Did the Greeks Discover the Irrationals?

PHILIP HUGLY AND CHARLES SAYWARD

A popular view is that the Greeks discovered that there were irrational numbers, e.g., the positive square root of two. The idea is that there is something in our geometry which *forces* us to acknowledge the square root of two. The 'forcing' is, of course, not some kind of physical compulsion. No one supposes it is. Rather the idea is that we *see* something which shows that there *must* be a number the square of which is two.

Bertrand Russell expresses this view:

> It was Pythagoras who discovered the existence of incommensurables, and, in particular, the incommensurability of the side of a square and the diagonal. If the length of the side is 1 inch, the number of inches in the diagonal is the square root of 2, which appeared not to be a number at all.[1]

Friedrich Waismann also expresses this view:

> ... that the rational numbers, even though sown infinitely dense, still do not cover the entire number axis, is the great discovery of Pythagoras. He first recognized that there are numbers which, though completely different from the rational numbers, are still related to them. We will illustrate this extraordinary discovery by constructing a square on an interval of length 1 and drawing a diagonal in this square. Our intuition tells us that this diagonal must have a very definite length. Let us try to compute it. According to the theorem of Pythagoras the square of the diagonal is equal to the sum of the squares of the two legs of the right triangle, therefore 2. Consequently the diagonal has a length $\sqrt{2}$.[2]

Against this we shall argue that geometry forces *nothing*, that nothing which one sees when one carries out geometrical demonstrations compels one to speak in terms of a number the square of which is two, and that this situation does not change when one brings what is seen from the mathematics of rational numbers.

[1] Russell, Bertrand, *Introduction to Mathematical Philosophy* (London: G. Allen & Unwin, 1919), 4.

[2] Waismann, Friedrich, *Introduction to Mathematical Thinking* (New–York: Harper Torchbooks, 1959), 5–6.

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I. What the Greeks Came to See

The Greeks did some geometry and mathematics. They found out some things about triangles and the rational solutions of certain equations. But none of their results logically entail the existence of irrational numbers.

The way the Greek discovery is sometimes put obscures this point. It is sometimes said that what they found out was that where a, b, and c are the lengths of the legs and hypotenuse of a right triangle, $a^2 + b^2 = c^2$. And from this it directly follows that if a=b=1, c is the square root of 2. And, of course, in some cases a=b=1. Clearly this reasoning occurs in Waismann's statement.

The reasoning is too quick. What the Greek geometricians demonstrated was that the area of the *square* on the hypotenuse was equal to the sum of the areas of the *square* on the sides. The equation did not come into it.

But did not they show that the area of the square on side a is a^2 and the area of the square on side b is b^2 and the area of the square on side c is c^2 and c^2 is the *sum* of those other two square numbers?

No. The letters which enter into the geometric proof stand for the *sides* of the triangle, not numbers. They are geometrical not arithmetical variables. And what gets shown is something *geometrical*—it concerns *squares* whose sides are related in point of length as are the sides of a right triangle. It is a proof about *figures*, not numbers. And so the notion of 'sum' which enters into the proof is not the arithmetical one.

It is the notion of 'sum' we employ when we draw a square and call it 'A' and then draw a diagonal and call the triangles thereby formed 'B' and 'C' and then say that the area of A is the *sum* of the areas of B and C. Arithmetic doesn't enter into it at all.

So what we are saying is that the demonstration is about geometric figures, whereas the proof that the formula lacks a solution in the rationals if a=b is about numbers, not figures.

We want to find out whether the Greeks discovered enough to *show* them that there were numbers in addition to the integers and fractions. So they didn't as yet have a conception of the continuum of real numbers. What they would have had available for connecting mathematics to geometry would have been the kind of counting by units which we shall now sketch.

One can compare lines as longer and shorter, and compare areas as greater or less, and that, so far, such comparisons are non-arithmetical. Also, we can and do combine lines with *counting* to yield a new type of comparison. This is measurement, and it brings in the arithmetical.

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Suppose A takes a cord, pulls it tight, and cuts a section. A then does the same, using the first cut section to cut a second section. A then lays these end to end and uses them to cut a third section, and says that this third cord is *twice* the length of the first cord. A could then introduce a unit term by saying that the first cord is one U long. Then the second cord, being equal in length to the first, is also one U long, and the third cord is *two* U long.

In this way we *count* in terms of *units*. That brings arithmetic to bear on lines.

Now suppose that four rods are lined up and prove equal in length, each being one U long. A then arranges the rods in the form of a square. This square outlines a certain area of ground. We might now introduce a unit term for that area by saying that it is one U-square. Suppose A next constructs a square using rods which are two U long. That square will contain four of U-squares. And so forth. In general, if A constructs a square and can mark off its sides into n equal portions each one U long, then the square will comprise n^2 U-squares.

In such ways as this we bring numbers into connection with figures as well as lines—and thereby form an amalgam of arithmetic and geometry.

Now let us consider the squares on the side of a right triangle, and—to keep things simple—the integers. *Suppose* that for each such triangle there is some unit U such that each side of the triangle is some integer multiple of U long. That is, suppose that there is a unit U and integers a, b and c such that the sides of the right triangle are respectively a U long and b U long and the hypotenuse is c U long. Then the areas of the squares on those legs will comprise a^2 U-squares and b^2 U-squares. Then the geometrical insight that the area of the square on the hypotenuse equals the sum of areas of the squares on the legs can be applied to yield the result that the square on the hypotenuse will comprise a^2+b^2 U-squares. But we have supposed also that the hypotenuse is c U long. If so, the square on the hypotenuse comprises c^2 U-squares. So, these integers a, b, and c are so related that $a^2+b^2=c^2$.

Consider next the case of a=b, that is, a right triangle with sides of equal length. It is easy to prove that if a=b, then there is *no* integer c such that $a^2+b^2=c^2$.

This refutes the claim that *in general* there will be a unit U and integers a, b, and c such that the legs and hypotenuse of a right triangle are respectively of lengths a U, b U and c U.

And this is what the Greeks came to see.

We can now say that given a unit U and integers a and b such that the sides are a U long and b U long, no integer measures the hypotenuse in terms of U.

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And since whatever can be expressed by a unit U and fractions n/i and m/j can also be expressed in terms of a unit U* and *integers* nj and mi, the argument holds for all rationals, not just the integers.

On this analysis the result is only that *no* rational number measures the hypotenuse in units in terms of which rational numbers measure the sides. It is just a *negative* result.

That cannot show us that there nevertheless *is* a number which measures the hypotenuse.

The central point can be put this way. No number measures anything on its own. Numbers measure only relative to units of measurement. And there certainly are units of measurement relative to which rational numbers measure the hypotenuse of a right triangle. *Any* line can be taken as the unit for a system of measurement, and *any* line can be divided into two lines of equal length which then can be taken as the unit for a system of measurement, and so forth.

So we certainly do not come to see that some lines are unmeasurable by rational numbers. Rather, what we come to see is that there is no unit of length relative to which *each* of the sides of a right triangle is measured by rational numbers.

That is the result.

II. Definite Lengths

Again reflecting on the diagonal of a square whose side is one unit in length, Waismann writes:

> ...there can be no doubt that the diagonal of the unit square has a very definite length. If we think of this length as laid on the number axis with one end at 0, we obtain a point which is the geometrical representative of $\sqrt{2}$.[3]

What is it for something to have 'a very definite length'? What would it be for it to have a length, but not a definite one? Are we here to think of a line which constantly changes in length? Clearly not. Perhaps, then, 'It has a definite length' comes to 'It has the length that it has' or, more simply, 'It has a length'. On either reading the assertion 'Every line has a definite length' is a truism. How does adding such an assertion to the results from geometry and mathematics relevant to the case form a set of propositions entailing 'There are irrational numbers'?

Waismann's claim perhaps comes to this. Suppose we have laid out a triangle, say with chalk on a blackboard. We carefully make two sides, each one inch long, and set them at right angles. We then

[3] Ibid., p.7

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connect them with a third line. That third chalk line is right there before our eyes. We *see* it. *It* has some one definite length. That length *is* the square root of two inches!

Yes, the line is there—right before our eyes. And, if you will, it has a definite length. But the question is whether it has a length *in inches.* What we see is *the line*. Do we *see* that the line has a length in inches? No—though the line is before our eyes, *that it has a length in inches* is not before our eyes.

We may *say* that it does, but nothing which we see *compels* that conclusion. So though one sees that it has some length, one does not see that it has a length in inches.

What would it be to see that, for that to be the case? Surely it would be to see that there is some number which measures its length *in inches*. And all we have seen *so far* is that no rational number measures its length in inches if there are rational numbers which measure the length of the sides *in inches*. Certainly, that it has a definite *length* is not enough to show that it has a definite length *in inches*.

III. Applied Geometry and Mathematics

So far we have argued that the geometry and mathematics of the situation are not enough to show that there is a positive square root of two. But would not various *practices* of measurement force the recognition of such irrational numbers as the square root of two? After all, there is more to mathematics and geometry than *pure* mathematics and geometry. There also are their applications.

We shall use a conversation to bring out the issues. In the following W is a person with the Russell/Waismann view. Our view is voiced by S.

W. Look, I must make my measurements in *some* way. If I make my measurement via a unit which measures the sides with rational numbers, then I *must* assign to the hypotenuse an irrational length if I stick with that unit. And if my measurement uses a unit in terms of which a rational number measures the hypotenuse, then I *must* assign to the sides an irrational length if I use that unit. So the irrational length cannot be avoided.

S. I think we've already dealt with this. What is true is that any way you choose leaves something *unmeasured* by any rational. But nothing forces you to speak in terms of measurement by another sort of number. You can simply work with an *approximate* measure. Nothing in our *practice* will fall to pieces if we do so.

Suppose A is cutting rods to one foot lengths in order to weld frames in the shape of isosceles right triangles. A asks B to cut rods

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for the hypotenuse. B applies to 2 an operation for extracting roots, carries it out three steps, and cuts the rods to the fractional length in feet thus specified. The rods will probably fit just fine.

If greater accuracy is needed the calculation can be carried out as many more steps as proves sufficient.

W. But won't it smooth matters to bring in irrationals such as the square root of 2?

S. Well, suppose that the rods for the sides are each one foot long. We can then say that the rod for the hypotenuse is to be *the square root of 2 feet long*. But we still can only approximate to the length of the hypotenuse for cutting purposes.

We could say: Since the sides are each one foot long, we need a rod 7/5 feet long.

Or we could say: Since the sides are each one foot long, we need a rod which is the square root of 2 feet long.

W. But it is clear that 'We need a rod 7/5 feet long' and 'We need a rod the square root of 2 feet long' say very different things.

S. We agree. The requests differ. But they play similar roles, and acting on either produces much the same result. In each case what we need is a rod *roughly* so and so many feet long.

W. But the hypotenuse has a specific length, not an approximate length. To give its *specific* length you need an irrational number.

S. But you can specify its length without bringing in another kind of number—you need only shift to another unit of measurement.

W. But look, given any unit of length U, we want to have a *definite* answer to the question 'How many Us long is this line?'
For the line has a *definite* length.

S. Well, it would be a definite answer to say that the line is 1·41 inches long.

W. But if the sides of the triangle are each one inch long, then the answer 'The hypotenuse is 1·41 inches long' might be disconfirmed, whereas the answer 'The hypotenuse is the square root of two inches long' would not be disconfirmed.

S. Yes, but then perhaps the answer 'The hypotenuse is 1·41421 inches long' will be confirmed.

W. What that comes to is this: We select an answer that will do for *practical purposes* and then say 'about'. But what we want is a *definite* answer which says how long the *line* is.

But without the irrationals we will sometimes be able to answer how many inches long the line is only with a statement of the form '*About* n/m inches long'. And since we are asking for *its* length, this is no answer at all.

S. We agree that you *want* an answer of a certain form. But what shows you that there *is* such an answer. The fact that the line has

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some definite length (it's just as long as it is!) doesn't show that there is a non-approximative answer in terms of inches.

IV. An Analogy

Mathematics and geometry, together with their applications, do not show us that there are irrational numbers, or compel our assent to that proposition. This does not mean that it is unnatural to speak in terms of a square root of two given what we *can* see. We suppose it is natural enough to do so. And some people might want to so speak because that would have further consequences they also want.

Here is an analogy.

It involves *no error whatsoever* to accept certain set theoretical axioms but *not* the 'power set' axiom which says that for each set S there is a set S´ the members of which are exactly the subsets of S. A set theory less this sort of axiom would be 'topped off' with sets which are one-one with the set of integers.

Of course, anyone who has felt the charm of working with higher level sets will *want* the power set axiom.

People say: But if we do not accept that axiom then there are enormously many results we cannot obtain! That of course is correct. Just as it is correct that if we remove the Queen from the game of Chess then there are enormously many games which can be played in standard Chess which we now cannot play. But the game lacking the Queen is not in any way *in error*. This new game can be said to lack something, but only in the way that *every* game lacks something. The new game is not *incomplete* in any sense in which standard Chess would be that game in a more *complete* form.

To this people will reply that games are only *play* whereas set theory is serious, it aims at *truth*. And in dropping the power set axiom we diminish our power to determine the truth about sets.

Perhaps, but there is nothing which we see *within* set theory less the power set axiom which *shows* us that for every set there is a set of all its subsets. The set theory without that axiom is not incomplete in any way which can be discerned from within.

To this it will be said that this *of course* is the case, that to see how set theory without that axiom is incomplete we must stand *outside* that theory and *independently* see that there is a power set for each set there is.

Perhaps, but we certainly would want to be *brought* to that position from which such a thing can be seen. If we put ourselves in the place of people who have set theory without that or any equivalent axiom and then ask ourselves whether there is anything we would

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then see which would show us that there must be, for each set, the set of all its subsets, we would answer that we see *nothing* which shows that.

The case with positive square root of two may seem different. One might agree that the usual axioms for geometry and arithmetic do not entail that two has a square root, but nonetheless insist that when one works through the geometry one sees something which shows us that *there is* that number. That is an illusion.

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