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RESEARCH ARTICLE

Small differences between cubic and rational numbers

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Abstract

In this paper, we show that for each sufficiently large integer H, there is a real cubic number α and a rational number r, both of height smaller than H, such that $0 < \alpha - r < 272/H^4$. The exponent 4 of H in this inequality is best possible. The numbers α and r are both constructed explicitly. We also show that a necessary and sufficient condition on (u, v) under which there is a positive constant $c_1 = c_1(d, u, v)$ such that the inequality $|\alpha - \beta| \geq c_1 H(\alpha)^{-u} H(\beta)^{-v}$ holds for all real algebraic numbers α of degree $d \geq 2$ and all real algebraic numbers β of degree $d - 1$ is $u \geq d - 1$ and $v \geq d$.

1. Introduction

A typical problem in Diophantine approximation is to evaluate the distance between the elements lying in two distinct sets of complex numbers A and B. For example, if the set A consists of a single irrational real algebraic number α and B is the set of all rational numbers $\mathbb{Q}$, then Roth's theorem asserts that for every $\delta > 0$, there exists a constant $c = c(\alpha, \delta) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^{2+\delta}} \tag{1}$$

for all coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$; see, for example, [12, 14, 16]. Generalizations of Roth's theorem can also be found in [6, 9] and most recently in [8]. On the other hand, by Hurwitz's theorem [16, p. 28], for any real irrational (algebraic or transcendental) number α, there are infinitely many pairs (p, q), with coprime $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}. \tag{2}$$

(A weaker Dirichlet's theorem asserts the same but without the factor $\sqrt{5}$ in the denominator.) A more general version of the Hurwitz theorem is given in [13]; see also a very recent paper [10].

Much less is known in the case when A and B are the sets of all real algebraic numbers of some degrees d and ℓ (where $d > \ell$), respectively. Recall that the *height* $H(\alpha)$ of an algebraic number α over $\mathbb{Q}$ is the maximal modulus of the coefficients of its minimal polynomial in $\mathbb{Z}[x]$. Since both above inequalities (1) and (2) depend on the height $H(p/q) = \max(|p|, q)$ of the involved rational number p/q, the heights of the involved algebraic numbers α of degree d and β of degree ℓ in such an estimate between α and β are expected as well.

A lower bound easily follows from an effective version of the Liouville theorem (see [4, Theorem A.1], [17, Lemma 3.7]) or from a simple estimate on the resultant of two algebraic numbers [7, Formula (15)]. For any positive integers $d > \ell$, there is a positive constant $c_0 = c_0(d, \ell)$ such that for any real algebraic number α of degree d over $\mathbb{Q}$ and any real algebraic number β of degree ℓ over $\mathbb{Q}$, we have

$$|\alpha - \beta| \geq \frac{c_0}{H(\alpha)^{\ell} H(\beta)^{d}}. \tag{3}$$

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Recently, in this direction, Badziahin [1] raised the following natural problem of possible extension of the inequality (3). Given positive integers $d > \ell$, describe all pairs (u, v), $u, v > 0$, for which there exists $c_1 = c_1(d, \ell, u, v) > 0$ such that the inequality

$$|\alpha - \beta| \geq \frac{c_1}{H(\alpha)^u H(\beta)^v} \tag{4}$$

holds for all pairs (α, β), where α and β are real algebraic numbers of degrees d and ℓ, respectively. We call such a pair (u, v) *suitable* with respect to (d, ℓ).

Evidently, by (3), each pair (u, v), where $u \geq \ell$ and $v \geq d$, is suitable with respect to (d, ℓ). (One can take, for instance, $c_1 = c_0$.) However, it is not clear if there are suitable pairs (u, v) other than those above.

From some known results, we will derive the following:

Proposition 1. *If the pair (u, v) is suitable with respect to (d, ℓ), where $d > \ell$, then*

$$u \geq \ell \quad \text{and} \quad v \geq \ell + 1. \tag{5}$$

This settles the above problem in the case when $d - \ell = 1$. Indeed, by Proposition 1, we see that then $u \geq \ell$ and $v \geq \ell + 1 = d$ is also a necessary condition on (u, v) to be suitable with respect to (d, ℓ). Hence

Theorem 2. *If d, ℓ are two positive integers satisfying $d - \ell = 1$, then the pair (u, v) is suitable with respect to (d, ℓ) if and only if*

$$u \geq \ell \quad \text{and} \quad v \geq d. \tag{6}$$

However, the question of whether there are other suitable pairs (u, v) with respect to (d, ℓ), other than those satisfying $u \geq \ell$, $v \geq d$, remains open for $d - \ell > 1$, as the situation is not clear in the case when $v \in [\ell + 1, d)$. For example, for $(d, \ell) = (3, 1)$, unlike in Theorem 2, the paper [1] indicates that there are certain series of examples of suitable pairs (u, v) with respect to $(3, 1)$, where $u > 1$ and $v \in (2, 3)$. Unfortunately, the corresponding results are conditional (see Theorems 2 and 3 in [1]). They depend on Hall's conjecture related to the difference between perfect squares and perfect cubes of integers or on versions of the *abc*-conjecture; see [2, 14]. Roughly, following the method in [1], in order to confirm any nontrivial suitable pair (u, v) with respect to $(3, 1)$, we need an estimate for the nonzero difference $|x^3 - y^2|$, where $x, y \in \mathbb{N}$, of the form $|x^3 - y^2| > c(\delta) x^{\delta}$ with some $\delta > 0$ and $c(\delta) > 0$, whereas the best known unconditional estimate is only $|x^3 - y^2| > (\log x)^{1-\delta}$ [15].

In [1], there is also an interesting example of an increasing sequence of positive integers v_n, $n = 1, 2, 3, \ldots$, such that for each $n \in \mathbb{N}$, there is a cubic algebraic number α_n of height $H(\alpha_n) \ll v_n$ and a rational number r_n of height $H(r_n) \ll v_n^3$ with distance between them satisfying $|\alpha_n - r_n| < c_2 / v_n^{10}$. Note that the exponent 10 here is best possible by (3) because $(d, \ell) = (3, 1)$. This example is (conditionally) sharp for all suitable pairs (u, v) with $2 < v \leq 3$. Badziahin's example relies on some nontrivial identities that can be verified using computer algebra packages.

In this paper, we will give a more natural construction. The heights of both cubic number α and rational number r are roughly of the same size and both are bounded above by a sufficiently large positive integer, while the difference between α and r is still as small as possible.

Theorem 3. *For each sufficiently large integer H, there is a real cubic algebraic number α and a rational number r such that*

$$\frac{H}{6} < H(\alpha) < H(r) < H \tag{7}$$

and

$$0 < \alpha - r < \frac{272}{H^4}. \tag{8}$$

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For α and r whose heights are bounded above as in (7), the exponent 4 for H in (8) is best possible by the inequality (3) with $(d, \ell) = (3, 1)$. To obtain a slightly better constant than c_0 in (3) that comes from [4, Theorem A.1], note that the difference $\alpha - r$ is a cubic number itself whose Weil's height is bounded above as

$$h(\alpha - r) \le \log 2 + h(\alpha) + h(r); \tag{9}$$

see [17, Lemma 3.7]. Recall that the *Weil height* $h(\beta)$ of an algebraic number β of degree n with conjugates $\beta_1 = \beta, \beta_2, \dots, \beta_n$ over $\mathbb{Q}$ and the leading coefficient of its minimal polynomial $b \in \mathbb{N}$ is defined as $n^{-1} \log M(\beta)$, where $M(\beta) = b \prod_{j=1}^{n} \max(1, |\beta_j|)$. This quantity is called the *Mahler measure* of β. For any $P \in \mathbb{C}[x]$ of degree n, we have

$$M(P) \le H(P)\sqrt{n+1}; \tag{10}$$

see [11]. From (9), it follows that

$$M(\alpha - r) \le 8M(\alpha)M(r)^3 = 8M(\alpha)H(r)^3,$$

since $M(r) = H(r)$. Next, by (10) with $n = 3$, we get $M(\alpha) \le 2H(\alpha)$, so that $M(\alpha - r) \le 16H(\alpha)H(r)^3$. So, if α is a cubic number satisfying $H(\alpha) < H$ and $r \in \mathbb{Q}$ satisfies $H(r) < H$, then $M(\alpha - r) < 16H^4$. Therefore, applying the trivial inequality $|\beta| M(\beta) \ge 1$ to $\beta = \alpha - r$, we obtain the lower bound

$$|\alpha - r| \ge \frac{1}{M(\alpha - r)} > \frac{1}{16H^4}$$

for any cubic number α and any rational r satisfying $H(\alpha), H(r) < H$.

In the proof of Theorem 3, we will use the following result.

Theorem 4. *Let $k > 1$ be an integer that is not a perfect square, and let $\delta > 0$. For each $n \in \mathbb{N}$, set*

$$\varepsilon_n = \sqrt{k} - \frac{p_n}{q_n}, \tag{11}$$

where p_n/q_n is the nth continued fraction to the number $\sqrt{k}$. Then there is $n_0 = n_0(k, \delta)$ such that for each integer $n \ge n_0$, the polynomial

$$P_n(x) = q_n(x^3 - k) - p_n(x^2 - x) \tag{12}$$

has a real root α_n that satisfies

$$(1 - \delta)\xi_k \varepsilon_n^2 < \alpha_n - \sqrt{k} + \frac{\sqrt{k} - 1}{\sqrt{k} + 1}\varepsilon_n < (1 + \delta)\xi_k \varepsilon_n^2, \tag{13}$$

with

$$\xi_k = \frac{3k - 4\sqrt{k} + 1}{\sqrt{k}(\sqrt{k} + 1)^3}. \tag{14}$$

In the next section, we will prove Proposition 1. As we already observed, the proposition implies Theorem 2. Then in Sections 3 and 4, we will prove Theorems 4 and 3, respectively.

2. Proof of Proposition 1

To show that $u \ge \ell$, we will first investigate the case when $\ell = 1$. For $d \ge 2$, we consider the polynomial

$$f(x) = x^d - H(x^{d-1} + \cdots + 1) \in \mathbb{Z}[x].$$

We claim that f is irreducible in $\mathbb{Z}[x]$ for each $H \in \mathbb{N}$.

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Indeed, since $f(H) < 0$ and $f(H+1) > 0$, f has a root β in the interval $(H, H+1)$. It remains to show that its other $d-1$ roots all lie in $|z| < 1$. (If so, then f cannot be reducible, since the product of its several roots excluding β is of modulus less than 1.) Consider the polynomial

$$g(x) = f(x)(x-1) = x^{d+1} - (H+1)x^d + H.$$

Assume that f has a root $\gamma \neq \beta$ satisfying $|\gamma| > 1$. Select a small positive $\varepsilon < \min(|\gamma| - 1, \beta - 1)$ for which $(H+1)(1+\varepsilon)^d > H + (1+\varepsilon)^{d+1}$, which is possible due to $(H+1)d > d+1$. Then, by Rouché's theorem applied to the disc with contour $|z| = 1+\varepsilon$, g has the same number of roots in $|z| < 1+\varepsilon$ as $(H+1)x^d$ does. Thus, g has d roots in $|z| < 1+\varepsilon$, which is not true, since the roots γ, β of g of degree $d+1$ have moduli larger than $1+\varepsilon$. Consequently, all roots of f other than β must lie in the disc $|z| \le 1$. Assume that some root γ of f has modulus 1. From $(H+1)\gamma^d = H + \gamma^{d+1}$, by the moduli consideration, we get $H+1 = |H + \gamma^{d+1}|$, which yields $\gamma^{d+1} = 1$. This implies $\gamma^d = (H + \gamma^{d+1})/(H+1) = 1$, and hence, $\gamma = \gamma^{d+1}/\gamma^d = 1$. However, $\gamma = 1$ is not a root of f, because $f(1) < 0$, a contradiction. Therefore, f has $d-1$ roots in $|z| < 1$ as claimed.

From the irreducibility of f, it follows that $\alpha = \beta^{-1}$ is an algebraic number of degree d and height H. From $\beta > H$, we get $0 < \alpha < 1/H$. Selecting $r = 0$, we find that $|\alpha - r| = \alpha < 1/H$. This shows that (4) cannot hold if $u < 1$, since we can select H arbitrarily large. Hence, $u \ge 1$ for each (u, v) satisfying (4).

Assume now that $\ell \ge 2$. The next lemma is a part of [4, Theorem 2.11]:

Lemma 5. *Let $d > \ell \ge 2$ be integers, and let β be a real algebraic number of degree ℓ. Then there is a constant $c_3 = c_3(\beta, d)$ and infinitely many real algebraic integers α of degree d such that*

$$|\alpha - \beta| < \frac{c_3}{H(\alpha)^\ell}.$$

As above, this shows that (4) cannot hold if $u < \ell$, since we can select $H(\alpha)$ arbitrarily large. Hence, $u \ge \ell$ for each (u, v) satisfying (4). This completes the proof of the first inequality in (5).

Now, we will prove the second inequality in (5), namely, $v \ge \ell + 1$. For this purpose, we can use the following result of Bugeaud [5]:

Lemma 6. *Let $n > \ell \ge 1$ be integers, and let α be a real algebraic number of degree n. Then there is a positive constant $c = c(\alpha)$ and infinitely many algebraic numbers β of degree ℓ such that*

$$|\alpha - \beta| < H(\beta)^{-\ell - 1 + c(\log\log(3H(\beta)))^{-1/(2\ell+6)}}.$$

Assume that the pair (u, v) is suitable with respect to (d, ℓ). Then, by (4), for any fixed real algebraic number α of degree d and infinitely many β of degree ℓ as indicated in Lemma 6, we deduce

$$\frac{c_1}{H(\alpha)^u} \le H(\beta)^v |\alpha - \beta| < H(\beta)^{v - \ell - 1 + c(\log\log(3H(\beta)))^{-1/(2\ell+6)}}. \tag{15}$$

Note that $c(\log\log(3H(\beta)))^{-1/(2\ell+6)}$ tends to zero as $H(\beta) \to \infty$. So, if $v < \ell + 1$, then the exponent on the right-hand side of (15) for $H(\beta)$ tends to the negative number $v - \ell - 1$ as $H(\beta) \to \infty$. Consequently, the right-hand side of (15) tends to zero as $H(\beta) \to \infty$, which is not the case. Hence $v \ge \ell + 1$.

3. Proof of Theorem 4

Fix an integer $k > 1$ that is not a perfect square and consider the polynomial

$$P(x, y) = x^3 - y(x^2 - x) - k. \tag{16}$$

In particular, for $y = \sqrt{k}$, the only real root of the polynomial $P(x, \sqrt{k})$ is $\sqrt{k}$, while its two complex roots are $\pm i\sqrt[4]{k}$. Since the roots of $P(x, y)$ as a polynomial in x can be found in terms of the Cardano formulas in parameters y, k, the real root $\alpha(\varepsilon)$ of $P(x, \sqrt{k} - \varepsilon)$ for real ε of sufficiently small modulus can be expanded as

$$\alpha(\varepsilon) = \alpha = \sqrt{k} + B_1\varepsilon + B_2\varepsilon^2 + B_3\varepsilon^3 + \dots, \tag{17}$$

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with some real numbers $B_1, B_2, B_3, \ldots$ depending on k.

By squaring this expression, we get

$$\alpha^2 = k + 2\sqrt{k}B_1\varepsilon + (2\sqrt{k}B_2 + B_1^2)\varepsilon^2 + \ldots.$$

Multiplying α and α^2, we further deduce

$$\alpha^3 = k\sqrt{k} + 3kB_1\varepsilon + 3(\sqrt{k}B_1^2 + kB_2)\varepsilon^2 + \ldots.$$

Since $P(\alpha, \sqrt{k}-\varepsilon) = \alpha^3 - (\sqrt{k}-\varepsilon)(\alpha^2-\alpha) - k = 0$, collecting the terms for ε and ε^2 in equality $\alpha^3 - k = (\sqrt{k}-\varepsilon)(\alpha^2-\alpha)$, we obtain

$$3kB_1 = -(k-\sqrt{k}) + B_1(2k-\sqrt{k})$$

and

$$3(\sqrt{k}B_1^2 + kB_2) = -(2\sqrt{k}-1)B_1 + (2k-\sqrt{k})B_2 + \sqrt{k}B_1^2.$$

The first equality of the two above gives

$$B_1 = -\frac{\sqrt{k}-1}{\sqrt{k}+1}, \tag{18}$$

while from the second, we get

$$(k+\sqrt{k})B_2 = -(2\sqrt{k}-1)B_1 - 2\sqrt{k}B_1^2. \tag{19}$$

Inserting B_1 from (18) into (19) multiplied by $(\sqrt{k}+1)^2$, we obtain

$$B_2 = \frac{3k - 4\sqrt{k} + 1}{\sqrt{k}(\sqrt{k}+1)^3}, \tag{20}$$

which is equal to ξ_k by (14). Consequently, for real ε of sufficiently small modulus and for $\delta > 0$, by (17), (18), (20), we derive that

$$\alpha(\varepsilon) = \sqrt{k} - \frac{\sqrt{k}-1}{\sqrt{k}+1}\varepsilon + B\varepsilon^2,$$

where $B = B(k,\varepsilon) \in ((1-\delta)\xi_k, (1+\delta)\xi_k)$. Using this formula for $\varepsilon = \varepsilon_n$ as in (11) and for the root $\alpha_n = \alpha(\varepsilon_n)$ of P_n, we deduce (13) for n large enough, because, by (11), (12) and (16),

$$q_nP(x, \sqrt{k}-\varepsilon_n) = q_nP(x, p_n/q_n) = q_nx^3 - p_n(x^2-x) - kq_n = P_n(x).$$

4. Proof of Theorem 3

We will apply Theorem 4 with $k=2$, integer $n \geq n_0$ and α_n being the real root of P_n defined in (12). Note that the coefficients of P_n are coprime, since so are the integers p_n and q_n. If P_n were reducible, then it would have a rational root, say $\mu_n \in \mathbb{Q}$. Since two other roots of P_n are complex for n large enough, this root μ_n must be equal to α_n. So $\alpha_n > 0$ is either rational (which will be ruled out below) or P_n is irreducible in $\mathbb{Z}[x]$. In the latter case, α_n is the cubic algebraic number with height

$$H(\alpha_n) = 2q_n,$$

because $p_n < 2q_n$. Note that $p_0/q_0 = 1, p_1/q_1 = 3/2, p_2/q_2 = 7/5$, etc. In fact, since $\sqrt{2} = [1,2,2,2,\ldots]$, from the formulas

$$p_n = 2p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = 2q_{n-1} + q_{n-2}$$

for $n = 2, 3, 4, \ldots$, where $p_0 = q_0 = 1$, $p_1 = 3$, $q_1 = 2$ [16, p. 38]), we obtain

$$p_n = \frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{2} \tag{21}$$

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and

$$q_n = \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{2\sqrt{2}}. \tag{22}$$

Multiplying (22) by $\sqrt{2}$ and subtracting (21), we get

$$q_n\sqrt{2} - p_n = -(1-\sqrt{2})^{n+1}. \tag{23}$$

Consider the rational number

$$r_n = \frac{2p_n + 2q_n}{p_n + 2q_n}. \tag{24}$$

By the above, we see that the number p_n is odd, so the denominator of r_n is odd. The integers $2p_n + 2q_n$ and $p_n + 2q_n$ are therefore coprime, since so are $p_n + q_n$ and $p_n + 2q_n$. Hence

$$H(r_n) = 2(p_n + q_n) > 2q_n = H(\alpha_n).$$

Fix a small number $\epsilon > 0$. For each integer $H > 10$, take the largest $n \in \mathbb{N}$ for which

$$2(p_n + q_n) < H. \tag{25}$$

Then $H(\alpha_n) < H(r_n) < H$. We claim that

$$H(\alpha_n) = 2q_n > (1-\epsilon)(\sqrt{2}-1)^2 H \tag{26}$$

for H large enough.

Indeed, if (26) does not hold, that is, $2q_n \le (1-\epsilon)(\sqrt{2}-1)^2 H$, then for H (and so n) large enough, by (21) and (22), we have

$$2p_n < 2(1+\epsilon)\sqrt{2}q_n \le (1-\epsilon^2)(\sqrt{2}-1)^2\sqrt{2}H.$$

Consequently,

$$2p_n + 2q_n < (1-\epsilon^2)(\sqrt{2}-1)^2\sqrt{2}H + (1-\epsilon)(\sqrt{2}-1)^2 H < (1-\epsilon^2)(\sqrt{2}-1)H.$$

But then, since n is large enough, using (21) and (22), we find that

$$2p_{n+1} + 2q_{n+1} < (1+\epsilon^2)(1+\sqrt{2})(2p_n + 2q_n) < (1-\epsilon^4)H < H,$$

contrary to the choice of n in (25). Since $(\sqrt{2}-1)^2 = 0.171\ldots > 1/6$, from (25) and (26), we deduce (7) for the pair $(\alpha, r) = (\alpha_n, r_n)$.

It remains to prove (8) for the same pair. By (11), (13) with $k = 2$ and (24), we deduce

$$\begin{aligned}\alpha_n - r_n &= \sqrt{2} - \frac{\sqrt{2}-1}{\sqrt{2}+1}\varepsilon_n + C_n\varepsilon_n^2 - \frac{2p_n/q_n + 2}{p_n/q_n + 2} \\ &= \sqrt{2} - (3-2\sqrt{2})\varepsilon_n + C_n\varepsilon_n^2 - \frac{2(\sqrt{2}-\varepsilon_n)+2}{2+\sqrt{2}-\varepsilon_n},\end{aligned}$$

where $(1-\delta)\xi_2 < C_n < (1+\delta)\xi_2$. Therefore, the product

$$(\alpha_n - r_n - C_n\varepsilon_n^2)(2+\sqrt{2}-\varepsilon_n)$$

is equal to

$$\sqrt{2}(2+\sqrt{2}-\varepsilon_n) - (3-2\sqrt{2})\varepsilon_n(2+\sqrt{2}-\varepsilon_n) - 2\sqrt{2} - 2 + 2\varepsilon_n = (\sqrt{2}-1)^2\varepsilon_n^2.$$

(The choice of r_n in (24) is such that here the constant term and the term with ε_n both cancel out.) For fixed $\delta > 0$ and n large enough, this implies

$$(1-\delta)\frac{(\sqrt{2}-1)^3}{\sqrt{2}}\varepsilon_n^2 < \alpha_n - r_n - C_n\varepsilon_n^2 < (1+\delta)\frac{(\sqrt{2}-1)^3}{\sqrt{2}}\varepsilon_n^2. \tag{27}$$

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By (14), we have

$$\xi_2 = \frac{3\cdot 2 - 4\sqrt{2}+1}{\sqrt{2}(\sqrt{2}+1)^3} = \frac{(3\sqrt{2}-1)(\sqrt{2}-1)}{\sqrt{2}(\sqrt{2}+1)^3} = (3-1/\sqrt{2})(\sqrt{2}-1)^4,$$

and hence

$$\xi_2 + \frac{(\sqrt{2}-1)^3}{\sqrt{2}} = \left(3-\frac{1}{\sqrt{2}}\right)(\sqrt{2}-1)^4 + \frac{(\sqrt{2}-1)^3}{\sqrt{2}} = 4(\sqrt{2}-1)^4.$$

Consequently,

$$\begin{aligned} C_n\varepsilon_n^2 + (1+\delta)\frac{(\sqrt{2}-1)^3}{\sqrt{2}}\varepsilon_n^2 &< (1+\delta)(\xi_2 + \frac{(\sqrt{2}-1)^3}{\sqrt{2}})\varepsilon_n^2 \\ &= 4(1+\delta)(\sqrt{2}-1)^4\varepsilon_n^2. \end{aligned}$$

It is also clear that

$$C_n\varepsilon_n^2 + (1-\delta)\frac{(\sqrt{2}-1)^3}{\sqrt{2}}\varepsilon_n^2 > (1-\delta)\xi_2\varepsilon_n^2 + (1-\delta)\frac{(\sqrt{2}-1)^3}{\sqrt{2}}\varepsilon_n^2 > 0.$$

Therefore, from (27), it follows that

$$0 < \alpha_n - r_n < 4(1+\delta)(\sqrt{2}-1)^4\varepsilon_n^2.$$

Observe that, by (23), we have

$$\varepsilon_n = \sqrt{2} - \frac{p_n}{q_n} = \frac{q_n\sqrt{2}-p_n}{q_n} = \frac{-(1-\sqrt{2})^{n+1}}{q_n}.$$

Thus, selecting small $\delta = \epsilon > 0$ and using (22), we derive that

$$0 < \alpha_n - r_n < \frac{4(1+\epsilon)(\sqrt{2}-1)^{2n+6}}{q_n^2} < \frac{(1+2\epsilon)(\sqrt{2}-1)^4}{2q_n^4}. \tag{28}$$

Using (26), we see that the latter expression is

$$\frac{8(1+2\epsilon)(\sqrt{2}-1)^4}{(2q_n)^4} < \frac{8(1+2\epsilon)(\sqrt{2}-1)^4}{(1-\epsilon)^4(\sqrt{2}-1)^8H^4} < \frac{8(1+7\epsilon)(\sqrt{2}+1)^4}{H^4}.$$

This is less than $272/H^4$ due to $8(\sqrt{2}+1)^4 = 271.764\ldots$, which proves inequality (8) for the pair $(\alpha, r) = (\alpha_n, r_n)$. This completes the proof of Theorem 3 under the assumption that α_n is irrational. However, if α_n were rational with minimal (linear) polynomial $Q_n \in \mathbb{Z}[x]$, then, by the multiplicativity of Mahler's measure and (10), we would get

$$H(\alpha_n) = M(\alpha_n) = M(Q_n) \le M(P_n) \le 2H(P_n) = 4q_n.$$

But then the denominators of the rational numbers α_n and r_n (see (24)) do not exceed $4q_n$, so $|\alpha_n - r_n|$ is either zero or greater than $1/(4q_n)^2$, which contradicts (28) for n large enough. The proof of Theorem 3 is thus completed.

In fact, inserting r_n as in (24) into the polynomial

$$P_n(x) = q_n(x^3-2) - p_n(x^2-x)$$

and multiplying by $(p_n+2q_n)^3$, we find that $(p_n+2q_n)^3P_n(r_n)$ equals

$$q_n((2p_n+2q_n)^3 - 2(p_n+2q_n)^3) - p_n^2(2p_n+2q_n)(p_n+2q_n),$$

which is $-2(p_n^2-2q_n^2)^2$. Thus

$$(p_n+2q_n)^3P_n(r_n) = -2(p_n^2-2q_n^2)^2. \tag{29}$$

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From (23), mapping $\sqrt{2}$ into $-\sqrt{2}$, we get

$$-q_n\sqrt{2}-p_n=-(1+\sqrt{2})^{n+1}.$$

Multiplying this equality with (23) yields $p_n^2-2q_n^2=(-1)^{n+1}$. Thus, by (29),

$$(p_n+2q_n)^3P_n(r_n)=-2.$$

Using this identity, we can give an alternative proof of the theorem. Indeed, from

$$q_n(r_n-\alpha_n)(r_n-\beta_n)(r_n-\gamma_n)(p_n+2q_n)^3=-2,$$

where α_n, β_n and $\gamma_n=\overline{\beta_n}$ are the roots of P_n, it follows that

$$q_n^4(\alpha_n-r_n)(r_n^2-(\beta_n+\gamma_n)r_n+\beta_n\gamma_n)(p_n/q_n+2)^3=2. \tag{30}$$

The root α_n of P_n tends to $\sqrt{2}$ as $n\to\infty$, and β_n and γ_n tend to $i\sqrt[4]{2}$ and to $-i\sqrt[4]{2}$ or vice versa. Also, $r_n\to\sqrt{2}$ as $n\to\infty$ and $p_n/q_n\to\sqrt{2}$ as $n\to\infty$. Thus

$$(r_n^2-(\beta_n+\gamma_n)r_n+\beta_n\gamma_n)(p_n/q_n+2)^3\to(2+\sqrt{2})^4\quad\text{as}\quad n\to\infty.$$

For each $\varepsilon>0$ and n large enough, this combined with (30) yields

$$2-\varepsilon<q_n^4(\alpha_n-r_n)(2+\sqrt{2})^4<2+\varepsilon.$$

Therefore,

$$0<\alpha_n-r_n<\frac{(1+\varepsilon/2)(\sqrt{2}-1)^4}{2q_n^4},$$

which implies (28) with $\varepsilon=4\epsilon$. This concludes the argument as above.

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