109.29 Reverse of the Weitzenböck inequality

In a triangle ABC, with the sides a, b, c, the semi-perimeter, circumradius and inradius are denoted by s, R and r respectively and Δ is its area.

The celebrated Weitzenböck inequality can be stated as follows:

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta, \tag{1}$$

Equality holds if, and only if, the triangle is equilateral.

This was first proved in 1919 [1]. It has received considerable attention from researchers in the field of geometrical inequalities and has motivated a number of papers providing various sharpening, generalisations and improvements. Among them, the following inequality of Finsler and Hadwiger (see [2])

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta + (a-b)^2 + (b-c)^2 + (c-a)^2,$$

is the most famous one. It seems to be a refinement of Weitzenböck's inequality, but it is actually equivalent to it [3]. A direct refinement of both inequalities is

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{4 - \frac{2r}{R}}\Delta + (a-b)^2 + (b-c)^2 + (c-a)^2.$$

which is equivalent to Kooi's inequality

There are relatively few articles in the literature on the upper bound of Weitzenböck's inequality. Recently, while studying a problem, the author unexpectedly obtained an upper bound of Weitzenböck's inequality. Our results may be stated as follows.

Theorem 1

In ΔABC, we have

$$a^2 + b^2 + c^2 \leqslant 4\Delta\sqrt{\frac{R^2}{r^2} - 1}. \tag{2}$$

Equality holds if, and only if, the triangle is equilateral.

Proof: Using the following basic identities in triangle ABC

$$a^2 + b^2 + c^2 = 2s^2 - 8Rr - 2r^2$$

and

$$\Delta = rs.$$

(2) is equivalent to

$$s^4 - \left(4R^2 + 8Rr - 2r^2\right)s^2 + r^2(4R + r)^2 \leqslant 0. \tag{3}$$

This is true since

$$s^4 - \left(4R^2 + 8Rr - 2r^2\right)s^2 + r^2(4R + r)^2$$
$$= \left(s^2 - 16Rr + 5r^2\right)\left(s^2 - 4R^2 - 4Rr - 3r^2\right)$$

https://doi.org/10.1017/mag.2025.10095 Published online by Cambridge University Press

$$+ 12Rr\left(s^2 - 4R^2 - 4Rr - 3r^2\right)$$
$$- 4r\left(R - 2r\right)\left(4R^2 + 3Rr + 2r^2\right),$$

and (3) follows from Gerretsen's inequality

$$4R^2 + 4Rr + 3r^2 \geqslant s^2 \geqslant 16Rr - 5r^2$$

and Euler's inequality $R \geqslant 2r$.

Finally, we present a second direct and shorter proof as follows. We use the equivalent form of the fundamental triangle inequality [5]

$$s^4 - 2\left(2R^2 + 10Rr - r^2\right)s^2 + r\left(4R + r\right)^3 \leqslant 0. \tag{4}$$

Now

$$s^4 - \left(4R^2 + 8Rr - 2r^2\right)s^2 + r^2(4R + r)^2 \leqslant s^4 - 2\left(2R^2 + 10Rr - r^2\right)s^2 + r(4R + r)^3$$

is true since it is equivalent to the known inequality $\sqrt{3}s \leqslant 4R + r$, which in turn is equivalent to the Finsler-Hadwiger inequality [6]. Hence (3) follows from (4) and so the reverse Weitzenböck's inequality (2) is proved.

Acknowledgments

The author is thankful to anonymous referees and the editor for their valuable comments on the original version of this paper. Special thanks to the reviewer for providing the second concise proof.

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10.1017/mag.2025.10095 © The Authors, 2025
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https://doi.org/10.1017/mag.2025.10095 Published online by Cambridge University Press