THE REVIEW OF SYMBOLIC LOGIC
Volume 6, Number 1, March 2013

ANTIREALISM AND CONSTRUCTIVISM: BROUWER'S WEAK COUNTEREXAMPLES

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Abstract. Strictly intuitionistic inferences are employed to demonstrate that three conditions—the existence of Brouwerian weak counterexamples to **Test**, the recognition condition, and the **BHK** interpretation of the logical signs—are together inconsistent. Therefore, if the logical signs in mathematical statements governed by the recognition condition are constructive in that they satisfy the clauses of the **BHK**, then every relevant instance of the classical principle **Test** is true intuitionistically, and the antirealistic critique of conventional logic, once thought to yield such weak counterexamples, is seen, in this instance, to fail.

§1. Antirealism, an incomplete project. When it comes to intuitionistic mathematics, Dummett's antirealism is at best a project and not, contrary to views widely held, a task completed. As formulated in Dummett (1975a, 1975b, 1977), that project was to engage philosophers and logicians in contriving kinds of semantics that take justified assertibility or construction or proof as primary and are (intuitionistically) provably adequate to the formal intuitionistic logic set out in Heyting (1930). In serving those ends, investigators imposed a number of assumptions on the imagined semantics.

First, it was to establish an intellectually firm basis on which the general *Tertium non Datur* or **TND**, the scheme

$$\phi \vee \neg\phi,$$

as well as such intuitionistically inadmissible forms as the *Principle of Testability* or **Test**,

$$\neg\phi \vee \neg\neg\phi,$$

could be proven—with full intuitionistic rigor—to be invalid. Often, enthusiasts for antirealism thought to see in such writings of Dutch mathematician L.E.J. Brouwer (1881–1966) as (Brouwer, 1908) a clear route to the desired proofs of invalidity, that route arising from Brouwer's *weak counterexample* arguments against theorems of conventional logic. In those arguments, Brouwer claimed to find, for the classical laws mentioned, instances of them lacking constructive truth.

Second, as a means to laying a philosophical foundation under said counterexamples, antirealistic semantics was to satisfy a *recognition condition*. According to that condition, speakers having a real grasp of a mathematical statement ϕ^A over domain A would have the ability in principle to recognize ϕ^A as holding whenever mathematical circumstances are such that ϕ^A does indeed hold. Antirealists argued that this feature would set the desired semantics apart from a conventional or realistic, truth-value semantics on which classical laws like **TND** and **Test** are valid. By their lights, the latter account of meaning violates the recognition condition. Truth-value semantics seems to permit mathematically sophisticated

Received: February 1, 2012.

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doi:10.1017/S1755020312000287

https://doi.org/10.1017/S1755020312000287 Published online by Cambridge University Press

speakers to accept as correct such a disjunction as "Either Goldbach's Conjecture is true or its negation is," while giving them no help when it comes to picking out, from the two disjoined options—Goldbach's Conjecture and its negation—the single true disjunct.

Third, the mathematics to which antirealism was to afford a semantical foundation and in terms of which antirealistic semantics was to be formulated would be constructive. Among other things, this meant to antirealists that the tenets of intuitionistic mathematics and semantical antirealism, once the latter got spelled out, were to conform to the *Brouwer–Heyting–Kolmogorov* or **BHK** interpretation of the familiar logical signs. On that interpretation, the assertion conditions or proof conditions of sentences had to interlace recursively with the logical signs in a fashion shortly to be described.

Here and throughout, the sign "$\Vdash$" stands for the relation 'proves' or 'justifies the assertion of', and "$\nVdash$" stands for its complement. In the display below, the variables "c" and "d" refer to or range over constructions supplying assertion or proof conditions. The result of applying c, conceived as an operation, to d is $c \circ d$; c_0 and c_1 are the results of dismantling c, conceived as an ordered pair, into its respective first and second components. The natural numbers 0 and 1 are also assumed to be proofs, perhaps nugacious. For mathematical sentences of the forms $\phi \vee \psi$, $\phi \to \psi$, and $\exists x.\phi(x)$, the **BHK** interpretation requires that

$$c \Vdash (\phi \vee \psi) \text{ iff } (c_0 = 0 \to c_1 \Vdash \phi) \wedge (c_0 \neq 0 \to c_1 \Vdash \psi),$$
$$c \Vdash (\phi \to \psi) \text{ iff } \forall d\, (d \Vdash \phi \to (c \circ d) \Vdash \psi), \text{ and}$$
$$c \Vdash \exists x.\phi(x) \text{ iff } c_1 \Vdash \phi(c_0).$$

This is an initial, orientative sample from the full **BHK** interpretation, which is set out in Section 4. The first clause is to guarantee *inter alia* that disjunctions, when interpreted constructively, can be decided: if $\phi \vee \psi$ is proven, one can, by determining whether $c_0 = 0$, find out which of ϕ or ψ is true. The second clause is to ensure that, if $c \Vdash (\phi \to \psi)$, then there is an operational connection, forged by c, between the assertion conditions of ϕ and those of ψ. The third clause is meant to enforce a constructive reading of $\exists$ on which $c \Vdash \exists x.\phi(x)$ obtains only if c is a constructive proof of $\exists x.\phi(x)$ in that from c, one can unpack a specific c_0 such that $\phi(c_0)$ provably holds thanks to c_1.

§2. Recognition and decidability. I shall now be more specific. Antirealists took the fundamental notion of their semantics to be a relation defined by associating with each suitable sentence ϕ of a fixed and delimited language a unary predicate

$$c \Vdash \phi$$

(sometimes denoted "$\Pi_\phi(c)$") on canonical justifications c. Their idea was that $c \Vdash \phi$ holds if and only if c affords a circumstance under which the assertion of ϕ would be fully justified. When it comes to mathematics, c is known as a construction or 'proof', and, if $c \Vdash \phi$, one says that c is a proof of ϕ. Lest troubling issues of reflexivity arise, $c \Vdash \phi$ will here be formed only for ϕ drawn from languages not containing the semantic predicates "$c \Vdash \phi$" or their equivalents. In the context "$c \Vdash \phi$," "ϕ" is not a variable admitting quantification.

Since, in this essay, values of "ϕ" are mathematical sentences exclusively, an assertion of ϕ is indefeasibly justified when ϕ is correctly proven. Hence, it should be the case that

$$\exists c(c \Vdash \phi) \to \phi^A.$$

The superscript "A" marks explicitly that ϕ^A is an interpreted sentence, interpreted in accord with the **BHK** over a suitable, inhabited mathematical domain A.

https://doi.org/10.1017/S1755020312000287 Published online by Cambridge University Press

In addition, the antirealists adopted a recognition condition to secure both the verificationistic credentials of antirealist semantics and the purported counterexamples to conventional logical laws, as exemplified in the 'weak counterexample' reasoning concerning Goldbach's Conjecture detailed below. As will be seen, that reasoning appears to require the converse of the foregoing, namely,

$$\phi^A \rightarrow \exists d(d \Vdash \phi).$$

Together, the two lines last displayed yield the biconditional scheme,

$$\phi^A \leftrightarrow \exists d\ (d \Vdash \phi).$$

Since our mathematics and metamathematics are to be thoroughly constructive, we must read the existential quantifier "$\exists$" and the conditional "$\rightarrow$" in the above display according to the **BHK** interpretation. On that reading, the foregoing scheme reports, at least, that operational means are available in principle such that ϕ^A turns out to be true only if those means can be employed to yield a specific condition d such that $d \Vdash \phi$ holds, and conversely.

To ensure that the biconditional scheme affords genuine recognition of all intuitionistic mathematical truths, the antirealist demands as well that the predicate

$$c \Vdash \phi$$

on proofs c be decidable for all ϕ, that is, he or she insists that

$$c \Vdash \phi \vee c \nVdash \phi$$

for each mathematical sentence ϕ and for all conditions c. On the **BHK** interpretation of $\vee$, "decidable" here means that a procedure is available in principle by which, given a ϕ and a suitable description of assertion condition c, one can determine which of $c \Vdash \phi$ or $c \nVdash \phi$ is true. Hence, the antirealist argues that, whenever $c \Vdash \phi$ obtains, one can, using the decision procedure, recognize that $c \Vdash \phi$. By the way, it is not commonly required that the predicate "$c \Vdash \phi$" be recursive in the sense of Turing, and this is not assumed herein. Nor is it essential that "$c \Vdash \phi$" be definable in any particular formalism.

Therefore, a reasonable expression of the antirealist's recognition condition seems to be the conjoint imposition of the biconditional scheme,

$$\phi^A \leftrightarrow \exists d.(d \Vdash \phi)$$

and the disjunctive scheme,

$$\forall c(c \Vdash \phi \vee\ c \nVdash \phi).$$

To check that this is indeed reasonable, assume that ϕ^A is true. Then, with the rightward half of the biconditional, it follows that

$$\exists d.\ d \Vdash \phi.$$

From the above, interpreted à la **BHK**, one extracts a c_0 such that $c_0 \Vdash \phi$. Because the predicate "$c \Vdash \phi$" is decidable, one can recognize that $c_0 \Vdash \phi$, the other option, that $c_0 \nVdash \phi$, being impossible in this case. In sum, the recognition condition, formulated as the biconditional and disjunctive schemes jointly, makes it such that, if ϕ^A is true, one can bring to hand a condition d under which ϕ is proven and be assured that d is indeed a requisite such condition.

To a dyed-in-the-wool constructivist, this encomium for the recognition condition may appear trivial, merely a longwinded rehearsal of the core constructivistic *Weltanschauung*.

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That is no problem; the current section was intended to set terminology and to aid those who may not be thoroughly familiar with the constructivist's outlook. Moreover, if the recognition condition is truly a constructivistic triviality, then the formal arguments of the present essay cut the constructivistic antirealist all the more deeply, since they purport to show that this 'triviality', when allied with the **BHK** interpretation, entails a result that is nontrivial, namely, that **Test** has no weak counterexamples. Apart from the statement of the recognition condition, no element from the informal exposition of the present section plays any role in the arguments for the Main Theorem, which commence in Section Six below.

Those acquainted with Goodman's Paradox (Goodman, 1968) for Kreisel's theory of constructions (Kreisel, 1962) may worry that this rendering of the recognition condition, added to a basic theory of constructions, leads to inconsistency. We avoid Goodman's paradoxical reasoning by restricting the expression of the recognition condition to closed ϕ.

§3. Brouwerian weak counterexamples. The antirealist seems to require the recognition condition, as formulated above, plus the **BHK** to reconstruct Brouwer's purported counterexamples to the **TND** (Brouwer, 1908). For example, Goldbach's Conjecture $G^{\mathfrak{N}}$ makes the statement, expressed by formula G under the standard interpretation $\mathfrak{N}$, that every even number larger than 3 is the sum of two prime numbers. At the time of writing, $G^{\mathfrak{N}}$ is neither proven nor refuted. Assume that **TND** is valid. Then, its instance provided by $G^{\mathfrak{N}}$,

$$G^{\mathfrak{N}} \vee \neg G^{\mathfrak{N}},$$

should be true. From the biconditional scheme of the recognition condition, it follows that

$$\exists c.\ c \Vdash (G \vee \neg G).$$

From the clause of the **BHK** governing $\vee$, this implies in turn that

$$\exists c.\, c \Vdash G \vee c \Vdash \neg G.$$

Since the existential quantifier $\exists c$ over proofs is to be constructive, there must be obtainable a specific construction d such that

$$d \Vdash G \vee d \Vdash \neg G.$$

According to the disjunctive scheme, either it is recognizable that $d \Vdash G$ or recognizable that $d \nVdash G$. If the former, $G^{\mathfrak{N}}$ is known to be true—but that is not the case, since $G^{\mathfrak{N}}$ represents an unsolved problem. If the latter,

$$d \Vdash \neg G.$$

Using the disjunctive scheme once again on $d \Vdash \neg G$, the latter can be recognized to obtain when it does. It follows that $\neg G^{\mathfrak{N}}$ is known to be true, which is also contrary to fact. Therefore, on this antirealistic reconstruction of Brouwer's reasoning, the **TND** fails to be valid, and $G^{\mathfrak{N}}$ is said to provide a weak counterexample to that law.

§4. The BHK interpretation. Loyal mathematical antirealists as Dummett (1977) imagined them need to prefer the constructively-interpreted logical signs over their classically-interpreted counterparts. They look to ensure a suitably constructive reading of those signs by imposing on mathematical language the *Brouwer–Heyting–Kolmogorov* or **BHK** interpretation, as described in Troelstra & van Dalen (1988, p. 9). To that end,

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they assume the **BHK** to spell out basal semantic relations that are presumed to hold between constructions or assertion conditions or so-called proofs c and formulae ϕ when the latter express their peculiarly constructive meanings.

The **BHK** interpretation is meant to enforce meanings on the connectives and quantifiers as constructive *logical* signs. Hence, it is intended to apply to any mathematical statements over any (constructively admissible) domain. In accord with that, we let A be an inhabited set. If the **BHK** is to be the all-purpose explanation for the constructive forces of the logical signs that it pretends to be, then A must be expandable into a collection $\mathfrak{C}$ by the addition of a set of constructions c and d such that $c \circ d$, $\langle c, d \rangle$, c_0, and c_1 are defined not only for all added constructions but also for members of the original set A. We further assume that $\mathfrak{C}$ includes a suitable null construction 0 and that the predicate "$c = 0$" is always decidable in that, for all c,

$$c = 0 \vee c \neq 0$$

obtains.

It seems reasonable to assume that such an expansion of domain A can be carried out. Starting with an inhabited set A, one forms the closure of A under finite sequences, obtaining A^*. Then, one defines the graph model $\mathcal{G}(A^*)$ over A^* as in Scott (1975). $\mathcal{G}(A^*)$ is a model of combinatory logic; it features a universal operation of application as well as pairing and unpairing.

Now, we make it all official. Let $\mathcal{L}$ be an arbitrary, one-sorted, first-order language with names, predicate, and relation signs to be interpreted over the domain A. Let $\mathcal{L}^e$ be a two-sorted extension of $\mathcal{L}$ with

1. a set of new names
$$c, d, \ldots,$$
used autonymously, one for each member of $\mathfrak{C}$,
2. a new set of atomic formulae of the forms
$$c = d \quad \text{and} \quad c \,\varepsilon\, A,$$
3. special function symbols for application
$$c \circ d,$$
and for left- and right- unpairing of members of $\mathfrak{C}$
$$c_0 \quad \text{and} \quad c_1,$$
and
4. allied with the new sort in $\mathcal{L}^e$, new quantifiers
$$\forall d \quad \text{and} \quad \exists d$$
to range over the set $\mathfrak{C}$.

That the letters
$$c, d, \ldots$$
function in $\mathcal{L}^e$ both as names used autonomously and as bound variables in quantifiers ranging over constructions should cause the reader no trouble. In the display immediately below, formulae ϕ and ψ are formulae from the original language $\mathcal{L}$ with autonymous names from $\mathcal{L}^e$ substituted for all their free variables. Therefore, those formulae are really

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sentences of $\mathcal{L}^e$. (In the sequel, the term 'sentence' will refer exclusively to sentences in $\mathcal{L}^e$ obtained by replacing all free variables of formulae from $\mathcal{L}$ with autonymous names for members of $\mathfrak{C}$.) Formulae ϕ in the context "$c \Vdash \phi$" should be treated as uninterpreted; formulae on the right-hand sides of the biconditionals and outside that context should be treated as interpreted over A. Here, and throughout, the superscript on 'ϕ^A' is to remind the reader of those occasions when ϕ is to be so understood.

With these preliminaries in place, a full statement of the **BHK** interpretation comprises the following biconditionals.

For ϕ an atomic sentence, $(c \Vdash \phi) \leftrightarrow \phi^A$
$c \Vdash (\phi \wedge \psi) \leftrightarrow c_0 \Vdash \phi \ \wedge \ c_1 \Vdash \psi$
$c \Vdash (\phi \vee \psi) \leftrightarrow (c_0 = 0 \rightarrow c_1 \Vdash \phi) \wedge (c_0 \neq 0 \rightarrow c_1 \Vdash \phi)$ **[1]**
$c \Vdash (\phi \rightarrow \psi) \leftrightarrow \forall d\ (d \Vdash \phi \rightarrow c \circ d \Vdash \psi)$
$c \Vdash \neg\phi \quad \leftrightarrow \forall d.\, d \nVdash \phi$
$c \Vdash \exists x.\phi(x) \leftrightarrow c_1 \Vdash \phi(c_0) \ \wedge \ c_0 \in A$
$c \Vdash \forall x.\phi(x) \leftrightarrow \forall d\, \varepsilon\, A.\ c \circ d \Vdash \phi(d).$

§5. Objections and replies. In intuitionistic logic, conjunction and conditional are *negative* propositional connectives. Some readers may object that the third of the clauses above displayed, clause **[1]** governing disjunction, appears here in negative form, specifically, with conjunction as the main connective on the right-hand side of the biconditional, and the conditional in a supporting role. Those readers would prefer a clause governing disjunction formulated strictly in terms of positive connectives, perhaps using disjunction itself. They might opt to replace **[1]** with **[2]**:

$$c \Vdash (\phi \vee \psi) \leftrightarrow (c \Vdash \phi) \vee (c \Vdash \psi) \quad \textbf{[2]}$$

En passant one remarks that version **[2]**, although similar to the formulation of Troelstra & van Dalen (1988, p. 9), seems insufficient in itself. According to **[2]**, a construction c can justify the assertion of a disjunction $(\phi \vee \psi)$ even though nobody can tell from c alone—and without recourse to other metamachinery—whether $c \Vdash \phi$ or $c \Vdash \psi$. Where it assumed throughout that the "$c \Vdash \phi$" predicate is intuitionistically decidable (for c, with ϕ fixed), there would be no difficulty in the long run.

A response to this objection is immediate. When "$c \Vdash \phi$" is assumed decidable for each ϕ, the clause **[2]**, formulated positively as above, is intuitionistically logically equivalent to a strictly negative formulation, **[3]**:

$$c \Vdash (\phi \vee \psi) \leftrightarrow \neg[(c \nVdash \phi) \wedge (c \nVdash \psi)] \quad \textbf{[3]}$$

Proof. It is obvious that

$$(c \Vdash \phi) \vee (c \Vdash \psi)$$

implies

$$\neg[(c \nVdash \phi) \wedge (c \nVdash \psi)].$$

For the other direction, assume that

$$\neg[(c \nVdash \phi) \wedge (c \nVdash \psi)]$$

obtains. From decidability, it follows that

$$c \Vdash \phi \ \vee \ c \nVdash \phi,$$

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and that

$$c \Vdash \psi \ \vee \ c \nVdash \psi.$$

There are now four combined cases. If the first of them holds, then

$$c \Vdash \phi \ \vee \ c \Vdash \psi$$

holds *a fortiori*. The second case, namely,

$$c \Vdash \phi \ \wedge \ c \nVdash \psi,$$

implies that

$$(c \Vdash \phi) \ \vee \ (c \Vdash \psi),$$

as does the analogous third case. The fourth and last of the possible cases,

$$(c \nVdash \phi) \ \wedge \ (c \nVdash \psi),$$

is explicitly ruled out by the original assumption. □

Therefore, **Proposition 6.2** *infra* continues to hold, with $c \Vdash (\phi \vee \psi)$ expressed positively. Subsequent proofs of the Lemmas and the Main Theorem go through as well.

Some readers might opt to replace clause [1] with the alternative,

$$c \Vdash (\phi \vee \psi) \ \leftrightarrow \ (c_0 = 0 \rightarrow c_1 \Vdash \phi) \wedge (c_0 = 1 \rightarrow c_1 \Vdash \phi).$$

If one adds the perfectly reasonable—if not requisite—assumption that, for all constructions c,

$$c = 1 \vee c \neq 1,$$

we can proceed precisely as before.

In addition, readers might object that clause [**1**] governing disjunction differs from [**2**] not only in that [**1**] is negative but also in that it applies explicit unpairing. According to [**1**], if $c \Vdash (\phi \vee \psi)$, then, by testing c_0 alone for equality with 0, one can determine which of the two disjunctions—ϕ or ψ—c_1 justifies. Once again, the response is straightforward. Clauses [**1**] and [**2**] are equivalent relative to a simple, primitive recursive theory of constructions, a theory essential to a proof that formal intuitionistic logic is sound with respect to the **BHK** interpretation.

Proof. Here, the two competing definitions of

$$\Vdash (\phi \vee \psi)$$

are tracked by subscripting the $\Vdash$ symbol. In accord with [**1**], we write,

$$c \Vdash_1 (\phi \vee \psi) \ \leftrightarrow \ (c_0 = 0 \rightarrow c_1 \Vdash_1 \phi) \wedge (c_0 \neq 0 \rightarrow c_1 \Vdash_1 \phi),$$

and, in accord with [**2**],

$$c \Vdash_2 (\phi \vee \psi) \ \leftrightarrow \ (c \Vdash_2 \phi) \vee (c \Vdash_2 \phi).$$

The inductive hypothesis is that there are functions f_ϕ, f_ψ, g_ϕ, and g_ψ, primitive recursive on constructions c, such that, for formulae ϕ and ψ, if

$$c \Vdash_1 \phi,$$

then

$$f_\phi(c) \Vdash_2 \phi,$$

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and similarly, if

$$c \Vdash_1 \psi,$$

then

$$f_\psi(c) \Vdash_2 \psi.$$

The converse conditionals are assumed to hold for g_ϕ and g_ψ with respect to $c \Vdash_2 \phi$ and $c \Vdash_2 \psi$.

If suffices to demonstrate that there are functions $f_{(\phi\vee\psi)}$ and $g_{(\phi\vee\psi)}$, primitive recursive on constructions, such that, for all constructions c, if

$$c \Vdash_1 (\phi \vee \psi),$$

then

$$f_{(\phi\vee\psi)}(c) \Vdash_2 (\phi \vee \psi),$$

and conversely, if

$$c \Vdash_2 (\phi \vee \psi),$$

then

$$g_{(\phi\vee\psi)}(c) \Vdash_1 (\phi \vee \psi).$$

For $f_{(\phi\vee\psi)}$, we note that, if intuitionistic propositional logic is to be provably sound with respect to constructions and the **BHK**, there must be a construction d such that, for all constructions x and y,

$$d(x, y, z) = \begin{cases} y, & \text{if } x = 0, \\ z, & \text{if } x \neq 0. \end{cases}$$

It is in terms of d, f_ϕ, and f_ψ that $f_{(\phi\vee\psi)}$ is defined as follows.

$$f_{(\phi\vee\psi)} = d(c_0, f_\phi(c_1), f_\psi(c_1)).$$

To check that this definition does the trick, assume that

$$c \Vdash_1 (\phi \vee \psi).$$

In accord with [**1**], it follows that

$$(c_0 = 0 \to c_1 \Vdash_1 \phi) \wedge (c_0 \neq 0 \to c_1 \Vdash_1 \phi).$$

Hence, if $c_0 = 0$, then by the inductive hypothesis,

$$f_\phi(c_1) \Vdash_2 \phi,$$

so that

$$f_{(\phi\vee\psi)} \Vdash_2 (\phi \vee \psi).$$

In the same way, if $c_0 \neq 0$, then by the inductive hypothesis again,

$$f_\psi(c_1) \Vdash_2 \psi,$$

and

$$f_{(\phi\vee\psi)} \Vdash_2 (\phi \vee \psi).$$

For the converse with g_ϕ and g_ψ, one makes the assumption (parallel to that imposed upon $\Vdash_1$) that the predicate

$$c \Vdash_2 \phi$$

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is decidable, as a predicate of c. In the present context, such decidability comes down to the existence of a construction e such that, for all constructions x,

$$e(x) = \begin{cases} 0, & \text{if } x \Vdash_2 \phi, \\ 1, & \text{if } x \nVdash_2 \phi. \end{cases}$$

Then, one defines $g_{(\phi \vee \psi)}(c)$ to be

$$d(e(c), \langle e(c), g_\phi(c) \rangle, \langle e(c), g_\psi(c) \rangle),$$

where $\langle x, y \rangle$ denotes standard pairing. It is easy to check that, so defined, $g_{(\phi \vee \psi)}$ has the desired property: if

$$c \Vdash_2 (\phi \vee \psi),$$

then

$$g_{(\phi \vee \psi)}(c) \Vdash_1 (\phi \vee \psi). \qquad \square$$

Therefore, up to a simple set of interchange functions, $\Vdash_1$ and $\Vdash_2$ are equivalent; formulae verified by the one are verified by the other and conversely. Were **Test** valid with respect to $\Vdash_1$, it would be valid with respect to $\Vdash_2$ as well. Were a weak counterexample forthcoming from $\Vdash_2$, the same would arise from $\Vdash_1$.

Next, some might object to the present statement of the clause governing negation, claiming that it is not sufficiently constructivistic. To repeat, the clause in question is

$$c \Vdash \neg\phi \leftrightarrow \forall d.\, d \nVdash \phi.$$

One replies that, if the above is indeed insufficiently constructivistic, then so is the equivalent clause given as (H4) at Troelstra & van Dalen (1988, p. 9). The Troelstra–van Dalen clause reads as follows in the notation of the current discussion.

$$\forall c.\, c \nVdash_{td} \bot.$$

For present purposes, the subscript "td" signals Troelstra and van Dalen's definition of $\Vdash$. Troelstra and van Dalen explain, also on their page 9, that $\bot$ represents a contradiction. On this approach, it is understood throughout that $\neg\phi$ is an abbreviation for

$$\phi \rightarrow \bot,$$

which is in accord with formal intuitionistic logic.
It is easy to show, exploiting intuitionistic inferences exclusively, that

$$c \Vdash_{td} \neg\phi \leftrightarrow \forall d.\, d \nVdash_{td} \phi,$$

that is, that the clause governing $\neg$ in the submitted article is equivalent intuitionistically to that in the book of Troelstra and van Dalen.

Proof. From

$$c \Vdash_{td} \neg\phi,$$

it follows, by the definition of $\neg$ just supplied, that $c \Vdash_{td} (\phi \rightarrow \bot)$. By the **BHK** interpretation of $\rightarrow$ (shared with the Troelstra–van Dalen monograph), we have that

$$\forall d(d \Vdash_{td} \phi \rightarrow c \circ d \Vdash_{td} \bot).$$

Since Troelstra and van Dalen insist that there can be no constructions c such that $c \Vdash_{td} \bot$, the latter entails intuitionistically that

$$\forall d.\, d \nVdash_{td} \phi.$$

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For the converse, assume that

$$\forall d.\ d \nVdash_{td} \phi.$$

From this assumption, in intuitionistic logic, it follows that

$$\forall d(d \Vdash_{td} \phi \rightarrow c \circ d \Vdash_{td} \bot).$$

Indeed, any formula could replace $\bot$ in the last display; formal intuitionistic logic validates *ex falso quodlibet*. Hence, $c \Vdash_{td} (\phi \rightarrow \bot)$ and, finally,

$$c \Vdash_{td} \neg\phi.$$

From the intuitionistic perspective, these chains of inferences appear unobjectionable. □

Finally, there are those who would object that the clauses of the **BHK** as set out at the close of the preceding section do not capture correctly or adequately the special contents or meanings thought to be allied to the intuitionistic connectives. To them, there are two responses. To begin with, I maintain that there are no such proprietary intuitionistic meanings, at least if they are supposed to differ from those familiar meanings of the logical connectives, if any, explained to undergraduates in baby logic courses almost everywhere. Cf. McCarty (2008, 2011). Second, this objection seems to miss the metamathematical mark. The principal question relevant to the present writing is not whether the clauses as here presented capture some imaginary and arcane meanings, but whether those statements follow, within an intuitionistic metatheory, from whatever formulations of the clauses the objector would prefer. If they do follow, the present arguments go through with the desired final consequences.

§6. Two propositions. The following propositions are of a technical nature.

PROPOSITION 6.1. *For any sentence* ϕ, $0 \Vdash \neg\phi$ *iff* $\exists c.\ c \Vdash \neg\phi$ *iff* $\forall d.\ d \nVdash \phi$.

Proof. Immediate from the **BHK** clause for $\neg\phi$. □

Definition: A formula is *almost negative* iff it is constructed from atomic formulae by means of the connectives and quantifiers $\neg$, $\wedge$, $\rightarrow$, $\forall x$, and $\forall d$ exclusively, or is equivalent intuitionistically to such a formula.

PROPOSITION 6.2. *For all sentences* ϕ, *the formula* $c \Vdash \phi$ *is an almost negative formula.*

Proof. By induction on formulae using the clauses of the **BHK**. □

§7. Two lemmas. With help from these two lemmas, the validity of **Test** under substitutions is seen to be a consequence of the recognition condition plus the **BHK**.

LEMMA 7.1. *For any almost negative sentence* ϕ *and for any construction c,* $c \Vdash \phi$ *if and only if* ϕ^A.

Proof. By induction on the constructions of almost negative formulae using the clauses of the **BHK**. One example should suffice. According to the **BHK**, $c \Vdash \forall x.\ \phi(x)$ if and only if

$$\forall x \in A.\ c \circ x \Vdash \phi(x).$$

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The latter holds just in case, for all $a \in A$,

$$c \circ a \Vdash \phi(a).$$

By induction, the latter holds just in case

$$\forall x \in A.\, \phi(x). \qquad \square$$

The proof of **Lemma 2** calls for the assumption of the recognition condition.

LEMMA 7.2. *Given the recognition condition, for any sentence ϕ, $\neg\phi$ is almost negative.*

Proof. First, one notes that, in intuitionistic logic, $\neg(\phi \vee \psi)$ is equivalent to $\neg\phi \wedge \neg\psi$ and $\neg\exists x.\, \phi(x)$ to $\forall x.\, \neg\phi(x)$. However, $\neg\forall x.\, \phi(x)$ is not equivalent constructively to $\neg\forall x.\neg\neg\phi(x)$. In this case, it suffices to use the biconditional scheme of the recognition condition to obtain

$$\neg\forall x.\, \phi(x) \leftrightarrow \exists c.\, c \Vdash \neg\forall x.\, \phi(x).$$

From **Proposition 6.1**, it follows that

$$\neg\forall x.\, \phi(x) \leftrightarrow 0 \Vdash \neg\forall x.\, \phi(x).$$

From **Proposition 6.2**, the latter is an almost negative formula. $\square$

§8. The main theorem.

THEOREM 8.1. *Given the recognition condition and **BHK**, every instance of **Test** is true.*

Proof. According to the disjunctive scheme,

$$c \Vdash \phi \vee\ c \nVdash \phi$$

holds for every sentence ϕ of $\mathcal{L}$. By **Lemma 7.1**, whenever the sentence ϕ is almost negative,

$$\phi^A \vee \neg\phi^A$$

obtains. By **Lemma 7.2**,

$$\neg\phi^A \vee \neg\neg\phi^A$$

is true for all sentences ϕ.

Throughout the reasoning, A is an inhabited set arbitrarily chosen, and $\mathcal{L}$ is an arbitrary first-order language to be interpreted over A. Therefore, **Test**

$$\neg\phi \vee \neg\neg\phi$$

is valid under arbitrary substitutions. $\square$

A final note may be in order. The results of the present essay are no obvious consequences of a correct solution to exercise 1.3.4 of Troelstra & van Dalen (1988, pp. 32–33). There, Troesltra and van Dalen give an abstract version of the **BHK** interpretation, a version consonant with the **BHK** as set out in this article and in the main body of their monograph. Troelstra and van Dalen then ask the reader to show that, if **TND** is adopted in the metatheory, all instances of **TND** and of the other familiar valid schemes of classical logic are true under the given interpretation. The abstract version of the **BHK**, as well as that presented here, can be adopted by the intuitionist. Neither entails intuitionistically

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the **TND** on its own. As Troelstra and van Dalen remark, a solution to their exercises proves that, if **TND** is to get a weak counterexample via the **BHK** interpretation, some further constraint, perhaps of a computational nature, must be imposed upon the concept of construction as it functions in the **BHK**. Certainly, one such possible constraint would be to demand that there are formulae $\phi(x)$ and inhabited mathematical domains A such that, for no construction c, is it the case that, for all $d \varepsilon A$,

$$\phi^A(d) \leftrightarrow c(d) = 0.$$

The Main Theorem and attendant propositions of the present essay, proved in a strictly intuitionistic metatheory, demonstrate that, even if further constraints—as just described—are added to the **BHK**, there is no hope for a counterexample to **Test**, a principle weaker, over formal intuitionistic logic, than **TND**, but rejected by constructivists and independent of formal intuitionistic logic.

§9. Conclusion and question. The recognition condition plus **BHK** imply that every relevant instance of the intuitionistically invalid scheme **Test** is intuitionistically true. It appears that no weak counterexample to **Test** can be constructed along strictly antirealist lines. It would seem therefore that the project of Dummett's antirealism cannot be completed for the usual formalization of constructive or intuitionistic logic, that bequeathed to us by Heyting (1930), and on which **Test** is not a theorem. If the antirealist wishes to avoid these conclusions, some new or alternative presentations of recognition or **BHK** would be required.

As Koss (2009) amply demonstrates, L.E.J. Brouwer, the father of mathematical intuitionism in the Twentieth Century, was hardly an antirealist of the Dummett stripe when he first (Brouwer, 1908) launched his critical assault on classical logic. Brouwer would not then have endorsed the verificationistic philosophy of language that Dummett took to be a propaedeutic to intuitionistic logic. It remains to ask whether Brouwer could have produced, in the period before his redeployment of intuitionistic analysis using the choice sequences discovered by Paul du Bois–Reymond, perhaps using methods alternative to those canvassed here, any cogent argument against **Test**.

§10. Acknowledgments. The author wishes to thank the members of the Indiana Foundations Group, especially Mr. Michael Koss, for many helpful discussions on the subject of this essay. Thanks also to the anonymous referees for their determined critical efforts on my behalf.

§11. Dedication. This article is dedicated to the memory of Mr. Neil E. McDarby (1951–2011), a true philosopher.

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