Probability in the Engineering and Informational Sciences, **20**, 2006, 363–379. Printed in the U.S.A.

REGRESSION DEPENDENCE IN LATENT VARIABLE MODELS

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Three new notions of positive dependence (positive regression dependence, positive left-tail regression dependence, and positive right-tail regression dependence) are studied in this article. Consider a latent variable model where the manifest random variables $T_1, T_2, \ldots, T_n$ given latent random variable/vector $(\Theta_1, \ldots, \Theta_m)$ are conditional independent. Conditions are identified under which $T_1, \ldots, T_n$ possesses the new dependence notions for different types of latent variable model. Applications of the results are also provided.

1. INTRODUCTION

Consider the following *multidimensional latent variable model*. Let $T_1, T_2, \ldots, T_n$ be n observable (or manifest) random variables, and let $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_m)$ be an unobservable (or latent) random vector. Given $\boldsymbol{\Theta} = \boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$, $T_1, \ldots, T_n$ are assumed to be conditionally independent. Let $F_i(\cdot|\boldsymbol{\theta})$ denote the conditional distribution function of T_i given $\boldsymbol{\Theta} = \boldsymbol{\theta}$, $i = 1, \ldots, n$, and by G denote the distribution function of $\boldsymbol{\Theta}$. Then the joint distribution function of $\mathbf{T} = (T_1, \ldots, T_n)$ is given by

$$F(\mathbf{t}) = \int \cdots \int_{\Re^m} \prod_{i=1}^{n} F_i(t_i|\boldsymbol{\theta})\, dG(\boldsymbol{\theta}), \qquad \mathbf{t} = (t_1, \ldots, t_n) \in \Re^n, \tag{1.1}$$

and the survival function of $\mathbf{T}$ is given by

$$\bar{F}(\mathbf{t}) = \int \cdots \int_{\Re^m} \prod_{i=1}^{n} \bar{F}_i(t_i|\boldsymbol{\theta})\, dG(\boldsymbol{\theta}), \qquad \mathbf{t} \in \Re^n. \tag{1.2}$$

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https://doi.org/10.1017/S0269964806060220 Published online by Cambridge University Press

If $F_i(\cdot|\boldsymbol{\theta})$ is absolutely continuous with respect to the Lebesgue measure on $\Re$ for each $\boldsymbol{\theta}$ in the support of $\boldsymbol{\Theta}$, with a density function $f_i(\cdot|\boldsymbol{\theta})$, then F is absolutely continuous with respect to the Lebesgue measure on $\Re^n$, and the joint density function of $\mathbf{T}$ is given by

$$f(\mathbf{t}) = \int \cdots \int_{\Re^m} \prod_{i=1}^{n} f_i(t_i|\boldsymbol{\theta})\, dG(\boldsymbol{\theta}), \qquad \mathbf{t} \in \Re^n. \tag{1.3}$$

Let $T_i(\boldsymbol{\theta})$ denote a random variable with distribution function $F_i(\cdot|\boldsymbol{\theta})$. If $m = n$ and the univariate marginal distribution of $T_i(\boldsymbol{\theta})$ depends on $\boldsymbol{\theta}$ only through θ_i for each i, then Eq. (1.1) reduces to the following *componentwise separable latent variable model*:

$$F(\mathbf{t}) = \int \cdots \int_{\Re^n} \prod_{i=1}^{n} F_i(t_i|\theta_i)\, dG(\boldsymbol{\theta}), \qquad \mathbf{t} \in \Re^n. \tag{1.4}$$

In this case, we will write $T_i(\theta_i)$ and $F_i(\cdot|\theta_i)$ instead of $T_i(\boldsymbol{\theta})$ and $F_i(\cdot|\boldsymbol{\theta})$ for $i = 1, \ldots, n$, respectively. In general, $\boldsymbol{\Theta}$ can be either a vector or a scalar; however, when we restrict attention to a scalar latent variable, we will allow the notation to emphasize the restriction by writing Θ in place of $\boldsymbol{\Theta}$. If Θ is a scalar, we call the model a *unidimensional latent variable model*.

Latent variable models for multivariate distributions arise in a wide variety of applications (see Holland and Rosenbaum [11]). Certain general classes of latent variable models imply that the manifest variables $T_1, \ldots, T_n$ exhibit some forms of positive dependence. For example, it is well known (see, e.g., Tong [26]) that if Θ is a scalar and $F_1(\cdot|\theta) = \cdots = F_n(\cdot|\theta)$, then $\mathbf{T}$ is *positive function dependent* (PFD); that is,

$$\mathbb{E}\left[\prod_{i=1}^{n} \phi(T_i)\right] \geq \prod_{i=1}^{n} \mathbb{E}[\phi(T_i)]$$

holds for all Borel-measurable functions $\phi : \Re \to \Re$ (if n is odd, there is an additional restriction that ϕ is nonnegative). It is natural to investigate conditions under which $\mathbf{T}$ possesses some other notions of multivariate positive dependence for general latent variable models. Several results of this kind can be found in the literature. Jogdeo [13], Linqvist [18], and Belzunce, Lillo, Pellerey, and Shaked [2] proved preservation results under latent variable models for *positive association* (PA), which was introduced by Esary, Proschan, and Walkup [9]. Shaked and Spizzichino [25] dealt with two notions of positive dependence: *weakened by failures* (WBF) studied by Norros [21] and by Shaked and Shanthikumar [23], and *multivariate totally positive of order 2* (MTP_2) studied by Karlin and Rinott [14]. Holland and Rosenbaum [11], Scarsini and Spizzichino [22], Khaledi and Kochar [16] and Belzunce and Semeraro [3], among others, established preservation results for some other notions of positive dependence. We state two such results below.

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The formal definitions of some notions of positive dependence and of some stochastic orders (e.g., $\leq_{st}$, $\leq_{hr}$, and $\leq_{lr}$) mentioned in this section can be found in Section 2. For two m-dimensional vectors $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ and $\boldsymbol{\theta}' = (\theta_1', \ldots, \theta_m')$, $\boldsymbol{\theta} \leq \boldsymbol{\theta}'$ means that $\theta_i \leq \theta_i'$ for $i = 1, \ldots, m$.

THEOREM 1.1 (Jogdeo [13]): *Let a latent variable model* $(\mathbf{T}, \boldsymbol{\Theta})$ *be as defined earlier. If*

$$T_i(\boldsymbol{\theta}) \leq_{st} [\geq_{st}] \, T_i(\boldsymbol{\theta}')$$

whenever $\boldsymbol{\theta} \leq \boldsymbol{\theta}'$ *for each* i *and if* $\boldsymbol{\Theta}$ *is* PA, *then* $\mathbf{T}$ *is* PA.

THEOREM 1.2 (Holland and Rosenbaum [11]; Shaked and Spizzichino [25]): *Let* $(\mathbf{T}, \boldsymbol{\Theta})$ *be as defined in Theorem 1.1. Assume that* $F_i(\cdot \mid \boldsymbol{\theta})$ *is absolutely continuous with respect to the Lebesgue measure on* $\Re$ *for every* $\boldsymbol{\theta}$ *in the support of* $\boldsymbol{\Theta}$ *and for each* i, *and denote the corresponding density function by* $f_i(\cdot \mid \boldsymbol{\theta})$. *If* $f_i(t \mid \boldsymbol{\theta})$ *is* MTP_2 *in* $(t, \boldsymbol{\theta})$ *for each* i, *and if* $\boldsymbol{\Theta}$ *is* MTP_2, *then* $\mathbf{T}$ *is* MTP_2.

An immediate consequence of Theorem 1.2 is the following corollary (see also Theorem 3.4 in [25]).

COROLLARY 1.1: *For a unidimensional latent variable model* $(\mathbf{T}, \Theta)$, *if*

$$T_i(\theta) \leq_{lr} [\geq_{lr}] \, T_i(\theta')$$

whenever $\theta \leq \theta'$ *for each* i, *then* $\mathbf{T}$ *is* MTP_2.

The purpose of this article is to introduce three new notions of positive dependence [*positive regression dependence* (PRD), *positive left-tail regression dependence* (PLTRD), and *positive right-tail regression dependence* (PRTRD)] and to identify conditions under which the manifest variables $T_1, \ldots, T_n$ possess the new dependence notions for different types of latent variable model.

The article is organized as follows. Section 2 recalls the definitions of some notions of positive dependence and of some stochastic orders, which will be used in this article. Section 3 introduces three new notions of positive dependence and describes relationships of the new notions to other commonly used notions of positive dependence. Useful preservation properties of the new dependence notions are proved in Section 4. Finally, applications that illustrate the power of the main results in Section 4 are detailed in Section 5.

Throughout the article, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively. A real function on $\Re^n$ will be called increasing [decreasing] if it is increasing [decreasing] in each variable when the other variables are held fixed. All expectations are implicitly assumed to exist whenever they are written. All ratios are well defined and $a/0$ is understood to be ∞ whenever $a > 0$. When an expectation or a probability is conditioned on an event such as $\boldsymbol{\Theta} = \boldsymbol{\theta}$, we assume that $\boldsymbol{\theta}$ is in the support of $\boldsymbol{\Theta}$. We denote by $[\mathbf{X} \mid A]$ any

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random vector/variable whose distribution is the conditional distribution of $\mathbf{X}$ given event A.

2. PRELIMINARIES

In this section, we first recall the definitions of some well-known stochastic orders and then give the definitions of some notions of multivariate positive dependence.

DEFINITION 2.1 (see Shaked and Shanthikumar [24, Chap. 1]): *Let X and Y be two random variables with distribution functions F and G, respectively. Then X is said to be smaller than Y for the following:*

(a) In the usual stochastic order, denoted by $X \leq_{\mathrm{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x, or if $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$ for all increasing functions h

(b) In the hazard rate order, denoted by $X \leq_{\mathrm{hr}} Y$, if $\bar{G}(x)/\bar{F}(x)$ is increasing in x

(c) In the reversed hazard rate order, denoted by $X \leq_{\mathrm{rh}} Y$, if $G(x)/F(x)$ is increasing in x

(d) In the likelihood ratio order, denoted by $X \leq_{\mathrm{lr}} Y$, if F and G have density functions f and g, respectively, and if $g(x)/f(x)$ is increasing in x

The relationships among these orders are shown in the following diagram (see, e.g., Shaked and Shanthikumar [24], Müller and Stoyan [20]):

$$\begin{array}{ccc} X \leq_{\mathrm{lr}} Y & \Rightarrow & X \leq_{\mathrm{hr}} Y \\ \Downarrow & & \Downarrow \\ X \leq_{\mathrm{rh}} Y & \Rightarrow & X \leq_{\mathrm{st}} Y \end{array}$$

A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be smaller than another random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ in the *usual stochastic order*, denoted by $\mathbf{X} \leq_{\mathrm{st}} \mathbf{Y}$, if $\mathbb{E}[\phi(\mathbf{X})] \leq \mathbb{E}[\phi(\mathbf{Y})]$ holds for all increasing functions ϕ (see [24, Sect. 4B]).

Let $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$ be a product of totally ordered spaces $\mathcal{X}_i$, $i = 1, \ldots, n$, endowed with the natural partial ordering. Let ψ_1, ψ_2 be two nonnegative functions defined on $\mathcal{X}$, satisfying

$$\psi_2(\mathbf{x} \vee \mathbf{y})\psi_1(\mathbf{x} \wedge \mathbf{y}) \geq \psi_1(\mathbf{x})\psi_2(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{X},$$

where $\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$. Then we use the following notation:

$$\psi_1 \prec_{\mathrm{TP}_2} \psi_2.$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with respective density functions f and g. $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the *multivariate likelihood ratio order*, denoted by $\mathbf{X} \leq_{\mathrm{lr}} \mathbf{Y}$, if $f \prec_{\mathrm{TP}_2} g$. It is well known that $\mathbf{X} \leq_{\mathrm{lr}} \mathbf{Y}$ implies $\mathbf{X} \leq_{\mathrm{st}} \mathbf{Y}$ (see Karlin and Rinott [14]).

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There are several notions of multivariate positive dependence in the literature; see Barlow and Proschan [1] and Tong [27]. Here we review some among them. A function $\psi : \mathcal{X} \to [0,\infty)$ is said to be *multivariate totally positive of order 2* (MTP_2) if $\psi \prec_{\mathrm{TP}_2} \psi$.

DEFINITION 2.2: *A random vector* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *is said to be the following:*

(a) MTP_2 *if* $\mathbf{X}$ *has an* MTP_2 *density function* $f(\mathbf{x})$ *(see Karlin and Rinott [14])*

(b) conditionally increasing in sequence (CIS) if

$$\mathbb{P}(X_i > x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$$

is increasing in $(x_1, \ldots, x_{i-1})$ *for all* x_i, $i = 2, \ldots, n$ *(see Barlow and Proschan [1, Chap. 5])*

(c) conditionally increasing (CI) if $(X_{i_1}, X_{i_2}, \ldots, X_{i_n})$ *is CIS for every permutation* $(i_1, \ldots, i_n)$ *of* $(1, \ldots, n)$ *(see Müller and Scarsini [19])*

(d) positively associated (PA) if

$$\mathbb{E}[\phi(\mathbf{X})\psi(\mathbf{X})] \geq \mathbb{E}[\phi(\mathbf{X})] \cdot \mathbb{E}[\psi(\mathbf{X})]$$

holds for all increasing functions ϕ *and* ψ *(see Esary et al. [9])*

(e) positively upper orthant dependent (PUOD) if

$$\mathbb{P}(X_1 > x_1, \ldots, X_n > x_n) \geq \prod_{i=1}^{n} \mathbb{P}(X_i > x_i), \qquad \mathbf{x} \in \Re^n$$

(f) positively lower orthant dependent (PLOD) if

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) \geq \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i), \qquad \mathbf{x} \in \Re^n.$$

It is well known that MTP_2 implies CIS and CI, each of which, in turn, implies PA and that PA implies both PUOD and PLOD; see Barlow and Proschan [1] or Müller and Stoyan [20]. For $n = 2$, MTP_2 reduces to TP_2.

Harris [10] introduced the following two notions of multivariate dependence.

DEFINITION 2.3: *Let* $\mathbf{X}$ *be an n-dimensional random vector. Then* $\mathbf{X}$ *is said to be the following:*

(a) right corner set increasing (RCSI) if $\mathbb{P}(\mathbf{X} > \mathbf{x} | \mathbf{X} > \mathbf{z})$ *is increasing in* $\mathbf{z}$ *for all* $\mathbf{x}$

(b) left corner set decreasing (LCSD) if $\mathbb{P}(\mathbf{X} \leq \mathbf{x} | \mathbf{X} \leq \mathbf{z})$ *is decreasing in* $\mathbf{z}$ *for all* $\mathbf{x}$

Lee [17] proved that if $\mathbb{P}(\mathbf{X} > \mathbf{x})$ is positive on a product space, then $\mathbf{X}$ is RCSI if and only if $\mathbb{P}(\mathbf{X} > \mathbf{x})$ is MTP_2 in $\mathbf{x}$ and that if $\mathbb{P}(\mathbf{X} \leq \mathbf{x})$ is positive on a product

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space, then $\mathbf{X}$ is LCSD if and only if $\mathbb{P}(\mathbf{X} \leq \mathbf{x})$ is MTP_2 in $\mathbf{x}$. It is known from Block and Ting [5] that MTP_2 implies both RCSI and LCSD. Example 1 in Colangelo, Scarsini, and Shaked [6] illustrates that neither RCSI nor LCSD implies PA for $n \geq 3$.

3. MULTIVARIATE NOTIONS OF POSITIVE DEPENDENCE

Three well-known notions of bivariate positive dependence are the *stochastically increasing* (SI), the *right-tail increasing* (RTI), and the *left-tail decreasing* (LTD); see Barlow and Proschan [1, Chap. 5]. Given a random vector (X_1, X_2), X_2 is said to be the following:

- Stochastically increasing in X_1, denoted by $\text{SI}(X_2|X_1)$, if $\mathbb{P}(X_2 > x_2|X_1 = x_1)$ is increasing in x_1 for all x_2
- Right-tail increasing in X_1, denoted by $\text{RTI}(X_2|X_1)$, if $\mathbb{P}(X_2 > x_2|X_1 > x_1)$ is increasing in x_1 for all x_2
- Left-tail decreasing in X_1, denoted by $\text{LTD}(X_2|X_1)$, if $\mathbb{P}(X_2 \leq x_2|X_1 \leq x_1)$ is decreasing in x_1 for all x_2

It is well known that SI implies both RTI and LTD, each of which, in turn, implies PA. Several notions of positive dependence were introduced in the literature to generalize SI, RTI, and LTD from the bivariate case to the multivariate case, which can be found in Colangelo et al. [6]. In this section, we introduce three notions of multivariate positive regression dependence generalizing bivariate notions of SI, RTI, and LTD.

For $\mathbf{x} \in \Re^n$ and $\mathbf{X} = (X_1, \ldots, X_n)$, let $\mathbf{x}_J$ and $\mathbf{X}_J$ denote a vector and a random vector constructed by using the coordinates in $J \subseteq \{1, \ldots, n\}$ of the indicated vectors, and let $\mathbf{X}_J \leq \mathbf{x}_J$ mean that $X_j \leq x_j$ for $j \in J$. The notations $\mathbf{X}_J = \mathbf{x}_J$ and $\mathbf{X}_\mathbf{J} > \mathbf{x}_J$ have a similar interpretation.

DEFINITION 3.1: *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector. $\mathbf{X}$ is said to be the following:*

(a) Positive regression dependent (PRD) if

$$[\mathbf{X}_I|\mathbf{X}_J = \mathbf{x}_J] \leq_{\text{st}} [\mathbf{X}_I|\mathbf{X}_J = \mathbf{x}_J^*], \tag{3.1}$$

where $\mathbf{x}_J \leq \mathbf{x}_J^*$ and I and J are any disjoint subsets of $\{1, \ldots, n\}$

(b) Positive left-tail regression dependent (PLTRD) if

$$[\mathbf{X}_I|\mathbf{X}_J \leq \mathbf{x}_J] \leq_{\text{st}} [\mathbf{X}_I|\mathbf{X}_J \leq \mathbf{x}_J^*], \tag{3.2}$$

where $\mathbf{x}_J \leq \mathbf{x}_J^*$ and I and J are any disjoint subsets of $\{1, \ldots, n\}$

(c) Positive right-tail regression dependent (PRTRD) if

$$[\mathbf{X}_I|\mathbf{X}_J > \mathbf{x}_J] \leq_{\text{st}} [\mathbf{X}_I|\mathbf{X}_J > \mathbf{x}_J^*], \tag{3.3}$$

where $\mathbf{x}_J \leq \mathbf{x}_J^*$ and I and J are any disjoint subsets of $\{1, \ldots, n\}$.

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The above notions of positive dependence reflect the intuition that if a set of positively dependent random variables is split into two subsets in some manner, then the random variables in two subsets tend to assume concordant values. For $n = 2$, PRD, PRTRD, and PLTRD reduce to SI, RTI, and LTD, respectively. Dubhashi and Ranjan [8] introduced the notions of *negative regression dependence* (NRD), *negative left-tail regression dependence* (NLTRD) and *negative right-tail regression dependence* (NRTRD) by reversing inequalities (3.1), (3.2), and (3.3) in Definition 3.1, and advocated their use as a simple and unifying paradigm for the analysis of random structures and algorithms. They claimed in their Proposition 24 [8] that the implication NRD $\Rightarrow$ NLTRD and NRTRD is in the positive. However, the proof of their Proposition 24 is not correct. There is one big gap there. For occupancy numbers in the balls and bins experiment, their NRD property was proved by Dubhashi and Ranjan [8]. They also stated the NLTRD and NRTRD properties of occupancy numbers by their wrong Proposition 24. Hu and Xie [12] gave a detailed proof of the NLTRD and NRTRD properties of occupancy numbers with applications to dependence structure of order statistics.

We now investigate relationships between the above-defined notions.

PROPOSITION 3.1: *PRD* $\Leftrightarrow$ *CI.*

PROOF: By the definition of CI, it suffices to prove that $\mathbf{X}$ being CIS is equivalent to

$$\begin{aligned}&[(X_{k+1},\dots,X_n)|X_1 = x_1,\dots,X_k = x_k]\\ &\quad \le_{\mathrm{st}} [(X_{k+1},\dots,X_n)|X_1 = x_1^*,\dots,X_k = x_k^*]\end{aligned} \tag{3.4}$$

for $x_1 \le x_1^*,\dots,x_k \le x_k^*$ and $k = 1,\dots,n-1$.

Fix $k < n$ and let $h(x_{k+1},\dots,x_n)$ be an increasing function. For each $i = k,\dots,n$, define

$$h_i(X_1,\dots,X_i) = \mathbb{E}[h(X_{k+1},\dots,X_n)|X_1,\dots,X_i].$$

Observe that

$$h_n(X_1,\dots,X_n) = h(X_{k+1},\dots,X_n) \tag{3.5}$$

$$\begin{aligned}h_i(X_1,\dots,X_i) &= \mathbb{E}\{\mathbb{E}[h(X_{k+1},\dots,X_n)|X_1,\dots,X_{i+1}]|X_1,\dots,X_i\}\\ &= \mathbb{E}[h_{i+1}(X_1,\dots,X_{i+1})|X_1,\dots,X_i]\end{aligned} \tag{3.6}$$

for $i = k,\dots,n-1$. We claim that, due to (3.6), if h_{i+1} is increasing, then h_i is also increasing. To see this, let $x_j \le x_j^*$ for $j = 1,\dots,i$. Then by (3.6),

$$\begin{aligned}h_i(x_1,\dots,x_i) &= \mathbb{E}[h_{i+1}(X_1,\dots,X_{i+1})|X_1 = x_1,\dots,X_i = x_i]\\ &= \mathbb{E}[h_{i+1}(x_1,\dots,x_i,X_{i+1})|X_1 = x_1,\dots,X_i = x_i]\\ &\le \mathbb{E}[h_{i+1}(x_1^*,\dots,x_i^*,X_{i+1})|X_1 = x_1,\dots,X_i = x_i]\\ &\le \mathbb{E}[h_{i+1}(x_1^*,\dots,x_i^*,X_{i+1})|X_1 = x_1^*,\dots,X_i = x_i^*]\\ &= h_i(x_1^*,\dots,x_i^*),\end{aligned}$$

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where the first inequality follows from the increasing property of h_{i+1} and the second one follows from the CIS property of $\mathbf{X}$. From (3.5), we see that h_n is increasing. Therefore, h_k is increasing. This completes the proof. ■

PROPOSITION 3.2: *$MTP_2 \Rightarrow PRD$, $PLTRD$, and $PRTRD$.*

PROOF: Since it is well known that MTP_2 implies CI (and hence PRD by Proposition 3.1), we only give the proof of the implication $\text{MTP}_2 \Rightarrow$ PRTRD; the proof of the implication $\text{MTP}_2 \Rightarrow$ PLTRD is similar. Let I and J be any two disjoint proper subsets of $\{1,\ldots,n\}$. Without loss of generality, assume that $I = \{1,\ldots,k\}$ and $J = \{k+1,\ldots,n\}$ with $1 \le k < n$. Then the conditional density function of $\mathbf{X}_I$, given $\mathbf{X}_J > \mathbf{x}_J$, is given by

$$f_1(\mathbf{x}_I) = \frac{\int \cdots \int_{[\mathbf{x}_J,\infty)} f(\mathbf{x}_I, \mathbf{v}_J)\, d\mathbf{v}_J}{\hat{f}(\mathbf{x}_J)},$$

where f is the density function of $\mathbf{X}$ and

$$\hat{f}(\mathbf{x}_J) = \int \cdots \int_{[\mathbf{x}_J,\infty)} \int \cdots \int_{\Re^k} f(\mathbf{u}_I, \mathbf{v}_J)\, d\mathbf{u}_I\, d\mathbf{v}_J$$

is a normalizing constant. Similarly, the conditional density function of $\mathbf{X}_I$, given $\mathbf{X}_J > \mathbf{x}_J^*$, is

$$f_2(\mathbf{x}_I) = \frac{\int \cdots \int_{[\mathbf{x}_J^*,\infty)} f(\mathbf{x}_I, \mathbf{v}_J)\, d\mathbf{v}_J}{\hat{f}(\mathbf{x}_J^*)}.$$

Fix $\mathbf{x}_J \le \mathbf{x}_J^*$ and notice that $1_{[\mathbf{x}_J,\infty)}(\mathbf{v}_J) <_{\text{TP}_2} 1_{[\mathbf{x}_J^*,\infty)}(\mathbf{v}_J)$, where 1_A denotes the indicator function of set A. Since $f(\mathbf{x})$ is MTP_2 in $\mathbf{x}$, it follows from Theorem 2.4 in Karlin and Rinott [14] that

$$\begin{aligned}
\int \cdots \int_{[\mathbf{x}_J,\infty)} f(\mathbf{x}_I, \mathbf{v}_J)\, d\mathbf{v}_J &= \int \cdots \int_{\Re^{n-k}} f(\mathbf{x}_I, \mathbf{v}_J) \cdot 1_{[\mathbf{x}_J,\infty)}(\mathbf{v}_J)\, d\mathbf{v}_J \\
&<_{\text{TP}_2} \int \cdots \int_{\Re^{n-k}} f(\mathbf{x}_I, \mathbf{v}_J) \cdot 1_{[\mathbf{x}_J^*,\infty)}(\mathbf{v}_J)\, d\mathbf{v}_J \\
&= \int \cdots \int_{[\mathbf{x}_J^*,\infty)} f(\mathbf{x}_I, \mathbf{v}_J)\, d\mathbf{v}_J,
\end{aligned}$$

and, hence, $f_1(\mathbf{x}_I) <_{\text{TP}_2} f_2(\mathbf{x}_I)$. By Theorem 2.2 in [14], we get that $f_1 \le_{\text{st}} f_2$; that is,

$$[\mathbf{X}_I | \mathbf{X}_J > \mathbf{x}_J] \le_{\text{st}} [\mathbf{X}_I | \mathbf{X}_J > \mathbf{x}_J^*]$$

for all $\mathbf{x}_J \le \mathbf{x}_J^*$. This completes the proof. ■

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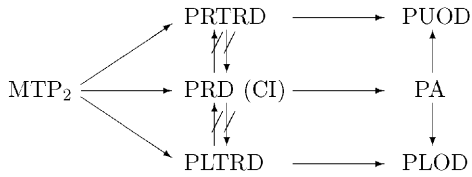


FIGURE 1. Implications among some notions of positive dependence for $n \geq 3$.

It is seen that PRD implies PRTRD and PLTRD in the bivariate case and that neither PRTRD nor PLTRD implies PRD. However, the next counterexample illustrates that PRD does not imply PRTRD or PLTRD in the multivariate case.

Example 3.1 (PRD $\not\Rightarrow$ PRTRD or PLTRD for $n \geq 3$; see Example 2 in [6]): Let (X_1, X_2, X_3) be a random vector with probability mass $\frac{1}{6}$ at the points $(-1,-1,0)$, $(0,-1,0)$, $(-1,0,1)$, $(1,0,1)$, $(0,2,2)$, and $(1,2,2)$. It can be checked that $\mathbf{X}$ is PRD. However, $\mathbf{X}$ is not PRTRD or PLTRD, since

$$\mathbb{P}(X_3 > -1 | X_1 > -1, X_2 > -1) = \frac{2}{3} \not\leq \frac{1}{2} = \mathbb{P}(X_3 > 1 | X_1 > 0, X_2 > -1)$$

and

$$\mathbb{P}(X_3 \leq 0 | X_1 \leq -1, X_2 \leq 0) = \frac{1}{2} \not\geq \frac{2}{3} = \mathbb{P}(X_3 \leq 0 | X_1 \leq 0, X_2 \leq 0).$$

Example 3.1 also illustrates that PRD does not imply RCSI or LCSD. Example 3 in [6] shows that neither PRTRD nor PLTRD implies RCSI or LCSD. Furthermore, it can be checked that Example 1 in [6] shows that neither RCSI nor LCSD implies PRTRD or PLTRD. Therefore, there is no relationship among PRD, PRTRD, PLTRD, RCSI, and LCSD.

The relationships among these notions defined in Definitions 2.2 and 3.1 are summarized in Figure 1. It is still an open problem whether PRTRD or PLTRD implies PA.

4. REGRESSION DEPENDENCE PROPERTIES IN LATENT VARIABLE MODELS

Let $(\mathbf{T}, \boldsymbol{\Theta})$ be a latent variable model as described in Section 1. Throughout, assume that either $\boldsymbol{\Theta}$ is absolutely continuous with respect to the Lebesgue measure on $\Re$, or that $\boldsymbol{\Theta}$ is a discrete random variable. Let $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_m$ denote a product measure on the support of $\boldsymbol{\Theta}$, where σ_i is the Lebesgue measure or the counting measure on the support of Θ_i, $i = 1, \ldots, m$. We will generally use the abbreviations $d\sigma_i(\theta_i) = d\theta_i$ and $d\sigma(\boldsymbol{\theta}) = d\boldsymbol{\theta}$ without ambiguity.

https://doi.org/10.1017/S0269964806060220 Published online by Cambridge University Press

4.1. Unidimensional Latent Variable Models

THEOREM 4.1: *Let* $(\mathbf{T}, \Theta)$ *be a unidimensional latent variable model.*

1. *If*

$$T_i(\theta) \leq_{\text{hr}} [\geq_{\text{hr}}]\, T_i(\theta') \text{ whenever } \theta \leq \theta' \text{ for each } i, \tag{4.1}$$

   then $\mathbf{T}$ *is PRTRD.*
2. *If*

$$T_i(\theta) \leq_{\text{rh}} [\geq_{\text{rh}}]\, T_i(\theta') \text{ whenever } \theta \leq \theta' \text{ for each } i, \tag{4.2}$$

   then $\mathbf{T}$ *is PLTRD.*
3. *If*

$$T_i(\theta) \leq_{\text{lr}} [\geq_{\text{lr}}]\, T_i(\theta') \text{ whenever } \theta \leq \theta' \text{ for each } i, \tag{4.3}$$

   then $\mathbf{T}$ *is PRD.*

PROOF: Part (3) follows from Corollary 1.1 and Proposition 3.2. We only give the proof of part (1), and the proof of part (2) is similar. To prove part (1), let I and J be any two disjoint proper subsets of $\{1,2,\ldots,n\}$ and let $\phi : \Re^{|I|} \to \Re$ be any increasing function. It suffices to prove that

$$\mathbb{E}[\phi(\mathbf{T}_I)|\mathbf{T}_J > \mathbf{t}_J] \text{ is increasing in } \mathbf{t}_J. \tag{4.4}$$

Notice that

$$\begin{aligned}\mathbb{E}[\phi(\mathbf{T}_I)|\mathbf{T}_J > \mathbf{t}_J] &= \mathbb{E}\{\mathbb{E}[\phi(\mathbf{T}_I)|\Theta, \mathbf{T}_J > \mathbf{t}_J]|\mathbf{T}_J > \mathbf{t}_J\} \\ &= \mathbb{E}\{\mathbb{E}[\phi(\mathbf{T}_I)|\Theta]|\mathbf{T}_J > \mathbf{t}_J\} \\ &= \int_{\Re} \mathbb{E}[\phi(\mathbf{T}_I)|\Theta = \theta]\, dG_\Theta(\theta|\mathbf{t}_J),\end{aligned} \tag{4.5}$$

where $G_\Theta(\cdot|\mathbf{t}_J)$ is the conditional distribution function of $[\Theta|\mathbf{T}_J > \mathbf{t}_J]$. On the other hand, it is easy to see from (4.1) and the conditional independence of $\mathbf{T}$ given $\Theta = \theta$ that $\mathbb{E}[\phi(\mathbf{T}_I)|\Theta = \theta]$ is increasing [decreasing] in θ. So, to prove (4.4), we need to verify that

$$[\Theta|\mathbf{T}_J > \mathbf{t}_J] \leq_{\text{st}} [\geq_{\text{st}}]\, [\Theta|\mathbf{T}_J > \mathbf{t}'_J] \tag{4.6}$$

whenever $\mathbf{t}_J \leq \mathbf{t}'_J$. In fact, denote by $g_\Theta(\cdot|\mathbf{t}_J)$ the conditional density or mass function of $G_\theta(\cdot|\mathbf{t}_J)$ and by $g(\theta)$ the density or mass function of Θ. Then

$$g_\Theta(\theta|\mathbf{t}_J) = c(\mathbf{t}_J) g(\theta) \prod_{j \in J} \bar{F}_j(t_j|\theta),$$

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where

$$c(\mathbf{t}_J) = \left[\int_{\Re} g(\theta) \prod_{j \in J} \bar{F}_j(t_j|\theta)\, d\theta \right]^{-1}$$

is a normalizing positive constant. Using the fact that the likelihood ratio order implies the hazard rate order, it follows from (4.1) that for any $\mathbf{t}_J \leq \mathbf{t}'_J$,

$$\frac{g_\Theta(\theta|\mathbf{t}'_J)}{g_\Theta(\theta|\mathbf{t}_J)} = \frac{c(\mathbf{t}'_J)}{c(\mathbf{t}_J)} \prod_{j \in J} \frac{\bar{F}_j(t'_j|\theta)}{\bar{F}_j(t_i|\theta)}$$

is increasing [decreasing] in θ; that is, $[\Theta|\mathbf{T}_J > \mathbf{t}_J] \leq_{\text{lr}} [\geq_{\text{lr}}] [\Theta|\mathbf{T}_J > \mathbf{t}'_J]$ whenever $\mathbf{t}_J \leq \mathbf{t}'_J$, which implies (4.6). This completes the proof. ■

Assumptions (4.1), (4.2), and (4.3) in Theorem 4.1 cannot be replaced by the weaker assumption that

$$T_i(\theta) \leq_{\text{st}} [\geq_{\text{st}}]\, T_i(\theta') \text{ whenever } \theta \leq \theta', \tag{4.7}$$

which can be seen from the following counterexample in Shaked and Spizzichino [25].

Example 4.1: Let $((T_1, T_2), \Theta)$ be a unidimensional latent variable model with Θ taking on only two values, θ_1 and θ_2, where $\theta_1 < \theta_2$, and suppose that $F_1(\cdot|\theta) = F_2(\cdot|\theta) = K(\cdot|\theta)$ for each θ. Denote $p = \mathbb{P}(\Theta = \theta_1)$ and $K_i(\cdot) = K(\cdot|\theta_i)$ for $i = 1,2$ and assume that $0 < p < 1$. Then the joint survival function of (T_1, T_2) can be written as

$$\bar{F}(t_1, t_2) = p\bar{K}_1(t_1)\bar{K}_1(t_2) + (1-p)\bar{K}_2(t_1)\bar{K}_2(t_2),$$

and, hence,

$$\begin{aligned} \mathbb{P}(T_2 > s|T_1 > t) &= \frac{p\bar{K}_1(t)\bar{K}_1(s) + (1-p)\bar{K}_2(t)\bar{K}_2(s)}{p\bar{K}_1(t) + (1-p)\bar{K}_2(t)} \\ &= \frac{p + (1-p)\dfrac{\bar{K}_2(s)}{\bar{K}_1(s)}\dfrac{\bar{K}_2(t)}{\bar{K}_1(t)}}{p + (1-p)\dfrac{\bar{K}_2(t)}{\bar{K}_1(t)}}\, \bar{K}_1(s) \end{aligned}$$

is increasing [decreasing] in $\eta(t) = \bar{K}_2(t)/\bar{K}_1(t)$ for any fixed s if

$$T_1(\theta_1) \leq_{\text{st}} [\geq_{\text{st}}]\, T_1(\theta_2).$$

Therefore, if $T_1(\theta_1) \leq_{\text{st}} [\geq_{\text{st}}]\, T_1(\theta_2)$, then $\mathbb{P}(T_2 > s|T_1 > t)$ is increasing in t for any fixed s if and only if $\eta(t)$ is increasing [decreasing] in t; that is, (T_1, T_2) is RTI if and only if $T_1(\theta_1) \leq_{\text{hr}} [\geq_{\text{hr}}]\, T_1(\theta_2)$.

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From the argument in Example 4.1, we can easily get the following theorem, which states that assumptions (4.1), (4.2), and (4.3) are respectively necessary and sufficient conditions for $\mathbf{T}$ to be PRTRD, PLTRD, and PRD under the additional assumption that $F_i(\cdot|\theta)$ does not depend on i and that G is an arbitrarily chosen distribution function of Θ. The proof is omitted here.

THEOREM 4.2: *Let* $(\mathbf{T}, \Theta)$ *be a unidimensional latent variable model with* $F_i(\cdot|\theta)$ *does not depend on i, and let G be the distribution function* Θ. *Then the following hold:*

1. $\mathbf{T}$ *is PRTRD for any G if and only if (4.1) holds.*
2. $\mathbf{T}$ *is PLTRD for any G if and only if (4.2) holds.*
3. $\mathbf{T}$ *is PRD for any G if and only if (4.3) holds.*

4.2. Multidimensional Latent Variable Models

Theorem 4.1 will be generalized to the case that $\mathbf{\Theta}$ is an m-dimensional random vector. The first result below gives sufficient conditions under which $\mathbf{T}$ is PRTRD.

THEOREM 4.3: *Let* $(\mathbf{T}, \mathbf{\Theta})$ *be a multidimensional latent variable model with the conditional survival function of* $T_i(\boldsymbol{\theta})$ *being* $\bar{F}_i(\cdot|\boldsymbol{\theta})$ *for each i. If*

(a)

$$T_i(\boldsymbol{\theta}) \leq_{\text{hr}} [\geq_{\text{hr}}]\, T_i(\boldsymbol{\theta}') \text{ whenever } \boldsymbol{\theta} \leq \boldsymbol{\theta}' \text{ for each } i \tag{4.8}$$

(b) $\bar{F}_i(t|\boldsymbol{\theta})$ *is* MTP_2 *in* $\boldsymbol{\theta}$ *for each t and i*
(c) $\mathbf{\Theta}$ *is* MTP_2,

then $\mathbf{T}$ *is PRTRD.*

PROOF: Let I and J be any two disjoint proper subsets of $\{1,2,\ldots,n\}$ and let $\phi : \Re^{|I|} \to \Re$ be any increasing function. It follows that

$$\begin{aligned}\mathbb{E}[\phi(\mathbf{T}_I)|\mathbf{T}_J > \mathbf{t}_J] &= \mathbb{E}\{\mathbb{E}[\phi(\mathbf{T}_I)|\mathbf{\Theta}]|\mathbf{T}_J > \mathbf{t}_J\} \\ &= \int \cdots \int_{\Re^m} \mathbb{E}[\phi(\mathbf{T}_I)|\mathbf{\Theta} = \boldsymbol{\theta}]\, dG_{\mathbf{\Theta}}(\boldsymbol{\theta}|\mathbf{t}_J),\end{aligned}$$

where $G_{\mathbf{\Theta}}(\cdot|\mathbf{t}_J)$ is the conditional distribution function of $[\mathbf{\Theta}|\mathbf{T}_J > \mathbf{t}_J]$. It is easy to see from (4.8) and the conditional independence of $\mathbf{T}$, given $\mathbf{\Theta} = \boldsymbol{\theta}$, that $\mathbb{E}[\phi(\mathbf{T}_I)|\mathbf{\Theta} = \boldsymbol{\theta}]$ is increasing [decreasing] in $\boldsymbol{\theta}$. To prove that $\mathbb{E}[\phi(\mathbf{T}_I)|\mathbf{T}_J > \mathbf{t}_J]$ is increasing in $\mathbf{t}_J$, it suffices to prove that

$$[\mathbf{\Theta}|\mathbf{T}_J > \mathbf{t}_J] \leq_{\text{st}} [\geq_{\text{st}}]\, [\mathbf{\Theta}|\mathbf{T}_J > \mathbf{t}'_J], \qquad \mathbf{t}_J \leq \mathbf{t}'_J. \tag{4.9}$$

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We now turn to prove that

$$[\boldsymbol{\Theta}|\mathbf{T}_J > \mathbf{t}_J] \leq_{\text{lr}} [\geq_{\text{lr}}] [\boldsymbol{\Theta}|\mathbf{T}_J > \mathbf{t}'_J], \qquad \mathbf{t}_J \leq \mathbf{t}'_J, \tag{4.10}$$

which implies (4.9).

Let $h_{\boldsymbol{\Theta}}(\cdot|\mathbf{t}_J)$ and $g(\cdot)$ denote the density or mass functions of $[\boldsymbol{\Theta}|\mathbf{T}_J > \mathbf{t}_J]$ and $\boldsymbol{\Theta}$, respectively. Then

$$g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|\mathbf{t}_J) = c(\mathbf{t}_J) g(\boldsymbol{\theta}) \prod_{j \in J} \bar{F}_j(t_j|\boldsymbol{\theta}),$$

where $c(\mathbf{t}_J)$ is a normalizing positive constant given by

$$c(\mathbf{t}_J) = \left[\int \cdots \int_{\Re^m} g(\boldsymbol{\theta}) \prod_{j \in J} \bar{F}_j(t_j|\boldsymbol{\theta})\, d\boldsymbol{\theta} \right]^{-1}.$$

Since a product of MTP_2 functions is still an MTP_2 function (see Proposition 3.3 in Karlin and Rinott [14]), it follows from assumptions (b) and (c) that $g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|\mathbf{t}_J)$ is MTP_2 in $\boldsymbol{\theta}$ for each $\mathbf{t}_J$. Let $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ be any two vectors in the support of $\boldsymbol{\Theta}$, and $\mathbf{t}_J \leq \mathbf{t}'_J$. Then

$$\begin{aligned}
g_{\boldsymbol{\Theta}}(\boldsymbol{\theta} \vee \boldsymbol{\theta}'|\mathbf{t}'_J) g_{\boldsymbol{\Theta}}(\boldsymbol{\theta} \wedge \boldsymbol{\theta}'|\mathbf{t}_J) &= \frac{g_{\boldsymbol{\Theta}}(\boldsymbol{\theta} \vee \boldsymbol{\theta}'|\mathbf{t}'_J)}{g_{\boldsymbol{\Theta}}(\boldsymbol{\theta} \vee \boldsymbol{\theta}'|\mathbf{t}_J)} g_{\boldsymbol{\Theta}}(\boldsymbol{\theta} \vee \boldsymbol{\theta}'|\mathbf{t}_J) g_{\boldsymbol{\Theta}}(\boldsymbol{\theta} \wedge \boldsymbol{\theta}'|\mathbf{t}_J) \\
&\geq \frac{c(\mathbf{t}'_J)}{c(\mathbf{t}_J)} \prod_{j \in J} \frac{\bar{F}_j(t'_j|\boldsymbol{\theta} \vee \boldsymbol{\theta}')}{\bar{F}_j(t_j|\boldsymbol{\theta} \vee \boldsymbol{\theta}')} g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|\mathbf{t}_J) g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}'|\mathbf{t}_J) \\
&\geq \frac{c(\mathbf{t}'_J)}{c(\mathbf{t}_J)} \prod_{j \in J} \frac{\bar{F}_j(t'_j|\boldsymbol{\theta})}{\bar{F}_j(t_j|\boldsymbol{\theta})} g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|\mathbf{t}_J) g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}'|\mathbf{t}_J) \\
&= g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|\mathbf{t}'_J) g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}'|\mathbf{t}_J),
\end{aligned}$$

implying (4.10), where the first inequality follows from the MTP_2 property of $g_{\boldsymbol{\Theta}}(\cdot|\mathbf{t}_J)$ for each $\mathbf{t}_J$ and the second one follows from assumption (a). This completes the proof. ■

By similar argument to that in the proof of Theorem 4.3, we can get the following two theorems concerning the dependence notions of PLTRD and PRD.

THEOREM 4.4: *Let* $(\mathbf{T}, \boldsymbol{\Theta})$ *be a multidimensional latent variable model with the conditional distribution function of* $T_i(\boldsymbol{\theta})$ *being* $F_i(\cdot|\boldsymbol{\theta})$ *for each i. If*

(a)

$$T_i(\boldsymbol{\theta}) \leq_{\text{rh}} [\geq_{\text{rh}}]\, T_i(\boldsymbol{\theta}') \text{ whenever } \boldsymbol{\theta} \leq \boldsymbol{\theta}' \text{ for each } i$$

(b) $F_i(t|\boldsymbol{\theta})$ *is MTP*$_2$ *in* $\boldsymbol{\theta}$ *for each t and i*

(c) $\boldsymbol{\Theta}$ *is MTP*$_2$,

then $\mathbf{T}$ *is PLTRD.*

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PROOF: The proof is similar to that of Theorem 4.3 with $g_{\boldsymbol{\Theta}}(\cdot|\mathbf{t}_J)$, denoting the density or mass function of $[\boldsymbol{\Theta}|\mathbf{T}_J \leq \mathbf{t}_J]$, given by

$$g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|\mathbf{t}_J) = c(\mathbf{t}_J)g(\boldsymbol{\theta}) \prod_{j\in J} F_j(t_j|\boldsymbol{\theta}),$$

where $c(\mathbf{t}_J)$ is a normalizing positive constant given by

$$c(\mathbf{t}_J) = \left[\int \cdots \int_{\Re^m} g(\boldsymbol{\theta}) \prod_{j\in J} F_j(t_j|\boldsymbol{\theta})\, d\boldsymbol{\theta} \right]^{-1}.$$

This completes the proof. ■

THEOREM 4.5: *Let* $(\mathbf{T}, \boldsymbol{\Theta})$ *be a multidimensional latent variable model with the conditional density function of* $T_i(\boldsymbol{\theta})$ *being* $f_i(\cdot|\boldsymbol{\theta})$ *for each i. If*

(a)

$$T_i(\boldsymbol{\theta}) \leq_{\text{lr}} [\geq_{\text{lr}}]\, T_i(\boldsymbol{\theta}') \text{ whenever } \boldsymbol{\theta} \leq \boldsymbol{\theta}' \text{ for each } i$$

(b) $f_i(t|\boldsymbol{\theta})$ *is* MTP_2 *in* $\boldsymbol{\theta}$ *for each t and i*
(c) $\boldsymbol{\Theta}$ *is* MTP_2,

then $\mathbf{T}$ *is PRD.*

PROOF: The proof is similar to that of Theorem 4.3, with $g_{\boldsymbol{\Theta}}(\cdot|\mathbf{t}_J)$, denoting the density or mass function of $[\boldsymbol{\Theta}|\mathbf{T}_J = \mathbf{t}_J]$, given by

$$g_{\boldsymbol{\Theta}}(\boldsymbol{\theta}|\mathbf{t}_J) = c(\mathbf{t}_J)g(\boldsymbol{\theta}) \prod_{j\in J} f_j(t_j|\boldsymbol{\theta}),$$

where $c(\mathbf{t}_J)$ is a normalizing positive constant given by

$$c(\mathbf{t}_J) = \left[\int \cdots \int_{\Re^m} g(\boldsymbol{\theta}) \prod_{j\in J} f_j(t_j|\boldsymbol{\theta})\, d\boldsymbol{\theta} \right]^{-1}.$$

This completes the proof. ■

For a componentwise separable latent variable model $(\mathbf{T}, \boldsymbol{\Theta})$, assumption (b) in Theorems 4.3, 4.4, and 4.5 can be omitted.

5. APPLICATIONS

In this section, we give two applications of the results that were developed in previous sections. The applications are not exhaustive. There are several other interesting applications and examples given in Shaked and Spizzichino [25], Khaledi and Kochar [16], and Belzunce and Semeraro [3] to which the results in Section 4 can be applied.

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5.1. Dependence Among Concomitants of Order Statistics

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample of size n from a continuous bivariate distribution. If we arrange the X's in ascending order as $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$, then the Y's associated with these order statistics are denoted by $Y_{[1]}, Y_{[2]}, \ldots, Y_{[n]}$ and are called concomitants of order statistics. They are also known as induced order statistics in the literature. Let $F(\cdot|x)$ denote the conditional distribution function of Y given $X = x$ and let G denote the distribution function of $\boldsymbol{\Theta} = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$. Then the distribution function of $\mathbf{Y}_{[\,]} = (Y_{[1]}, Y_{[2]}, \ldots, Y_{[n]})$ is given by

$$H(\mathbf{y}) = \int \cdots \int_{\Re^n} \prod_{i=1}^{n} F(y_i|\theta_i)\, dG(\boldsymbol{\theta}), \qquad \mathbf{y} \in \Re^n$$

(see David and Nagaraja [7]); that is, $(\mathbf{Y}_{[\,]}, \boldsymbol{\Theta})$ is a componentwise separable latent variable model.

Khaledi and Kochar [15] obtained several results on dependence structure of $\mathbf{Y}_{[\,]}$ under different types of dependence between the parent random variables X and Y. Blessinger [4] extended some results in [15]. He proved that if (X, Y) is TP_2 or RR_2 (reverse regular of order 2) then $\mathbf{Y}_{[\,]}$ is MTP_2, and that if $\bar{F}(y|x)$ is TP_2 or RR_2 in (x, y), then $\mathbf{Y}_{[\,]}$ is RCSI. Similarly, if $F(y|x)$ is TP_2 or RR_2 in (x, y), then $\mathbf{Y}_{[\,]}$ is LCSD. Recall that (X, Y) is RR_2 if and only if $(X, -Y)$ is TP_2, and that a function $\psi(x, y)$ is RR_2 if and only if $\psi(x, -y)$ is TP_2 in (x, y). Notice that $\boldsymbol{\Theta}$ is MTP_2 (see [14, Prop. 3.11]). By Proposition 3.2 and Theorems 4.3 and 4.4, we conclude the following.

THEOREM 5.1:

1. *If (X, Y) is TP_2 or RR_2, then $\mathbf{Y}_{[\,]}$ is PRD.*
2. *If $\bar{F}(y|x)$ is TP_2 or RR_2 in (x, y), then $\mathbf{Y}_{[\,]}$ is PRTRD.*
3. *If $F(y|x)$ is TP_2 or RR_2 in (x, y), then $\mathbf{Y}_{[\,]}$ is PLTRD.*

5.2. Dependence Among Concomitants of Record Values

Let $\{(X_i, Y_i), i \ge 1\}$ be a sequence of independent and identically distributed random vectors from a continuous bivariate distribution and let $\{\Theta_n, n \ge 1\}$ be the sequence of upper record values of the sequence $\{X_i, i \ge 1\}$ with $\Theta_1 = X_1$. Then the Y's associated with Θ_n are denoted by $Y_{[n]}$ and is called the concomitant of the nth record value. Let $F(\cdot|x)$ denote the conditional distribution function of Y_1 given $X_1 = x$, and let $\bar{W}(\cdot)$ and $w(\cdot)$ denote the survival function and the density function of X_1, respectively. It is well known that the joint density function of $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_n)$ is given by

$$g(\boldsymbol{\theta}) = w(\theta_n) \prod_{i=1}^{n-1} \frac{w(\theta_i)}{\bar{W}(\theta_i)} \cdot 1_{\{\theta_1 < \theta_2 < \cdots < \theta_n\}} \tag{5.1}$$

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and that the joint survival function of the concomitants $\mathbf{Y}_{[\,]} = (Y_{[1]}, \ldots, Y_{[n]})$ is given by

$$\bar{H}(\mathbf{y}) = \int \cdots \int_{\Re^n} \left\{ \prod_{i=1}^{n} \bar{F}(y_i|\theta_i) \right\} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}$$

(see Khaledi and Kochar [16]). Hence, $(\mathbf{Y}_{[\,]}, \boldsymbol{\Theta})$ is a componentwise separable latent variable model.

Khaledi and Kochar [16] proved that if (X,Y) is TP_2 or RR_2, then $\mathbf{Y}_{[\,]}$ TP_2 in pairs, and that if $\bar{F}(y|x)$ is TP_2 or RR_2 in (x, y), then $\bar{H}(\mathbf{y})$ is TP_2 in pairs. Similarly, if $F(y|x)$ is TP_2 or RR_2 in (x, y), then $H(\mathbf{y})$ is TP_2 in pairs. Since $1_{\{\theta_1<\theta_2<\cdots<\theta_n\}}$ is MTP_2 in $\boldsymbol{\theta}$, it follows from (5.1) that $\boldsymbol{\Theta}$ is MTP_2. Therefore, by Theorems 4.3–4.5, we conclude that Theorem 5.1 also holds.

Acknowledgments

This work was supported by the Program for New Century Excellent Talents in University (NCET-04-0569), a PhD Program Foundation of Ministry of Education of China, and two grants from the University of Science and Technology of China and the Chinese Academy of Sciences.

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