_1(1-\mu_1)p(0)p(1) \leq \frac{\alpha_1(1-\mu_1)}{2}[(p(0))^2 + (p(1))^2],\ -\alpha_2(1-\mu_2)q(0)q(1) \leq \frac{\alpha_2(1-\mu_2)}{2}[(q(0))^2 + (q(1))^2].$$

From (2.9), we get that $\alpha_1\mu_1 - \frac{\eta_1}{2\tau_1} - \frac{\alpha_1(1-\mu_1)}{2}$, $\frac{\eta_1}{2\tau_1} - \frac{\alpha_1(1-\mu_1)}{2}$, $\alpha_2\mu_2 - \frac{\eta_2}{2\tau_2} - \frac{\alpha_2(1-\mu_2)}{2}$ and $\frac{\eta_2}{2\tau_2} - \frac{\alpha_2(1-\mu_2)}{2}$ are all nonnegative. Hence $(\mathcal{A}W, W)_1 \leq 0$, which implies that $\mathcal{A}$ is dissipative in $(\mathcal{H},\ \|\cdot\|_1)$. □

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By Lemmas 2.1 and 2.2, the Lumer-Phillips Theorem (see Pazy [13]) asserts the following result.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and the condition (2.8) be fulfilled. Then $\mathcal{A}$ generates a C_0 semigroup $T(t)$ of contractions on $\mathcal{H}$. Hence the closed loop system (1.7) is well-posed.*

3. Spectral analysis of $\mathcal{A}$

In this section, we shall discuss the asymptotic distribution of the spectrum of $\mathcal{A}$. From Corollary 2.1, we have known that the spectrum of $\mathcal{A}$ only consists of isolated eigenvalues of finite multiplicity. So $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. Thus we only need to discuss the eigenvalues of $\mathcal{A}$. Let $\lambda \in \sigma_p(\mathcal{A})$, $(w, z, p, \varphi, \psi, q)^\tau \in \mathcal{D}(\mathcal{A})$ be an eigenvector of $\mathcal{A}$ corresponding to λ. Then $z(x) = \lambda w(x)$ and $\psi(x) = \lambda\varphi(x)$. Thus, $w(x)$, $\varphi(x)$, p and q satisfy the following differential equations

$$\begin{cases} \rho\lambda^2 w - k(w_{xx} - \varphi_x) = 0, \quad x \in (0,1), \\ I_\rho\lambda^2\varphi - EI\varphi_{xx} - k(w_x - \varphi) = 0, \quad x \in (0,1), \\ \tau_1\lambda p + p_x = 0, \quad \tau_2\lambda q + q_x = 0, \quad x \in (0,1), \\ w(0) = \varphi(0) = 0, \quad p(0) = \lambda w(1), \quad q(0) = \lambda\varphi(1), \\ k(w_x - \varphi)(1) = -\alpha_1\mu_1\lambda w(1) - \alpha_1(1-\mu_1)p(1), \\ EI\varphi_x(1) = -\alpha_2\mu_2\lambda\varphi(1) - \alpha_2(1-\mu_2)q(1). \end{cases} \tag{3.1}$$

3.1. Fundamental matrix solution

In order to calculate the eigenvalues of $\mathcal{A}$, we shall find the fundamental matrix solution to the following differential equation:

$$\begin{cases} \rho\lambda^2 w - k(w_{xx} - \varphi_x) = 0, \quad x \in (0,1), \\ I_\rho\lambda^2\varphi - EI\varphi_{xx} - k(w_x - \varphi) = 0, \quad x \in (0,1). \end{cases} \tag{3.2}$$

Setting

$$Y_1 := (w, \varphi)^\tau, \; Y_2 := \frac{Y_1'}{\lambda}, \; Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \tag{3.3}$$

we have

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \widetilde{A}Y, \quad \widetilde{A} := \begin{bmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ \frac{\rho}{k}\lambda & 0 & 0 & 1 \\ 0 & \frac{I_\rho}{EI}\lambda + \frac{k}{EI\lambda} & -\frac{k}{EI} & 0 \end{bmatrix}. \tag{3.4}$$

Under the following transformation:

$$Y(x) = T_0 Z(x), \quad T_0 := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sqrt{\frac{\rho}{k}} & 0 & -\sqrt{\frac{\rho}{k}} & 0 \\ 0 & \sqrt{\frac{I_\rho}{EI}} & 0 & -\sqrt{\frac{I_\rho}{EI}} \end{bmatrix}, \tag{3.5}$$

(3.4) is transformed into the following equation:

$$\frac{\mathrm{d}Z(x)}{\mathrm{d}x} = T_0^{-1}\widetilde{A}T_0 Z(x). \tag{3.6}$$

A direct calculation leads to $T_0^{-1}\widetilde{A}T_0 := \lambda\widetilde{\Lambda}_1 + \widetilde{\Lambda}_0 + \frac{1}{\lambda}\widetilde{\Lambda}_{-1}$, where

$$\widetilde{\Lambda}_1 := (\widetilde{\Lambda}_1^{ij})_{4\times 4} = \begin{bmatrix} \sqrt{\frac{\rho}{k}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{I_\rho}{EI}} & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{\rho}{k}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{I_\rho}{EI}} \end{bmatrix}, \quad \widetilde{\Lambda}_{-1} := (\widetilde{\Lambda}_{-1}^{ij})_{4\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{k}{2\sqrt{I_\rho EI}} & 0 & \frac{k}{2\sqrt{I_\rho EI}} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{k}{2\sqrt{I_\rho EI}} & 0 & -\frac{k}{2\sqrt{I_\rho EI}} \end{bmatrix}$$

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and

$$\widetilde{\Lambda}_0 := (\widetilde{\Lambda}_0^{ij})_{4\times 4} = \begin{bmatrix} 0 & \frac{\sqrt{I_\rho k}}{2\sqrt{\rho EI}} & 0 & -\frac{\sqrt{I_\rho k}}{2\sqrt{\rho EI}} \\ -\frac{\sqrt{k\rho}}{2\sqrt{I_\rho EI}} & 0 & \frac{\sqrt{k\rho}}{2\sqrt{I_\rho EI}} & 0 \\ 0 & -\frac{\sqrt{I_\rho k}}{2\sqrt{\rho EI}} & 0 & \frac{\sqrt{I_\rho k}}{2\sqrt{\rho EI}} \\ \frac{\sqrt{k\rho}}{2\sqrt{I_\rho EI}} & 0 & -\frac{\sqrt{k\rho}}{2\sqrt{I_\rho EI}} & 0 \end{bmatrix}.$$

Therefore, we have

$$\frac{\mathrm{d}Z(x)}{\mathrm{d}x} = \left[\lambda\widetilde{\Lambda}_1 + \widetilde{\Lambda}_0 + \frac{1}{\lambda}\widetilde{\Lambda}_{-1}\right] Z(x). \tag{3.7}$$

Set

$$Z(x,\lambda) := \sum_{k=0}^{\infty} \frac{P_{-k}(x)}{\lambda^k} E(x,\lambda), \tag{3.8}$$

where

$$E(x,\lambda) := \exp(x\lambda\widetilde{\Lambda}_1). \tag{3.9}$$

We shall identify $P_{-k},\ k = 0, 1, 2, \ldots$ so as to get the expression of $Z(x,\lambda)$. Substituting (3.8) into (3.7) leads to

$$\sum_{k=0}^{\infty} \frac{P'_{-k}(x)}{\lambda^k} E(x,\lambda) + \sum_{k=0}^{\infty} \frac{P_{-k}(x)}{\lambda^k} E'(x,\lambda) = [\lambda\widetilde{\Lambda}_1(x) + \widetilde{\Lambda}_0 + \lambda^{-1}\widetilde{\Lambda}_{-1}] \sum_{k=0}^{\infty} \frac{P_{-k}(x)}{\lambda^k} E(x,\lambda).$$

Then each coefficient with the same power of λ in both sides of above equation must be equal, *i.e.*,

$$\lambda \ : \quad P_0(x)\widetilde{\Lambda}_1(x) = \widetilde{\Lambda}_1(x)P_0(x), \tag{3.10}$$

$$1 \ : \quad P'_0(x) + P_{-1}(x)\widetilde{\Lambda}_1(x) = \widetilde{\Lambda}_1(x)P_{-1}(x) + \widetilde{\Lambda}_0 P_0(x), \tag{3.11}$$

$$\lambda^{-1} \ : \quad P'_{-1}(x) + P_{-2}(x)\widetilde{\Lambda}_1(x) = \widetilde{\Lambda}_1(x)P_{-2}(x) + \widetilde{\Lambda}_0 P_{-1}(x) + \widetilde{\Lambda}_{-1}P_0(x), \tag{3.12}$$

$$\lambda^{-k} \ : \quad P'_{-k}(x) + P_{-(k+1)}(x)\widetilde{\Lambda}_1(x) = \widetilde{\Lambda}_1(x)P_{-(k+1)}(x) + \widetilde{\Lambda}_0 P_{-k}(x) + \widetilde{\Lambda}_{-1}P_{-k+1}(x). \tag{3.13}$$

(3.10) implies that P_0 is a diagonal matrix. By (3.11), we have $P'_0 = \mathrm{diag}\widetilde{\Lambda}_0 \cdot P_0(x)$. From the expression of $\widetilde{\Lambda}_0$, a direct calculation yields $P_0 = I$.

Now let us calculate P_{-1}. Assume that

$$\sqrt{\frac{I_\rho}{EI}} \neq \sqrt{\frac{\rho}{k}}. \tag{3.14}$$

Firstly, by (3.11), we get the expression of the non-diagonal entries of $P_{-1} = (P_{-1}^{ij})_{4\times 4}$ given as follows

$$\begin{cases} P_{-1}^{12} = \dfrac{\frac{\sqrt{I_\rho k}}{\sqrt{EI\rho}}}{2\left(\sqrt{\frac{I_\rho}{EI}} - \sqrt{\frac{\rho}{k}}\right)}, \quad P_{-1}^{13} = 0, \quad P_{-1}^{14} = \dfrac{\frac{\sqrt{I_\rho k}}{\sqrt{EI\rho}}}{2\left(\sqrt{\frac{I_\rho}{EI}} + \sqrt{\frac{\rho}{k}}\right)}, \\ P_{-1}^{21} = \dfrac{\frac{\sqrt{\rho k}}{\sqrt{I_\rho EI}}}{2\left(\sqrt{\frac{I_\rho}{EI}} - \sqrt{\frac{\rho}{k}}\right)}, \quad P_{-1}^{23} = \dfrac{\frac{\sqrt{\rho k}}{\sqrt{I_\rho EI}}}{-2\left(\sqrt{\frac{I_\rho}{EI}} + \sqrt{\frac{\rho}{k}}\right)}, \quad P_{-1}^{24} = 0, \\ P_{-1}^{31} = 0, \quad P_{-1}^{32} = \dfrac{\frac{\sqrt{I_\rho k}}{\sqrt{EI\rho}}}{-2\left(\sqrt{\frac{I_\rho}{EI}} + \sqrt{\frac{\rho}{k}}\right)}, \quad P_{-1}^{34} = \dfrac{\frac{\sqrt{I_\rho k}}{\sqrt{EI\rho}}}{-2\left(\sqrt{\frac{I_\rho}{EI}} - \sqrt{\frac{\rho}{k}}\right)}, \\ P_{-1}^{41} = \dfrac{\frac{\sqrt{\rho k}}{\sqrt{I_\rho EI}}}{2\left(\sqrt{\frac{I_\rho}{EI}} + \sqrt{\frac{\rho}{k}}\right)}, \quad P_{-1}^{42} = 0, \quad P_{-1}^{43} = \dfrac{\frac{\sqrt{\rho k}}{\sqrt{I_\rho EI}}}{-2\left(\sqrt{\frac{I_\rho}{EI}} - \sqrt{\frac{\rho}{k}}\right)}. \end{cases} \tag{3.15}$$

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Then, let us calculate diagP_{-1}. From (3.12), we have

$$\frac{\mathrm{d}P_{-1}^{ii}}{\mathrm{d}x} = \widetilde{\Lambda}_0^{ii} P_{-1}^{ii} + \sum_{j=1, j\neq i}^{4} \widetilde{\Lambda}_0^{ij} P_{-1}^{ji} + \widetilde{\Lambda}_{-1}^{ii} P_0^{ii}$$

which yields that the diagonal entries of P_{-1} are

$$\begin{cases} P_{-1}^{11} = x\left[\sum\limits_{j=1,j\neq 1}^{4} \frac{\widetilde{\Lambda}_0^{1j}\widetilde{\Lambda}_0^{j1}}{\sqrt{\frac{\rho}{k}}-\widetilde{\Lambda}_1^{jj}}\right], \quad P_{-1}^{22} = x\left[\sum\limits_{j=1,j\neq 2}^{4} \frac{\widetilde{\Lambda}_0^{2j}\widetilde{\Lambda}_0^{j2}}{\sqrt{\frac{I_\rho}{EI}}-\widetilde{\Lambda}_1^{jj}} + \frac{k}{2\sqrt{I_\rho EI}}\right], \\ P_{-1}^{33} = x\left[\sum\limits_{j=1,j\neq 3}^{4} \frac{\widetilde{\Lambda}_0^{3j}\widetilde{\Lambda}_0^{j3}}{-\sqrt{\frac{\rho}{k}}-\widetilde{\Lambda}_1^{jj}}\right], \quad P_{-1}^{44} = x\left[\sum\limits_{j=1,j\neq 4}^{4} \frac{\widetilde{\Lambda}_0^{4j}\widetilde{\Lambda}_0^{j4}}{-\sqrt{\frac{I_\rho}{EI}}-\widetilde{\Lambda}_1^{jj}} - \frac{k}{2\sqrt{I_\rho EI}}\right]. \end{cases} \tag{3.16}$$

Thus, all entries of P_{-1} have been identified. Similarly, we can calculate P_{-i}, $i = 2, 3, \ldots$ by (3.13). Therefore, the following result holds.

Lemma 3.1. *Suppose that condition (3.14) is fulfilled. Under the transformation $Y(x) = T_0 Z(x)$, the expression of the asymptotic fundamental matrix solution to (3.6) is given as follows*

$$\widetilde{E}(x,\lambda) = \sum_{k=0}^{\infty} \frac{P_{-k}(x)}{\lambda^k} E(x,\lambda), \tag{3.17}$$

where $E(x,\lambda)$ is defined by (3.9), $P_0 = I$, P_{-1} are given by (3.15) and (3.16) and $P_{-k,\ k=2,3,\ldots}$ can be calculated similarly. Thus, by the inverse transformation, the fundamental matrix solution to (3.4) is $T_0\widetilde{E}(x,\lambda)$.

3.2. Asymptotic spectrum of $\mathcal{A}$

In this subsection, we shall consider the distribution of the spectrum of $\mathcal{A}$ using the Birkhöff asymptotic technique (see [7]). According to (3.1), we have

$$p(1) = p(0)\mathrm{e}^{-\tau_1\lambda} = \lambda w(1)\mathrm{e}^{-\tau_1\lambda}, \quad q(1) = q(0)\mathrm{e}^{-\tau_2\lambda} = \lambda\varphi(1)\mathrm{e}^{-\tau_2\lambda}. \tag{3.18}$$

Substituting (3.18) into the last two boundary conditions of (3.1) yields

$$k(w_x - \varphi)(1) = -\alpha_1\mu_1\lambda w(1) - \alpha_1(1-\mu_1)\lambda w(1)\mathrm{e}^{-\tau_1\lambda},\ EI\varphi_x(1) = -\alpha_2\mu_2\lambda\varphi(1) - \alpha_2(1-\mu_2)\lambda\varphi(1)\mathrm{e}^{-\tau_2\lambda}. \tag{3.19}$$

Let Y be defined by (3.3). Together with (3.4), we translate (3.1) into the following matrix form

$$\begin{cases} \frac{\mathrm{d}Y}{\mathrm{d}x} = \widetilde{A}Y, \\ B_1Y(0) + B_2Y(1) = 0 \end{cases}$$

where $\widetilde{A}$ is given by (3.4) and

$$B_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_1\mu_1 + \alpha_1(1-\mu_1)\mathrm{e}^{-\tau_1\lambda} & -\frac{k}{\lambda} & k & 0 \\ 0 & \alpha_2\mu_2 + \alpha_2(1-\mu_2)\mathrm{e}^{-\tau_2\lambda} & 0 & EI \end{bmatrix}.$$

Set

$$H(\lambda) := B_1T_0 + B_2T_0\widetilde{E}(1,\lambda)$$

where T_0 and $\widetilde{E}(1,\lambda)$ are given by (3.5) and (3.17), respectively.

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Then, similarly with the proof in [23], we get the following result:

Lemma 3.2. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\lambda \in \sigma(\mathcal{A})$ if and only if λ satisfies*

$$\Delta(\lambda) := \det H(\lambda) = 0. \tag{3.20}$$

Since all coefficients in $\Delta(\lambda)$ are real constants, we have:

Corollary 3.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the eigenvalues of $\mathcal{A}$ distribute in conjugate pairs on the complex plane,* i.e., $\sigma(\mathcal{A}) = \overline{\sigma(\mathcal{A})}$.

In order to get the spectrum of $\mathcal{A}$, from Lemma 3.2, it is sufficient to identify the zeros of $\Delta(\lambda)$. For convenience, set

$$[B]_1 := B + \mathcal{O}(\lambda^{-1}).$$

Then $\widetilde{E}(1,\lambda)$ can be rewritten as follows

$$\widetilde{E}(1,\lambda) = \begin{bmatrix} [1]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} & [0]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [0]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & [1]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} & [0]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [0]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} & [1]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [0]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} & [0]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & [1]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \end{bmatrix}.$$

Hence,

$$\begin{aligned} H(\lambda) &= B_1T_0 + B_2T_0\widetilde{E}(1,\lambda) \\ &= \begin{bmatrix} [1]_1 + [0]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [0]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & [1]_1 + [0]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [\alpha_1\mu_1 + \alpha_1(1-\mu_1)e^{-\tau_1\lambda} + \sqrt{\rho k}]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & -[0]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [0]_1 e^{\lambda\sqrt{\frac{\rho}{k}}} & [\alpha_2\mu_2 + \alpha_2(1-\mu_2)e^{-\tau_2\lambda} + \sqrt{I_\rho EI}]_1 e^{\lambda\sqrt{\frac{I_\rho}{EI}}} \end{bmatrix} \\ &\quad \begin{matrix} [1]_1 + [0]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [0]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & [1]_1 + [0]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [\alpha_1\mu_1 + \alpha_1(1-\mu_1)e^{-\tau_1\lambda} - \sqrt{\rho k}]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & -[0]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\ [0]_1 e^{-\lambda\sqrt{\frac{\rho}{k}}} & [\alpha_2\mu_2 + \alpha_2(1-\mu_2)e^{-\tau_2\lambda} - \sqrt{I_\rho EI}]_1 e^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \end{matrix}_{4\times 4}. \end{aligned}$$

Therefore, when $\Re\lambda \to +\infty$,

$$\left| \lim_{\Re\lambda\to+\infty} \frac{\Delta(\lambda)}{e^{\lambda\left(\sqrt{\frac{\rho}{k}}+\sqrt{\frac{I_\rho}{EI}}\right)}} \right| = (\alpha_1\mu_1 + \sqrt{\rho k})(\alpha_2\mu_2 + \sqrt{I_\rho EI}). \tag{3.21}$$

Similarly, when $\Re\lambda \to -\infty$,

$$\left| \lim_{\Re\lambda\to-\infty} \frac{\Delta(\lambda)}{e^{-\lambda\left(\sqrt{\frac{\rho}{k}}+\sqrt{\frac{I_\rho}{EI}}+\tau_1+\tau_2\right)}} \right| = \alpha_1\alpha_2(1-\mu_1)(1-\mu_2). \tag{3.22}$$

Since $\mu_i < 1,\ i = 1, 2$, (3.21) and (3.22) together with the dissipativity of $\mathcal{A}$ imply the following result:

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Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the spectrum of $\mathcal{A}$ is contained in a strip parallel to imaginary axis,* i.e.*, there exists a sufficiently large positive constant h such that*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} | \Delta(\lambda) = 0\} \subset \{\lambda \in \mathbb{C} | -h \leq \Re\lambda \leq 0\}.$$

Thus, when $|\Re\lambda| \leq h$ and $|\lambda|$ is large enough, we have

$$\begin{aligned}
\Delta(\lambda) &= \det(B_1(0)T_0 + B_2(1)T_0\widetilde{E}(1,\lambda)) \\
&= \det \begin{bmatrix}
[1]_1 + [0]_1 \mathrm{e}^{\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 \mathrm{e}^{\lambda\sqrt{\frac{I_\rho}{EI}}} \\
[0]_1 \mathrm{e}^{\lambda\sqrt{\frac{\rho}{k}}} & [1]_1 + [0]_1 \mathrm{e}^{\lambda\sqrt{\frac{I_\rho}{EI}}} \\
[\alpha_1\mu_1 + \alpha_1(1-\mu_1)\mathrm{e}^{-\tau_1\lambda} + \sqrt{\rho k}]_1 \mathrm{e}^{\lambda\sqrt{\frac{\rho}{k}}} & -[0]_1 \mathrm{e}^{\lambda\sqrt{\frac{I_\rho}{EI}}} \\
[0]_1 \mathrm{e}^{\lambda\sqrt{\frac{\rho}{k}}} & [\alpha_2\mu_2 + \alpha_2(1-\mu_2)\mathrm{e}^{-\tau_2\lambda} + \sqrt{I_\rho EI}]_1 \mathrm{e}^{\lambda\sqrt{\frac{I_\rho}{EI}}}
\end{bmatrix} \\
&\quad \begin{bmatrix}
[1]_1 + [0]_1 \mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}} & [0]_1 \mathrm{e}^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\
[0]_1 \mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}} & [1]_1 + [0]_1 \mathrm{e}^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\
[\alpha_1\mu_1 + \alpha_1(1-\mu_1)\mathrm{e}^{-\tau_1\lambda} - \sqrt{\rho k}]_1 \mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}} & -[0]_1 \mathrm{e}^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \\
[0]_1 \mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}} & [\alpha_2\mu_2 + \alpha_2(1-\mu_2)\mathrm{e}^{-\tau_2\lambda} - \sqrt{I_\rho EI}]_1 \mathrm{e}^{-\lambda\sqrt{\frac{I_\rho}{EI}}}
\end{bmatrix}_{4\times 4} \\
&= \Delta_1(\lambda)\Delta_2(\lambda) + \mathcal{O}(\lambda^{-1})
\end{aligned}$$

where

$$\Delta_1(\lambda) := \det \begin{bmatrix} 1 & 1 \\ [\alpha_1\mu_1 + \alpha_1(1-\mu_1)\mathrm{e}^{-\tau_1\lambda} + \sqrt{\rho k}]\mathrm{e}^{\lambda\sqrt{\frac{\rho}{k}}} & [\alpha_1\mu_1 + \alpha_1(1-\mu_1)\mathrm{e}^{-\tau_1\lambda} - \sqrt{\rho k}]\mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}} \end{bmatrix} \tag{3.23}$$

and

$$\Delta_2(\lambda) := \det \begin{bmatrix} 1 & 1 \\ [\alpha_2\mu_2 + \alpha_2(1-\mu_2)\mathrm{e}^{-\tau_2\lambda} + \sqrt{I_\rho EI}]\mathrm{e}^{\lambda\sqrt{\frac{I_\rho}{EI}}} & [\alpha_2\mu_2 + \alpha_2(1-\mu_2)\mathrm{e}^{-\tau_2\lambda} - \sqrt{I_\rho EI}]\mathrm{e}^{-\lambda\sqrt{\frac{I_\rho}{EI}}} \end{bmatrix}. \tag{3.24}$$

According to the discussion above, we have:

Lemma 3.3. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the asymptotic values of the spectrum of $\mathcal{A}$ can be determined by $\Delta_1(\lambda) = 0$ and $\Delta_2(\lambda) = 0$ which are given by* (3.23) *and* (3.24).

Hence, let us discuss the zeros of $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$, which can imply the asymptotic distribution of the spectrum of $\mathcal{A}$.

Lemma 3.4. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and the condition* (3.14) *be fulfilled. Then the zeros of $\Delta_i(\lambda)$, $i = 1, 2$ are at most of degree two and separated. Furthermore, if $\sqrt{\frac{\rho}{k}}/\tau_1$ and $\sqrt{\frac{I_\rho}{EI}}/\tau_2$ are irrational constants, then all of the zeros of $\Delta_i(\lambda)$, $i = 1, 2$ are simple.*

Proof. Firstly, we consider the zeros of $\Delta_1(\lambda)$. A direct calculation yields

$$\Delta_1(\lambda) = -2\alpha_1\mu_1 \sinh\lambda\sqrt{\frac{\rho}{k}} - 2\sqrt{\rho k}\cosh\lambda\sqrt{\frac{\rho}{k}} - 2(1-\mu_1)\alpha_1\mathrm{e}^{-\tau_1\lambda}\sinh\lambda\sqrt{\frac{\rho}{k}}. \tag{3.25}$$

Then differentiating (3.25), we have

$$\begin{cases}
\Delta_1'(\lambda) = -2\alpha_1\mu_1\sqrt{\frac{\rho}{k}}\cosh\lambda\sqrt{\frac{\rho}{k}} - 2\rho\sinh\lambda\sqrt{\frac{\rho}{k}} - 2\sqrt{\frac{\rho}{k}}(1-\mu_1)\alpha_1\mathrm{e}^{-\tau_1\lambda}\cosh\lambda\sqrt{\frac{\rho}{k}} \\
\qquad\quad + 2(1-\mu_1)\alpha_1\tau_1\mathrm{e}^{-\tau_1\lambda}\sinh\lambda\sqrt{\frac{\rho}{k}}, \\
\Delta_1''(\lambda) = -2\alpha_1\mu_1\frac{\rho}{k}\sinh\lambda\sqrt{\frac{\rho}{k}} - 2\frac{\rho}{k}\sqrt{\rho k}\cosh\lambda\sqrt{\frac{\rho}{k}} - 2(1-\mu_1)\alpha_1\frac{\rho}{k}\mathrm{e}^{-\tau_1\lambda}\sinh\lambda\sqrt{\frac{\rho}{k}} \\
\qquad\quad + 4\sqrt{\frac{\rho}{k}}(1-\mu_1)\alpha_1\tau_1\mathrm{e}^{-\tau_1\lambda}\cosh\lambda\sqrt{\frac{\rho}{k}} - 2(1-\mu_1)\alpha_1\tau_1^2\mathrm{e}^{-\tau_1\lambda}\sinh\lambda\sqrt{\frac{\rho}{k}}.
\end{cases} \tag{3.26}$$

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We shall show that the zeros of $\Delta_1(\lambda)$ are at most of degree two. It only needs to prove that for any $\widetilde{\lambda} \in \mathbb{C}$, if $\Delta_1(\widetilde{\lambda}) = \Delta_1'(\widetilde{\lambda}) = 0$, it must hold that $\Delta_1''(\widetilde{\lambda}) \neq 0$.

Since $\Delta_1(\widetilde{\lambda}) = 0$ implies $\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}} \neq 0$, we have

$$\frac{\Delta_1(\widetilde{\lambda})}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} = -2\alpha_1\mu_1 - 2\sqrt{\rho k}\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} - 2(1-\mu_1)\alpha_1 \mathrm{e}^{-\tau_1\widetilde{\lambda}}, \tag{3.27}$$

$$\frac{\Delta_1'(\widetilde{\lambda})}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} = -2\alpha_1\mu_1\sqrt{\frac{\rho}{k}}\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} - 2\rho - 2(1-\mu_1)\alpha_1\sqrt{\frac{\rho}{k}}\mathrm{e}^{-\tau_1\widetilde{\lambda}}\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} + 2(1-\mu_1)\alpha_1\tau_1\mathrm{e}^{-\tau_1\widetilde{\lambda}}. \tag{3.28}$$

Hence, $\Delta_1(\widetilde{\lambda}) = 0$ yields

$$(1-\mu_1)\alpha_1\mathrm{e}^{-\tau_1\widetilde{\lambda}} = -\alpha_1\mu_1 - \sqrt{\rho k}\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}. \tag{3.29}$$

Then, substituting (3.29) into (3.28) leads to

$$\frac{\Delta_1'(\widetilde{\lambda})}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} = 2\rho\left(\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}\right)^2 - 2\tau_1\sqrt{\rho k}\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} - 2\rho - 2\tau_1\alpha_1\mu_1.$$

Thus, from $\Delta_1'(\widetilde{\lambda}) = 0$, it holds that

$$\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} = \frac{\tau_1\sqrt{\rho k} \pm \sqrt{\tau_1^2\rho k + 4\rho(\rho + \tau_1\alpha_1\mu_1)}}{2\rho}. \tag{3.30}$$

According to (3.26), we have

$$\begin{aligned}\frac{\Delta_1''(\widetilde{\lambda})}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} &= \frac{\rho}{k}\frac{\Delta_1(\widetilde{\lambda})}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} + 4(1-\mu_1)\alpha_1\tau_1\mathrm{e}^{-\tau_1\widetilde{\lambda}}\sqrt{\frac{\rho}{k}}\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} - 2(1-\mu_1)\alpha_1\tau_1^2\mathrm{e}^{-\tau_1\widetilde{\lambda}} \\ &= 2(1-\mu_1)\alpha_1\tau_1\mathrm{e}^{-\tau_1\widetilde{\lambda}}\left[2\sqrt{\frac{\rho}{k}}\frac{\cosh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} - \tau_1\right].\end{aligned}$$

Therefore, (3.30) leads to $\frac{\Delta_1''(\widetilde{\lambda})}{\sinh \widetilde{\lambda}\sqrt{\frac{\rho}{k}}} \neq 0$, which implies that the zeros of $\Delta_1(\lambda)$ are at most of degree two.

Now, we suppose that $\widetilde{\lambda}$ is a zero of $\Delta_1(\lambda)$ of degree two. Then (3.29) and (3.30) must hold.

From (3.30), a direct calculation yields

$$\mathrm{e}^{2\widetilde{\lambda}\sqrt{\frac{\rho}{k}}} = \begin{cases} \dfrac{-\rho - \frac{1}{2}\left(\tau_1\sqrt{\rho k} + \sqrt{\tau_1^2\rho k + 4\rho(\rho + \tau_1\alpha_1\mu_1)}\right)}{\rho - \frac{1}{2}\left(\tau_1\sqrt{\rho k} + \sqrt{\tau_1^2\rho k + 4\rho(\rho + \tau_1\alpha_1\mu_1)}\right)}, & \text{take} + \text{sign in (3.30)}, \\ \dfrac{-\rho - \frac{1}{2}\left(\tau_1\sqrt{\rho k} - \sqrt{\tau_1^2\rho k + 4\rho(\rho + \tau_1\alpha_1\mu_1)}\right)}{\rho - \frac{1}{2}\left(\tau_1\sqrt{\rho k} - \sqrt{\tau_1^2\rho k + 4\rho(\rho + \tau_1\alpha_1\mu_1)}\right)}, & \text{take} - \text{sign in (3.30)}. \end{cases} \tag{3.31}$$

Substituting (3.30) into (3.29), we get

$$\mathrm{e}^{\tau_1\widetilde{\lambda}} = \begin{cases} \dfrac{2\rho(1-\mu_1)\alpha_1}{-2\alpha_1\mu_1\rho - \sqrt{\rho k}\left(\tau_1\sqrt{\rho k} + \sqrt{\tau_1^2\rho k + 4\rho(\rho + \tau_1\alpha_1\mu_1)}\right)}, & \text{take} + \text{sign in (3.30)}, \\ \dfrac{2\rho(1-\mu_1)\alpha_1}{-2\alpha_1\mu_1\rho - \sqrt{\rho k}\left(\tau_1\sqrt{\rho k} - \sqrt{\tau_1^2\rho k + 4\rho(\rho + \tau_1\alpha_1\mu_1)}\right)}, & \text{take} - \text{sign in (3.30)}. \end{cases} \tag{3.32}$$

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Set $\widetilde{\lambda} = x+yi,\ x, y \in \mathbb{R}$. Then by (3.31) and (3.32), we get that $e^{\tau_1\widetilde{\lambda}},\ e^{2\widetilde{\lambda}\sqrt{\frac{\rho}{k}}} \in \mathbb{R}$. Thus, $\sin 2y\sqrt{\frac{\rho}{k}} = \sin y\tau_1 = 0$, which leads to

$$2y\sqrt{\frac{\rho}{k}} = n\pi, \quad \tau_1 y = m\pi$$

for some integers n and m. Hence $\sqrt{\frac{\rho}{k}}/\tau_1 = \frac{n}{2m}$ is a rational number. Therefore, if $\sqrt{\frac{\rho}{k}}/\tau_1$ is an irrational constant, then all zeros of $\Delta_1(\lambda)$ are simple.

Let us now show the separability of the zeros of $\Delta_1(\lambda)$, *i.e.*, there is a positive constant $\delta > 0$ such that

$$\inf_{\widetilde{\xi}\neq\widetilde{\lambda},\ \widetilde{\xi},\widetilde{\lambda}\in\{\lambda|\Delta_1(\lambda)=0\}} |\lambda - \xi| \geq \delta.$$

We shall show the separability by the following two cases:

Case 1: $\sqrt{\frac{\rho}{k}}/\tau_1 = \frac{n}{2m}$ is rational with some integers n, m.

Set $z = e^{\frac{\lambda\sqrt{\frac{\rho}{k}}}{n}}$. Then $\Delta_1(\lambda) = 0$ is equivalent to the following equation:

$$(\alpha_1\mu_1 + \sqrt{\rho k})z^{2m+2n} + (1-\mu_1)\alpha_1 z^{2n} - (\alpha_1\mu_1 + \sqrt{\rho k})z^{2m} - (1-\mu_1)\alpha_1 = 0.$$

Since the number of the zeros of the above polynomial equation is at most $(2m + 2n)$ and these zeros are separated, the zeros of $\Delta_1(\lambda)$ are also separated.

Case 2: $\sqrt{\frac{\rho}{k}}/\tau_1$ is irrational.

From (3.25), we can get that $\widetilde{\lambda} \in \{\lambda|\Delta_1(\lambda) = 0\}$ implies

$$0 < \inf_{\widetilde{\lambda}\in\{\lambda|\Delta_1(\lambda)=0\}} \left|\sinh\widetilde{\lambda}\sqrt{\frac{\rho}{k}}\right| \leq \sup_{\widetilde{\lambda}\in\{\lambda|\Delta_1(\lambda)=0\}} \left|\sinh\widetilde{\lambda}\sqrt{\frac{\rho}{k}}\right| < \infty. \tag{3.33}$$

(3.33) together with (3.31) and (3.32) yields that when $\sqrt{\frac{\rho}{k}}/\tau_1$ is irrational, $\inf_{\widetilde{\lambda}\in\{\lambda|\Delta_1(\lambda)=0\}} |\Delta_1'(\lambda)| \neq 0$, which yields that the zeros of $\Delta_1(\lambda)$ are separated by [20].

A similar method can be used to deal with the zeros of $\Delta_2(\lambda)$ and the corresponding result follows. The proof is complete. □

Applying the Rouché Theorem (see [1]), we have the following result about the asymptotic spectrum of $\mathcal{A}$:

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and the condition (3.14) be fulfilled. Then the asymptotic eigenvalues of $\mathcal{A}$ are at most of degree two and separated. Furthermore, if $\sqrt{\frac{\rho}{k}}/\tau_1$ and $\sqrt{\frac{I_\rho}{EI}}/\tau_2$ are irrational constants, then the asymptotic eigenvalues of $\mathcal{A}$ are simple.*

4. Riesz basis property of the (generalized) eigenvectors

In this section we shall discuss the Riesz basis property of the eigenvectors and generalized eigenvectors of $\mathcal{A}$. Firstly, let us establish the completeness of the eigenvectors and generalized eigenvectors of $\mathcal{A}$.

To this end, we define an auxiliary operator $\mathcal{A}_0$ in $\mathcal{H}$:

$$\mathcal{A}_0(w, z, p, \varphi, \psi, q)^\tau = \begin{pmatrix} z \\ \rho^{-1}kw_{xx} \\ -\tau_1^{-1}p_x \\ \psi \\ I_\rho^{-1}EI\varphi_{xx} \\ -\tau_2^{-1}q_x \end{pmatrix} \tag{4.1}$$

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with domain

$$\mathcal{D}(\mathcal{A}_0) = \left\{ (w,z,p,\varphi,\psi,q)^\tau \in V^2(0,1)\times V^1(0,1)\times H^1(0,1)\times V^2(0,1)\times V^1(0,1)\times H^1(0,1) \;\middle|\; \begin{array}{l} kw'(1) = -\alpha_1\mu_1 z(1) - \alpha_1(1-\mu_1)p(1),\ z(1) = p(0) \\ EI\varphi'(1) = -\alpha_2\mu_2\psi(1) - \alpha_2(1-\mu_2)q(1),\ v(1) = q(0) \end{array} \right\}.$$

Note that there is no coupled term of $(w,z,p)^\tau$ and $(\varphi,\psi,q)^\tau$ in $\mathcal{A}_0$ and $\mathcal{D}(\mathcal{A}_0)$. Thus, it becomes two wave systems, *i.e.*,

$$\mathcal{A}_0^1(w,z,p)^\tau = \begin{pmatrix} z \\ \rho^{-1}kw_{xx} \\ -\tau_1^{-1}p_x \end{pmatrix}, \qquad \mathcal{A}_0^2(\varphi,\psi,q)^\tau = \begin{pmatrix} \psi \\ I_\rho^{-1}EI\varphi_{xx} \\ -\tau_2^{-1}q_x \end{pmatrix}, \tag{4.2}$$

where

$$\mathcal{D}(\mathcal{A}_0^1) = \left\{ (w,z,p)^\tau \in V^2(0,1)\times V^1(0,1)\times H^1(0,1) \;\middle|\; kw'(1) = -\alpha_1\mu_1 z(1) - \alpha_1(1-\mu_1)p(1),\ z(1) = p(0) \right\}$$

and

$$\mathcal{D}(\mathcal{A}_0^2) = \left\{ (\varphi,\psi,q)^\tau \in V^2(0,1)\times V^1(0,1)\times H^1(0,1) \;\middle|\; EI\varphi'(1) = -\alpha_2\mu_2\psi(1) - \alpha_2(1-\mu_2)q(1),\ v(1) = q(0) \right\}.$$

From Xu *et al.* [24], we get the following lemma:

Lemma 4.1. *Let $\mathcal{A}_0^1$ and $\mathcal{A}_0^2$ be defined by (4.2). Then $\mathcal{A}_0^1$ and $\mathcal{A}_0^2$ generate a C_0 group on $V^1[0,1]\times L^2[0,1]\times L^2[0,1]$, respectively.*

Therefore, we have:

Lemma 4.2. *Let $\mathcal{H}$ and $\mathcal{A}_0$ be defined by (2.1) and (4.1). Then $\mathcal{A}_0$ generates a C_0 group on $\mathcal{H}$.*

Now, we shall use the property of $\mathcal{A}_0$ to discuss the corresponding property of $\mathcal{A}$. The following result holds.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\mathcal{A}$ generates a C_0 group on $\mathcal{H}$. Furthermore, the system of the eigenvectors and generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

Proof. We define the transform operator $\mathcal{P}:\mathcal{H}\to\mathcal{H}$ as follows:

$$\mathcal{P}(\hat{w},z,p,\varphi,\psi,q)^\tau = (w,z,p,\varphi,\psi,q)^\tau,$$

where $w(x) = \hat{w}(x) + x\varphi(1)$. Then

$$w_x = \hat{w}_x + \varphi(1).$$

Hence, the operator $\mathcal{P}$ maps $\mathcal{D}(\mathcal{A}_0)$ onto $\mathcal{D}(\mathcal{A})$. Obviously, $\mathcal{P}$ is a bijective operator. Then, for $(\hat{w},z,p,\varphi,\psi,q)^\tau \in \mathcal{D}(\mathcal{A}_0)$, we have

$$\mathcal{P}^{-1}\mathcal{A}\mathcal{P}\begin{pmatrix} \hat{w} \\ z \\ p \\ \varphi \\ \psi \\ q \end{pmatrix} = \begin{pmatrix} z - x\varphi(1) \\ \rho^{-1}k(\hat{w}_{xx} - \varphi_x) \\ -\tau_1^{-1}p_x \\ \psi \\ \mathcal{I}_\rho^{-1}EI\varphi_{xx} + \mathcal{I}_\rho^{-1}k(\hat{w}_x + \varphi(1) - \varphi) \\ -\tau_2^{-1}q_x \end{pmatrix}. \tag{4.3}$$

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Define the operator $\mathcal{B}$ on $\mathcal{H}$ as follows:

$$\mathcal{B}\begin{pmatrix}\hat{w}\\ z\\ p\\ \varphi\\ \psi\\ q\end{pmatrix}=\begin{pmatrix}-x\varphi(1)\\ -\rho^{-1}k\varphi_x\\ 0\\ 0\\ \mathcal{I}_\rho^{-1}k(\hat{w}_x+\varphi(1)-\varphi)\\ 0\end{pmatrix}. \tag{4.4}$$

Then,

$$\mathcal{A}_0+\mathcal{B}=\mathcal{P}^{-1}\mathcal{A}\mathcal{P}. \tag{4.5}$$

A direction calculation yields that there exists a positive constant $\widehat{M}$, such that

$$\|B(\hat{w},z,p,\varphi,\psi,q)^\tau\|\le\widehat{M}\|(\hat{w},z,p,\varphi,\psi,q)^\tau\|$$

which implies that $\mathcal{B}$ is a bounded linear operator. Transforming (4.5), we get

$$\mathcal{A}=\mathcal{P}\mathcal{A}_0\mathcal{P}^{-1}+\mathcal{P}\mathcal{B}\mathcal{P}^{-1}. \tag{4.6}$$

Thus, $\mathcal{A}$ is a bounded perturbation of $\mathcal{P}\mathcal{A}_0\mathcal{P}^{-1}$. We know from Lemma 4.2 that $\mathcal{A}_0$ generates a C_0 group on $\mathcal{H}$. Therefore, it holds that $\mathcal{A}$ also generates a C_0 group on $\mathcal{H}$ due to perturbation theorem.

Now let us show the completeness of the eigenvectors and generalized eigenvectors of $\mathcal{A}$. We assume that $\sigma(\mathcal{A})=\{\lambda_k,\ k\in\mathbb{N}\}$ due to Corollary 2.1. The completeness of the eigenvectors and generalized eigenvectors of $\mathcal{A}$ is that

$$\mathrm{Span}(\mathcal{A})=\overline{\left\{\sum_k y_k, y_k\in E(\lambda_k,\mathcal{A})\mathcal{H},\lambda_k\in\sigma(\mathcal{A})\right\}}=\mathcal{H},$$

where $E(\lambda_k,\mathcal{A})$ is the Riesz projection corresponding to λ_k.

Assume that $Y_0=(y_1,y_2,y_3,y_4,y_5,y_6)^\tau\in\mathcal{H}$, $Y_0\perp\mathrm{Span}(\mathcal{A})$. Then for $\lambda\in\mathbb{C}$, $R^*(\lambda,\mathcal{A})Y_0$ is a $\mathcal{H}$-valued entire function. Thus for any $F=(f^1,f^2,f^3,f^4,f^5,f^6)^\tau\in\mathcal{H}$, the function

$$G(\lambda)=(F,R^*(\lambda,\mathcal{A})Y_0)_{\mathcal{H}},\quad\forall\lambda\in\mathbb{C} \tag{4.7}$$

is an entire function. Since $\mathcal{A}$ generates a C_0 group, we have that $\lim\limits_{\Re\lambda\to\infty}G(\lambda)=0$. In particular, for $\lambda\in\rho(\mathcal{A})$, it holds that $G(\lambda)=(R(\lambda,\mathcal{A})F,Y_0)_{\mathcal{H}}$. Then the dissipativity of $\mathcal{A}$ ensures that $|G(\lambda)|$ is bounded on the domain $\Re\lambda\ge\widehat{\alpha}>0$. Since $G(\lambda)$ is an entire function of finite exponential type, the Phragmén-Lindelöf Theorem (see [25]) asserts that $|G(\lambda)|$ is bounded on the sectors region: $\widehat{\Omega}_1=\{\lambda\in\mathbb{C}\mid\Re\lambda\le\widehat{\alpha},\ \Im\lambda\ge0\}$ and $\widehat{\Omega}_2=\{\lambda\in\mathbb{C}\mid\Im\lambda\le0,\ \Re\lambda\le\widehat{\alpha}\}$ due to $G(\lambda)$ is bounded on their boundary lines $\Re\lambda=\widehat{\alpha}$ and $\Im\lambda=0$. Therefore, $|G(\lambda)|$ is uniformly bounded on $\mathbb{C}$, *i.e.*, $|G(\lambda)|\le M$, $\forall\lambda\in\mathbb{C}$.

Then by the Liouville's theorem, we have $G(\lambda)\equiv0$ since $\lim\limits_{\Re\lambda\to\infty}G(\lambda)=0$. Note that $G(\lambda)=(F,R^*(\lambda,\mathcal{A})Y_0)_{\mathcal{H}}$ holds for any $F\in\mathcal{H}$. It must be $R^*(\lambda,\mathcal{A})Y_0=0$, which means $Y_0=0$. Therefore, $\mathrm{Span}(\mathcal{A})=\mathcal{H}$. The desired result follows. □

In order to get the Riesz basis property of the (generalized) eigenvectors, we need the following result from [22].

Theorem 4.2. *Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{A}$ be the generator of a C_0 semigroup $T(t)$ on $\mathcal{H}$. Suppose that:*

(1) *$\sigma(\mathcal{A})=\sigma_1(\mathcal{A})\cup\sigma_2(\mathcal{A})$, where $\sigma_2(\mathcal{A})=\{\lambda_k\}_{k=1}^\infty$ consists of isolated eigenvalues of $\mathcal{A}$ with finite multiplicity;*

(2) *$\sup\limits_{k\ge1}m_a(\lambda_k)<\infty$, where $m_a(\lambda_k)=\dim E(\lambda_k,\mathcal{A})\mathcal{H}$ and $E(\lambda_k,\mathcal{A})$ is the Riesz projector associated with λ_k;*

(3) *There is a constant α such that $\sup\{\mathrm{Re}\lambda|\lambda\in\sigma_1(\mathcal{A})\}\le\alpha\le\inf\{\mathrm{Re}\lambda|\lambda\in\sigma_2(\mathcal{A})\}$ and $\inf\limits_{n\ne m}|\lambda_n-\lambda_m|>0$.*

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Then the following assertions are true.

(i) *There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ with the property that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$, $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$, $E(\lambda_k, \mathcal{A})\mathcal{H}_2$ forms a subspace Riesz basis for $\mathcal{H}_2$, and $\mathcal{H} = \overline{\mathcal{H}_1 \bigoplus \mathcal{H}_2}$.*

(ii) *If $\sup_{k\geq 1} ||E(\lambda_k, \mathcal{A})|| < \infty$, then $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}$.*

(iii) *$\mathcal{H}$ has the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ (topological direct sum) if and only if*

$$\sup_{n\geq 1} \left\| \sum_{k=1}^{n} E(\lambda_k, \mathcal{A}) \right\| < \infty.$$

Applying Corollary 2.1, and Theorems 3.1, 3.2, 4.1, 4.2 to our problem, we get the following result:

Theorem 4.3. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before and the condition (2.8) be fulfilled. If (3.14) is satisfied, then there is a sequence of the eigenvectors and generalized eigenvectors of $\mathcal{A}$ which forms a Riesz basis for $\mathcal{H}$.*

Proof. Set $\sigma_1(\mathcal{A}) = \emptyset$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. Theorems 3.1, 3.2 and Corollary 2.1 show that all hypotheses in Theorem 4.2 are fulfilled. So the results of Theorem 4.2 are true. Hence, there is a sequence of the eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a subspace Riesz basis for $\mathcal{H}_2$. Theorem 4.1 says that the (generalized) eigenvectors is complete in $\mathcal{H}$, *i.e.*, $\mathcal{H}_2 = \mathcal{H}$. Therefore, the sequence is also a Riesz basis for $\mathcal{H}$. The proof is complete. □

The Riesz basis property together with the uniform boundedness of multiplicities of the eigenvalues of $\mathcal{A}$ implies that:

Corollary 4.1. *Under the conditions (2.8) and (3.14), the closed loop system (1.7) satisfies the spectrum determined growth condition,* i.e., *$\omega(\mathcal{A}) = S(\mathcal{A})$, where $\omega(\mathcal{A}) = \lim_{t\to\infty} \frac{1}{t} \ln \|e^{\mathcal{A}t}\|$ is the growth order of $e^{\mathcal{A}t}$ and $S(\mathcal{A}) = \sup\{\Re\lambda | \lambda \in \sigma(\mathcal{A})\}$ is the spectral bound of $\mathcal{A}$.*

5. Stability of the system

In this section, the stability of the closed loop system (1.7) is discussed. Under the choice of μ_i, $i = 1, 2$, the asymptotic stability and exponential stability of the system (1.7) are gotten under certain conditions.

When $\mu_1 < \frac{1}{2}$ or $\mu_2 < \frac{1}{2}$, the following result holds.

Theorem 5.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. When $\mu_1 < \frac{1}{2}$ or $\mu_2 < \frac{1}{2}$, the system (1.7) is always unstable.*

Proof. We shall show that there always exists $\lambda \in \sigma(\mathcal{A})$ such that $\Re\lambda > 0$ when $\mu_1 < \frac{1}{2}$ or $\mu_2 < \frac{1}{2}$, which leads to the instability of the system (1.7).

Let us consider the case $\mu_1 < \frac{1}{2}$. Set $\widehat{y} := \frac{\tau_1}{\sqrt{\frac{\rho}{k}}}$. If $\widehat{y} = \frac{2(2m+1)}{2n+1}$, $n, m \in \mathbb{N}$, we set

$$\lambda := \frac{1}{\sqrt{\frac{\rho}{k}}} \left[\sqrt{\frac{k}{\rho}} \widehat{\eta} + i \left(n + \frac{1}{2} \right) \pi + i2s(2n+1)\pi \right], \ s \in \mathbb{Z} \tag{5.1}$$

with some parameter $\widehat{\eta}$. Then a direct calculation yields

$$2\sqrt{\frac{\rho}{k}}\lambda = 2\widehat{\eta} + i(2n+1)\pi + i4s(2n+1)\pi, \ \lambda\tau_1 = \frac{\lambda\sqrt{\frac{\rho}{k}}}{\sqrt{\frac{\rho}{k}}}\tau_1 = \widehat{\eta}\tau_1 + i(2m+1)\pi + i4s(2m+1)\pi, \ s \in \mathbb{Z}.$$

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Hence,

$$\begin{aligned}\Delta_1^0(\widehat{\eta}) &:= \mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}}\Delta_1(\lambda)\\ &= \mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}}\left[-2\alpha_1\mu_1\sinh\lambda\sqrt{\frac{\rho}{k}}-2\sqrt{\rho k}\cosh\lambda\sqrt{\frac{\rho}{k}}-2(1-\mu_1)\alpha_1\mathrm{e}^{-\tau_1\lambda}\sinh\lambda\sqrt{\frac{\rho}{k}}\right]\\ &= -\sqrt{\rho k}(1-\mathrm{e}^{-2\widehat{\eta}})-\left(\alpha_1\mu_1-\alpha_1(1-\mu_1)\mathrm{e}^{-\sqrt{\frac{k}{\rho}}\widehat{\eta}\tau_1}\right)(1+\mathrm{e}^{-2\widehat{\eta}})\end{aligned}$$

where $\Delta_1(\lambda)$ is given by (3.23). Since $\Delta_1^0(0) = -2(2\alpha_1\mu_1-\alpha_1) > 0$ and $\lim_{\widehat{\eta}\to+\infty}\Delta_1^0(\widehat{\eta}) = -\sqrt{\rho k}-\alpha_1\mu_1 < 0$, there is at least one $\widehat{\eta} > 0$ such that $\Delta_1^0(\widehat{\eta}) = 0$. Thus, for this $\widehat{\eta}$, the complex values λ given by (5.1) are the zeros of $\Delta_1(\lambda)$.

Now let $\widehat{y}$ be any positive real value. We can choose a sequence of rational numbers, $\widehat{y}_{n,m} := \frac{2(2m+1)}{2n+1}$, such that $\lim_{n,m\to\infty}\widehat{y}_{n,m} = \widehat{y}$. Set $\widehat{y} = \widehat{y}_{n,m}+\varepsilon_{n,m}$, in which $\varepsilon_{n,m}\to 0$ when $n,m\to\infty$. We compare

$$\Delta_1^{\widehat{y}}(\lambda) := \frac{1}{\lambda}\left[-2\alpha_1\mu_1\sinh\lambda\sqrt{\frac{\rho}{k}}-2\sqrt{\rho k}\cosh\lambda\sqrt{\frac{\rho}{k}}-2(1-\mu_1)\alpha_1\mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}\widehat{y}}\sinh\lambda\sqrt{\frac{\rho}{k}}\right]$$

with

$$\Delta_1^{\widehat{y}_{n,m}}(\lambda) := \frac{1}{\lambda}\left[-2\alpha_1\mu_1\sinh\lambda\sqrt{\frac{\rho}{k}}-2\sqrt{\rho k}\cosh\lambda\sqrt{\frac{\rho}{k}}-2(1-\mu_1)\alpha_1\mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}\widehat{y}_{n,m}}\sinh\lambda\sqrt{\frac{\rho}{k}}\right].$$

Suppose that $\lambda_{n,m}$ is a zero for $\Delta_1^{\widehat{y}_{n,m}}(\lambda)$ with positive real part and given by (5.1). Then for $|\lambda-\lambda_{n,m}| \le \frac{1}{2}\Re\lambda_{n,m}$, using Taylor formula, we have

$$|\Delta_1^{\widehat{y}}(\lambda)-\Delta_1^{\widehat{y}_{n,m}}(\lambda)| = 2(1-\mu_1)\alpha_1\left|\lambda^{-1}\right|\left|\mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}\widehat{y}_{n,m}}-\mathrm{e}^{-\lambda\sqrt{\frac{\rho}{k}}\widehat{y}}\right|\sinh\lambda\sqrt{\frac{\rho}{k}}\le|\Delta_1^{\widehat{y}_{n,m}}(\lambda)|.$$

Hence, the Rouché theorem says that $\Delta_1^{\widehat{y}_{n,m}}(\lambda)$ and $\Delta_1^{\widehat{y}}(\lambda)$ have the same number of zeros in $|\lambda-\lambda_{n,m}| \le \frac{1}{2}\Re\lambda_{n,m}$. Since $\Delta_1^{\widehat{y}_{n,m}}(\lambda)$ has at least one zero with positive real part, the same is true for $\Delta_1^{\widehat{y}}(\lambda)$ which implies that $\Delta_1(\lambda)$ has zeros with positive real part. Let n in (5.1) be sufficiently large. By the Rouché theorem again, there exists at least one eigenvalue of $\mathcal{A}$ with positive real part. Therefore, the system (1.7) is unstable.

For the case $\mu_2 < \frac{1}{2}$, by the similar discussion, there exists at least one eigenvalue of $\mathcal{A}$ with positive real part which also lead to the instability of the system (1.7). The proof is complete. □

Now, let us consider the stability of the system (1.7) when $\mu_i \ge \frac{1}{2}$, $i = 1,2$. Firstly, from [18], we introduce the fundamental solution to (3.2) as follows

$$w(x) = w(0)w_1(x,\lambda)+\varphi(0)w_2(x,\lambda)+w_x(0)w_3(x,\lambda)+\varphi_x(0)w_4(x,\lambda), \tag{5.2}$$

$$\varphi(x) = w(0)\varphi_1(x,\lambda)+\varphi(0)\varphi_2(x,\lambda)+w_x(0)\varphi_3(x,\lambda)+\varphi_x(0)\varphi_4(x,\lambda). \tag{5.3}$$

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The expressions of $w_i,\ \varphi_i,\ i=1,2,3,4$ can be gotten directly from [18] and given as follows

$$\begin{cases} w_1(x,\lambda) &= \frac{1}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\left(\widetilde{\zeta_1}-\widetilde{a}\right)\cosh\sqrt{\widetilde{\zeta_2}}x-\left(\widetilde{\zeta_2}-\widetilde{a}\right)\cosh\sqrt{\widetilde{\zeta_1}}x\right),\\ \varphi_1(x,\lambda) &= \frac{\widetilde{a}\widetilde{c}}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\frac{1}{\sqrt{\widetilde{\zeta_1}}}\sinh\sqrt{\widetilde{\zeta_1}}x-\frac{1}{\sqrt{\widetilde{\zeta_2}}}\sinh\sqrt{\widetilde{\zeta_2}}x\right),\\ w_2(x,\lambda) &= \frac{\widetilde{b}}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\frac{1}{\sqrt{\widetilde{\zeta_1}}}\sinh\sqrt{\widetilde{\zeta_1}}x-\frac{1}{\sqrt{\widetilde{\zeta_2}}}\sinh\sqrt{\widetilde{\zeta_2}}x\right),\\ \varphi_2(x,\lambda) &= \frac{1}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\left(\widetilde{\zeta_1}-\widetilde{b}\right)\cosh\sqrt{\widetilde{\zeta_2}}x-\left(\widetilde{\zeta_2}-\widetilde{b}\right)\cosh\sqrt{\widetilde{\zeta_1}}x\right),\\ w_3(x,\lambda) &= \frac{1}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\left(\widetilde{\zeta_1}-\widetilde{b}\right)\frac{1}{\sqrt{\widetilde{\zeta_1}}}\sinh\sqrt{\widetilde{\zeta_1}}x-\left(\widetilde{\zeta_2}-\widetilde{b}\right)\frac{1}{\sqrt{\widetilde{\zeta_2}}}\sinh\sqrt{\widetilde{\zeta_2}}x\right),\\ \varphi_3(x,\lambda) &= \frac{\widetilde{c}}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\cosh\sqrt{\widetilde{\zeta_1}}x-\cosh\sqrt{\widetilde{\zeta_2}}x\right),\\ w_4(x,\lambda) &= \frac{1}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\cosh\sqrt{\widetilde{\zeta_1}}x-\cosh\sqrt{\widetilde{\zeta_2}}x\right),\\ \varphi_4(x,\lambda) &= \frac{1}{\widetilde{\zeta_1}-\widetilde{\zeta_2}}\left(\left(\widetilde{\zeta_1}-\widetilde{a}\right)\frac{1}{\sqrt{\widetilde{\zeta_1}}}\sinh\sqrt{\widetilde{\zeta_1}}x-\left(\widetilde{\zeta_2}-\widetilde{a}\right)\frac{1}{\sqrt{\widetilde{\zeta_2}}}\sinh\sqrt{\widetilde{\zeta_2}}x\right), \end{cases} \tag{5.4}$$

where

$$\widetilde{a}=\frac{\rho}{k}\lambda^2,\ \widetilde{b}=\frac{I_\rho}{EI}\lambda^2+\frac{k}{EI},\ \widetilde{c}=-\frac{k}{EI}, \tag{5.5}$$

and $\widetilde{\zeta_i},\ i=1,2$ are the two roots of the following quadratic equation:

$$\widetilde{\zeta}^2-(\widetilde{a}+\widetilde{b}+\widetilde{c})\widetilde{\zeta}+\widetilde{a}\widetilde{b}=0. \tag{5.6}$$

We have the following result:

Theorem 5.2. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then,*

(1) *when $\mu_1>\frac{1}{2},\ \mu_2>\frac{1}{2}$, the system (1.7) is asymptotically stable;*

(2) *when $\mu_1=\frac{1}{2},\ \mu_2>\frac{1}{2}$, if there exists $k\in\mathbb{Z}$ such that the vectors*

$$\left(w_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right),\varphi_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right),\varphi_j\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right)\right)^\tau,\quad j=3,4$$

are linearly dependent, then the system (1.7) is unstable; otherwise, this system is asymptotically stable;

(3) *when $\mu_1>\frac{1}{2},\ \mu_2=\frac{1}{2}$, if there exists $k\in\mathbb{Z}$ such that*

$$\left(w_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right)-\varphi_j\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right),\varphi_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right),w_j\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right)\right)^\tau,\quad j=3,4$$

are linearly dependent, the system (1.7) is unstable; otherwise, this system is asymptotically stable.

Proof. When $\mu_1\geq\frac{1}{2},\ \mu_2\geq\frac{1}{2}$, from Section 2, we get that $\mathcal{A}$ is dissipative in $\mathcal{H}$. Therefore, according to the Lyubich and Phóng theorem [6], we only need to verify that there is no eigenvalue of $\mathcal{A}$ on the imaginary axis to get the asymptotic stability of the system (1.7).

If there exists $\sigma\in\mathbb{R}$ such that $\lambda=i\sigma$ is an eigenvalue of $\mathcal{A}$ and $W=(w,\lambda w,p,\varphi,\lambda\varphi,q)^\tau$ is an eigenvector of $\mathcal{A}$ corresponding to λ, then we have

$$0=\Re\lambda(W,W)_1=\Re(\mathcal{A}W,W)_1. \tag{5.7}$$

We consider the asymptotic stability of the system (1.7) under these following cases.

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Case 1: $\mu_1 > \frac{1}{2},\ \mu_2 > \frac{1}{2}$.

From (2.11) and (5.7), we get $p(0) = q(0) = p(1) = q(1) = 0$. Hence, according to (3.1), we have that $\lambda = i\sigma$ satisfy

$$\begin{cases} \rho\lambda^2 w - k(w_{xx} - \varphi_x) = 0, \quad x \in (0,1), \\ I_\rho\lambda^2\varphi - EI\varphi_{xx} - k(w_x - \varphi) = 0, \quad x \in (0,1), \\ w(0) = \varphi(0) = w(1) = \varphi(1) = 0, \quad w_x(1) = \varphi_x(1) = 0. \end{cases} \tag{5.8}$$

Therefore, $w(1) = \varphi(1) = w_x(1) = \varphi_x(1) = 0$ together with the fundamental solution (5.2)–(5.6) implies that

$$w(x) = \varphi(x) = 0.$$

Thus, $(w, \lambda w, p, \varphi, \lambda\varphi, q)^\tau = 0$ which contradicts the assumption that $\lambda = i\sigma$ is an eigenvalue of $\mathcal{A}$. So there is no eigenvalue on the imaginary axis. The Lyubich and Phóng theorem [6] asserts that the system (1.7) is asymptotically stable.

Case 2: $\mu_1 = \frac{1}{2},\ \mu_2 > \frac{1}{2}$.

Similarly, if $\lambda = i\sigma$ is an eigenvalue of $\mathcal{A}$, we can get

$$\begin{aligned} 0 &= 2\Re(\mathcal{A}W, W)_1 \\ &= -2\left(\alpha_1\mu_1 - \frac{\eta_1}{2\tau_1}\right)p(0)\overline{p(0)} - 2\left(\alpha_2\mu_2 - \frac{\eta_2}{2\tau_2}\right)q(0)\overline{q(0)} - \frac{\eta_1}{\tau_1}p(1)\overline{p(1)} - \frac{\eta_2}{\tau_2}q(1)\overline{q(1)} \\ &\quad -\alpha_1(1-\mu_1)p(0)\overline{p(1)} - \alpha_1(1-\mu_1)p(1)\overline{p(0)} - \alpha_2(1-\mu_2)q(0)\overline{q(1)} - \alpha_2(1-\mu_2)q(1)\overline{q(0)}. \end{aligned}$$

Since (2.9) implies that $\eta_1 = \frac{\tau_1\alpha_1}{2}$,

$$\begin{aligned} 2\Re(\mathcal{A}W, W)_1 &= -\frac{\alpha_1}{2}(p(0)+p(1))\overline{(p(0)+p(1))} - 2\left(\alpha_2\mu_2 - \frac{\eta_2}{2\tau_2}\right)q(0)\overline{q(0)} \\ &\quad -\frac{\eta_2}{\tau_2}q(1)\overline{q(1)} - \alpha_2(1-\mu_2)q(0)\overline{q(1)} - \alpha_2(1-\mu_2)q(1)\overline{q(0)} \\ &\le -\frac{\alpha_1}{2}(p(0)+p(1))\overline{(p(0)+p(1))} - 2\left(\alpha_2\mu_2 - \frac{\eta_2}{2\tau_2} - \frac{\alpha_2(1-\mu_2)}{2}\right)|q(0)|^2 \\ &\quad -2\left(\frac{\eta_2}{2\tau_2} - \frac{\alpha_2(1-\mu_2)}{2}\right)|q(1)|^2. \end{aligned}$$

From (5.7), we get

$$p(0) + p(1) = 0, \quad q(0) = \lambda\varphi(1) = 0, \quad q(1) = 0. \tag{5.9}$$

This implies that $\lambda = i\sigma$ is an eigenvalue if and only if the following equation has nonzero solution:

$$\begin{cases} \rho\lambda^2 w - k(w_{xx} - \varphi_x) = 0, \quad x \in (0,1), \\ I_\rho\lambda^2\varphi - EI\varphi_{xx} - k(w_x - \varphi) = 0, \quad x \in (0,1), \\ \tau_1\lambda p(x) + p_x(x) = 0, \quad \tau_2\lambda q(x) + q_x(x) = 0, \\ w(0) = \varphi(0) = \varphi(1) = 0, \\ k(w_x - \varphi)(1) = -\frac{\alpha_1}{2}(\lambda w(1) + p(1)) = 0, \quad EI\varphi_x(1) = 0. \end{cases} \tag{5.10}$$

Then, using the fundamental solution to Timoshenko beam (5.2)–(5.6), we have

$$w = cw_3(x,\lambda) + dw_4(x,\lambda), \quad \varphi = c\varphi_3(x,\lambda) + d\varphi_4(x,\lambda), \tag{5.11}$$

where $c = w_x(0)$ and $d = \varphi_x(0)$. By (5.9) and the third equation in (5.10), a direct calculation yields

$$p(0)(1 + \mathrm{e}^{-\tau_1\lambda}) = 0.$$

From the proof in Case 1, we know that if $p(0) = 0$, then $\lambda = i\sigma \notin \sigma(\mathcal{A})$. Hence, if $\lambda = i\sigma$ is an eigenvalue of $\mathcal{A}$, it must satisfy $1 + \mathrm{e}^{-\tau_1\lambda} = 0$, which implies

$$\lambda = \frac{(2k+1)\pi i}{-\tau_1}, \quad k \in \mathbb{Z}. \tag{5.12}$$

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Then, substituting (5.11) and (5.12) into the boundary conditions in (5.10) yields

$$\begin{bmatrix} w_{3x}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right) & w_{4x}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right) \\ \varphi_{3x}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right) & \varphi_{4x}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right) \\ \varphi_3\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right) & \varphi_4\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right) \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = 0. \tag{5.13}$$

Therefore, (5.10) has nonzero solution if and only if there exists $k \in \mathbb{Z}$ such that the vectors $\left(w_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right), \varphi_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right), \varphi_j\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right)\right)^\tau$, $j = 3,4$ are linearly dependent, which leads to the instability of the system (1.7). Otherwise, the system (1.7) is asymptotically stable.

Case 3: $\mu_1 > \frac{1}{2}$, $\mu_2 = \frac{1}{2}$.

If $\lambda = i\sigma$ is an eigenvalue of $\mathcal{A}$, we have

$$p(0) = p(1) = 0, \quad q(0) + q(1) = 0.$$

Similarly with the proof in Case 2, we have that if there exists $k \in \mathbb{Z}$ such that $\left(w_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right) - \varphi_j\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right), \varphi_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right), w_j\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right)\right)^\tau$, $j = 3,4$ are linearly dependent, the system (1.7) is unstable. Otherwise, this system is asymptotically stable. The proof is complete. □

Remark 5.1. "$\mu_1 = \mu_2 = \frac{1}{2}$" is a critical point for the stability of the system (1.7). Under this condition, the stability of the system (1.7) is very complicated. It can be divided into the following four cases:

(a) $p(0) = q(0) = 0$.

According to Case 1 in Theorem 5.2, the system is asymptotically stable.

(b) $q(0) = 1 + \mathrm{e}^{-\tau_1\lambda} = 0$.

From Case 2 in Theorem 5.2, if there exists $k \in \mathbb{Z}$ such that vectors $\left(w_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right), \varphi_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right), \varphi_j\left(1,\frac{(2k+1)\pi i}{-\tau_1}\right)\right)^\tau$, $j = 3,4$ are linearly dependent, the system (1.7) is unstable. Otherwise, this system is asymptotically stable.

(c) $p(0) = 1 + \mathrm{e}^{-\tau_2\lambda} = 0$.

By Case 3 in Theorem 5.2, if there exists $k \in \mathbb{Z}$ such that $\left(w_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right) - \varphi_j\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right), \varphi_{jx}\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right), w_j\left(1,\frac{(2k+1)\pi i}{-\tau_2}\right)\right)^\tau$, $j = 3,4$ are linearly dependent, the system (1.7) is unstable. Otherwise, this system is asymptotically stable.

(d) $1 + \mathrm{e}^{-\tau_1\lambda} = 1 + \mathrm{e}^{-\tau_2\lambda} = 0$.

A direct calculation leads to $\lambda = i\sigma = \frac{i(2m+1)\pi}{-\tau_1} = \frac{i(2n+1)\pi}{-\tau_2}$, $m, n \in \mathbb{Z}$. From (3.1) and (5.7), we also find that this $\lambda = i\sigma$ must satisfy $k(w_x - \varphi)(1) = EI\varphi_x(1) = 0$. By the fundamental solution to Timoshenko beam, it yields

$$\widetilde{\Delta}(\lambda) := \begin{vmatrix} w_{3x}(1,\lambda) - \varphi_3(1,\lambda) & w_{4x}(1,\lambda) - \varphi_4(1,\lambda) \\ \varphi_{3x}(1,\lambda) & \varphi_{4x}(1,\lambda) \end{vmatrix} = 0. \tag{5.14}$$

Therefore, if $\frac{\tau_1}{\tau_2}$ is a rational number and $\widetilde{\Delta}(\frac{(2m_j+1)\pi i}{-\tau_j}) = 0$ for some $m_j \in \mathbb{Z}$, $j = 1,2$, the system (1.7) is unstable. Otherwise, this system is asymptotically stable.

In order to discuss the exponential stability of the system (1.7), we have the following result:

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Lemma 5.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before and the conditions (2.8), (3.14) be fulfilled. Then the system (1.7) is exponentially stable if and only if*

$$\inf_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| > 0. \tag{5.15}$$

Proof. Necessity: Under the condition (2.8), we know that $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$, which implies that there is no eigenvalue of $\mathcal{A}$ on the right half complex plane. If $\inf\limits_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| = 0$, there always exists a sequence $\lambda_j,\ j = 1, 2, \ldots,\ \Re\lambda_j \to 0,\ \Im\lambda_j \to \infty$ such that $|\Delta(\lambda_j)| \to 0,\ j \to \infty$, which implies that the imaginary axis is the asymptote of $\sigma(\mathcal{A})$. Since the system (1.7) satisfies spectrum determined growth condition, this system is not exponentially stable, which is a contradiction. Therefore, (5.15) holds.

Sufficiency: According to the proof of Necessity, it is easy to check that (5.15) implies that the imaginary axis is not an asymptote of $\sigma(\mathcal{A})$. Thus, the spectrum determined growth condition of the system (1.7) implies that this system is exponentially stable. □

Based on this lemma, we get the following result about the exponential stability of the system (1.7).

Theorem 5.3. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and the condition (3.14) be fulfilled. If $\mu_1 > \frac{1}{2},\ \mu_2 > \frac{1}{2}$, then the closed loop system (1.7) is exponentially stable.*

Proof. From Theorem 5.2, we know under the conditions $\mu_1 > \frac{1}{2},\ \mu_2 > \frac{1}{2}$, the system (1.7) is asymptotically stable. From Lemma 5.1, we only need to check (5.15). To this end, set $\xi \in \mathbb{R}$. Since $\Delta(\lambda) = \Delta_1(\lambda)\Delta_2(\lambda) + \mathcal{O}(\lambda^{-1})$, it only needs to show that

$$\Delta_j(i\xi) \neq 0, \quad \xi \to \infty,\ j = 1, 2$$

where $\Delta_j(\lambda),\ j = 1, 2$ are given by (3.23) and (3.24). A direct calculation leads to

$$\begin{aligned}
\Delta_1(\lambda) &= 2i\left[-\alpha_1\mu_1 \sin\sqrt{\frac{\rho}{k}}\xi - \alpha_1(1-\mu_1)\cos\tau_1\xi \sin\sqrt{\frac{\rho}{k}}\xi\right] + 2\Bigg[-\alpha_1(1-\mu_1)\sin\tau_1\xi\sin\sqrt{\frac{\rho}{k}}\xi \\
&\quad -\sqrt{\rho k}\cos\sqrt{\frac{\rho}{k}}\xi\Bigg], \\
\Delta_2(\lambda) &= 2i\left[-\alpha_2\mu_2 \sin\sqrt{\frac{I_\rho}{EI}}\xi - \alpha_2(1-\mu_2)\cos\tau_2\xi \sin\sqrt{\frac{I_\rho}{EI}}\xi\right] + 2\Bigg[-\alpha_2(1-\mu_2)\sin\tau_2\xi\sin\sqrt{\frac{I_\rho}{EI}}\xi \\
&\quad -\sqrt{I_\rho EI}\cos\sqrt{\frac{I_\rho}{EI}}\xi\Bigg].
\end{aligned}$$

Then $\Delta_1(i\xi) = 0$ is equivalent to

$$\alpha_1\mu_1 \sin\sqrt{\frac{\rho}{k}}\xi + \alpha_1(1-\mu_1)\cos\tau_1\xi\sin\sqrt{\frac{\rho}{k}}\xi = 0, \tag{5.16}$$

$$\alpha_1(1-\mu_1)\sin\tau_1\xi\sin\sqrt{\frac{\rho}{k}}\xi + \sqrt{\rho k}\cos\sqrt{\frac{\rho}{k}}\xi = 0. \tag{5.17}$$

Similarly, $\Delta_2(i\xi) = 0$ is equivalent to

$$\alpha_2\mu_2 \sin\sqrt{\frac{I_\rho}{EI}}\xi + \alpha_2(1-\mu_2)\cos\tau_2\xi\sin\sqrt{\frac{I_\rho}{EI}}\xi = 0, \tag{5.18}$$

$$\alpha_2(1-\mu_2)\sin\tau_2\xi\sin\sqrt{\frac{I_\rho}{EI}}\xi + \sqrt{I_\rho EI}\cos\sqrt{\frac{I_\rho}{EI}}\xi = 0. \tag{5.19}$$

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Now we show $\Delta_1(i\xi) \neq 0$, $\xi \to \infty$ in the following two cases:

Case 1: $\mu_1 = 1$.

Under this condition, (5.16) and (5.17) imply that $\sin\sqrt{\frac{\rho}{k}}\xi = \cos\sqrt{\frac{\rho}{k}}\xi = 0$, which is a contradiction. Thus, $\Delta_1(i\xi) \neq 0$ for any $\xi \in \mathbb{R}$, $\xi \to \infty$.

Case 2: $\frac{1}{2} < \mu_1 < 1$.

From (5.16), we have $\alpha_1\mu_1 + \alpha_1(1-\mu_1)\cos\tau_1\xi = 0$ or $\sin\sqrt{\frac{\rho}{k}}\xi = 0$. It is easy to check that $\sin\sqrt{\frac{\rho}{k}}\xi \neq 0$. Hence, we can get

$$\cos\tau_1\xi = -\frac{\mu_1}{1-\mu_1}. \tag{5.20}$$

Substituting (5.20) into (5.17) leads to

$$\pm\alpha_1(1-\mu_1)\sqrt{1-\left(\frac{-\mu_1}{1-\mu_1}\right)^2}\sin\sqrt{\frac{\rho}{k}}\xi - \sqrt{\rho k}\cos\sqrt{\frac{\rho}{k}}\xi = 0.$$

From (5.16) and (5.17), we have $\cos\sqrt{\frac{\rho}{k}}\xi \neq 0$. Therefore,

$$\tan\sqrt{\frac{\rho}{k}}\xi = \pm\frac{\sqrt{\rho k}}{\alpha_1(1-\mu_1)\sqrt{1-(\frac{\mu_1}{1-\mu_1})^2}} = \pm\frac{\sqrt{\rho k}}{\alpha_1\sqrt{1-2\mu_1}}.$$

If $\frac{1}{2} < \mu_1 < 1$, $\frac{\sqrt{\rho k}}{\alpha_1\sqrt{1-2\mu_1}}$ is an imaginary number, which contradicts the fact that $\tan\sqrt{\frac{\rho}{k}}\xi \in \mathbb{R}$ for $\xi \in \mathbb{R}$. Therefore, $\Delta_1(i\xi) \neq 0$ for any $\xi \in \mathbb{R}$, $\xi \to \infty$ when $\mu_1 \neq 1$.

By the similar method, we get $\Delta_2(i\xi) \neq 0$ for any $\xi \in \mathbb{R}$, $\xi \to \infty$ when $\mu_2 > \frac{1}{2}$. Based on the discussion above, by Lemma 5.1, the system (1.7) is exponentially stable. The proof is complete. □

When $\mu_1 = \frac{1}{2}$ or $\mu_2 = \frac{1}{2}$, from Theorem 5.2 and Remark 5.1, the system (1.7) can be asymptotically stable, based on which this system is only exponentially stable under strict conditions.

Theorem 5.4. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and the conditions (2.8), (3.14) be fulfilled. Suppose that the system (1.7) is asymptotically stable. Then*

if $\mu_1 = \frac{1}{2}$, $\mu_2 > \frac{1}{2}$ and $\sqrt{\frac{\rho}{k}} = \tau_1$, the system (1.7) is exponentially stable;

if $\mu_1 > \frac{1}{2}$, $\mu_2 = \frac{1}{2}$ and $\sqrt{\frac{I_\rho}{EI}} = \tau_2$, the system (1.7) is exponentially stable;

if $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2}$ and $\sqrt{\frac{\rho}{k}} = \tau_1$, $\sqrt{\frac{I_\rho}{EI}} = \tau_2$, the system (1.7) is exponentially stable.

Proof. When $\mu_1 = \frac{1}{2}$, from (5.16) and (5.17), $\Delta_1(i\xi) = 0$ is equivalent to

$$\frac{1}{2}\alpha_1\sin\sqrt{\frac{\rho}{k}}\xi + \frac{1}{2}\alpha_1\cos\tau_1\xi\sin\sqrt{\frac{\rho}{k}}\xi = 0, \tag{5.21}$$

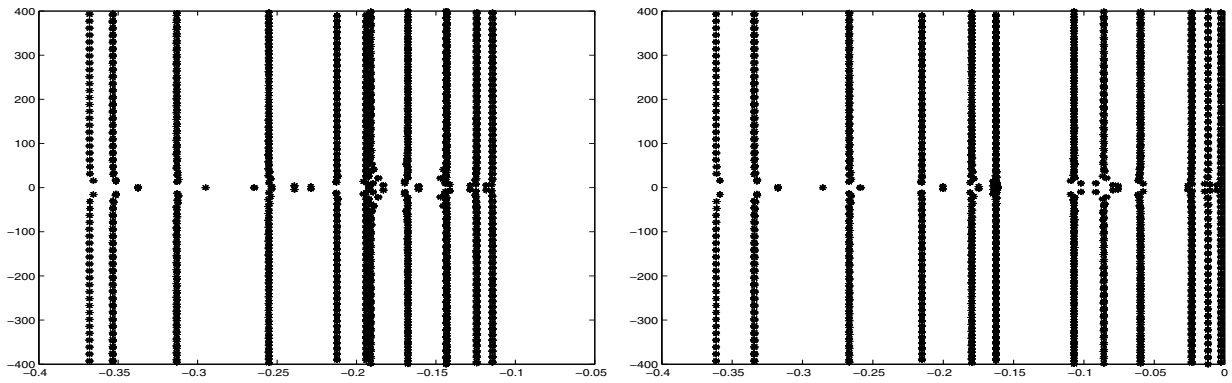
$$\frac{1}{2}\alpha_1\sin\tau_1\xi\sin\sqrt{\frac{\rho}{k}}\xi + \sqrt{\rho k}\cos\sqrt{\frac{\rho}{k}}\xi = 0. \tag{5.22}$$

A direct calculation yields

$$\sin\sqrt{\frac{\rho}{k}}\xi \neq 0, \quad \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_1\cos\tau_1\xi = \sin\tau_1\xi = \cos\sqrt{\frac{\rho}{k}}\xi = 0,\ \xi \in \mathbb{R}. \tag{5.23}$$

Then, if $\sqrt{\frac{\rho}{k}} = \tau_1$, we can not find a sequence $\xi_n \in \mathbb{R}$, such that $\sin\tau_1\xi_n \to 0$, $\cos\sqrt{\frac{\rho}{k}}\xi_n \to 0$ simultaneously, which leads to $\inf\limits_{\xi\in\mathbb{R}}|\Delta_1(i\xi)| \neq 0$. Similarly, when $\mu_2 = \frac{1}{2}$ we obtain that if $\sqrt{\frac{I_\rho}{EI}} = \tau_2$, then $\inf\limits_{\xi\in\mathbb{R}}|\Delta_2(i\xi)| \neq 0$.

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(a) $\mu_1 = 0.8,\ \mu_2 = 0.6$ (b) $\mu_1 = 0.5,\ \mu_2 = 0.5$

FIGURE 1. Distribution of the spectrum.

We have known from the proof of Theorem 5.3 that if $\mu_j > \frac{1}{2}$, then $\Delta_j(i\xi) \neq 0,\ \xi \to \infty,\ \xi \in \mathbb{R}$ holds for each $j,\ j = 1, 2$. Thus, if $\mu_1 = \frac{1}{2},\ \mu_2 > \frac{1}{2}$ and $\sqrt{\frac{\rho}{k}} = \tau_1$,

$$\inf_{\xi\in\mathbb{R}} |\Delta(i\xi)| = \inf_{\xi\in\mathbb{R}} |\Delta_1(i\xi)\Delta_2(i\xi)| > 0.$$

Therefore, Lemma 5.1 asserts that the system (1.7) is exponentially stable.

Similarly, we can get that if $\mu_1 > \frac{1}{2},\ \mu_2 = \frac{1}{2}$ and $\sqrt{\frac{I_\rho}{EI}} = \tau_2$, the system (1.7) is exponentially stable as well.

If $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{2}$, under the conditions $\sqrt{\frac{\rho}{k}} = \tau_1,\ \sqrt{\frac{I_\rho}{EI}} = \tau_2$, the system (1.7) is also exponentially stable by the similar discussion. The proof is complete. □

6. SIMULATIONS

In this section, we shall give some simulations of the system (1.7) to support our results. Firstly, we give each parameter in this system a numerical value as follows:

ρ	k	I_ρ	EI	τ_1	τ_2	α_1	α_2
27	3	8	2	0.5	0.4	5	6

Then let us discuss the distribution of the spectrum of $\mathcal{A}$ under the system parameters chosen above. Based on the fundamental solution to Timoshenko beams (5.2)–(5.6) and the Matlab scientific calculation, we get many simulations for the distribution of the spectrum of $\mathcal{A}$ by changing the value of $\mu_i,\ i = 1, 2$. We find that in these simulations the eigenvalues of $\mathcal{A}$ distribute in conjugate pairs in the complex plane and there are many "vertical" lines which the eigenvalues approach. These lines are in fact the asymptotes of the eigenvalues of $\mathcal{A}$. Here we only show two figures. Figures 1(a) and 1(b) denote the distribution of the spectrum of $\mathcal{A}$ when $\mu_1 = 0.8,\ \mu_2 = 0.6$ and $\mu_1 = 0.5,\ \mu_2 = 0.5$, respectively, in which " * " denotes the eigenvalues of $\mathcal{A}$.

We find that if we choose $\mu_i > \frac{1}{2},\ i = 1, 2$, the eigenvalues of $\mathcal{A}$ are always located in the left hand of the complex plane and far away from the imaginary axis, which implies the exponential stability of the system. We see in Figure 1(a) that the eigenvalues of $\mathcal{A}$ satisfy that $\Re\lambda < -0.1$. Since the spectrum determined growth condition holds, this system is exponentially stable when $\mu_1 = 0.8,\ \mu_2 = 0.6$. Furthermore, the exponential decay rate is less than -0.1.

However, when $\mu_i \to 0.5,\ i = 1, 2$, the distance between the maximum of the real part of the eigenvalues and the imaginary axis becomes smaller, which implies that the system (1.7) decays more slowly when $\mu_i \to 0.5,\ i = 1, 2$. Furthermore, when $\mu_1 = 0.5,\ \mu_2 = 0.5$, we see in Figure 1(b) that there is no vertical line to divide the spectrum and the imaginary axis, which shows that the imaginary axis may be the asymptote

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of the eigenvalues of $\mathcal{A}$. Therefore, the system (1.7) is possibly asymptotically stable but not exponentially stable when $\mu_i = 0.5,\ i = 1, 2$.

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