and

$$2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{\ddots}}}} = e.$$

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102.07 Evaluating the probability integral by approximating $\exp(-x^2)$

In a recent issue of the *Gazette* [1], Gustavsson and Sundqvist construct two monotone sequences that converge to the exponential function. The purpose of this Note is to apply the argument of [1] to the function e^{-x^2} and then evaluate the probability integral using a method inspired by Stieltjes [2]. Interestingly, we obtain the well-known Wallis product at the same time. There are various methods for evaluating the probability integral via the use of the Wallis product (see [2], [3], [4], [5], and [6]).

From the treatment of e^{-y} in [1] we find, with $y = x^2$, that

$$\left(1 - \frac{x^2}{n}\right)^n \leqslant e^{-x^2} \leqslant \left(1 + \frac{x^2}{n}\right)^{-n}, \; n = 1, 2, \ldots .$$

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Integration over $0 \leqslant x \leqslant \sqrt{n}$, together with the substitutions $x = \sqrt{n}z$ and $x = \sqrt{n}\dfrac{z}{\sqrt{1-z^2}}$ in the bounds, then yields

$$\sqrt{n}\int_0^1 (1-z^2)^n\,dz \leqslant \int_0^{\sqrt{n}} e^{-x^2}dx \leqslant \sqrt{n}\int_0^{1/\sqrt{2}} (1-z^2)^{n-\frac{3}{2}}dz < \sqrt{n}\int_0^1 (1-z^2)^{n-\frac{3}{2}}dz.$$

Let $I_t = \int_0^1 (1-z^2)^t\,dz$, where $t = \frac{1}{2}M$ with $M = 0, 1, 2, \ldots$. The above inequality can be written as

$$\sqrt{n}\,I_n \leqslant \int_0^{\sqrt{n}} e^{-x^2}dx < \sqrt{\frac{n}{n-\frac{3}{2}}}\sqrt{n-\frac{3}{2}}\,I_{n-\frac{3}{2}}.$$

Hence, the evaluation of the probability integral $\int_0^\infty e^{-x^2}dx$ amounts to the evaluation of $\lim_{t\to\infty} \sqrt{t}\,I_t$ for such t.

The definition of I_t immediately gives $I_0 = 1$ and $I_{1/2} = \frac{\pi}{4}$ since $I_{1/2}$ is the definite integral which computes the area of a quarter of a unit circle or a quadrant. Together with the identity $z^2 = 1 - (1-z^2)$, integration by parts for simplifying both I_t and $I_{t+1/2}$ readily gives the reduction formulae

$$I_t = \frac{2t}{2t+1}I_{t-1} \qquad \text{and} \qquad I_{t+1/2} = \frac{2t+1}{2t+2}I_{t-1/2}. \tag{1}$$

On writing $G(t) = (t+1)I_{t+1/2}I_t$, we find that $G(t+1) = G(t)$, and hence $G(t) = G(0) = G\left(\frac{1}{2}\right) = \frac{\pi}{4}$ for all $t = \frac{1}{2}M \geqslant 0$.

Next, inspired by Stieltjes [2], we note that the quadratic expression

$$I_t u^2 + 2I_{t+1/2}u + I_{t+1} = \int_0^1 (1-z^2)^t \left[u + \sqrt{1-z^2}\right]^2 dz$$

is positive for all real u, so that its discriminant is negative; thus

$$I_{t+1/2} < \sqrt{I_t\,I_{t+1}} \qquad \text{and} \qquad I_t < \sqrt{I_{t-1/2}\,I_{t+1/2}}. \tag{2}$$

By combining (1) and (2) it follows that

$$\sqrt{\frac{2t+3}{2t+2}}\,I_{t+\frac{1}{2}} < I_t < \sqrt{\frac{2t+2}{2t+1}}\,I_{t+\frac{1}{2}}. \tag{3}$$

Multiplying by tI_t gives

$$\frac{t}{t+1}\sqrt{\frac{2t+3}{2t+2}}\,G(t) \leqslant tI_t^2 \leqslant \frac{t}{t+1}\sqrt{\frac{2t+2}{2t+1}}\,G(t),$$

where we add the equality sign to accommodate the case $t = 0$.

The desired result

$$\int_0^\infty e^{-x^2}dx = \lim_{t\to\infty}\sqrt{t}\,I_t = \frac{\sqrt{\pi}}{2}$$

$\left(t = \frac{1}{2}M\right)$ now follows. The substitution $x = \dfrac{w}{\sqrt{2}}$ and the symmetry of e^{-w^2} lead to

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\,e^{-w^2/2}dw = 1,$$

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which gives the standard normal density function. Reflecting the anecdotes and subjects mentioned in [7], here we justify π in the standard normal density function to come from the area of a quadrant of a unit circle.

Another straightforward implication of (3) is

$$\lim_{n\to\infty} \frac{I_n}{I_{n+1/2}} = 1, \text{ where } n \text{ is a natural number.}$$

Furthermore, the recurrence relations of (1) show that

$$\frac{I_n}{I_{n+1/2}} = \frac{(2n+2)(2n)}{(2n+1)(2n+1)} \frac{I_{n-1}}{I_{n-1/2}}$$
$$= \frac{(2n+2)(2n)(2n)(2n-2)\ldots(4)(2)}{(2n+1)(2n+1)(2n-1)(2n-1)\ldots(3)(3)} \frac{I_0}{I_{1/2}}.$$

Consequently, we obtain the famous Wallis product

$$\lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot \ldots \cdot (2n)\cdot(2n)}{1\cdot 3\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)\cdot(2n+1)} = \frac{\pi}{2}.$$

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