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THE CONCEPT *HORSE* IS A CONCEPT

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Abstract. I offer an analysis of the sentence 'the concept *horse* is a concept'. It will be argued that the grammatical subject of this sentence, 'the concept *horse*', indeed refers to a concept, and not to an object, as Frege once held. The argument is based on a criterion of proper-namehood according to which an expression is a proper name if it is so rendered in Frege's ideography. The predicate 'is a concept', on the other hand, should not be thought of as referring to a function. It will be argued that the analysis of sentences of the form '*C* is a concept' requires the introduction of a new form of statement. Such statements are not to be thought of as having function–argument form, but rather the structure subject–copula–predicate.

> **rosskur**, *f.* eine gewagte kur mit ungeheuerlichen mitteln.
>
> DEUTSCHES WÖRTERBUCH

Certain difficulties are caused by the logical analysis of the sentence 'the concept *horse* is a concept'. (I shall use italics to indicate that 'horse' in the context 'the concept horse' stands for a concept.[1]) The most salient difficulties associated with the grammatical subject of this sentence go under the name of the problem—or, the paradox—of the concept *horse*. Frege, in his 'On concept and object' (1892a), seemed to have felt that he was forced by his own logico-semantic views to hold that 'the concept *horse*' does not refer to a concept, but rather to an object in his sense. 'By a certain linguistic necessity' says Frege (ibid. p. 204), 'my expression at times, taken quite literally, miss my thought, in that an object is mentioned, where a concept is meant'. That there are complications associated with the logical analysis of the predicate 'is a concept' is perhaps less obvious. But it should be clear that this and similar predicates, such as 'is a function', are in some sense special, since by using them we do not intend merely to ascribe a property to a thing, but venture to say of a thing of which type, or kind, or category it is. It may be that the peculiar nature of such predicates requires that different tools be used in the analysis of 'the concept *horse* is a concept' from those used in the analysis of, for instance, 'the concept *horse* is easily attained'.

In the first part of this paper I shall argue that 'the concept *horse*' does in fact refer to a concept, as our ordinary understanding of language expects, and not to an object, as Frege seemed to have held at the time of writing 'On concept and object'. Indeed, I shall

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[1] Since italics are not a part of English grammar, I take it that if 'the concept *horse*' is grammatical, then so is 'the concept horse'. But the italics might help to indicate that 'horse' here has simple supposition, in mediaeval terminology (cf. e.g., Kretzmann, 1966, p. 107). Mulligan (2006) calls the modification effected by the word 'concept' in the context 'the concept horse' *formal ascent*.

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argue that Frege himself could have seen that 'the concept *horse*' refers to a concept had he only attempted to formulate in his ideography[2] sentences in which the phrase 'the concept *horse*' occurs. Once one makes use of the ideography, one sees that 'the concept *horse*' will in general be represented by the name of a concept there, from which it seems natural to conclude that 'the concept *horse*' itself must be the name of a concept, and so refer to a concept.

I can think of two reasons why Frege did not appeal to his ideography in getting to grips with the sentence 'the concept *horse* is a concept' and in particular with its grammatical subject, 'the concept *horse*'. The first is that the ideography was designed primarily as a language for general logical and mathematical reasoning, whence it may not be immediately clear how to apply it in reasoning about quite specific empirical notions, such as the notion of a horse. Appeal to the ideography may seem more natural when analyzing a sentence of mathematics than it does when analyzing a sentence of zoology or psychology. Instead of 'the concept *horse*' I shall therefore concentrate below on the phrase 'the sine function', an ordinary phrase of mathematical English. It is clear that the considerations that led Frege to think that 'the concept *horse*' must refer to an object would also have led him to think that 'the sine function' must refer to an object. I shall argue that once we render in the ideography sentences in which 'the sine function' occurs, we are led to see that 'the sine function' is in fact a function name and therefore refers to a function.

The other reason I can see why Frege did not appeal to his ideography in contemplating the logical form of 'the concept *horse* is a concept' is that it may not have seemed obvious to him how to render there the predicate 'is a concept'. This difficulty appears to be less serious than those associated with the analysis of 'the concept *horse*': can we not simply render this predicate as a second-order predicate true precisely of all first-order concepts?[3] Recall that for Frege, a concept is a function whose range is a subset of the set of the two truth-values, the True and the False. It is therefore natural to think of concepts as forming a subtype of the type of functions. I shall for the most part concentrate on the predicate 'is a function' and only towards the end of the paper apply the analysis provided of this predicate to the analysis of the predicate 'is a concept'. Let U be a second-order function that yields the True for all of its admissible arguments. It would seem that we could render 'φ is a function' as the ordinary function–argument formula '$U(\varphi)$'. Thus, in analogy with the suggested treatment of the predicate 'is a concept', we would render the predicate 'is a function' as an ordinary predicate in the ideography true precisely of all unary functions of objects. I shall argue that the logical analysis of predicates of the likes of 'is a function' may not be as straightforward as this. It is namely a condition on the well-formedness of '$U(\varphi)$' that φ be a function. It could therefore be argued that the truth of 'φ is a function' is a presupposition for the well-formedness of '$U(\varphi)$'. Hence it would seem that sentences whose predicate is 'is a function' are in a sense more fundamental than any sentence we can form by means of the ordinary function–argument syntax of Frege's ideography and ordinary predicate logic.

I shall develop this view in more detail in the second part of this paper. It will there be argued that we should follow type theorists and computer scientists in representing

[2] Following Barnes (2002) I shall use 'ideography' as the English for 'Begriffsschrift'.

[3] I shall use 'order' in the sense of Russell & Whitehead (1910, pp. 51–54): a unary function whose admissible arguments are all individuals is first order; a unary function whose admissible arguments are all n-th order is $n+1$-th order. (The special case of functions all of whose arguments are of the same type is the only one that interests us here.) Frege (1893, §21) used 'Stufe', viz. level, for the same notion, a terminology that is still in use in German today.

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sentences whose predicate is 'is a function' as having a novel logical form. More precisely, I shall suggest that we represent sentences of the form 'a is of type τ', or more simply 'a is a τ', by formulae '$a : \tau$'. Such formulae have a novel logical form, since they are not to be thought of as having function–argument structure: neither the colon nor the concatenation '$: \tau$' is to be thought of as the name of a function residing somewhere in the type hierarchy to which τ belongs.

To avoid the complications thus caused by the predicate 'is a function' I shall use as my paradigm sentence in the first part of this paper not 'the sine function is a function', but rather 'the sine function is periodic'. A function f on the real line is periodic if it repeats its value at a regular interveral, that is if there is a real number $r > 0$ such that for all real numbers x, $f(x + r) = f(x)$. This is true of the sine function, hence 'the sine function is periodic' is an ordinary mathematical theorem.

I shall begin by setting out the argument that must have moved Frege to hold that the phrase 'the concept *horse*' does not refer to a concept and that would likewise have moved him to hold that the phrase 'the sine function' does not refer to a function. This conclusion will be shown to contradict the deliveries of a straightforward logical analysis of 'the sine function is periodic' by means of Frege's ideography. The fault of Frege's argument will be found to lie in its assumption that 'the sine function' is a proper name in his sense. It will be argued that, at least in the case of mathematical English, to decide the logical type of an expression e we should render a sentence in which e occurs in the ideography and see which type the ideographical expression representing e then receives. Some objections to this procedure for determining the logical type of a natural language expression will be considered.

The second part of the paper is concerned with the predicate 'is a function'. I first consider strategies for representing this predicate in the ideography as a first-order or second-order predicate. Although both of these strategies, and perhaps especially the second, may seem quite natural, I shall suggest that we render sentences whose predicate is 'is a τ', where τ is a type in a given type hierarchy, as having a novel logical form, written '$a : \tau$'. In particular, I shall suggest that we analyze 'the sine function is a function' as '$\sin(x) : (\iota)$', where (ι) is the type of unary functions of objects; and that we analyze 'the concept *horse* is a concept' as '$horse(x) : \mathbf{C}$', where $\mathbf{C}$ is the subtype of the type (ι) consisting of all first-order concepts.

For the most part of the paper I shall use a functional notation close to Frege's, namely as exemplified by '$\sin(x)$' in contrast to 'sin' as a name of the sine function. How to understand the relation between these two styles of functional notation is an important question, but not one I shall enter into here. Second-order functions will accordingly be written after the fashion of '$F_\beta(f(\beta))$', some explanation of which will be given below. The Fregean notation is used here not because it is found to be an especially good one, but because at least part of what I am concerned with is exegesis of Frege's thinking about functions and how to speak about them.

§1. The concept *horse*.

1.1. The argument. Frege's argument to the conclusion that the phrase 'the concept *horse*' does not refer to a concept rests on three premisses. The first premiss is the maxim that a proper name in Frege's sense refers, if it refers at all, to an object. The second premiss is the principle that no object is a concept. The third premiss says that 'the concept *horse*' is a proper name in Frege's sense. Since 'the concept *horse*' is thus a proper name, it refers (if it refers at all) to an object by the first premiss, whence it does not refer to a concept by

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the second premiss. Let us convince ourselves that each premiss would seem to be justified by Fregean doctrine.

The first premiss, that a proper name refers, if it refers at all, to an object, follows from Frege's definition of a proper name as the sign of an object (Frege 1892b, p. 27; GG §2).[4] The first premiss is thus true by definition. We may, however, ask whether Frege's definition of the term 'proper name' is a reasonable one. 'Proper name' was of course a well-entrenched term of logic and grammar at the time Frege was writing,[5] so one may ask what justifies Frege in using precisely this term for expressions whose reference is an object in his sense. Frege, in response to this challenge, could begin by referring to his idea that the logical types of expressions and the ontological types of things are in a sense mirror images of each other.[6] There is a one–one correspondence between the types of expression at the level of language and the types of things at the level of reference; moreover, expressions of a given type share certain characteristics with the things to which they refer.[7] It is in order to describe these characteristics that Frege invokes the metaphors of saturation and completeness. Signs of objects are saturated, or complete, just as the objects themselves are saturated; whereas signs of other types of things, namely functions of various types, are unsaturated, or incomplete, to various degrees, just as the functions themselves are unsaturated to various degrees. The use of 'proper name' for all signs of objects can then be justified on the grounds that proper names in the traditional sense, such as 'Socrates' and 'Ludwig', are prototypes of saturated expressions.

The second premiss, that no object is a concept, follows from the definition of concepts as a kind of function and the principle that types are disjoint. Frege emphasized that objects are fundamentally different (*grundverschieden*) from unary functions of objects and that these in turn are fundamentally different from binary functions of objects as well as from second-order functions whose arguments are unary functions of objects (Frege, 1891, pp. 26–27; GG §21). These types of functions are fundamentally different from each other because either the number or the type (or types) of their admissible arguments differ. Given how the simple type hierarchy implicit in the logic of the *Grundgesetze* is generated—the type of a function is determined by the number and types of its admissible arguments—it is clear that Frege would regard all types, not only those he explicitly mentions, as fundamentally different from each other. And indeed, it seems difficult to make sense of a simple type hierarchy in which a function of one type is also a function of another type or an object.

The third premiss says that 'the concept *horse*' is a proper name in Frege's sense. Frege did not provide a general characterization of proper names apart from the characterization relying on the metaphor of saturation: a proper name is a saturated expression. Since the notion of proper name plays such an important role in what has been dubbed the Fregean

[4] 'GG' refers to the first volume of the *Grundgesetze* (Frege, 1893). 'GLA' refers to the *Grundlagen* (Frege, 1884).

[5] The distinction between general and proper name can be traced back at least to the Stoics with their distinction between *prosēgoria* and *onoma* (cf. e.g., Robins, 1966, p. 12). The term 'proper name' itself can be traced back to the earliest works on grammar in the Graeco-Roman tradition. Dionysius Thrax, for instance, says that the noun is either general (*koinōs legomenon*), such as 'man' or 'horse', or proper (*idiōs legomenon*), such as 'Socrates' (cf. Kemp, 1987, p. 176).

[6] Textor (2010) provides a detailed study of this 'mirroring principle' and its role in the problem of the concept *horse*.

[7] See GG §1, §2, and more explicitly (Frege, 1903, p. 371) and (Frege, 1904, p. 665).

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approach to ontology (cf. e.g., Hale, 2010)—according to which questions concerning ontological categories are to be answered by appeal to a prior analysis of the types of expression in language—a more rigorous characterization is wanted. A characterization primarily in terms of inference patterns that proper names are to validate has been proposed by Dummett (1973, chap. 4) and developed further by Hale (1994, 1996) and more recently by Schwartzkopff (2016).[8] I shall not rehearse that characterization here; suffice it to say that 'the concept *horse*', 'the sine function', 'the Dirichlet function' and similar phrases all count as proper names according to this characterization. Frege relied in several places, including his discussion of 'the concept *horse*', on a special criterion of proper-namehood. The criterion says that any singular noun phrase governed by a definite article is a proper name (Frege 1892a, p. 195; cf. GLA §51, §66).[9] Thus, by prefixing a definite article to a noun in the singular, which itself may be modified by nouns, adjectives, relative clauses, prepositional phrases, etc., one obtains a proper name. It is clear that not only 'the snubnosed philosopher from Athens' and 'the author of *Tractatus Logicus-Philosophicus*', but also 'the concept *horse*' and 'the sine function', meet this criterion. Again one may ask what justifies relying on this criterion for proper-namehood. In response one could note that noun phrases of the kind the criterion singles out would seem to be apt for flanking the identity sign; hence, that the reference of such phrases may be associated with criteria of identity; and that, according to the Fregean philosophy, it is characteristic of objects that they are associated with such criteria.

1.2. The argument cannot be correct. Adding to the above argument a reasonable principle of semantic descent allows one to infer the koan 'the concept *horse* is not a concept'. This further inference step may strengthen the impression that Frege has a problem here. But it should be clear that the argument as it stands already has a discomforting conclusion. In order to say something about, for instance, the sine function, one would naturally employ a sentence whose subject is the noun phrase 'the sine function'. The sine function is periodic, is analytic, is smooth, etc. The argument, if it were correct, would show, however, that in such sentences we do in fact not manage to pick out the sine function, since its subject 'the sine function' does not refer to a function. Thus, quite ordinary and innocent ways of speaking—involving no semantic concepts, for instance, nor any self-reference—would fail, as a matter of linguistic necessity, to pick out what one who uses such ways of speaking wishes to talk about.

Such a conclusion would certainly baffle a mathematician. He would be puzzled by someone telling him that when he uses phrases such as 'the sine function', 'the Dirichlet function', 'the Ackermann function', 'Euler's totient function', and so on, he in fact fails to pick out a function. For, he would probably have taken it for granted that in asserting, for instance, that the totient function is multiplicative he picks out a function and says something about it.[10]

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[8] Sentences count as proper names for Frege (cf. Frege, 1892b, p. 34). Dummett's characterization concerns what Hale and others call singular terms. A singular term in this sense we may regard as a Fregean proper name that is not a sentence.

[9] Frege (1892a, p. 196) notes that 'the Turk besieged Vienna' (where 'the Turk' is semantically plural, standing for, say, the Ottomans; this was a common use of the singular *der Türke* in German, as is noted in the lemma *türke* in the Grimm dictionary) and 'the horse is a quadruped' are counterexamples to this criterion. In response he says that 'these cases are so easily seen to be peculiar that the value of our rule is hardly impaired by their occurrence'.

[10] Trueman (2015) argues at length that phrases of the form 'the concept…' cannot refer to a concept. Trueman notices (ibid. p. 1905) that this leads to a form of quietism, but does not notice

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Even from a Fregean point of view one can see that there must be something wrong with the above argument. One sees this as soon as one renders a sentence such as 'the sine function is periodic' in Frege's ideographical notation. A formula along the following lines immediately suggests itself:[11]

$$(\exists r \in \mathbb{R}^{+})(\forall x \in \mathbb{R})\, \sin(x + r) = \sin(x). \tag{i}$$

That a function $\varphi(x)$ on the real numbers is periodic means that it repeats its value at a regular interval, in other words, that there is a positive real number r such for any real number Δ, $\varphi(\Delta + r) = \varphi(\Delta)$. This is precisely what is expressed in formula (i), hence it is natural to view it as an ideographical rendering of 'the sine function is periodic'. In this rendering, however, the phrase 'the sine function' is represented by the function name '$\sin(x)$' and not by a proper name. Hence what represents 'the sine function' in the ideographical rendering of 'the sine function is periodic' refers to a function, and not to an object. But then it seems reasonable to infer that also the natural language phrase 'the sine function' must refer to a function.

It could be objected that in the suggested ideographical rendering (i) what represents 'the sine function', namely '$\sin(x)$', does not occur as subject. In 'the sine function is periodic' the phrase 'the sine function' occurs as subject, and this must be reflected in a truthful rendering. It is well known that Frege in the *Begriffsschrift* (1879) replaced the pair of notions of subject and predicate with the pair of notions of function and argument. In Fregean terms, therefore, what the objection asks for is an ideographical rendering of 'the sine function is periodic' in which what represents 'the sine function' appears as an argument to a function. This can easily be provided. Let us define a second-order function P as follows:

$$P_{\beta}(f(\beta)) \equiv (\exists r \in \mathbb{R}^{+})(\forall x \in \mathbb{R})\, f(x + r) = f(x). \tag{Def}$$

We have here followed Frege's general notation for second-order functions (GG §25), where it is indicated that the argument place x in a first-order function $\varphi(x)$ gets saturated when we apply P to it; or, in non-Fregean terms, that the free variable x gets bound by the application of P. Thus we write '$P_{\beta}(f(\beta))$' rather than 'P', where 'f' is the argument place indicator and the use of 'β' is meant to indicate that a function $\varphi(x)$ gets saturated when P is applied to it. If $\varphi(x)$ is a function on the real numbers, then '$P_{\beta}(\varphi(\beta))$' says that $\varphi(x)$ is periodic. Hence the sentence 'the sine function is periodic' can be rendered in simple function–argument form as

$$P_{\beta}(\sin(\beta)). \tag{i*}$$

What represents 'the sine function' here is, as it is in (i), the function name '$\sin(x)$', though what this function name refers to now appears explicitly as the argument to the higher-level function $P_{\beta}(f(\beta))$. Thus we have provided what the objection asked for without having to sacrifice the conclusion that the phrase 'the sine function' must refer to a function. The difference between (i) and (i*) will not play any role in what follows. The latter is obtained from the former by replacing the *definiens* of (Def) by its *defiendum*, so one could argue that these two formulae therefore differ in sense. What matters for current purposes, however,

that not only reflective claims such as 'no object is a concept' but also quite ordinary ways of speaking in mathematics would seem to be trapped by this quietism.

[11] The purist will know how to translate this linear formula into Frege's two-dimensional cavitary notation.

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is the representation of 'the sine function' as a function name, and in this respect the two formulae agree.

The foregoing considerations do of course not depend on any special property of the phrase 'the sine function' (nor, of course, on any special property of the sine function). In the ideographical renderings of, say, 'the Dirichlet function is nowhere differentiable' or 'Euler's totient function is multiplicative' the singular noun phrases 'the Dirichlet function' and 'the totient function' are represented by function names and not by proper names. These renderings would therefore seem to show that these apparently problematic noun phrases do refer to functions, as common sense says they do, but as Frege's doctrine denies. In the argument leading to the conclusion that 'the sine function' cannot refer to a function we must have been misled by the surface-grammatical structure of English. The surface-grammatical structure does not reflect the true logical grammar of sentences such as 'the sine function is periodic'. Once written in a language that obeys logical grammar, one sees that 'the sine function' in this context does in fact refer to a function. Frege's ideography is such a language (let us assume), and it reveals that the 'the sine function' and similar phrases refer to unary functions of objects, and not to objects.

It is surprising that this quite simple observation seems not to have occurred to Frege at the time of writing 'On concept and object'. Frege had emphasized in the introduction to the *Begriffsschrift* that the ideography can be of help in laying bare confusions caused by natural language.[12] But the argument to the conclusion that 'the sine function' and similar phrases do not refer to functions seems to be precisely a confusion of this kind. The true logical grammar of such phrases is not evident in the surface structure of natural language (English or German, at least): it is seen only when we write sentences in which these phrases occur in Frege's ideography.

This insight is, in fact, implicit in at least one of Frege's earlier writings. Frege there considers the formalization of a number of mathematical sentences (Frege, 1881, pp. 23–29). Three of the sentences Frege considers have as their grammatical subject a phrase of the form 'the function…', hence precisely of the form that causes trouble in 'On concept and object'. In the ideographical rendering of the sentences in question these phrases are, however, always represented by a function name and not by a proper name. For instance, in the ideographical rendering of the sentence 'the real-valued function $f(x)$ is continuous on the interval from a to b' what represents 'the real-valued function $f(x)$' is a function name and not a proper name (ibid. p. 27). Hence, about a decade before writing 'On concept and object' Frege had in effect himself shown, by means of his ideography, that 'the sine function' and similar phrases are in fact function names. In light of Frege's notion of concept as a kind of function it is difficult to see any obstacle to his inferring that then also 'the concept *horse*' must be a function name.

In later writings from Frege's *Nachlass* this insight is not only implicit, but explicitly stated. In a note from 1906 Frege remarks that with phrases such as 'the concept prime number' natural language stamps something as an object which in fact is a concept

[12] Frege (1879, Preface, p. vi):

> If it is a task of philosophy to break the power of the word over the human mind, by laying bare the deceptions that often almost inevitably arise when we speak about the relation of concepts; by freeing the thought from that with which only the peculiarities of the linguistic means of expression saddles it; then my ideography, further developed for these purposes, can also be a useful tool for philosophers.

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(Frege, 1906, p. 192). In a longer piece dated 1914 Frege notes that the definite article in 'the function…' and 'the concept…' gives such phrases the form of

> proper names in the sense of logic, as if they were to designate objects, although that is precisely what they should not do (Frege, 1914, pp. 257–258).

It seems clear from the context of the cited passages that Frege takes phrases of the form 'the function…' and 'the concept…' to refer to something, to be meaningful, for he emphasizes that they can hardly be avoided and that the best we can do when being forced to use them is to be aware of their misleading form. Since such phrases do not refer to objects, we must take them to refer to functions. Thus, Frege would here seem to be saying that phrases of the form 'the function…' and 'the concept…' do refer to functions, and not to objects, as their grammatical form suggests, and as Frege had assumed in 'On concept and object'. The following passage from the late *Erkenntnisquellen der Mathematik und Naturwissenschaften* (Frege, 1924, pp. 288–292) reads as a criticism of the author of 'On concept and object':

> Here one sees how language—by the use of the definite article stamping something as an object which is a function, that is, not an object—can be the source of distorted and deceiving expressions and therefore also of errors of thought. Here, probably, lies the cause of most contaminations of the logical sources of knowledge.

The presence of the definite article deceives one into thinking that one is dealing with an object, when in fact one deals with a function. This was the mistake of 'On concept and object'. Indeed, Frege adds, mistakes of this kind may be the cause of 'most contaminations of the logical sources of knowledge'.

1.3. Proper names via the ideography. The diagnosis proposed here of the argument to the conclusion that 'the sine function' and like phrases do not refer to functions in effect attacks this argument's third premiss, according to which such phrases are proper names in Frege's sense. The ideographical rendering of sentences in which 'the sine function' occurs shows that this phrase in fact is a function name. That the third premiss is the weakest of the argument seems quite clear. We noted that the first premiss, that a proper name refers, if it refers at all, to an object, is true by the definition of 'proper name'. The second premiss, that types are disjoint, is a fundamental assumption of type theory.[13] It is not clear whether Frege's ideography will survive in any recognizable form the denial of this assumption. All complex expressions of Frege's ideography are formed by applying a function to one or more arguments, and it is difficult to make sense of the idea of function application when a function is taken to be of the same type as its argument. Indeed, if a function is regarded as an object one needs to appeal to some notion external to the notion of object to account for function application, and the most natural such account introduces an application function. On pain of a Bradley-style regress, however, one cannot regard this application function as an object.[14]

[13] This premiss is denied by Wright (1998). The consequences of this denial for the architecture of type-theoretical languages are noted, but not dealt with, at (Hale & Wright, 2012, pp. 119–121).

[14] This is, in effect, noted by Frege (1892a, pp. 204–205). See also the argument in Dummett (1973, pp. 251–252). Gaskin (2008, chap. 6) and Liebesman (2015, p. 551)—both of whom hold that

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The fault of the third premiss must lie in the characterization of proper names that it relies on. Contrary to what the Dummettian characterization of proper names conjectures, we cannot count 'the concept *horse*', 'the sine function' and similar phrases as proper names in Frege's sense, for then the argument forces us to hold that such phrases cannot refer to functions. A natural question is then whether some other characterization of proper names can be given that does not count 'the concept *horse*' and 'the sine function' as proper names. In our considerations above we have in effect relied on such a characterization of proper names, or logical type more generally. The assumed characterization can be expressed as the following rule: to find out the logical type of a suitable English expression, translate a sentence in which it occurs into the ideography and see what is the type of the ideographical expression representing the English expression in question. Thus we determined that 'the sine function' must refer to a function, hence that it cannot be a proper name in Frege's sense, by rendering in the ideography the sentence 'the sine function is periodic'. Of course not all English sentences can readily be translated into the ideography; but we are here primarily concerned with the problem of reference to functions, and this problem arises primarily for mathematical English, at least a significant fragment of which can be so translated. Someone proficient in both mathematical English and the ideography will therefore in most cases know how to translate between these two languages. According to the characterization we have assumed, the ideography should have the last word in settling what is a proper name and what is a function name.

That the ideography should play some role in characterizing the types of expressions seems reasonable enough when one reflects that the notion of logical type is, initially at least, a technical notion proper to type-theoretical languages such as Frege's ideography. This is perhaps especially clear in the case of function names. Frege insisted that a function name always include a suitable number of argument place indicators (or, free variables, in non-Fregean terminology), marking the unsaturatedness of the function referred to. Indeed, Frege calls what to some may seem like a function name, namely 'sin', without an argument place indicator, an *Unding*, a monstrosity (GG §147 fn.). In a letter to Russell, Frege says that leaving out the argument place indicator in a function name, as has been done in 'sin', 'contradicts the essence of the function, which consists in its unsaturatedness' (Frege, 1976, p. 243). An argument place indicator is, however, a technical accessory of Frege's ideography and is not found in natural language outside of specific mathematical contexts (namely where names of ordinary mathematical functions are given in the form '$f(x)$'). But then it is not clear whether Frege's idea of a function name can readily be transposed from the ideography to natural language. This is indeed suggested by our conclusions regarding 'the sine function'. This phrase, we have concluded, refers to a function, whence it is natural to infer that it must be a function name; but it certainly does not contain any argument place indicator.

It is worth noticing that our assumed characterization of proper-namehood, or logical type more generally, supports a version of the context principle. The original context principle says that one must not ask for the meaning of a word in isolation, but only in the context of a sentence (GLA p. x). The version of the principle supported by our characterization of logical type says that one must not ask for the type of an expression in isolation, but only in the context of a sentence. The characterization supports this principle, since if we are to determine the logical type of an expression according to it, we must in general find a sentence in which that expression occurs and then render this sentence in

all subsentential expressions refer, if they refer at all, to objects—accept that regress ensues, but insist that the regress is not vicious.

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the ideography. In the case of 'the sine function', for instance, we considered the sentence 'the sine function is periodic'. In general, it may not be clear how to render an expression in the ideography before we embed it in a sentence. Indeed, someone may be misled into thinking that 'the sine function' is a proper name rather than a function name precisely by failing to consider it in the context of a sentence.

Hale (2010, pp. 411–416), in his response to the problem of the concept *horse*, can be seen as also proposing that we revise the Dummettian characterization of proper names so that 'the sine function' and other apparent proper names referring to functions are not captured by it.[15] More precisely, Hale suggests that we think of 'the concept *horse*', 'the sine function', and other names of functions, such as 'addition' and 'multiplication', as nominalizations of function names; and further, that a nominalization should be taken to have the same reference as the original function name. A nominalization will therefore refer to a function and not to an object. It seems to me that the characterization of logical type in terms of the ideography assumed here is quite consistent with Hale's suggestion; indeed, the two can be seen as complementing each other. On both accounts 'the sine function' is taken to refer to a function. However, instead of relying on our intuitive understanding of 'the sine function' in isolation, as Hale seems to do, I have provided an argument why this phrase should be so taken; namely it is to be taken to refer to a function, since in the ideographical rendering of sentences in which this phrase occurs it is represented by a function name. Hale rather concentrates on the idea that there should be some systematic relationship between the noun phrase 'the sine function' and the function name '$\sin(x)$'. Such an idea does not feature in the present account. I am simply assuming that someone proficient in the ideography and in mathematical English will know how to translate between the two and thus see that what represents 'the sine function' in such a translation is a function name. I have been silent about whether any systematic account can be given of a reference-preserving passage from '$\sin(x)$' to 'the sine function'. But it is clear that, were Hale or someone else to provide it,[16] such an account would be a valuable addition to the characterization of logical type assumed here.

1.4. An objection and a response. A natural objection to the assumed characterization of logical type is that one and the same natural language sentence may be rendered by several ideographical sentences differing in the type of the ideographical expression representing a given natural language expression. Let S be a sentence of English containing an expression e. It may be that in one ideographical rendering of S the expression representing e is of one type and in another ideographical rendering of S the expression representing e is of another type. In that case the ideography will not yield a unique answer to the question of which logical type to assign to e.

Indeed, even our example 'the sine function is periodic' may be given ideographical renderings differing in the type of the expression representing 'the sine function'. Let us assume that a consistent version of Frege's ideography can be worked out in which there are proper names referring to objects corresponding to functions. In Frege's original

[15] Hale does not phrase his proposal this way. Hale's way of phrasing his proposal rather relies on a distinction between primary and derivative modes of reference. Proper names for Hale are precisely what are picked out by the Dummettian characterization. A proper name in this sense may refer to a function, albeit only derivatively.

[16] Hale has offered the beginnings of such an account in Hale & Wright (2012, pp. 126–129). He suggests to understand nominalization in terms of abstraction: the nominalization 'the sine function' should be regarded as obtained by abstraction from the function name '$\sin(x)$'.

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ideography there are names '$\dot{\alpha}.f(\alpha)$' purporting to refer to certain objects Frege calls courses of values. It follows from Basic Law V that the binary function of objects $x^\frown y$, satisfying $\dot{\alpha}.f(\alpha)^\frown\Delta = f(\Delta)$ for any object Δ, is well-defined (GG §34).[17] Since a function $\varphi(x)$ may take its own course of values $\dot{\alpha}.\varphi(\alpha)$ as an argument, Russell's contradiction is then easily derivable.[18] Let us, however, assume that reparations have been made to Frege's ideography so that the contradiction is no longer derivable.[19] One could then render the sentence 'the sine function is periodic' in the ideography as follows:

$$(\exists r \in \mathbb{R}^+)(\forall x \in \mathbb{R})(\dot{\alpha}.\sin(\alpha + r)^\frown x = \dot{\alpha}.\sin(\alpha)^\frown x). \tag{ii}$$

We obtain (ii) from (i) by replacing the function name '$\sin(x)$' by the proper name '$\dot{\alpha}.\sin(\alpha)$' embedded in the context '$\dot{\alpha}.\sin(\alpha)^\frown x$'. In particular, it seems natural to say that what in rendering (ii) represents the phrase 'the sine function' is a proper name, namely '$\dot{\alpha}.\sin(\alpha)$', and not a function name.

Frege (GG §22) noted that a proper name in the traditional sense, such as 'Socrates' or '2', may be rendered both by an ideographical proper name and by a quantifier, that is (in Frege's type theory), a unary function of unary functions of objects. This insight does not depend on any particular feature of proper names and quantifiers. There is a general principle of so-called type-raising which says that if an expression can be assigned a type τ, then it may also be assigned the type $((\tau))$, namely the type of unary functions of unary functions of entities of type τ.[20] In particular, we may introduce a third-order function **sin** (which I will not attempt to write in Fregean notation) satisfying

$$\mathbf{sin}(\Phi) = \Phi_\beta(\sin(\beta)).$$

Thus **sin** applied to a second-order concept $\Phi_\beta(f(\beta))$ yields the True if and only if $\Phi_\beta(f(\beta))$ applied to $\sin(x)$ yields the True. We therefore obtain yet another rendering of 'the sine function is periodic', where what represents 'the sine function' is a third-order function name, as follows:

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[17] Namely, if $\dot{\alpha}.f(\alpha) = \dot{\alpha}.g(\alpha)$, then by Basic Law V, $f(\Delta) = g(\Delta)$ holds for any Δ; therefore $\dot{\alpha}.f(\alpha)^\frown\Delta$ depends only on the identity and not on the form, 'the mode of givenness', of $\dot{\alpha}.f(\alpha)$.

[18] To derive Russell's contradiction also the property of the negation function $\neg x$ must be assumed that $\neg\Delta$ is never identical to Δ (cf. GG §6). Considering the unary function of objects $\neg(x^\frown x)$, we have

$$\big(\dot{\alpha}.\neg(\alpha^\frown\alpha)\big)^\frown\big(\dot{\alpha}.\neg(\alpha^\frown\alpha)\big) = \neg\big((\dot{\alpha}.\neg(\alpha^\frown\alpha))^\frown(\dot{\alpha}.\neg(\alpha^\frown\alpha))\big),$$

which is of the form

$$(\Delta^\frown\Delta) = \neg(\Delta^\frown\Delta).$$

This contradicts the assumed property of the negation function.

[19] Consistent theories of courses of values can be found, for instance, in Aczel (1980) and in Martin-Löf (1984). In Martin-Löf's type theory courses of values are put into separate types, something Frege himself suggested in a letter to Russell (Frege, 1976, p. 217). Aczel's theory, by contrast, is 'type-free' in the sense that it has only one type of individuals.

[20] The first to formalize principles of type-raising may have been Lambek (1958) in his syntactic calculus, in particular with its two rules:

$$y \to (z/y)\backslash z,$$
$$y \to z/(y\backslash z).$$

The intuitive meaning of the assertions of the Lambek calculus, $x \to y$, is that any expression that may be assigned the type x may also be assigned the type y. Note that in general there are no corresponding principles of 'type-lowering' (a simple cardinality consideration should convince one of that).

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$$\mathbf{sin}(P). \tag{iii}$$

Here P is the function defined above, namely such that $P_\beta(\varphi(\beta))$ is the True if and only if $\varphi(x)$ is periodic.

On what grounds, then, can we say that 'the sine function' is the name of a unary function of objects? I think we can say that on the grounds that our first rendering (i) is in a clear sense more original than renderings (ii) and (iii), since we obtain (ii) and (iii) from (i) by means of certain transformations. In fact, relying on these transformations we can define a preference relation on type assignments, thereby making more precise the idea of 'originality' appealed to here. Namely, if a type assignment is made on the strength of a type-raised expression or the name of a course of values, then it is natural to prefer the type of the ideographical expression on which the procedure of type-raising was applied or from which the the name of the course of values was obtained. In more detail the preference relation may be described as follows. If $\mathcal{A}$ is a function of type $((\tau))$ and there is an a of type τ such that $\mathcal{A}_\beta(\varphi(\beta)) = \varphi(a)$ is true for any function $\varphi(x)$ of type (τ); and if a natural language expression e is assigned the type $((\tau))$ on the strength of its being represented in the ideography by $\mathcal{A}$; then e should rather be assigned the type of a, namely τ. And, if a natural language expression e is assigned the type of proper names on the strength of its being represented in the ideography by a course of values $\grave{\alpha}.\varphi(\alpha)$, but this course of values occurs everywhere in the given sentence in a context '$\grave{\alpha}.\varphi(\alpha)^\frown\Delta$'; then e should rather be assigned the type of $\varphi(x)$, namely the type of unary functions of objects.

In many cases this preference relation will yield a unique answer to the question of what is the preferred type assignment of a given expression. It determines, for instance, that the preferred type of 'the sine function' is the type of unary functions of objects. It has not been established, and it probably cannot be established, that there always will be a unique preferred type assignment. Yet, at least if we restrict ourselves to mathematical English, it is not easy to see what the form of a sentence should be that contains an expression whose type cannot be uniquely specified using the stated preference relation. Let us consider two forms of sentence that could be suggested as problematic cases.

If an expression e is assigned different types on the grounds of two renderings of a given English sentence in which e occurs, then for the preference relation to apply the renderings must be associated to each other as either (ii) or (iii) is to (i). The preference relation does therefore not apply to two standard renderings of sentences containing definite descriptions. Let us consider the sentence 'the smallest prime number is even'. Following Russell (1905) this sentence may be rendered as follows:

$$(\exists x \in \mathbb{N})(\mathrm{Prime}(x) \wedge (\forall y \in \mathbb{N})(\mathrm{Prime}(y) \supset x \leq y) \wedge \mathrm{Even}(x)).$$

Following Frege (GG §11) in making use of a definite description operator $\imath$, however, the sentence may be rendered as follows:

$$\mathrm{Even}(\imath x.(\mathrm{Prime}(x) \wedge (\forall y \in \mathbb{N})(\mathrm{Prime}(y) \supset x \leq y))).$$

These two formulae, it would seem, are not related to each other as either (ii) or (iii) is to (i). Nor, it seems, can we argue that any of these is in a natural sense more original than the other and therefore presents one with the preferred type assignment. But these two formulae do not present one with different type assignments of the definite description 'the smallest prime number'. Rather, they present one with different answers to the question of whether this phrase is to be regarded as a self-standing expression or as what Russell & Whitehead (1910, p. 66) called an incomplete symbol, an expression that has no

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meaning—and therefore no logical type—in isolation. Only when the phrase is treated as a self-standing expression does the question of which type to assign to the phrase occur, and in that case the answer is uniquely specified.

A similar remark applies to sentences in which a number is assigned to a concept, for instance 'the number of Platonic solids is five'. Suppose we have available a second-order function $Nx.f(x)$ such that for any first-order function $\varphi(x)$ we have: $Nx.\varphi(x) = n$ iff the number of objects Δ such that $\varphi(\Delta)$ is the True is n. Then we can render our sentence as

$$Nx.S(x) = 5.$$

But some will argue that numbers are best thought of as second-order concepts, in effect as quantifiers.[21] Thus, each number n is to be identified with a quantifier $\exists_n$ such that $\exists_n x\varphi(x)$ is true if and only if there are precisely n objects Δ such that $\varphi(\Delta)$ is the True. According to this conception of number our sentence should be rendered along the lines of

$$\exists_5 x S(x).$$

Again the two displayed formulae are not related to each other as either (ii) or (iii) is to (i).[22] In particular, the quantifier $\exists_n$ is not simply the result of type-raising the object n. Had that been the case, the equation $\exists_5 x\varphi(x) = \varphi(5)$ would hold for any $\varphi(x)$, which it clearly does not (take, for instance, $\varphi(x)$ to be $x = 5$). But, again the two renderings do not give conflicting answers to the question of which type to assign to 'the number of Platonic solids' (or to 'five' for that matter), since according to the latter rendering this phrase is not to be assigned any type at all.

I therefore claim that, provided we decide to treat a given phrase as a self-standing part of a given sentence, then the ideography is a reliable guide in assigning a type to that phrase. It might be asked on what grounds the expressions of the ideography are assigned their types; on what grounds, for instance, do we assign the sine function the type of unary function of objects? This we do, I would answer, on the basis of our understanding of the ideography and of mathematics. We know that the sinus function is a function that takes real numbers as arguments and yields real numbers as values (or complex numbers, in complex analysis). Assuming that real numbers are objects, it is then immediate from our understanding of Frege's type hierarchy that the sine function must there be assigned the type of functions of objects.

1.5. Parsons' worry. Parsons (1986) has given an account of the problem of the concept *horse* quite similar to the account proposed here. In particular, Parsons argues that the mistake in Frege's argument must lie in the classification of 'the concept *horse*' as a proper name (ibid. pp. 455–459); and he uses simple type theory to show that we can assign to this phrase the type of unary functions of objects (ibid. pp. 459–463).[23] Parsons wonders (ibid. p. 463), however, whether such an account—which drives a wedge between

[21] Versions of this view have been defended by Bostock (1974) and Hodes (1984).

[22] This is not to say that no translation between the two can be defined: the translation defined by Rayo (2002, p. 1626) allows one to go from $Nx.S(x) = 5$ to a second-order sentence that is logically equivalent to $\exists_5 x S(x)$. The sentence in question, written without Fregean argument place indicators, is $\exists Z(S \approx Z \wedge \exists W(\exists_5(W) \wedge W \approx Z))$, where $W \approx Z$ says that there is a one–one correspondence between the concepts $W(x)$ and $Z(x)$.

[23] More precisely, Parsons employs what may be called a 'two-dimensional' type system, namely where a word is classified according to both semantic type and syntactic category; the system is developed in more detail in Parsons (1979). It seems to me that this type system may provide Hale with what he needs to develop his idea of nominalization.

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the surface syntax of natural language and the logical syntax of the ideography—undercuts the Fregean argument that numbers are objects.[24] Frege's argument, at least as interpreted in the Neo-Fregean tradition, goes roughly as follows (cf. e.g., Hale & Wright, 2001, p. 8). Number terms such as 'five' and 'the number of Platonic solids' are proper names; whence they refer, if they refer at all, to objects. For a member of a suitably homogenous class of proper names to refer it is sufficient that it occurs in a suitably large class of true sentences; but the theorems of arithmetic are true, and number terms occur in plenty in these theorems; hence number terms refer to objects. Thus the conclusion that numbers are objects is taken to follow from the two premisses that number terms are proper names and that the theorems of arithmetic are true. But how are we to establish that number terms are proper names if we cannot do so on the basis of their surface-grammatical appearance in English and other natural languages? It is precisely the automatic inference from surface-grammatical appearance to logical type that is blocked by the account of the concept *horse* paradox proposed by Parsons and defended here.

The answer is that we establish that number terms are proper names by arguing that in the best theoretical development of arithmetic in a formal language number terms are represented by proper names. Frege wished to demonstrate all of arithmetic within his ideography; the Neo-Logicists have done so within second-order logic to which a function $Nx.f(x)$ has been added as a primitive constant and Hume's Principle has been added as an axiom (cf. e.g., Wright, 1983, sec. xix). And within both of these theoretical developments numbers belong to the type of objects, whence their names are proper names. The same holds for the development of arithmetic in axiomatic set theory as well as for the treatment of arithmetic in Martin-Löf's type theory (Martin-Löf, 1984, pp. 71–76).

That arithmetic may be developed within a theory in which numbers are treated as objects provides justification for that type assignment. This mode of justification is sometimes encountered in mathematics texts, where the author justifies a definition by saying that 'it works', it does what it is supposed to do. But this justification for treating numbers as objects cannot be conclusive, since arithmetic can also be developed within a theory in which numbers are assigned another type, for instance the type of quantifiers. One can, however, compare the merits of these competing type assignments by comparing the merits of the theories of arithmetic based upon them. It is, for instance, often considered a weakness of the second-order treatment of numbers that it seems to require the assumption of an axiom of infinity or else the introduction of modal notions.[25] The merits of a given type assignment can also be assessed by drawing upon considerations of natural language. Although we have severed the close tie between grammatical surface structure and logical syntax assumed at times by Frege, as well as by the Neo-Fregeans, I think we can still appeal to considerations of natural language in assessing a given type assignment. It is, for instance, natural to argue that identity sentences are especially central in arithmetic and that we should therefore strive to render as identity sentences in our theoretical development of arithmetic what appears to be identity sentences between number terms in natural language. Since in most systems of logic identity is a binary relation between objects, we thereby have the outlines of an argument that we should prefer a type assignment that renders numbers as objects.

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[24] A similar worry is in effect raised by Hale & Wright (2012, p. 99).

[25] See Wright (1983, pp. 36–40), Hodes (1984, pp. 148–149), and Dummett (1991, pp. 131–134). Kim (2013, 2015), who argues that numbers are neither objects nor concepts, but rather a kind of modes of existence of objects, relies on modal logic in his formal development of arithmetic.

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On the basis of considerations such as these I think that the Neo-Logicist argument that numbers are objects survives the proposed solution to the problem of the concept *horse*.

§2. Is a concept. I now turn to the logical analysis of the predicate 'is a concept'. I shall first concentrate on the predicate 'is a function', or more generally, on predicates of the form 'is a τ', where τ is a type. The predicate 'is a concept' can be regarded as a special case of this general form if, as seems quite natural, we view first-order concepts as forming a subtype of the type of unary functions of objects.

2.1. As a first-order predicate. Someone who insists that 'the concept *horse*' refers to an object will naturally hold that 'is a concept' is an ordinary first-order predicate true precisely of the objectifications of concepts. Likewise, if one holds that 'the sine function' refers to an object, then 'is a function' (or, more precisely 'is a unary function of objects') should be regarded as a first-order predicate true precisely of the objectifications of unary functions of objects. To implement this idea in Frege's ideography we should introduce a function of objects, **fun**(x), that yields the True precisely for objects equal to an object of the form $\grave{\alpha}.f(\alpha)$, where $f(x)$ is a unary function of objects.[26]

Given a consistent theory of courses of values, this strategy can be generalized. In *Grundgesetze* §36 Frege shows how to define the course of values of a binary function of objects. Namely, if $g(x, y)$ is a binary function, then $\grave{\varepsilon}.g(x, \varepsilon)$ is a unary function whose course of values is $\grave{\alpha}\grave{\varepsilon}.g(\alpha, \varepsilon)$. We have

$$(\grave{\alpha}\grave{\varepsilon}.g(\alpha, \varepsilon)^\frown\Gamma)^\frown\Delta = \grave{\varepsilon}.g(\Gamma, \varepsilon)^\frown\Delta = g(\Gamma, \Delta).$$

The course of values of the binary function $g(x, y)$ is therefore $\grave{\alpha}\grave{\varepsilon}.g(\alpha, \varepsilon)$. (Frege here employs, in effect, the technique of 'Currying'.) By repeating this procedure we obtain for each natural number n objects representing n-ary functions.

In fact, all functions in Frege's type hierarchy can be represented as courses of values (cf. GG ibid. §25, §34–§35). A second-order function $\Phi_\beta(f(\beta))$ can be mimicked by a first-order function $\varphi(x)$ satisfying

$$\varphi(\grave{\alpha}.\psi(\alpha)) = \Phi_\beta(\psi(\beta))$$

for every first-order function $\psi(x)$. Thus, the value of the function $\varphi(x)$ on the course of values $\grave{\alpha}.\psi(\alpha)$ is identical to the value of $\Phi_\beta(f(\beta))$ on the function $\psi(x)$ from which that course of values was obtained. We may therefore regard $\grave{\alpha}.\varphi(\alpha)$ as the course of values of the second-order function $\Phi_\beta(f(\beta))$. A third-order function can now be mimicked by a first-order function having the right values on courses of values of second-order functions; and this first-order function can itself be turned into a course of values. And thus one can continue, up through the hierarchy.

Combining these two methods, one obtains for each function in the hierarchy an object representing that function. But then one can also introduce for each type τ a first-order predicate true precisely of the objects representing functions of type τ. Thus we can render

[26] A function yielding the True precisely for the courses-of-values of *concepts* is easily definable in Frege's language, for instance as $\forall x(y^\frown x = -(y^\frown x))$, where '$-$' is Frege's horizontal stroke. A function such as **fun** seems, however, not to be definable, so it would have to be introduced as a primitive. That **fun** would seem not to be definable is related to the fact that we lack a 'criterion of application' for courses of values.

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any sentence of the form 'the function f is of type τ' as an ordinary first-order sentence $\varphi(a)$, where a is an object and φ a unary function of objects. (The reader may worry, for instance in light of Cantor's theorem on the size of the power set, whether there are enough objects around to do all this representing. That there are enough such objects is a presupposition of Frege's doctrine of courses of values.[27])

A very different strategy for rendering predicates such as 'is a concept' and 'is a function' as first order is suggested by Carnap's treatment in the *Logical Syntax of Language* (Carnap, 1937) of what he there calls 'universal predicates'. A predicate is universal if it is true of everything in its range of significance.[28] Carnap thought that by employing such predicates one easily gets entangled in pseudo-problems (ibid. §80). For instance, in employing predicates such as 'is a function' or a 'is a number'—both of which are universal according to Carnap—we may take ourselves to be saying something about the essence of functions or numbers, and thus be led to ask pseudo-questions about essences. To avoid getting entangled in pseudo-questions we must, according to Carnap, regard universal predicates as metalinguistic predicates. Sentences such as 'the sine function is a function' and '7 is a number' therefore fall into a broader class of sentences that Carnap calls 'quasi-syntactic': they seem to be about objects, but in fact they are about language (ibid. §64, §74). These sentences should thus properly be rendered as ' '$\sin(x)$' is a function-symbol' and ' '7' is a numeral' (ibid. §77), where the universal predicate has been replaced by a metalinguistic predicate, which is not universal. If we regard signs as being of the type of objects, then universal predicates such as 'is a function' are thus to be regarded as confused forms of non-universal first-order predicates.[29]

I shall not investigate either of these strategies any further. It has been argued at length above that phrases of the form 'the concept…' and 'the function…' should be taken to refer to functions and not to objects. If we accept that line of argument, then it is clear that the predicates 'is a concept' and 'is a function' cannot be taken to be first-order. Let us therefore consider how to render these predicates if we are to regard the reference of phrases of the form 'the concept…' and 'the function…' as functions.

2.2. As a second-order predicate. Provided we are to treat, for instance, 'the sine function' as referring to the function $\sin(x)$, the obvious way of rendering 'the sine function is a function' in a type-theoretical language such as Frege's ideography is by means of a universal second-order concept. Namely, let $U_\beta(f(\beta))$ be a function that yields the True for every admissible argument; it can for instance be defined as

$$U_\beta(f(\beta)) \equiv \forall x(f(x) = f(x)).$$

Then we may render 'the sine function is a function' as

$$U_\beta(\sin(\beta)).$$

Since $U_\beta(f(\beta))$ yields the True when applied to any unary function of objects, the formula '$U_\beta(\varphi(\beta))$', provided it is well-formed, can be thought of as saying that $\varphi(x)$ is a

[27] But see (Paseau, 2015).

[28] A more detailed discussion of Carnap's approach to this matter can be found in Klein (2004).

[29] This strategy was also defended by Russell. When considering the question of how to formulate type theory without contradicting the theory itself, Russell was led to the conclusion that one has to regard type theory as 'a theory of symbols, not of things' (Russell, 1919, p. 267); see also Russell (1944, p. 692).

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function. Similar concepts can be defined for all types. Namely, we can define for each type τ a function $\mathcal{U}^{\tau}$ of type (τ) that yields the True for each of its admissible arguments. The formula '$\mathcal{U}^{\tau}_{\beta}(\varphi(\beta))$' can therefore be thought of as saying that $\varphi(x)$ is a τ.

That 'is a function' should be rendered as a second-order function in the ideography is explicitly stated by Frege in a letter to Russell. Frege says that 'the concept of a function must be a second-order concept' (Frege, 1976, p. 218), and he introduces the notation $\grave{\alpha}.f(\alpha)$ for this function. Thus, $\grave{\alpha}.f(\alpha)$, with the so-called *spiritus asper*, or 'rough breathing', rather than the *spiritus lenis*, or 'smooth breathing', is extensionally equivalent to the function $U_{\beta}(f(\beta))$ defined above: it yields the True for each admissible argument.[30] It can therefore be taken to express functionhood. Frege's view seems to be shared by most commentators, at least most of the commentators who think that 'the sine function' and like phrases refer to functions. Dummett (1973, pp. 211–217), for instance, argues at length that the phrase 'is something which everything either is or is not' is a second-order predicate that we should use instead of the apparently first-order—and therefore misleading—predicate 'is a concept' in saying that something is a concept. Likewise, Parsons (1986, p. 461) suggests to render 'f is a concept' as $\forall x(f(x) = -f(x))$, where '$-$' is Frege's content stroke, referring to a first-order function that sends the True to itself and every other object to the False. More recently, Jones (2016, pp. 149–150) has proposed a formula employing a reference function (roughly 'expression e refers to concept C') for rendering 'is a concept', the result of which is again a universal predicate.

There are, however, reasons to look for alternative accounts of sentences like 'the sine function is a function'. The predicate 'is a function' would namely seem to be a predicate of a kind quite different from, for instance, 'is periodic'. One way of seeing the peculiar nature of 'is a function' is to note that the truth of the sentence 'the sine function is a function' is a presupposition for the well-formedness of 'the sine function is periodic', since an admissible argument to 'is periodic' must be a function. At least this is so in the ideographical renderings of these sentences. Questions of well-formedness in natural language are moot, but in a type-theoretical language a definite answer is usually available. And the second-order function $P_{\beta}(f(\beta))$, which renders 'is periodic', can meaningfully be applied only to unary functions of objects. Thus the meaningfulness of '$P_{\beta}(\varphi(\beta))$' presupposes the truth of '$\varphi(x)$ is a unary function of objects'. The same holds for a universal concept such as $U_{\beta}(f(\beta))$. It is a presupposition for the well-formedness of '$U_{\beta}(\varphi(\beta))$' that $\varphi(x)$ is a unary function of objects, in other words, that the sentence '$\varphi(x)$ is a unary function of objects' is true. In general, where F is of type (τ), the well-formedness of '$F(a)$' presupposes the truth of 'a is a τ'. Predicates of the form 'is a τ' therefore seem to be peculiar, since they feature in presuppositions for the well-formedness of sentences of function–argument form. And we do not capture this peculiar nature of 'is a τ' if we render it as the universal concept $\mathcal{U}^{\tau}$, since also the well-formedness of '$\mathcal{U}^{\tau}_{\beta}(\varphi(\beta))$' presupposes the truth of '$\varphi(x)$ is a τ'.

2.3. Type predications. Let us call a sentence of the form 'a is a τ', where τ is a type in a given type hierarchy, a type predication. I am urging that type predications are special and that the best way of rendering them in a type-theoretical language may not be by means

[30] In a later letter to Russell (Frege, 1976, pp. 243–244), Frege again employs the spiritus asper, but then for a different purpose, namely as something along the lines of a higher-order abstraction operator (cf. Martin-Löf, 2006). J.J. Green must be thanked for adding the spiritus asper to his already splendid package for typesetting Frege's ideography in LaTeX.

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of formulae of function–argument form. Before considering what an alternative rendering may look like, it is worth remarking that the thought that type predications are special is not new. One of the clearest applications of the famous saying/showing distinction in Wittgenstein's *Tractatus* (Wittgenstein, 1922) concerns, precisely, type predications.[31] Wittgenstein held that something's falling under what he called a *formal concept* cannot be asserted by means of propositions (*Sätze*), but is rather shown in propositions (ibid. 4.122, 4.124, 4.126). It is quite clear from Wittgenstein's discussion that what he calls formal concepts are the types of the Tractarian system; thus he mentions both 'object' and 'function' as examples of formal concepts (ibid. 4.1272).[32] The Tractarian position thus seems to have been that what we try to say in uttering, for instance, 'the sine function is a function' cannot in fact be said; and, moreover, that this is not primarily because the subject 'the sine function' fails to pick out a function, but because there is no propositional function answering to the verb phrase 'is a function'. Wittgenstein says (ibid. 4.126):

> That something falls under a formal concept as one of its objects cannot be expressed by means of a proposition. [. . .]
>
> Formal concepts cannot, in fact, be represented by means of a function, as concepts proper can.

Replacing 'formal concept' here with 'type' we get: that something falls under a type cannot be expressed by means of a proposition; in fact, a type cannot be represented by means of a function, as a concept proper (*eigentlicher Begriff*) can.

The author of the *Tracatus* thus seems to have held that one cannot express by means of a sentence of the form '$F(a)$' that a is of such and such type. This negative doctrine is accompanied by the positive doctrine that the type of an entity is shown by the logical behaviour of symbols signifying it. There are no functions representing formal concepts, but there are variables whose ranges are formal concepts. There are, for instance, no functions representing predicates such as 'is an object' or 'is a function', but there are variables ranging just over objects and variables ranging just over functions. Which type an entity belongs to is then thought to be shown by the fact that a symbol signifying it can be substituted for a certain variable (ibid. 4.126–4.1272). That a symbol can be substituted for a variable is, in turn, a matter of the symbol's 'logico-syntactic employment' (ibid. 3.326–3.327). The type of an entity will therefore 'come out' through the logical behaviour of symbols signifying it. It is in this sense that the type of an entity is shown, although it cannot be said.

Holding that type predications do not have function–argument form does, however, not automatically commit one to Tractarian quietism. For it may be that what one attempts to say in a type predication in fact can be said provided one introduces a new logical form. That what one attempts to say in a type predication cannot be said by means of a sentence of function–argument form does not exclude that there may be some other form of statement by means of which type predications can be made. In that case type predications would have a logical form not recognized by standard predicate logic. It is a fundamental assumption of Frege's ideography as well as of the standard understanding of predicate-

[31] Many commentators have seen a connection between Frege's wrestling with 'the concept *horse*' and Wittgenstein's saying/showing distinction; a classical example is Geach (1976); for a critical discussion of this connection, see Proops (2013).

[32] The identification of formal concepts and types is made, for instance, by Hamlyn (1959), Hacker (1986, e.g., p. 20), and Klein (2004, p. 302).

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logical languages that all of their complex expressions are formed by applying a function to a suitable number of arguments.[33] If the form of type predications is not to be thought of as function–argument, then an extension of the forms of statement recognized by predicate logic is required if it is to accommodate such predications.

That type predications are not felicitously rendered by function–argument formulae is indeed suggested by a consideration of how to express such predications in a language containing type symbols, that is, names of types. In Frege's original ideography there are no type symbols, but such symbols can certainly be introduced.[34] Thus, let ι be the type of Fregean objects; and if $\alpha_1, \ldots, \alpha_n$ are types, let $(\alpha_1, \ldots, \alpha_n)$ be the type of n-ary functions whose k-th argument is of type α_k and whose values are of type ι (recall that in Frege's type theory all functions take values in ι). Given these type symbols, how should we express that a certain entity is of a certain type? It is clear that one could not do so by means of function–argument formulae: one could not, for instance, express that f is a unary function of objects by '$(\iota)(f)$' or that a is an object by '$\iota(a)$'. One could not do so since type symbols are not functors, just as types themselves are not functions. There are functions belonging to, for instance, (ι); but (ι) itself is not a function, it is a type. Strings such as '$(\iota)(f)$' and '$\iota(a)$' are therefore not well-formed, hence they cannot be used to say that f is of type (ι) and a of type ι, respectively.

Church (1940) employed subscripts on the simple expressions (variables and constants) of his language to indicate their type. Thus, the subscripted 'ι' on '7_ι' indicates that 7 is an object, and the subscripted '(ι)' on '$\sin_{(\iota)}$' indicates that sin is a unary function of objects.[35] This strategy does, however, not quite give us a way of expressing type predications in our language. We do not, for instance, say that 7 is of type ι simply by writing '7_ι', for '7_ι' is not a sentence, but a proper name. By writing '7_ι' we are merely naming the number 7, not saying that 7 is an object. Church's notation explicitly displays the type of a simple expression, hence if it is adopted one does not have to attend to the 'logico-syntactic employment' of such an expression to find out its type: one can then simply look at the type index. But Church's notation does not let us make type predications. A type predication is made not by merely showing the type of a entity in a type index, but by saying that an entity is of a certain type.

2.4. The form of type predications. Let us therefore introduce a new form of statement. If τ is a type let us write '$a : \tau$' for the type predication that a is of type τ. From the point of view of predicate logic this is a new form of statement since it is not of function–argument form. In particular, we are not to think of '$a : \tau$' as having the structure argument–function–argument, for the colon does not stand for a function. More precisely, there is no function in a given type hierarchy expressing the relation between a and τ that is expressed by '$a : \tau$', where τ is a type in that hierarchy. To the extent that the colon in '$a : \tau$' signifies a relation at all, it is thus not a relation found in a type hierarchy to which τ belongs. In terms of Fregean doctrine this is to say that the colon, used to express that an entity belongs to a type of Frege's type hierarchy, does not refer at all. If a sign

[33] By the qualification 'standard' I mean to exclude an interpretation such as Quine's, according to which the connectives are particles or syncategoremata, and not functions (Quine, 1986, cf. pp. 15–30).

[34] The first to introduce a systematic notation for simple types may have been (Carnap, 1929, pp. 30–33). A simpler notation was given by Church (1940). In Montague grammar one often uses 'e' instead of 'ι' for the type of individuals.

[35] In Church's notation, $\sin(x)$ would be assigned the type ι; hence we must write the function as 'sin' without the variable.

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refers, for Frege, then it refers to an entity in his type hierarchy; but there is no relation in Frege's type hierarchy to which the colon thus employed refers; whence it does not refer at all. The colon can therefore better be thought of as a syncategorematic symbol, like the copula of traditional logic. Indeed, it is natural to think of 'a' as the subject and 'τ' as the predicate of '$a : \tau$'. The whole type predication '$a : \tau$' may therefore be analyzed as having subject–copula–predicate structure.[36]

It may be useful to compare type predications '$a : \tau$' with set-theoretical formulae '$a \in b$'. These two forms of statement differ of course in that what is to the right of the colon, 'τ', signifies a type, whereas what is to the right of the epsilon signifies a set. Much can be said about the relation between sets and types,[37] but it should be clear that the two notions differ. One powerful intuition about set formation says that whenever we have some objects, then we can form the set of them.[38] In particular, assuming that 7 and Caesar are already recognized as objects, then we can form the set {7, Caesar} containing the number 7 and the person Caesar as its only members; but no reasonable type system will recognize a type under which only 7 and Caesar fall.

More important for current purposes, however, is the difference between the colon and the epsilon. This difference is perhaps most easily seen if we consider a language in which both forms of statement, '$a : \tau$' and '$a \in b$', can be made. We get such a language by embedding set theory into a type hierarchy such as the following. There is a ground type V of (pure) sets. Higher types are generated by the rule: if $\alpha_1, \ldots, \alpha_n$ are types, then so is $(\alpha_1, \ldots, \alpha_n)$, namely the type of n-ary relations whose k-th argument is of type α_k. Examples of type predications that now can be made are '$\emptyset : V$' and '$\in : (V, V)$'. It is clear that we cannot here replace the colon by the epsilon: since neither V nor $\in$ is a set, they cannot feature as the a or b in '$a \in b$'. Indeed, whereas $\in$ is a relation belonging to a given type, namely (V, V), the colon is used to express that an entity, such as $\in$ or $\emptyset$, thus belongs to a type.

In writing type predications as '$a : \tau$' and analyzing these into the structure of subject–copula–predicate rather than argument–function–argument, I am in fact following an already established usage in the more technical literature on type theory. In particular, the idea of expressing type predications by means of a novel form of statement is not new, but already well entrenched in certain parts of mathematical logic and theoretical computer science. The term 'type predication' is not common; instead one often says 'type assignment' (e.g., Hindley, 1997, p. 14) or, following Martin-Löf (1982), 'judgement' or 'typing judgement'. I will continue to speak of type predications. The colon seems now to be the established sign for the copula in a type predication. As such it may have originated with De Bruijn's AUTOMATH languages for formalizing mathematical reasoning (e.g., de Bruijn, 1980),[39] though it is of course found also in the common notation '$f : A \to B$' for indicating that f is a function from A to B.[40]

2.5. Putting type predications to use. That type theorists have seen the need to invent a notation for type predications suggests that there is work to be done for the notion of type

[36] The similarity of '$a : \tau$' to the traditional form of judgement 'S is P' is noted by Sundholm (2009, p. 291).

[37] See for instance Granström (2011, pp. 53–63).

[38] See for instance the definition of set provided by Dedekind (1888, art. 2) or the discussion in Linnebo (2010).

[39] In earlier work De Bruijn used **E**, thus writing $a \,\mathbf{E}\, \tau$ for the type predication $a : \tau$ (cf. de Bruijn, 1975).

[40] The earliest instance of this notation I have found is (Eilenberg & MacLane, 1942).

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predication and a notation for such predications. A notation for type predications is for instance needed if one wishes to reason about which entities belong to which types. Let us consider two examples of how such reasoning can be carried out for Frege's ideography. Using type predications we can formulate a rule for how courses of values are formed. Recall that ι is the type of objects, and (ι) the type of unary functions of objects.

$$\frac{\varphi(x) : (\iota)}{\grave{\alpha}.\varphi(\alpha) : \iota}.$$

Given a function $\varphi(x)$, there is a an object $\grave{\alpha}.\varphi(\alpha)$, which we call the course of values of $\varphi(x)$. More interesting perhaps is a rule that tells us how functions are formed. Let us write '$\Phi[e]$' to indicate that 'Φ' is an expression (of Frege's ideography) in which the expression 'e' occurs, and let '$\Phi[x/e]$' be the same expression, but with the argument place indicator 'x' substituted for some of the occurrences of 'e' in 'Φ'.[41] We can formulate the following rule for the formation of unary functions:

$$\frac{\Phi[e] : \iota \qquad e : \tau}{\Phi[x/e] : (\tau)}.$$

If $\Phi[e]$ is an object, and e is of type τ, then $\Phi[x/e]$ is a function of type (τ). In the formal mode of speech: if from the name of an object you eject one or more occurrences of a name of an entity of type τ, then you are left with a name of a function of type (τ).

Let us also consider examples of the uses of type predications in type systems other than Frege's. Assume that our type hierarchy allows the formation of product types. That is, if α and β are types, then there is a type $\alpha \times \beta$. We want to say that if we have have an entity a of type α and an entity b of type β, then there is a pair $\langle a, b\rangle$ of type $\alpha \times \beta$; and that if we have an entity c of type $\alpha \times \beta$, then there is an entity $\mathrm{fst}(c)$ of type α and an entity $\mathrm{snd}(c)$ of type β. We can do that by means of the following rules.

$$\frac{a : \alpha \qquad b : \beta}{\langle a, b\rangle : \alpha \times \beta} \qquad \frac{c : \alpha \times \beta}{\mathrm{fst}(c) : \alpha} \qquad \frac{c : \alpha \times \beta}{\mathrm{snd}(c) : \beta}.$$

It is not uncommon in type theory to regard the natural numbers as a separate type. Thus the natural numbers will not belong to a universal type of objects, along Fregean lines, but will form their own type. If that is assumed, then the following rules are natural:

$$0 : \mathbb{N} \qquad \frac{n : \mathbb{N}}{s(n) : \mathbb{N}}.$$

These rules postulate that 0 is a natural number, and that whenever n is a natural number, then so is its successor $s(n)$. To phrase less trivial rules involving type predications more notions and theoretical background would be needed. (For instance the important notion of a context would have to be discussed.) But the few examples given here should suffice to indicate that type predications can be of use in type theories.

2.6. The sine function is a function. It should now be clear how to analyze 'the sine function is a function'. Assuming Frege's type hierarchy as well as his functional notation, we do it as follows:

$$\sin(x) : (\iota).$$

[41] I trust that the reader will understand this even if no corner quotes are used.

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Although I shall not argue the matter here, a functional notation without argument place indicators, as in 'sin', seems to me preferable to the Fregean notation.[42] One may also find it natural to single out the real numbers, $\mathbb{R}$, as a separate type. Let us write the type of functions from $\mathbb{R}$ to $\mathbb{R}$ as $(\mathbb{R})\mathbb{R}$. Our type predication would then be written as

$$\sin : (\mathbb{R})\mathbb{R}.$$

Recall that we may render 'the sine function is periodic' as

$$P_\beta(\sin(\beta)),$$

where '$P_\beta(f(\beta))$' says that f is periodic. We can, however, not write this as '$\sin : P$', for P is not a type, but a higher-order function; in fact, $P : ((\iota))$. Thus again we see the difference between type predications and ordinary function–argument formulae: whereas '$P_\beta(\sin(\beta))$' says, in Fregean terms, that the function P, when applied to the sine function, yields the True, the type predication '$\sin(x) : (\iota)$' says that the sine function belongs to the type (ι) of unary functions of objects.

2.7. The concept **horse** *is a concept.* There is no type of concepts in Frege's ideography, so it is not immediately obvious how to render 'the concept *horse* is a concept' as a type predication. It seems, however, natural to think of concepts as forming a subtype of the type of unary functions of objects, just as it seems natural to think of the two truth-values as forming a subtype of the type of objects. Also the notion of a subtype has been investigated by type theorists. Following Reynolds (1998, chap. 16) we may write

$$\alpha \leq \beta$$

to say that α is a subtype of β. Let us think of the two truth-values True and False as forming their own type **bool**.[43] Thus we have

$$\mathbf{bool} \leq \iota.$$

We may then introduce a type **C** under which fall just those functions whose range is included in **bool**. Thus, **C** will be the type of concepts in Frege's sense. Using type predications we can specify **C** as follows:

$$\mathbf{C} \leq (\iota) \qquad \frac{\varphi(x) : \mathbf{C} \qquad \Delta : \iota}{\varphi(\Delta) : \mathbf{bool}} \qquad \frac{\varphi(x) : (\iota) \qquad \Delta : \iota \vdash \varphi(\Delta) : \mathbf{bool}}{\varphi(x) : \mathbf{C}}.$$

We first specify that **C** is a subtype of (ι). The first rule of inference states a necessary condition for being a function of type **C**; namely, if $\varphi(x)$ is of type **C** then it yields a truth-value for any argument. The expression '$\Delta : \iota \vdash \varphi(\Delta) : \mathbf{bool}$' in the second rule of inference means that $\varphi(\Delta) : \mathbf{bool}$ for any $\Delta : \iota$. This rule therefore states a sufficient condition for being a function of type **C**; namely, if $\varphi(x)$ is a function of objects that yields a truth-value for any argument, then $\varphi(x)$ is of type **C**. We may now formalize 'the concept *horse* is a concept' as follows:

$$horse(x) : \mathbf{C}.$$

[42] This notation requires the use of lambda-abstraction; for instance, the Fregean notation $x^2 + x$ would be replaced by $\lambda x.x^2 + x$.

[43] Church (1940) used 'o' for this type; in Montague grammar the common notation is 't'.

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Thus, by regarding concepts as forming a subtype of the type of objects, we have found a way of rendering 'the concept *horse* is a concept' as a type predication.

This completes the analysis of 'the concept *horse* is a concept'. It is natural to ask now whether the notions and notations we have introduced also allow us to express, for instance, that the concept *horse* is *not* an object in Frege's sense. Indeed, in the literature on the concept *horse* paradox it is sometimes suggested that the deeper problem Frege struggled with in his response to Kerry was the problem of expressing what may be called *negative* type predications.[44] A negative type predication is a predication in which one says, or attempts to say, of an entity that it is not of a certain type. I shall not try to get to the bottom of this problem here, but only make a few remarks in concluding this paper. Since '$a : \tau$' falls outside the compass of standard predicate-logical syntax, we cannot simply attach the ordinary negation operator to '$a : \tau$', as we can to a formula of the form '$F(a)$'. In a type-theoretical setting negation, $\neg$, is a function defined on a certain type, say Frege's $\imath$ or the computer scientist's **bool**; but '$a : \tau$' does not name an entity of a type, hence '$\neg(a : \tau)$' will not be well-formed. We can of course postulate a form of statement, say '$a \ddagger \tau$', with the intuitive meaning that a is not of type τ. But one must then clarify more precisely how such a statement is to be understood, for instance by stating certain rules like those given for '$a : \tau$' above, which would tell us under what conditions we can assert such a thing as '$a \ddagger \tau$'. To investigate whether this can be done in a satisfactory manner is work for another occasion.

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[44] See in particular Geach (1976, pp. 57–58), Noonan (2006, pp. 165–166), and Proops (2013, pp. 89–94).

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