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Integer difference covers which are not k-sum covers, for $k = 6, 7$

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Constructions are given of sets $F(k)$, $(k = 6, 7)$ of residues mod $q(k)$, (for a suitably chosen integer $q(k)$) such that $F(k) - F(k)$ contains all residues, while

$$(k)F(k) = F(k) + \ldots + F(k),$$

has a gap of an assigned number of consecutive residues.

Introduction. These results for $k = 6$ and 7 are an extension of the results of Jackson, Williamson and Woodall(1), where constructions for the cases $k = 3, 4, 5$ were given. The results $k = 6$ and 7 are interesting in themselves and show some of the difficulties likely to be encountered in treating larger values of k, using the methods of (1) involving Rademacher arrays.

Three new techniques have been introduced in the present paper to extend the results of (1). The first is to define and use n-inflated complemented Rademacher arrays, which are a generalization, in a sense, of inflated complemented Rademacher assays defined in (1). These n-inflated arrays are defined in section 1·2 and will be used in sections 2 and 4. The second, is to form pairs of sets (in sections 2 and 4) and use them (in sections 3 and 5) in a comparatively simple manner to construct $F(6)$ and $F(7)$. By symmetry considerations which will become apparent in sections 3 and 5, this technique has the significant effect of halving the demands of the ϕ mappings of (1) which were seen to become rapidly complex as k went from 3 to 5. Suitable ϕ mappings for $k = 6$ and 7, akin to those used in cases $k = 3, 4, 5$ appear to be extremely complex, maybe even non-existent. This second technique also allows us to consider $[\frac{1}{2}k]$ simpler pairs of mappings, rather than a single pair of mappings satisfying $[\frac{1}{2}k]$ particular sets of restrictions, this aspect is exemplified in section 2. Thirdly, augmented Rademacher arrays are (necessarily) used to reduce further the requirements of the ϕ mappings.

Some of the above techniques have been applied to the constructions in (1) and reduced the minimum estimates for $q(k)$, $k = 3, 4, 5$ (see Jackson(2)). It is also hoped that they may give results for $k = 8$ or even a little higher and ultimately lead to a recursive construction for each k, of a set F (if it exists) such that $F - F$ contains all residues mod $q(k)$, but $(k)F$ has a gap of consecutive residues.

1. *Preliminary definitions and results.* It is assumed that the reader is familiar with (1); whenever possible, we shall use notation that is the same, or at least consistent

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with that paper. For the convenience of the reader some notation and definitions are set out again.

1·1. (*a*) A complete set of residues mod k, for positive integer k, will be identified with the set $\{0, 1, 2, \ldots, k-1\}$ and will be denoted by $Z(k)$.

(*b*) For positive integers k and l, $F(k, l)$ denotes an integer set of residues mod some corresponding integer $q(k, l)$, say, where $F(k, l)$ has the properties: $F(k, l) - F(k, l)$ contains $Z(q(k, l))$ but $(k)\, F(k, l)$ has a gap of l consecutive residues. Often $F(k)$, $q(k)$ are written for $F(k, l)$, $q(k, l)$ respectively, particularly when the size of l is not significant.

For arbitrary l, $F(5, l)$ (and a corresponding $q(5, l)$) has been established in proposition 3·6 of (1), and will be used in the construction of $F(6)$. It should be noted, for the purposes of Lemma 1·4 (to follow) that the elements of $F(5, l)$, $\xi = \xi_{P(5)} \ldots \xi_0$ (written in generalized decimal mod $(\kappa_{P(5)+1}, \ldots, \kappa_1)$) are such that $\kappa_{P(5)+1} = 200$ and $0 \leqslant \xi_{P(5)} \leqslant 100$.

1·2. *Generalized decimal development of integers.* Let $\kappa = (\kappa_1, \kappa_2, \ldots)$ with $\kappa_r \geqslant 2$ be a sequence of positive integers, and write:

$$\delta_0 = 1,$$
$$\delta_r = \kappa_1 . \kappa_2 \ldots \kappa_r \quad (r \in N).$$

Then any non-negative integer ξ can be written uniquely as a finite sum:

$$\xi = \sum_{r=0}^{N} \xi_r \delta_r \quad (0 \leqslant \xi_r < \kappa_{r+1}).$$

We shall usually just write $\xi = \xi_N \xi_{N-1} \ldots \xi_0$ and refer to it as the generalized decimal of ξ mod $(\kappa_{N+1}, \kappa_N, \ldots, \kappa_1)$.

1·3. *Rademacher arrays.* Two of the following arrays are defined exactly as in (1) and the third is a generalization of the inflated array. These arrays are very useful in describing and visualizing the various sets to be constructed in sections 2 and 4.

A *simple Rademacher array* of order N, is a set of N rows of symbols 0, X; each row is of length 2^N, where the rth row consists of 2^{N-r}0's followed by $2^{N-r}X$'s, followed by 2^{N-r}0's, ...

A *complemented Rademacher array* of order N, is a set of $2N$ rows of symbols 0, X, where with each row r (as in a simple array), there is associated another (complementary) row r', having 0's where row r has X's, and X's where row r has 0's.

An *n-inflated complemented Rademacher array* of order N, is a set of $2N$ rows of symbols 0, X, Y each row of length $\binom{N}{n} 2^N$, obtained from a complemented Rademacher array of order N by replacing each of its 2^N columns by a set of $\binom{N}{n}$ columns: each column of the set being the same as the one it is replacing, except that $\frac{Y}{Y}$ replaces n complemented pairs $\frac{0}{X}$ or $\frac{X}{0}$ in each new column. The pairs of Y's (n pairs to a column) are placed so that every possible combination of pairs occurs exactly once.

It is evident that n cannot exceed the order N; and that for $n = 1$, the above array corresponds to the inflated array defined in (1), § 3. A 2-inflated array is described as a doubly inflated array, a 3-inflated array as triply inflated, and so on.

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Notation: if A is an n-inflated complemented Rademacher array of order N, then it will have $P(N) = \binom{N}{n} 2^N$ columns and $2N$ rows; we denote the elements of A by $A_{(i,j)}$; it is convenient to label the rows $1, 1', 2, 2', \ldots, N, N'$ and the columns, ordered from left to right: $P(N), P(N)-1, \ldots, 3, 2, 1$.

LEMMA 1·3. *Any set of k distinct rows from an n-inflated complemented Rademacher array of order N such that $2r$ ($\leqslant k \leqslant 2n$) of the rows are k complementary rows, will contain a column consisting of $2r$ Y's and $(k-2r)$ 0's provided that $(k-2r) \leqslant (N-n)$.*

Proof. Consider first a non-inflated array of order N. Each column has N X's and N 0's. In passing to the n-inflated array, each column is replaced by $\binom{N}{n}$ columns containing $2n$ Y's, $(N-n)$ X's and $(N-n)$ 0's. Since the Y's are placed in n pairs, in every possible combination, there will be at least one column with Y's in the $2r$ complemented rows and $(k-2r)$ 0's in the remaining $(k-2r)$ uncomplemented rows in question; provided of course that $(k-2r) \leqslant N-n$.

1·4. *Augmented Rademacher arrays.*

LEMMA 1·4. *Let $\xi^1, \xi^2, \ldots, \xi^k$ be k arbitrary elements of $F(5, l)$. Then*

$$\sum_{i=1}^{k} \xi^i \leqslant [\tfrac{1}{2}k] \times q(5, l)$$

where $[\frac{1}{2}k]$ is the integer part of $\frac{1}{2}k$.

Proof. Each ξ of $F(5, l)$ can be expressed in generalized decimal notation

$$\xi = \xi_{P(N)} \ldots \xi_0 \quad \text{where} \quad \xi_{P(N)} \leqslant 100.$$

Since $\kappa_{P(N)+1} = 200$ we see that when summing k ξ's, we can carry at most

$$[k \times 100] \bmod 200 = [\tfrac{1}{2}k]$$

from place $P(N)$ to place $P(N)+1$ in the general decimal expansion of ξ.

This lemma will suggest in sections 2·2, 2·3, 4·2, 4·3 that we augment the Rademacher arrays (order N) when necessary, by prefixing the array with a single column of Y's. In accordance with the notation above (section 1·3) this new column will be number $P(N)+1$; we denote the augmented array by $A' = [A'_{(i,j)}]$.

It will prove a considerable advantage to carry at most 3, rather than 5 (or 6) from place $P(N)$ to $P(N)+1$ when summing six (or seven) elements of various sets to be defined in sections 2·2, 2·3, 4·2 and 4·3.

2. In this section we construct pairs of integer sets (F_{i1}, F_{i2}), $i = 1, 2, 3$ with properties:

(*a*) $(F_{i1} - F_{i2}) = (F_{i2} - F_{i1})$ and each set contains $Z(q_i(6))$.

(*b*) (*i*) $F_{i1} + (6-i) F_{i2}$ omits at least six consecutive residues mod $q_i(6)$: $\zeta^*_{i6}, \ldots, \zeta^*_{i1}$ say.

The elements of these sets will be placed in a simple symmetric array, which is illustrated in section 3, to construct $F(6)$.

2·1. $i = 1$. We begin by considering two set valued mappings Ψ_{11}, Ψ_{12} defined on $\{1, 2\}$, $\{1', 2'\}$ respectively, as follows:

$$\Psi_{11}(h) = \{6h+1, 6h, \ldots, 6h-5\},$$
$$\Psi_{12}(h') = \{-h\}.$$

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It is readily seen that the set of integers

$$\{[\psi_{11}(h)-\psi_{12}(h')] \bmod 14/\psi_{ij}\in\Psi_{ij},(h,h')\in\{(1,1'),(2,2')\}\}$$

contains all the integers: $1, 2, \ldots, 14$. However, the set

$$\{[\psi_{11}(h)+5\psi_{12}(h')] \bmod 14/(h,h')\in\{(1,1'),(2,2')\}\}$$

does not contain the integers: 4, 5, 6, 7, 8, 9.

Now take an inflated (i.e. 1-inflated) complemented Rademacher array A of order $N = 2$ (hence having eight columns); then define a (finite) integer sequence $\kappa_1, \ldots, \kappa_9$ by

$$\kappa_r = q(5,6) \quad (1 \leqslant r \leqslant 8)$$

$$\kappa_q = 14$$

write
$$q_1(6) = \kappa_1\kappa_1\ldots\kappa_8\kappa_9 \quad \text{(product)}.$$

Next, using generalized decimal notation for ξ, define the following (two) integer sets F_{11h} for $h = 1, 2$:

$$F_{11h} = \{\xi : \xi = \xi_8\xi_7\ldots\xi_r\ldots\xi_0 \quad \text{where} \quad \xi_8\in\Psi_{11}(h)$$

and
$$\xi_r = 0, \in F(5,6), \in Z(\kappa_r) \quad \text{if} \quad A_{h,r+1} = 0, Y, X \text{ resp. } 0 \leqslant r \leqslant 7\}.$$

Similarly, we define the sets $F_{12h'}$ for $h' = 1', 2'$ by replacing $\Psi_{11}(h)$ by $\Psi_{12}(h')$ in the above definitions of F_{11h}.

Write $F_{11} = F_{111} \cup F_{112}$ and $F_{12} = F_{121'} \cup F_{122'}$. It can be shown, using the same argument used in the proof of Proposition 3·3 of (1), that $F_{11} - F_{12} = F_{12} - F_{11}$ and that both differences contain $Z(q_1(6))$.

Now consider the set $F_{11} + (5)\,F_{12}$. Let

$\zeta = \zeta_8\zeta_7\ldots\zeta_0$ be an arbitrary element of $F_{11} + (5)\,F_{12}$.

$\eta = \eta_8\eta_7\ldots\eta_0$ be an arbitrary element of F_{11}.

$\xi^i = \xi^i_8\xi^i_7\ldots\xi^i_0$ $(i = 1, 2, 3, 4, 5)$ be five arbitrary elements of F_{12}, each of the above elements being expressed in general decimal notation. By writing $\eta_8 = \psi_{11}(h)\in\Psi_{11}(h)$

$$\xi^i_8 = \psi_{12}(h'_{j_i})\in\Psi_{12}(h'_{j_i}) \quad (1 \leqslant h, h_{j_i} \leqslant 2),$$

the elements $\zeta = \eta + \xi^1 + \xi^2 + \ldots + \xi^5 \in [F_{11} + (5)\,F_{12}]$ may be characterized by one of the following (two) exhaustive cases:

(*a*) $(h, h'_{j_i})\in\{(1,1'),(2,2')\}$ for $i = 1, 2, 3, 4$ and 5.

(*b*) $(h, h'_{j_i})\notin\{(1,1'),(2,2')\}$ for at least one value of i.

Consider ζ formed by method (*a*).

$\psi_{11}(h) + 5\psi_{12}(h')$, for either value of h, never takes the values 4, 5, 6, 7, 8, 9 and since we can carry at most 5 from place seven to place eight, it follows that $\zeta_8 \neq 9$.

Consider ζ formed by method (*b*).

In this case, there corresponds a column in A with entries, which are either 0 or Y, in the rows h, and h'_{j_i} $(i = 1, 2, 3, 4, 5)$ with at most five of the entries being Y's. If it is column 1, then the corresponding digit ζ_0 belongs to (5) $F(5, 6)$, so that ζ_0 will not take any of six consecutive values: $(a_1 - 5), (a_1 - 4), \ldots, a_1$ say. If it is any other column (column r, $2 \leqslant r \leqslant 8$), then similarly $\eta_r + \sum_{n=1}^{5} \xi^i_r$ belongs to (5) $F(5, 6)$, and since we can carry at most 5 from place $(r-1)$ to place r, it follows that $\zeta_r \neq a_1$.

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Since the cases (*a*) and (*b*) above, are exhaustive regarding the formation of

$$\zeta \in F_{11} + (5)\, F_{12},$$

it follows that ζ can never take the six consecutive values:

$$9a_1a_1\ldots(a_1-5), \ldots, 9a_1a_1\ldots a_1; \quad \zeta^*_{16}, \zeta^*_{15}, \ldots, \zeta^*_{11}$$

say.

2·2. $i = 2$. Consider this time two set valued mappings Ψ_{21}, Ψ_{22} defined on

$$\{1, 2, \ldots, 7\}, \{1', 2', \ldots, 7'\}$$

respectively, as follows:

$$\Psi_{21}(h) = \{2h, 2h-1, 2h-2\},$$
$$\Psi_{22}(h') = \{(1-h)\}.$$

It can be seen that the set of integers:

$$\{[\psi_{21}(h) - \psi_{22}(h')] \bmod 21 / \psi_{21} \in \Psi_{2i}, (h, h') \in \{(1, 1'), \ldots, (7, 7')\}\}$$

contains all the integers of $Z(21)$. However,

$$\{[\psi_{21}(h) + \psi_{21}(h_2) + 2\psi_{22}(h'_1) + \psi_{22}(h'_2) + \psi_{22}(h'_3)] \bmod 21 / h_3 = h_1 \text{ or } h_2,$$

h_1 not necessarily distinct from h_2, $(h_i, h'_i) \in \{(1, 1'), (2, 2'), \ldots, (7, 7')\}\}$, does not contain the integers 11, 12, 13 or 14.

We next proceed as in the above section for $i = 1$, making appropriate minor changes. We consider an inflated complemented Rademacher array of order $N = 7$ (hence with $P(7)$ columns) and augment this with a final (no. $P(7)+1$) column of Y's, as was suggested in section 1·4; then define an integer sequence $\kappa_1, \kappa_2, \ldots, \kappa_{P(7)}, \kappa_{P(7)+1}, \kappa_{P(7)+2}$ by:

$$\kappa_r = q(5, 6) \quad 1 \leqslant r \leqslant P(7)+1,$$
$$\kappa_{P(7)+2} = 21,$$

and write

$$q_2(6) = \kappa_1\kappa_2\ldots\kappa_{P(7)+2} \quad \text{(product)}.$$

Next, using generalized decimal notation for ξ, define F_{21h} and $F_{21h'}$ for $h = 1, 2, \ldots, 7$, by

$$F_{21h} = \{\xi : \xi = \xi_{P(7)+1}\ldots\xi_r\ldots\xi_0, \quad \text{where} \quad \xi_{P(7)+1} \in \Psi_{21}(h),$$
$$\xi_r = 0, \in F(5, 6), \in Z(\kappa_r) \quad \text{if} \quad A'_{h, r+1} = 0, Y, X \text{ resp. } 0 \leqslant r \leqslant P(7)\},$$
$$F_{22h'} = \{\xi : \xi = \xi_{P(7)+1}\ldots\xi_r\ldots\xi_0 \quad \text{where} \quad \xi_{P(7)+1} \in \Psi_{22}(h')$$
$$\xi_r = 0, \in F(5, 6), \in Z(\kappa_r) \quad \text{if} \quad A'_{h', r+1} = 0, Y, X \text{ resp. } 0 \leqslant r \leqslant P(7)\}.$$

Write

$$F_{21} = \bigcup_{h=1}^{7} F_{21h} \quad \text{and} \quad F_{22} = \bigcup_{h=1}^{7} F_{22h'}.$$

It can be shown that $F_{21} - F_{22} = F_{22} - F_{21}$ and that each difference contains $Z(q_2(6))$. Also it can be shown with an argument similar to that used for $i = 1$, that $(2)\, F_{21} + (4) F_{22}$ omits the six consecutive residues: $(14)\, a_1a_1\ldots(a_1-5), \ldots, (14)\, a_1a_1\ldots a_1$; $\zeta^*_{26}, \zeta^{*!}_{25}, \ldots, \zeta^*_{21}$ say.

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2·3. $i = 3$. This case is slightly more difficult than the above two cases, and we must use the results of section 2·2 in this section.

Consider the two set valued mappings Ψ_{31}, Ψ_{32} defined on $\{1, 2, 3, 4\}$, $\{1', 2', 3', 4'\}$ respectively, as follows:

$$\left.\begin{aligned}\Psi_{31}(h) &= \{h, h-1\},\\ \Psi_{32}(h') &= \{1-h\}.\end{aligned}\right\} \quad (h = 1, 2, 3, 4).$$

It can be seen that the set

$$\{[\psi_{31}(h) - \psi_{32}(h')] \bmod 8 / \psi_{3i} \in \Psi_{3i}, (h, h') \in \{(1, 1), \ldots, (4, 4')\}\}$$

contains all the integers of $Z(8)$. However,

$$\left\{\sum_{j=1}^{3} [\psi_{31}(h_j) + \psi_{32}(h'_j)] \bmod 8 / (h_j, h'_j) \in \{(1, 1), \ldots, (4, 4')\}\right\}$$

does not contain the integers 4, 5, 6, 7.

We now consider an array $A = [A_{(i,j)}]$ comprising three doubly inflated complemented Rademacher arrays, each of order 4, placed end to end (so that there will be now $3 . \binom{4}{2} . 2^4 = 288$ columns); further, we augment A with a final column (column 289) of Y's. Define an integer sequence $\kappa_1, \kappa_2, \ldots, \kappa_{290}$ by

$$\begin{aligned}\kappa_r &= q_2(6) \quad (1 \leqslant r \leqslant 2 . \tbinom{4}{2} . 2^4 = 192),\\ \kappa_r &= q(5, 6) \quad (193 \leqslant r \leqslant 289),\\ \kappa_{290} &= 8,\end{aligned}$$

and write $q_3(6) = \kappa_1 \kappa_2 \ldots \kappa_{290}$ (product).

Now, using generalized decimal notation for ξ, define the following integer sets F_{31h}, $h = 1, 2, 3, 4$:

$$F_{31h} = \{\xi : \xi = \xi_{289} \xi_{288} \ldots \xi_r \ldots \xi_0, \xi_{289} \in \Psi_{31}(h)$$

and for

$$\begin{aligned}0 \leqslant r \leqslant 95, \quad & \xi_r = 0, \in Z(\kappa_r), \in F_{21} \quad \text{if} \quad A'_{h,r+1} = 0, X, Y \text{ resp.},\\ 96 \leqslant r \leqslant 191, \quad & \xi_r = 0, \in Z(\kappa_r), \in F_{22} \quad \text{if} \quad A'_{h,r+1} = 0, X, Y \text{ resp.},\\ 192 \leqslant r \leqslant 288, \quad & \xi_r = 0, \in, Z(\kappa_r), \in F(5, 6) \quad \text{if} \quad A'_{h,r+1} = 0, X, Y \text{ resp.}\}.\end{aligned}$$

Similarly, we define the sets $F_{32h'}, h' \in \{1', 2', 3', 4'\}$ by replacing $\Psi_{31}(h)$ by $\Psi_{32}(h')$ and interchanging F_{21} and F_{22} in the above definitions of F_{31h}.

Finally, we write $F_{31} = \bigcup_{h=1}^{4} F_{31h}$ and $F_{32} = \bigcup_{h=1}^{4} F_{32h'}$. It can be shown, as in section 2·1, that $F_{31} - F_{32} = F_{32} - F_{31}$ and that each difference contains $Z(q_3(6))$.

Now consider the set $(3) F_{31} + (3) F_{32}$.

Let

$$\begin{aligned}\zeta = \zeta_{289} \zeta_{288} \ldots \zeta_0 \quad & \text{be an arbitrary element of } (3)F_{31} + (3) F_{32}.\\ \eta^i = \eta^i_{289} \eta^i_{288} \ldots \eta^i_0 \quad & (i = 1, 2, 3) \text{ be arbitrary elements of } F_{31}.\\ \xi^i = \xi^i_{289} \xi^i_{288} \ldots \xi^i_0 \quad & (i = 1, 2, 3) \text{ be arbitrary elements of } F_{32}.\end{aligned}$$

Each of the above elements being expressed in general decimal.

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By writing

$$\left.\begin{aligned}\eta^i_{289} &= \psi_{31}(h) \in \Psi_{31}(h)\\ \xi^i_{289} &= \psi_{32}(h'_{j_i}) \in \Psi_{32}(h'_{j_i})\end{aligned}\right\} \quad (1 \leqslant h, h_{j_i} \leqslant 3),$$

the elements

$$\zeta = [\eta^1+\eta^2+\eta^3+\xi^1+\xi^2+\xi^3] \in [(3)F_{31}+(3)F_{32}]$$

may be characterized by one of the following two exhaustive cases:

(a) $(h, h'_{j_i}) \in \{(1,1'), \ldots, (4,4')\}$ for $i = 1, 2, 3$;

(b) $(h, h'_{j_i}) \notin \{(1,1'), \ldots, (4,4')\}$ for at least one value of i.

Consider case (a). This can be broken down into two further cases:

($a.1$) (h_1, h'_1), (h_2, h'_2), (h, h'_3) each $\epsilon \{(1,1'), \ldots (4,4')\}$;

($a.2$) $(h_1, h_1, h'_1) \in \{(1,1,1'), \ldots, (4,4,4')\}$

and $(h_2, h'_2, h'_2) \in \{(1,1',1'), \ldots, (4,4',4')\}$ $(h_1 \neq h_2)$.

In the case of ($a.1$), by choice of the mappings Ψ_{31} and Ψ_{32}, ζ_{289} never takes the value 7, for we can carry at most 3 from the preceding place (in the generalized decimal representation of ζ), by Lemma 1·4.

In the case of ($a.2$), by Lemma 1·3, there is a column in one of two of the component arrays, column $r+1$ say, which consists of $2Y$'s representing elements from F_{21} and $4Y$'s representing elements of F_{22} (in the four rows h_1, h'_1, h_2, h'_2). Hence ζ_r cannot take the value ζ^*_{21}, some integer value of r between (inclusively) 0 and 191.

In the case of (b), there is by Lemma 1·3, a column, column $(m+1)$ say, in the third component array (columns 193 to 288), which has zeros and at most $5Y$'s in the corresponding rows concerned. The Y's representing elements from $F(5,6)$. Hence ξ_m cannot take the value a_1, for some value of m, between (inclusively) 192 and 287.

Since the above cases, ($a.1$), ($a.2$) and (b) are exhaustive in describing

$$\zeta \in [(3)F_{31}+(3)F_{32}],$$

it follows that ζ can never take the six consecutive values:

$$7sa_1a_1\ldots a_1\zeta^*_{21}\ldots\zeta^*_{21}\zeta^*_{26}, \ldots, 7sa_1a_1\ldots a_1\zeta^*_{21}\ldots\zeta^*_{21}\zeta^*_{21}$$

$\zeta^*_{36}, \zeta^*_{35}, \ldots, \zeta^*_{31}$ say, where s can be any element of $Z(q(5,6))$.

3. *The construction of* $F(6)$. We use the three pairs of sets (F_{i1}, F_{i2}) $(i = 1, 2, 3)$ of section 2, to construct $F(6)$, such that $F(6)$–$(F(6)$ contains $Z(q(6))$, but $(6)F(6)$ has a gap of l consecutive residues mod $q(6)$. The integer $q(6)$ will appear in the construction and it will be seen that l can be chosen as large as we please.

The construction is relatively simple. Write F_{10} for $Z(l+1)$. Then in conjunction with the sequence $\kappa_1, \kappa_2, \ldots, \kappa_7$ where

$$\kappa_1 = \kappa_2 = (l+1), \quad \kappa_5 = \kappa_6 = q_2(6);$$
$$\kappa_3 = \kappa_4 = q_1(6), \quad \kappa_7 = q_3(6).$$

Write x in generalized decimal and define two sets:

$F_1(6) = \{x: x = \zeta_{32}\xi_{21}\xi_{22}\xi_{11}\xi_{12}\xi_{10}0$ where $\xi_{ir} \in F_{ir}$, $i = 1, 2, 3$ and $r = 0, 1, 2\}$,

$F'_1(6) = \{x: x = \xi_{31}\xi_{22}\xi_{21}\xi_{12}\xi_{11}0\,\xi_{10}$ where $\xi_{ir} \in F_{ir}$, $i = 1, 2, 3$ and $r = 0, 1, 2\}$.

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Then $F(6) = F_1(6)\, UF_1'(6)$ has the properties we seek:

(a) $F(6) - F(6)$ contains $Z(q(6))$ where $q(6) = \kappa_7\kappa_6 \ldots \kappa_1$. This is readily seen in view of the properties of (F_{i1}, F_{i2}), $i = 1, 2, 3$ and lemma 1·1 of (1).

(b) $(6)\,F(6)$ has a gap of l consecutive residues $\bmod q(6)$, namely (expressed in generalized decimal)

$$\zeta^*_{31}\zeta^*_{21}\zeta^*_{21}\zeta^*_{11}\zeta^*_{11}61, \ldots, \zeta^*_{31}\zeta^*_{21}\zeta^*_{21}\zeta^*_{11}\zeta^*_{11}6l,$$

(where ζ^*_{i1} is the last of the six consecutive residues known to be absent from

$$(i)\,F_{i1} + (6-i)\,F_{i2},$$

as shown in section 2).

To see this, note that the elements of $y \in (6)\,F(6)$ are formed by one of the following seven methods; $(i = 0, 1, 2, \ldots, 6)$, (a_i): y is the sum of $(6-i)$ elements of $F_1(6)$ plus the sum of i elements of $F_1'(6)$.

Method (a_0). If $\zeta_7\zeta_6 \ldots \zeta_0$ is the generalized decimal for arbitrary $y \in (6)\,F(6)$, then for the elements formed by this method (a_0), we see $\zeta_0 = 0$, i.e. $\zeta_0 \neq 1, 2, 3, \ldots, (l-1)$ or l.

(a_1). If $y \in (6)\,F(6)$ is formed by method (a_1), then $\zeta_1 \neq 6$; for when summing six digits $\bmod (l+1)$ from the first position of y, we can carry at most five to the second place of y, i.e. $0 \leqslant \zeta_1 \leqslant 5$.

(a_i) $2 \leqslant i \leqslant 6$. Similarly, it follows that if y is formed by method (a_i) then

$$\zeta_i = \zeta^*_{j_i 1} \quad (j_2 = j_3 = 1; j_4 = j_5 = 2; j_6 = 3).$$

Since every $y \in (6)\,F(6)$ is formed by one and only one of the above seven methods, it follows that $y \in (6)\,F(6)$ cannot have any of the l consecutive residues $\bmod q(6)$:

$$\zeta^*_{31}\zeta^*_{21}\zeta^*_{21}\zeta^*_{11}\zeta^*_{11}61, \ldots, \zeta^*_{31}\zeta^*_{21}\zeta^*_{21}\zeta^*_{11}\zeta^*_{11}6l.$$

or simply $b_1 - l + 1, b_1 - l, \ldots, b_1$ say.

4. This section is analogous to section 2. We construct pairs of integer sets

$$(F_{i1}, F_{i2}) \quad i = 1, 2, 3$$

such that

(a) $(F_{i1} - F_{i2}) = (F_{i2} - F_{i1})$ each difference containing $Z(q_i(7))$,

(b) $(i)\,F_{i1} + (7-i)\,F_{i2}$ omits at least seven consecutive residues

$$\bmod q_i(7): \quad \zeta^\dagger_{i7}, \ldots, \zeta^\dagger_{i1} \quad \text{say.}$$

These sets will be used in the following section, section 5, in a similar fashion to section 3, to construct $F(7)$.

4·1. $i = 1$. As in section 2·1, we begin by considering two set valued mappings Ψ_{11}, Ψ_{12} defined on $\{1, 2\}$, $\{1', 2'\}$ respectively, as follows:

$$\Psi_{11}(h) = \{6h, 6h-1, \ldots, 6h-6\};$$
$$\Psi_{12}(h') = \{-h\}.$$

It is readily seen that the set of integers

$$\{[\psi_{11}(h) - \psi_{12}(h')] \bmod 14 / \psi_{ij} \in \Psi_{ij}, (h, h') \in \{(1, 1'), (2, 2')\}\}$$

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contains all the integers $1, 2, 3, \ldots, 14$. However, the set

$$\{[\psi_{11}(h) + 6\psi_{12}(h')] \bmod 14/(h, h') \in \{(1, 1'), (2, 2')\}\}$$

does not contain the integers $1, 2, 3, \ldots, 7$.

Now take an inflated complemented Rademacher array $A = [A_{(i,j)}]$ of order $N = 2$ (hence having $P(2) = 8$ columns); then define a (finite) integer sequence $\kappa_1, \ldots, \kappa_9$ by

$$\kappa_r = q(6, 7) \quad (1 \leqslant r \leqslant 8)$$

$$\kappa_q = 14$$

and write

$$q_1(7) = \kappa_1 \kappa_2 \ldots \kappa_9 \quad \text{(product)}.$$

Next, using generalized decimal notation for ξ, define the following integer sets F_{11h} for $h = 1, 2$.

$F_{11h} = \{\xi : \xi = \xi_8 \xi_7 \ldots \xi_r \ldots \xi_0$ where $\xi_8 \in \Psi_{11}(h)$ and

$$\xi_r = 0, \in F(6, 7), \in Z(\kappa_r) \text{ if } A_{h,r+1} = 0, Y, X \text{ resp. } 0 \leqslant r \leqslant 7\}.$$

Similarly, we define the sets $F_{12h'}$, for $h' = 1', 2'$ by replacing $\Psi_{11}(h)$ by $\Psi_{12}(h')$ in the above definitions.

Write $F_{11} = F_{111} \cup F_{112}$ and $F_{12} = F_{121'} \cup F_{122'}$. It can be shown exactly as in section 2·1, that (F_{11}, F_{12}) have the desired properties (a) and (b) stated at the beginning of this section. The missing residues of $F_{11} + (6) F_{12}$ will be (in generalized decimal):

$$7b_1 b_1 \ldots b_1 (b_1 - 6), \ldots, 7b_1 b_1 \ldots b_1 b_1$$

or simply $\zeta^{\dagger}_{17}, \zeta^{\dagger}_{16} \ldots, \zeta^{\dagger}_{11}$ say; (b_1 described in section 3).

4·2. $i = 2$. Consider the two set valued mappings Ψ_{21}, Ψ_{22} defined on $\{1, 2, \ldots, 6\}$, $\{1', 2', \ldots, 6\}$ respectively as follows:

$$\Psi_{21}(h) = \{4h, 4h - 1, \ldots, 4h - 4\},$$

$$\Psi_{22}(h') = \{-h\}.$$

It can be seen that the set of integers

$$\{[\psi_{21}(h) - \psi_{22}(h')] \bmod 30/\psi_{21} \in \Psi_{2j}, (h, h') \in \{(1, 1'), \ldots, (6, 6)\}\}$$

contains all the integers of $Z(30)$. However

$\{[\psi_{21}(h_1) + \psi_{21}(h_2) + 3\psi_{22}(h_1') + \psi_{22}(h_2') + \psi_{22}(h_3')] \bmod 30/$
$(h_1, h_i') \in \{(1, 1'), \ldots, (6, 6')\}; h_3 = h_1$ or h_2; h_1 & h_2 not necessarily distinct$\}$,
does not contain the integers 19, 20, 21, 22, 23, 24.

We next proceed as in the above section for $i = 1$, making appropriate minor changes. We consider an inflated complemented Rademacher array of order $N = 5$, and augment this with a final column of Y's; then define an integer sequence $\kappa_1, \kappa_2, \ldots, \kappa_{P(5)+1}, \kappa_{P(5)+2}$ by:

$$\kappa_r = q(6, 7) \quad (1 \leqslant r \leqslant P(5) + 1)$$

$$\kappa_{P(5)+2} = 30$$

and write

$$q_2(7) = \kappa_1 \kappa_2 \ldots \kappa_{P(5)+2} \quad \text{(product)}.$$

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Next using generalized decimal for ξ, define F_{21h} and $F_{22h'}$ for $h = 1, 2, \ldots, 5$, by

$$F_{21h} = \{\xi : \xi = \xi_{P(5)+1} \ldots \xi_r \ldots \xi_0 \text{ where } \xi_{P(5)+1} \in \Psi'_{21}(h) \text{ and } \xi_r = 0, \in F(5,6), \in Z(\kappa_r) \text{ if } A_{h,r+1} = 0,\ Y,\ X \text{ resp. } 0 \leqslant r \leqslant P(5)\}$$

$$F_{22h'} = \{\xi : \xi = \xi_{P(5)+1} \ldots \xi_r \ldots \xi_0 \text{ where } \xi_{P(5)+1} \in \Psi'_{22}(h') \text{ and } \xi_r = 0, \in F(5,6), \in Z(\kappa_r) \text{ if } A'_{h',r+1} = 0,\ Y,\ X \text{ resp. } 0 \leqslant r \leqslant P(5)\}.$$

Write

$$F_{21} = \bigcup_{h=1}^{5} F_{21h} \quad \text{and} \quad F_{22} = \bigcup_{h=1}^{5} F_{22h'}.$$

It can be shown that $F_{21} - F_{22} = F_{22} - F_{21}$ and that each difference contains $Z(q_2(7))$. Also it can be shown, with an argument similar to that used for $i = 1$, that $(2)F_{21} + (5)F_{22}$ omits the seven consecutive residues:

$$(14)b_1b_1 \ldots b_1(b_1 - 6), \ldots, (14)b_1 \ldots b_1b_1, \quad \text{or} \quad \zeta^{\dagger}_{27}, \zeta^{\dagger}_{26}, \ldots, \zeta^{\dagger}_{21} \quad \text{say.}$$

4·3. $i = 3$. Analogous to section 2·3, this case is slightly more difficult than the above two cases for $i = 1, 2$. We require the results of section 4·2 in this section. Consider the two set valued mappings Ψ'_{31}, Ψ'_{32} defined on $\{1, 2, \ldots, 7\}$, $\{1', 2', \ldots, 7'\}$ respectively as follows:

$$\Psi'_{31}(h) = \{h, h-1\},$$
$$\Psi'_{32}(h') = \{(1-h)\}.$$

It can be seen that the set

$$\{[\psi_{31}(h) - \psi_{32}(h')] \bmod 14 / \psi_{3i} \in \Psi_{3i},\ (h, h') \in \{(1,1'), \ldots, (7,7')\}\}$$

contains all integers of $Z(14)$. However, the set

$$\{[\psi_{31}(h_1) + \psi_{31}(h_4) + \psi_{31}(h_3) + 2\psi_{32}(h'_1) + \psi_{32}(h'_2) + \psi_{32}(h'_3)] \bmod 14 / (h_i, h'_i) \in \{(1,1'), \ldots, (7,7')\};\ h_4 = h_1, h_2 \text{ or } h_3;$$

h_i not necessarily distinct$\}$ does not contain the integers 4, 5, 6, 7.

We now (as in section 2·3) consider an array $A = [A_{(i,j)}]$ comprising two doubly inflated complemented Rademacher arrays followed by a triply inflated array each of order 7, placed end to end (so that there will be now $(2\binom{7}{2} + \binom{7}{3})2^7 = P'(7)$ (say) columns); further, we augment A with a final column of Y's; define an integer sequence $\kappa_1, \kappa_2, \ldots, \kappa_{P'(7)+2}$ by

$$\kappa_r = q_2(7) \quad (1 \leqslant r \leqslant 2.\tbinom{7}{2}2^7),$$
$$\kappa_r = q(6,7) \quad (2.\tbinom{7}{2}2^7 + 1 \leqslant r \leqslant P'(7)),$$
$$\kappa_{P'(7)+1} = q(5,6)$$
$$\kappa_{P'(7)+2} = 14$$

and write $$q_3(7) = \kappa_1\kappa_2 \ldots \kappa_{P'(7)+2} \quad \text{(product)}.$$

Now using generalized decimals for ξ, define the following integer sets

$$F_{31h}, \quad h = 1\ 2, \ldots, 7: \quad F_{31h} = \{\xi : \xi = \xi_{P'(7)+1}\xi_{P'(7)} \ldots \xi_r \ldots \xi_0$$

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$\xi_{P'(7)+1} \in \Psi_{31}(h)$ and $\xi_{P'(7)} \in F(5,6)$ and for

$0 \leqslant r \leqslant \binom{7}{2} 2^7 - 1$, $\xi_r = 0, \in Z(\kappa_r), \in F_{21}$ if $A'_{h,r+1} = 0$, X, Y resp.

$\binom{7}{2} 2^7 \leqslant r \leqslant 2\binom{7}{2} 2^7 - 1$, $\xi_r = 0, \in Z(\kappa_r), \in F_{22}$ if $A'_{h,r+1} = 0$, X, Y resp.

$(2)\binom{7}{2} 2^7 \leqslant r \leqslant P'(7) - 1$, $\xi_r = 0, \in Z(\kappa_r), \in F(6,7)$ if $A'_{h,r+1} = 0$, X, Y resp.$\}$.

Similarly, we define the sets $F_{32h'}$, $h' \in \{1', 2', \ldots, 7'\}$ by replacing $\Psi_{31}(h)$ by $\Psi_{32}(h')$ and interchanging F_{21} and F'_{22} in the above definitions of F_{31h}.

Finally, we write $F_{31} = \bigcup_{h=1}^{7} F_{31h}$ and $F_{32} = \bigcup_{h=1}^{7} F_{32h'}$. It can be shown, as in section 2·1, that $F_{31} - F_{32} = F_{32} - F_{31}$ and that each difference contains $Z(q_3(7))$.

Now consider the set $(3)F_{31} + (4)F_{32}$.

We show, as in the last part of section 2·3 that this set has a gap of seven consecutive residues mod $q_3(7)$.

Let $\zeta = \zeta_{P'(7)+1}\zeta_{P'(7)} \ldots \zeta_0$ be arbitrary elements of $(3)F_{31} + (4)F_{32}$,

$\eta^i = \eta^i_{P'(7)+1}\eta^i_{P'(7)} \ldots \eta^i_0$ $(i = 1, 2, 3)$ be arbitrary elements of F_{31},

$\xi^i = \xi^i_{P'(7)+1}\xi^i_{P'(7)} \ldots \xi^i_0$ $(i = 1, 2, 3, 4)$ be arbitrary elements of F_{32}.

Each of the above elements is expressed in generalized decimal.

By writing

$$\eta^i_{P'(7)+1} = \psi_{31}(h) \in \Psi_{31}(h) \quad (1 \leqslant h, h_{j_i} \leqslant 7)$$

$$\xi^i_{P'(7)+1} = \psi_{32}(h'_{j_i}) \in \Psi_{32}(h'_{j_i})$$

the elements

$$\zeta = [\eta^1 + \eta^2 + \eta^3 + \xi^1 + \xi^2 + \xi^3 + \xi^4] \in [(3)F_{31} + (4)F_{32}]$$

may be characterized by one of the following two cases:

(a) $(h, h'_{j_i}) \in \{(1,1'), \ldots, (7,7')\}$ all i

(b) $(h, h'_{j_i}) \notin \{(1,1'), \ldots, (7,7')\}$ for at least one value of i.

Consider case (a). This can be broken down into two further cases:

$(a.1)$ (h_1, h'_1, h'_1) and (h_2, h'_2), (h_3, h'_3) belonging to $\{(1,1',1'), \ldots, (7,7',7')\}$ and $\{(1,1'), \ldots, (7,7')\}$ respectively; h_i's not necessarily distinct.

$(a.2)$ $(h_1, h_1, h'_1) \in \{(1,1,1'), \ldots, 7,7,7')\}$ $(h_1 \neq h_2)$ and

$$(h_2, h'_2, h'_2\ h'_2) \in \{(1\ 1'\ 1'\ 1') \ \ldots, (7,7',7',7')\}.$$

In the case of $(a.1)$ by choice of mappings Ψ_{31}, Ψ_{32}, $\zeta_{P'(7)+1}$ never takes the value 7, for we can carry at most 3 from the preceding place (in the general decimal representation of ζ).

In the case of $(a.2)$, by Lemma 1·3, there is a column in one of the two component arrays, column $r+1$ say $(0 \leqslant r \leqslant 2\binom{7}{2} 2^7 - 1)$, of which consists of $2Y$'s representing elements from F_{21} and $5Y$'s representing elements of F_{22} (in the four rows h_1, h'_1, h_2, h'_2). Hence ζ_r cannot take the value $\zeta^\dagger_{21}$ (recall that $\zeta^\dagger_{21}$ is the last of seven consecutive residues known to be absent from $(2)F_{21} + (5)F_{22}$, shown in section 4·2).

In the case (b), there is by Lemma 1·3, a column, column $(m+1)$ say, in the third (triply-inflated array) component array (cols. $2.\binom{7}{2} 2^7$ to $P'(7) - 1$), which has zeros,

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and at most $6Y$'s in the corresponding rows concerned. The Y's representing elements from $F(6, 7)$. Hence ξ_m cannot take the value b_1, for some value of m,

$$2\binom{7}{2}2^7 - 1 \leqslant m \leqslant P'(7) - 2.$$

Since the above cases, $(a.1)$, $(a.2)$ and (b) are exhaustive in describing

$$\zeta \in [(3)F_{31} + (4)F_{32}],$$

it follows that ζ can never take the seven consecutive values:

$$7sb_1b_1\ldots b_1\zeta^\dagger_{21}\zeta^\dagger_{21}\ldots\zeta^\dagger_{21}\zeta^\dagger_{27}, \ldots, 7sb_1\ldots b_1\zeta^\dagger_{21}\ldots\zeta^\dagger_{21}\zeta^\dagger_{21}$$

or $\zeta^\dagger_{37}, \zeta^\dagger_{36}, \ldots, \zeta^\dagger_{31}$ say; where s can be any element of $Z(q(5, 6))$, section 5.

5. The construction of $F(7)$.

Using the three pairs of sets $(F_{i1}, F_{i2}), i = 1, 2, 3$ found in section 4, the construction of $F(7)$ is relatively simple; it is almost identical to the construction of $F(6)$ in section 3. We give the outline only.

We write F_{10} for the complete set of residues mod $(l+1)$. Then, using generalized decimal for x, define the two sets:

$$F_1(7) = \{x : x = \xi_{31}\xi_{32}\xi_{21}\xi_{22}\xi_{11}\xi_{12}\xi_{10}0;\ \xi_{ij} \in F_{ij}\ 1 \leqslant i \leqslant 3\ \&\ 0 \leqslant j \leqslant 2\},$$

$$F_2(7) = \{x : x = \xi_{32}\xi_{31}\xi_{22}\xi_{21}\xi_{12}\xi_{11}0\xi_{10};\ \xi_{ij} \in F_{ij}\ 1 \leqslant i \leqslant 3\ \&\ 0 \leqslant j \leqslant 2\}.$$

Then $F(7) = F_1(7) \cup F_2(7)$ is the required set with properties:

(a) $F(7) - F(7)$ contains all the residues mod $q(7)$ where

$$q(7) = [q_3(7)\,q_2(7)\,q_1(7)\,(l+1)]^2$$

(b) $(7)F(7)$ omits the l consecutive residues mod $q(7)$, expressed in general decimal

$$\zeta^\dagger_{31}\zeta^\dagger_{31}\zeta^\dagger_{21}\zeta^\dagger_{21}\zeta^\dagger_{11}\zeta^\dagger_{11}71, \ldots, \zeta^\dagger_{31}\zeta^\dagger_{31}\zeta^\dagger_{21}\zeta^\dagger_{21}\zeta^\dagger_{11}\zeta^\dagger_{11}7l$$

($\zeta^\dagger_{i1}$ being the last of seven consecutive residues shown to be absent from

$$(i)F_{i1} + (7-i)F_{i2}$$

in section 4, $i = 1, 2, 3$).

The above methods for finding difference covers apparently terminate at this point with $k = 7$. Although no progress has been made for $k \geqslant 8$, it is hoped that some recursive method will emerge to establish for every positive integer k, difference covers $F(k)$ which are not k-sum covers. Such covers would have applications in measure theory and would confirm the suspected existence of nonsymmetric maximal Raikov systems (see section 5 of reference (4)).

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