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Effective high-temperature estimates for intermittent maps

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Abstract. Using quantitative perturbation theory for linear operators, we prove a spectral gap for transfer operators of various families of intermittent maps with almost constant potentials ('high-temperature' regime). Hölder and bounded p-variation potentials are treated, in each case under a suitable assumption on the map, but the method should apply more generally. It is notably proved that for any Pommeau–Manneville map, any potential with Lipschitz constant less than 0.0014 has a transfer operator acting on $\mathrm{Lip}([0, 1])$ with a spectral gap; and that for any two-to-one unimodal map, any potential with total variation less than 0.0069 has a transfer operator acting on $\mathrm{BV}([0, 1])$ with a spectral gap. We also prove under quite general hypotheses that the classical definition of spectral gap coincides with the formally stronger one used in Giulietti *et al* [The calculus of thermodynamical formalism. *J. Eur. Math. Soc.*, to appear. *Preprint*, 2015, arXiv:1508.01297], allowing all results there to be applied under the high-temperature bounds proved here: analyticity of pressure and equilibrium states, central limit theorem, etc.

1. *Introduction*

The thermodynamical formalism, which provides a deep understanding of invariant measures of some topological dynamical systems, is by now quite well understood in the uniformly hyperbolic setting with sufficiently regular potentials. However, the non-uniformly hyperbolic setting presents many challenges and is under a lot of scrutiny. Here we shall restrict ourselves to 'intermittent' maps, which are expanding in certain zones but have a neutral fixed point or even a contracting behavior in other regions.

For some intermittent maps such as the Pommeau–Manneville family, on the one hand the absolutely continuous measure only exhibits polynomial decay of correlation [**Hu04**, **Sar02**], but on the other hand Ruelle–Perron–Frobenius† measures of potentials

† The term *Gibbs measure* is often used, including in some of my previous works, but conflicts with many related but subtly different concepts.

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with sufficiently small Hölder or C^k norm ('high-temperature regime') have exponential decay of correlation, following from a spectral gap for their transfer operator [**CV13**].

In view of this diversity of behavior, it is desirable to better understand where the frontier lies between these two regimes, polynomial versus exponential decay of correlations. As a small step in this direction, we shall prove completely explicit lower bounds on the size of the high-temperature regime. These bounds are certainly far from being sharp for most single maps we consider, but they are uniform over rather large families of maps.

The idea is simply to use spectral theory of perturbed linear operators: one only has to prove a spectral gap for a *constant* potential, which is usually easy, and then conclude by stability of the spectral gap property. The important additional point is that we use here an *effective* perturbation theory, leading to effective bounds. The present approach also has the advantage of simplicity (see the proofs of Theorems 3.3 and 4.2).

1.1. *Transfer operators.* Recall that given a finite-to-one dynamical system $T : \Omega \to \Omega$, where Ω is a metric space, and a potential $\varphi : \Omega \to \mathbb{R}$ in a suitable Banach algebra of functions $\mathcal{X}(\Omega)$, one defines a *transfer operator* $\mathrm{L}_{T,\varphi}$ which is a bounded linear operator from $\mathcal{X}(\Omega)$ to itself, mapping f to the sum (or average) of $e^{\varphi} f$ along inverse images under T. When T is k-to-one for some $k \in \mathbb{N}$ we will take the 'average' point of view and set

$$\mathrm{L}_{T,\varphi} f(x) = \frac{1}{k} \sum_{y \in T^{-1}(x)} e^{\varphi(y)} f(y).$$

More generally, we will consider the case when there is some probability transition kernel $\mathsf{M} = (m_x)_{x \in \Omega}$ (i.e. for all $x \in \Omega$, m_x is a probability measure) such that m_x is concentrated on $T^{-1}(x)$ (sometimes up to some 'negligible' set; see Remark 1.1 below) and set

$$\mathrm{L}_{\mathsf{M},\varphi} f(x) = \int_{\Omega} e^{\varphi(y)} f(y)\, dm_x(y).$$

The above k-to-one case is included in this framework by setting $m_x = (1/k) \sum_{y \in T^{-1}(x)} \delta_y$. The subscripts M and T will most often be kept implicit or interchanged depending on the focus we want to choose, and it should be clear from the context what the implicit object is.

The dual L_{φ}^* of L_{φ} preserves the set of finite positive measures, and under suitable assumptions L_{φ} can be shown to have a maximal eigenvalue λ_{φ}, a unique positive eigenfunction h_{φ}, and L_{φ}^* to have a unique positive eigenprobability ν_{φ}. They can then be used to construct a T-invariant positive measure $d\mu_{\varphi} = h_{\varphi}\, d\nu_{\varphi}$ (the normalization of h_{φ} being taken to ensure μ_{φ} is a probability) which we shall call the *Ruelle–Perron–Frobenius* (RPF) measure of the potential φ.

The transfer operator is thus an important tool in the study of invariant measures, and its spectral properties are crucial. In particular, if $\mathrm{L}_{T,\varphi}$ has a spectral gap below λ_{φ} one easily obtains an exponential decay of correlations,

$$\left| \int f \circ T^n \cdot g\, d\mu_{\varphi} - \left(\int f\, d\mu_{\varphi} \right) \left(\int g\, d\mu_{\varphi} \right) \right| = O(e^{-an}) \quad \text{for all } f, g \in \mathcal{X}(\Omega),$$

which means that if X is a random point drawn according to μ_{φ}, for any sufficiently regular f, g the number $g(X)$ is 'almost independent' of $f(T^n(X))$ when n is large. Properties of

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this kind have countless applications in the study of ergodic properties of T, and it is thus desirable to know when a spectral gap occurs.

Remark 1.1. The invariance of μ_φ is proved as follows. For all $f \in \mathcal{X}(\Omega)$, where $\mathcal{X}(\Omega)$ is assumed to contain enough functions to approximate uniformly all continuous functions,

$$\begin{aligned}\int f\, dT_*(\mu_\varphi) &= \int f \circ T \cdot h_\varphi \, d\nu_\varphi \\ &= \int f \circ T \cdot h_\varphi \, d\left(\frac{1}{\lambda_\varphi} \mathrm{L}_\varphi^* \nu_\varphi\right) \\ &= \frac{1}{\lambda_\varphi} \int \mathrm{L}_\varphi(f \circ T \cdot h_\varphi)\, d\nu_\varphi, \end{aligned} \tag{1}$$

where

$$\mathrm{L}_\varphi(f \circ T \cdot h_\varphi)(x) = \int e^{\varphi(y)} f(T(y)) h_\varphi(y)\, dm_x(y) = f(x) \int e^{\varphi(y)} h_\varphi(y)\, dm_x(y),$$

the last equality using the fact that m_x is concentrated on $T^{-1}(x)$. But to plug this into (1), *we only need this for ν_φ-almost all x*. Then we get

$$\int f\, dT_*(\mu_\varphi) = \frac{1}{\lambda_\varphi} \int f \mathrm{L}_\varphi(h_\varphi)\, d\nu_\varphi = \int f\, d\mu_\varphi.$$

The relevance of this relaxation comes from examples such as the Pommeau–Manneville family $(T_q)_{q>0}$ below, which are not, strictly speaking, k-to-one; see Example 3.2.

1.2. *Intermittent maps.* It is known in broad generality that if T is uniformly expanding (or hyperbolic) and φ is Hölder, then there is a spectral gap for the transfer operator acting on Hölder functions. Investigations have thus turned toward non-uniformly expanding maps (or less regular potentials). One particular class of such maps often plays the role of test case: the Pommeau–Manneville family

$$\begin{aligned} T_q : [0, 1] &\to [0, 1] \\ x &\mapsto \begin{cases} x(1 + (2x)^q) & \text{if } x \in [0, \tfrac{1}{2}), \\ 2x - 1 & \text{if } x \in [\tfrac{1}{2}, 1], \end{cases} \end{aligned}$$

where $q \in (0, +\infty)$ is a parameter quantifying the tangency to identity at the neutral point 0†. This type of maps is sometime called 'intermittent', since the dynamics is close to the dynamics of a uniformly expanding map until the orbit gets close to the neutral fixed point: then the orbit takes a long time to go away from the fixed point.

Let us state a specialized version of our first main result (we obtain below similar bounds for all α-Hölder potentials, and for more general maps, possibly acting on higher-dimensional spaces).

THEOREM 1.2. *For any $q > 0$ and any potential $\varphi \in \mathrm{Lip}(\Omega)$ such that $\mathrm{Lip}(\varphi) \leq 0.0014$, the transfer operator $\mathrm{L}_{T_q,\varphi}$ has a spectral gap when acting on* $\mathrm{Lip}(\Omega)$.

† Variants acting on the circle, which can then be made C^1 with derivative of the form $1 + O(x^q)$ at the fixed point 0, might be preferred by some readers, and this makes no difference in our result.

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The independence of q in the case of the Pommeau–Manneville family is already known (see the dependencies of ε_ϕ in [**CV13**]), and comes from the fact that these maps exhibit a backward *average* contraction rate uniformly bounded by the presence of a strictly contracting branch. Note that for any value of the Lipschitz constant less than 0.0014 the method provides an explicit lower bound for the size of the spectral gap.

Another important class of maps is the class of unimodal interval maps. We shall consider only maps with an (almost) constant number of inverse images, and thus restrict to unimodal maps which are two-to-one (counted with multiplicity). For this class, the bounded (p-)variation class of regularity will prove extremely efficient. There, the derivative plays little role, since bounded p-variation is 'insensitive to stretching'. We instead rely on its 'extensiveness' (total variation is no more than the sum of variations over tiles of a tiling of the phase space) to deduce a general result from which we extract this simply stated version (we use $\mathrm{BV}([a,b])$ to denote the space of the bounded variation function defined on $[a,b]$, and $\mathrm{BV}(\varphi)$ to denote the total variation of such a function φ).

THEOREM 1.3. *Let $T:[a,b]\to[a,b]$ be a continuous interval map which is increasing from a subinterval $[a,c]$ onto $[a,b]$ and decreasing from $[c,b]$ onto $[a,b]$. For all potentials $\varphi\in\mathrm{BV}([a,b])$ such that*

$$\mathrm{BV}(\varphi)\le 0.0069,$$

the transfer operator $\mathrm{L}_{T,\varphi}$ has a spectral gap when acting on the space of bounded variation functions.

The main appeal of this result is that there is no contraction assumption. As before, we get a more general result, valid for p-BV potentials, which include in particular $1/p$-Hölder functions. It follows that we get a spectral gap for Hölder potentials without any contraction assumption (and on a larger ball). The catch is that the spectral gap is in the p-BV norm; if φ, f are $1/p$-Hölder with $\mu_\varphi(f)=0$ and $\|\varphi\|_{\mathrm{BV}_p}$ small enough, then $\|\mathrm{L}_\varphi^n(f)\|_{\mathrm{BV}_p}$ decays to zero exponentially fast (in particular, the same holds for its uniform norm), but its Hölder norm could be arbitrarily large. However this is a rather small price to pay since many applications of spectral gap (e.g. exponential decay of correlations) do not rely so much on the norm, but more on the nature of the potentials one is allowed to consider.

To conclude this article, in §5 we prove that for transfer operators, having a spectral gap in the sense used in [**Klo17b**] (which is a common definition which we also use here) is equivalent to the seemingly stronger condition used in [**GKLMF15**] (see §2.4). In particular, under the assumptions of Theorems 1.2 or 1.3 we can apply the results of [**GKLMF15**]: there is an analytic dependency of pressure and μ_φ on φ, the RPF measures in the high-temperature regime (in particular, the maximal entropy measure) satisfy a central limit theorem, etc.

As further applications of spectral gap, let us mention limit theorems: in the setting of Theorems 1.2 and 1.3, if X_0 is a random variable with law μ_φ, $X_{k+1}:=T(X_k)$ and ψ is an observable in the corresponding functional space (α-Hölder or p-BV), then the random process with 'hidden deterministic dependency' $(\psi(X_k))_{k\ge0}$ satisfies a law of large numbers, quantified by concentration inequalities, and the central limit theorem,

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quantified by Berry–Esséen bounds. The proofs can be found, for example, in [**HH01**], or in [**Klo17a**] in a completely effective version (i.e. explicit non-asymptotic constants are obtained).

2. *Banach spaces and Wasserstein metric*

In this section we give the functional analytic setup in which we will work. The phase space Ω will always be a metric space, and its metric will be denoted by $d(\cdot\,,\cdot)$.

2.1. *Hölder norms.*

The Hölder classes are among the most usual regularity classes considered for potentials. For any $\alpha \in (0, 1]$, one defines the α-Hölder constant of a function $f : \Omega \to \mathbb{R}$ as the number

$$\mathrm{Hol}_\alpha(f) := \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}.$$

A function is α-Hölder if its α-Hölder constant is finite, and the set of such functions forms a Banach space when endowed with the usual Hölder norm. We prefer to assume Ω to be bounded and use the slightly modified norm

$$\|f\|_{\mathrm{Hol}_\alpha} := \|f\|_\infty + (\mathrm{diam}\,\Omega)^\alpha\, \mathrm{Hol}_\alpha(f)$$

which, thanks to the $(\mathrm{diam}\,\Omega)^\alpha$ factor, is more homogeneous: the two terms have the same physical dimension (they are to be expressed in the same physical unit as f), and the norm is invariant if one rescales the distance by a constant. This homogeneity will be slightly more natural and effective in the computations later on.

We denote by $\mathrm{Hol}_\alpha(\Omega)$ the space of α-Hölder functions, which is in fact a Banach algebra, i.e. $\|fg\|_{\mathrm{Hol}_\alpha} \leq \|f\|_{\mathrm{Hol}_\alpha}\|g\|_{\mathrm{Hol}_\alpha}$. Indeed, for all $f, g \in \mathrm{Hol}_\alpha(\Omega)$, we have $\|fg\|_\infty \leq \|f\|_\infty\|g\|_\infty$, and for all $x, y \in \Omega$,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &\leq \|f\|_\infty\, \mathrm{Hol}_\alpha(g) d(x, y)^\alpha + \mathrm{Hol}_\alpha(f) d(x, y)^\alpha \|g\|_\infty, \\ \mathrm{Hol}_\alpha(fg) &\leq \|f\|_\infty\, \mathrm{Hol}_\alpha(g) + \mathrm{Hol}_\alpha(f)\|g\|_\infty. \end{aligned}$$

2.2. *Wasserstein metric.*

The 1-Wasserstein metric is defined on the set $\mathcal{P}(\Omega)$ of Borel probability measures on Ω by

$$W_1(\mu, \nu) = \inf_{\pi \in \Gamma(\mu,\nu)} \int_{\Omega\times\Omega} d(x, y)\, d\pi(x, y)$$

where $\Gamma(\mu, \nu)$ is the set of measures on $\Omega \times \Omega$ whose marginals are μ and ν. Elements of $\Gamma(\mu, \nu)$ are called *transport plans* or *couplings* from μ to ν. For any $\alpha \in (0, 1]$, $d(\cdot\,,\cdot)^\alpha$ is also a metric and the corresponding Wasserstein metric is denoted by W_α.

We will only use a few basic properties. W_α is indeed a metric; the infimum in its definition is always attained by some transport plan, then called optimal and generally not unique. The topology induced by W_α is the weak-$*$ topology as soon as Ω is compact. The Kantorovich duality enables a reformulation of W_α as

$$W_\alpha(\mu, \nu) = \sup_f \left| \int f\, d\mu - \int f\, d\nu \right|$$

where the supremum is on all functions $f : \Omega \to \mathbb{R}$ such that $\mathrm{Hol}_\alpha(f) = 1$.

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General references on transport theory and the Wasserstein distance are [**Vil03**] and [**Gig11**].

2.3. *Bounded p-variation norms.* When Ω is an interval or the circle, we can consider functions of bounded variation or generalizations with great efficiency. For the sake of notational simplicity we shall only consider intervals here, but the circle case is handled very similarly. Higher-dimensional analogues also exist, but are less elementary and have different properties (in particular, they are no longer 'insensitive to stretching').

Assume $\Omega \subset \mathbb{R}$ is an interval, fix $p \in [1, \infty)$ and let $f : \Omega \to \mathbb{R}$. An increasing sequence $x_0 < x_1 < \cdots < x_n$ of elements of Ω is called a *partition* and denoted by $\underline{x}$. The p-variation of f along the partition $\underline{x}$ is defined by

$$v_p(f, \underline{x}) := \left(\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|^p \right)^{1/p},$$

and the total p-variation of f is defined by

$$\mathrm{BV}_p(f) := \sup_{\underline{x}} v_p(f, \underline{x})$$

where the supremum is taken over all partitions of Ω. If this number is finite, f is said to be of *bounded p-variation*.

When $p = 1$, this is the usual total variation, but $p > 1$ is interesting because it allows for much more irregular potentials. Many α-Hölder functions ($\alpha < 1$) are not of bounded variation, but when Ω is bounded all α-Hölder functions are of bounded p-variation for all $p \geq 1/\alpha$. Indeed for $p\alpha = 1$ we have

$$\begin{aligned} v_p(f, \underline{x}) &= \left(\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|^p \right)^{1/p} \\ &\leq \mathrm{Hol}_\alpha(f) \left(\sum_{j=1}^{n} |x_j - x_{j-1}|^{\alpha p} \right)^{1/p} \\ &\leq \mathrm{Hol}_\alpha(f) \left(\sum_{j=1}^{n} x_j - x_{j-1} \right)^{1/p} \\ &\leq \mathrm{Hol}_\alpha(f) (\operatorname{diam} \Omega)^\alpha \end{aligned}$$

for all partitions $\underline{x}$. For larger p, simply use the fact that when Ω is bounded, α-Hölder functions are also β-Hölder for all $\beta < \alpha$.

We endow the space $\mathrm{BV}_p(\Omega)$ of functions of bounded p-variation with the norm

$$\|f\|_{\mathrm{BV}_p} := \|f\|_\infty + \mathrm{BV}_p(f).$$

It is more usual to replace the supremum norm in the above definition by an integral norm, but:

- we do not want to give such a special role to Lebesgue measure in our context;
- the two definitions yield equivalent norms;
- the above choice of norm makes $\mathrm{BV}_p(\Omega)$ a Banach algebra (the proof is pretty much the same as for the Hölder norm).

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Two important interlaced differences between the bounded variation and Hölder classes of regularity are that:

- Hölder is *intensive* while BV is *extensive*, i.e. if one partitions Ω into two intervals I_1, I_2, then the sum of the total variation of f on I_1 and I_2 is not greater than the total variation of f over Ω (it can be less if f has a jump precisely at the interface between the two intervals);
- the Hölder constant is diminished when one stretches the space, i.e. if a map Y is θ-contracting, then $\mathrm{Hol}_\alpha(f \circ Y) \le \theta^\alpha \mathrm{Hol}_\alpha(f)$, while BV is insensitive to stretching.

This is to be kept in mind to understand the difference in the assumptions of our main results in these two cases.

2.4. *Positive eigendata and spectral gap.* Let us now make precise the notion of spectral gap, which appears under slightly different definitions in the literature. We consider a bounded linear operator L acting on a Banach space $\mathcal{X}$. The operator norm is denoted, like the norm on $\mathcal{X}$, by $\|\cdot\|$.

The first two definitions of spectral gap will make sense for a general Banach space $\mathcal{X}$, but the third will need $\mathcal{X}$ to be a space of functions, which we therefore assume from now on. This immediately provides us with an additional structure: the cone of positive functions.

Definition 2.1. We say that an operator L *has positive eigendata* if it has a positive eigenvalue λ and a positive eigenfunction $h \in \mathcal{X}$, bounded away from 0, for this eigenvalue.

Let us state three notions of spectral gap; the first two are classical and easily seen to be equivalent, and the third is used in [**GKLMF15**] and, under some general hypotheses, will be proved to be equivalent to the other two in §5. We assume that L has an eigenvalue $\lambda > 0$ with eigenvector u_0; the spectral gap is to be understood with respect to λ.

(SG1) There is a complement G to $\langle u_0 \rangle$ in $\mathcal{X}$ (i.e. G is a closed subspace and $\langle u_0 \rangle \oplus G = \mathcal{X}$) which is stable under the action of L, and there exist numbers $C > 0$, $\delta \in (0, 1)$ such that

$$\|\mathrm{L}^n_{|G}\| \le C\lambda^n(1-\delta)^n.$$

(SG2) There exist two continuous linear operators $\mathrm{P}, \mathrm{R} : \mathcal{X} \to \mathcal{X}$ such that $\mathrm{L} = \lambda\mathrm{P} + \mathrm{R}$, P is a rank-one projection ($\dim \mathrm{Im}\,\mathrm{P} = 1$ and $\mathrm{P}^2 = \mathrm{P}$), $\mathrm{PR} = \mathrm{RP} = 0$, and there exist numbers $C > 0$ and $\delta \in (0, 1)$ such that:

$$\|\mathrm{R}^n\| \le C\lambda^n(1-\delta)^n.$$

The spectral gap assumption used in [**GKLMF15**] seems slightly more precise:

(SG3) The dual operator L^* has an eigenmeasure $\nu \in \mathcal{P}(\Omega)$ for the eigenvalue λ, in particular

$$\int \mathrm{L}(f)\, d\nu = \lambda \int f\, d\nu \quad \text{for all } f \in \mathcal{X},$$

and there exist positive constants $C > 0$, $\delta \in (0, 1)$ such that for all $n \in \mathbb{N}$ and all $f \in \mathcal{X}$ such that $\int f\, d\nu = 0$,

$$\|\mathrm{L}^n(f)\| \le C\lambda^n(1-\delta)^n \|f\|.$$

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In particular, (SG3) implies some identification between measures and linear forms on $\mathcal{X}$, and identifies the stable complement G.

In all three definitions, C is called the *constant* and δ the *size* of the spectral gap. In the main theorems below, we shall use the first definition; this only matters when we are interested in the value of the constant, which may vary from one definition to another, but the mere statement that there is a spectral gap will hold for all definitions, thanks to Theorem 5.1.

2.5. *Quantitative perturbation theory.* Let us now state the main perturbative tool we will use. We consider $\mathrm{L}_0 : \mathcal{X} \to \mathcal{X}$ a bounded linear operator of a Banach space $\mathcal{X}$, with an eigenvalue λ_0, eigenvector u_0 and eigenform ϕ_0 (i.e. ϕ_0 is an eigenvector of the dual operator L_0^* for the same eigenvalue). We assume a spectral gap, and consider the *condition number* $\tau_0 := \|\phi_0\|\|u_0\|/|\phi_0(u_0)|$. We denote by π_0 the projection on $\ker \phi_0$ in the direction $\langle u_0 \rangle$.

THEOREM 2.2. [**Klo17b**, Corollary 2.12] *If* $\lambda_0 = \|\mathrm{L}_0\| = 1$ *and* L_0 *has a spectral gap of size* δ_0 *and constant* 1 *(according to definition (SG1), then all* L *such that*

$$\|\mathrm{L} - \mathrm{L}_0\| \leq \frac{\delta_0(\delta_0 - \delta)}{6(1 + \delta_0 - \delta)\tau_0\|\pi_0\|}$$

have a spectral gap of size δ *below* λ_L*, with constant* 1*. In particular, all* L *such that*

$$\|\mathrm{L} - \mathrm{L}_0\| < \frac{\delta_0^2}{6(1 + \delta_0)\tau_0\|\pi_0\|}$$

have some spectral gap, with constant 1.

Results of this flavor are quite old (see, for example,[**Bau84**, **DS88**, **Kat95**]). However, it is hard to find such explicit radius bounds written for spectral gaps, and not only for having a simple isolated eigenvalue.

To apply this result, we will need to estimate the quantities δ_0, τ_0 and $\|\pi_0\|$. We will first use the following lemma of Doeblin–Fortet/Lasota–Yorke type, which also appears in [**Klo17a**] and which we reproduce with its proof for the sake of completeness.

Consider a normed space $\mathcal{X}(\Omega)$ of (Borel measurable, bounded) functions $\Omega \to \mathbb{R}$, with norm $\|\cdot\| = \|\cdot\|_\infty + V(\cdot)$, where V is a semi-norm.

LEMMA 2.3. *Assume that for some constant* $D > 0$*, for all probability* μ *on* Ω *and for all* $f \in \mathcal{X}$ *such that* $\mu(f) = 0$*,* $\|f\|_\infty \leq DV(f)$*.*

Let $\mathrm{L}_0 \in \mathcal{B}(\mathcal{X}(\Omega))$ *such that for some* $\theta \in (0, 1)$ *and all* $f \in \mathcal{X}$*,*

$$\|\mathrm{L}_0 f\|_\infty \leq \|f\|_\infty \quad \textit{and} \quad V(\mathrm{L}_0 f) \leq \theta V(f)$$

and having eigenvalue 1 *with an eigenprobability* μ_0*, i.e.* $\mathrm{L}_0^*\mu_0 = \mu_0$*.*

Then L_0 *has a spectral gap (for the eigenvalue* 1*, the contraction being on the stable space* $\ker \mu_0$*) with constant* 1*, of size*

$$\delta_0 = \frac{1 - \theta}{1 + D\theta}.$$

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Of course, the hypotheses of the lemma also ensure that $\|L_0\| \le 1$; and since we will only consider operators of the form

$$L_0 f(x) = \frac{1}{k} \sum_{j=1}^{k} f(b_j(x))$$

we will always have $L_0 \mathbf{1} = \mathbf{1}$ and thus $\lambda_0 = \|L_0\| = 1$.

Proof. Let $f \in \ker \mu_0$. Then $\|L_0 f\|_\infty \le \|f\|_\infty$ and $L_0 f \in \ker \mu_0$, so that $\|L_0 f\|_\infty \le DV(L_0 f) \le D\theta V(f)$.

Denote by $t \in [0, 1]$ the number such that $\|f\|_\infty = t\|f\|$ (and therefore $V(f) = (1-t)\|f\|$). The above two controls on $\|L_0(f)\|_\infty$ can then be written as $\|L_0(f)\|_\infty \le \min(t, D\theta(1-t))\|f\|$ and, using $V(L_0 f) \le \theta V(f)$ again, we get

$$\|L_0(f)\| \le \min(t + \theta(1-t), (D+1)\theta(1-t))\|f\|,$$
$$\|(L_0)_{|\ker \mu_0}\| \le \max_{t\in[0,1]} \min(t + \theta(1-t), (D+1)\theta(1-t)).$$

The maximum is reached when $t + \theta(1-t) = (D+1)\theta(1-t)$, i.e. when $t = D\theta/(1+D\theta)$, at which point the value in the minimum is $(D+1)\theta/(D\theta+1) \in (0, 1)$. Therefore there is a spectral gap with constant 1 and size $1 - (D+1)\theta/(D\theta+1)$, as claimed. □

LEMMA 2.4. *Assume again that for some constant $D > 0$, for all probability μ on Ω and for all $f \in \mathcal{X}$ such that $\mu(f) = 0$, $\|f\|_\infty \le DV(f)$.*

If the semi-norm V is invariant under translation by a constant (i.e. $V(f + c) = V(f)$ for all $f \in \mathcal{X}(\Omega)$ and all constant c), then all operators $\pi \in \mathcal{B}(\mathcal{X}(\Omega))$ of the form $\pi f = f - \mu(f)$ satisfy

$$\|\pi\| \le \frac{2D+2}{D+2}.$$

This lemma will apply below, as both Hol_α and BV_p are invariant by translation by a constant.

Proof. The proof proceeds like the previous one. We consider $f \in \mathcal{X}(\Omega)$ and let $t \in [0, 1]$ be such that $\|f\|_\infty = t\|f\|$ and $V(f) = (1-t)\|f\|$. On the one hand we have $V(\pi f) = V(f) = (1-t)\|f\|$, and on the other hand we have both

$$\|\pi f\|_\infty \le 2\|f\|_\infty \quad \text{and} \quad \|\pi f\|_\infty \le DV(f),$$

the first inequality being a simple triangle inequality, while the second follows from a hypothesis since $\mu(\pi f) = 0$. Then we get $\|\pi f\| \le \min(1 + t, (D+1)(1-t))\|f\|$ and therefore

$$\|\pi\| \le \max_{t\in[0,1]} \min(1 + t, (D+1)(1-t)).$$

The maximum is reached at $t = D/(D+2)$, from which the result follows. □

3. *Hölder potentials and maps that are backward contracting on average*

Let (Ω, d) be a metric space which we assume to be of finite diameter.

Definition 3.1. We say that a map $T : \Omega \to \Omega$ is of *class* $H(\alpha, \theta)$ where $\alpha \in (0, 1]$ and $\theta \in (0, 1)$ if there exist an integer $k \ge 2$ and 'inverse branches' $(b_j)_{1\le j\le k}$ such that:

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(i) each b_j is a (Borel) measurable map $\Omega \to \Omega$;
(ii) for all x except possibly countably many, $T^{-1}(x) = \{b_1(x), b_2(x), \dots, b_k(x)\}$;
(iii) T is *backward θ-contracting on average* with respect to the metric d^α, i.e.

$$\text{for all } y, z \in \Omega, \text{ there exists } \sigma \in \mathfrak{S}_k, \text{ such that}$$
$$\frac{1}{k}\sum_{j=1}^{k} d(b_j(y), b_{\sigma(j)}(z))^\alpha \le \theta d(y, z)^\alpha.$$

If T is of class $H(\alpha, \theta)$, for each α-Hölder potential we define a transfer operator by

$$\mathrm{L}_\varphi f(x) = \frac{1}{k}\sum_{j=1}^{k} e^{\varphi(b_j(x))} f(b_j(x))$$

i.e. we consider the backward random walk $\mathsf{M} = ((1/k)\sum_j \delta_{b_j(x)})_x$.

The permutation σ is meant to account for the fact that one may not be able to define inverse branches globally in a continuous fashion (hence the choice to define them globally but only as measurable maps). The second condition is not strictly necessary for our main Theorem to hold, but it ensures that as soon as the RPF measures are atomless, they are indeed T-invariant (see Remark 1.1).

Example 3.2. Let us consider three variations on the doubling map that only differ in the details that led us to the definition above:

$$D_c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \quad x \mapsto 2x \mod 1,$$
$$D_t : [0,1] \to [0,1] \quad x \mapsto \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2-2x & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$
$$D_i : [0,1] \to [0,1] \quad x \mapsto \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}), \\ 2x-1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

For each of them, let us define inverse branches showing that these three maps are of class $H(1, \frac{1}{2})$ (and thus $H(\alpha, 1/2^\alpha)$ for all $\alpha \in (0, 1]$). For $x \in \mathbb{R}/\mathbb{Z}$, denote by $\{x\}$ its representative in $[0, 1)$. Then we take:

- for D_c, $b_1 = x \mapsto \{x\}/2 \mod 1$ and $b_2 = x \mapsto \{x\}/2 + 1/2 \mod 1$. Both branches have one discontinuity point at 0, while $D_c^{-1}(x) = \{b_1(x), b_2(x)\}$ for all x.
- for D_t, $b_1 = x \mapsto x/2$ and $b_2 = x \mapsto 1 - x/2$. Then both branches are continuous and $D_t^{-1}(x) = \{b_1(x), b_2(x)\}$ for all x, but D_t is not two-to-one since $b_1(1) = b_2(1)$.
- for D_i, $b_1 = x \mapsto x/2$ and $b_2 = x \mapsto x/2 + 1/2$. Then both branches are continuous but $D_i^{-1} = \{b_1(x), b_2(x)\}$ only holds for $x \neq 1$, since $D_i(b_1(1)) = D_i(1/2) = 0$.

We now state our first main result.

THEOREM 3.3. *Let T be a map of class $H(\alpha, \theta)$ acting on a space Ω of finite diameter. For all α-Hölder potentials φ such that*

$$\mathrm{Hol}_\alpha(\varphi) \le \frac{2}{3(\operatorname{diam}\Omega)^\alpha} \log\left(1 + \frac{(1-\theta)^2}{16(1+\theta)}\right),$$

the transfer operator L_φ has a spectral gap (with constant 1).

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Proof. We consider the transfer operator L_0 associated to the null potential, i.e.

$$\mathrm{L}_0 f(x) = \frac{1}{k}\sum_{j=1}^{k} f(b_j(x)),$$

which under the above assumption acts continuously on $\mathrm{Hol}_\alpha(\Omega)$: indeed $\|\mathrm{L}_0 f\|_\infty \le \|f\|_\infty$ and, for all $x, y \in \Omega$ and some permutation σ depending on x and y, we have

$$\begin{aligned}|\mathrm{L}_0 f(x) - \mathrm{L}_0 f(y)| &\le \frac{1}{k}\sum_{j=1}^{k} |f(b_j(x)) - f(b_{\sigma(j)}(y))| \\ &\le \mathrm{Hol}_\alpha(f)\cdot\frac{1}{k}\sum_{j=1}^{k} d(b_j(x), b_{\sigma(j)}(y))^\alpha \\ &\le \theta\,\mathrm{Hol}_\alpha(f) d(x, y)^\alpha,\end{aligned}$$

so that $\mathrm{Hol}_\alpha(\mathrm{L}_0 f) \le \theta\,\mathrm{Hol}_\alpha(f)$. It follows that $\|\mathrm{L}_0\|_{\mathrm{Hol}_\alpha} \le 1$.

The same computation shows that the dual operator L_0^*, which acts on probability measures over Ω since $\mathrm{L}_0\mathbf{1} = \mathbf{1}$, is a θ-contraction in the metric W_α:

$$\begin{aligned}W_\alpha(\mathrm{L}_0^*\mu, \mathrm{L}_0^*\nu) &= \sup_{\mathrm{Hol}_\alpha(f)=1}\left|\int \mathrm{L}_0 f\, d\mu - \int \mathrm{L}_0 f\, d\nu\right| \\ &\le \sup_{\mathrm{Hol}_\alpha(f)=1} \mathrm{Hol}(\mathrm{L}_0 f) W_\alpha(\mu, \nu) \\ &\le \theta W_\alpha(\mu, \nu).\end{aligned}$$

Since the Wasserstein metric is complete, this ensures that there is exactly one probability measure which is invariant by L_0^*, which we denote by μ_0.

Let us show that we can use Lemma 2.3. If μ is any probability measure and $f \in \mathrm{Hol}_\alpha(\Omega)$ satisfies $\mu(f) = 0$, then, f being continuous, f must vanish at some point $x \in \Omega$. Then for all $y \in \Omega$ we have $f(y) \le (\mathrm{diam}\,\Omega)^\alpha\,\mathrm{Hol}_\alpha(f)$. It follows that we can apply Lemmas 2.3 and 2.4 with $D = 1$, so that L_0 has a spectral gap of size $(1-\theta)/(1+\theta)$ with constant 1 and $\|\pi_0\| \le \frac{4}{3}$.

We shall apply Theorem 2.2. First observe that $\mathrm{L}_\varphi = \mathrm{L}_0(e^\varphi \cdot)$ so that, using the Banach algebra property, we get

$$\begin{aligned}\|\mathrm{L}_\varphi - \mathrm{L}_0\|_{\mathrm{Hol}_\alpha} = \|\mathrm{L}_0((e^\varphi - \mathbf{1})\cdot)\|_{\mathrm{Hol}_\alpha} &\le \left\|\sum_{j=1}^{\infty}\frac{1}{j!}\varphi^j\right\|_{\mathrm{Hol}_\alpha} \le \sum_{j=1}^{\infty}\frac{1}{j!}\|\varphi\|_{\mathrm{Hol}_\alpha}^j \\ &\le e^{\|\varphi\|_{\mathrm{Hol}_\alpha}} - 1.\end{aligned}$$

Then, since $\mathbf{1}$ is the eigenfunction of L_0 and μ_0 is a probability measure, we have $\tau_0 = 1$. Also observe that adding a constant to φ simply multiplies L_φ by a scalar, not changing its spectral gap. Let $\varphi \in \mathrm{Hol}_\alpha(\Omega)$ be a potential such that, for some constant c,

$$\|\varphi - c\|_{\mathrm{Hol}_\alpha} \le \log\left(1 + \frac{((1-\theta)/(1+\theta))^2}{6\cdot 4/3(1+(1-\theta)/(1+\theta))}\right) = \log\left(1 + \frac{(1-\theta)^2}{16(1+\theta)}\right).$$

Theorem 2.2 then implies that the transfer operator $\mathrm{L}_\varphi = \mathrm{L}_0(e^\varphi\cdot)$ has a spectral gap with constant 1.

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To conclude, we only have to observe that, taking $c = (\sup\varphi + \inf\varphi)/2$, we must have $\|\varphi - c\|_\infty \le \frac{1}{2}(\operatorname{diam}\Omega)^\alpha \operatorname{Hol}_\alpha(\varphi)$ and thus $\|\varphi - c\|_{\operatorname{Hol}_\alpha} \le \frac{3}{2}(\operatorname{diam}\Omega)^\alpha \operatorname{Hol}_\alpha(\varphi)$. □

Remark 3.4. If we only want to know that there is some neighborhood of the line of constant potentials where the transfer operator has a spectral gap, without effective estimates, then we only need to prove a spectral gap for L_0 and dispense with the last half of the above proof.

Proof of Theorem 1.2. In the case of the Pommeau–Maneville family, we have $\operatorname{diam}\Omega = 1$ and a contraction $\theta = \frac{1}{2} + 1/2^{1+\alpha}$ in W_α (one of the branches is 1/2-contracting in the Euclidean metric, the other one is 1-contracting). For $\alpha = 1$, we obtain $\theta = 3/4$ and observe

$$\frac{2}{3}\log\left(1 + \frac{(1/4)^2}{16 \times (7/4)}\right) \ge 0.0014.$$ □

4. *Potentials with bounded p-variation*

For BV potentials and their relatives, we need assumptions that are much less geometric.

Definition 4.1. We say that a map T of a compact interval $\Omega \subset \mathbb{R}$ is of class V if there are k maps $b_1, \dots, b_k : \Omega \to \Omega$ such that:

(i) each b_j is monotonic from Ω to an interval $I_j \subset \Omega$;
(ii) for all x except possibly countably many, $T^{-1}(x) = \{b_1(x), b_2(x), \dots, b_k(x)\}$;
(iii) the I_j have disjoint interior.

As an example, any unimodular map whose restriction to both its monotonicity intervals is onto is of class V, irrespective of its derivative and of the behavior of its post-critical orbit.

Here again we use the definition of the transfer operators using the inverse branches:

$$\mathrm{L}_\varphi f(x) = \frac{1}{k}\sum_{j=1}^{k} e^{\varphi(b_j(y))} f(b_j(x)),$$

keeping in mind that RPF measures are, as expected, T-invariant as soon as they are atomless.

THEOREM 4.2. *If T is of class V, any potential $\varphi \in \mathrm{BV}_p(\Omega)$ such that*

$$\mathrm{BV}_p(\varphi) \le \frac{2}{3}\log\left(1 + \frac{(k^{1/p}-1)^2}{16k^{1/p}(k^{1/p}+1)}\right)$$

gives the transfer operator L_φ acting on $\mathrm{BV}_p(\Omega)$ a spectral gap.

Remark that this condition becomes *less* stringent with large k; as $k \to \infty$ the bound goes to the limit $\frac{2}{3}\log(1 + 1/16) \approx 0.04$. Meanwhile, for $p = 1$ and $k = 2$ we get the bound $\frac{2}{3}\log(1 + 1/96) \ge 0.0069$. In particular, Theorem 1.3 follows immediately.

Proof. As above, we only need to consider the zero potential, whose corresponding transfer operator is again denoted by L_0. Given $f \in \mathrm{BV}_p(\Omega)$, it is again clear that $\|\mathrm{L}_0 f\|_\infty \le \|f\|_\infty$; the core of the argument is then to prove that

$$\mathrm{BV}_p(\mathrm{L}_0 f) \le \frac{1}{k^{1/p}}\mathrm{BV}_p(f). \tag{2}$$

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This is easily achieved thanks to the 'extensiveness' of the total p-variation: if $\underline{x} = (x_0 < x_1 < \cdots < x_n)$ is any partition of Ω, then

$$b_1(x_0) < \cdots < b_1(x_n) < b_2(x_0) < \cdots < b_2(x_n) < \cdots < b_k(x_0) < \cdots < b_k(x_n)$$

is a partition of Ω, in the case where the b_j are all increasing and their images are in increasing order. In any other case, we only have to permute the b_j and change the order of each subsequence $b_j(x_0), \ldots, b_j(x_n)$ to obtain a partition, which we denote by

$$\underline{y} = (y_0 < \cdots < y_{k(n+1)-1}).$$

Then, using Minkowski's inequality and concavity of $x \mapsto x^{1/p}$, we have

$$\begin{aligned} v_p(\mathrm{L}_0 f, \underline{x}) &= \frac{1}{k}\left(\sum_{j=1}^{n}\left|\sum_{i=1}^{k} f(b_i(x_j)) - f(b_i(x_{j-1}))\right|^p\right)^{1/p} \\ &\le \frac{1}{k}\sum_{i=1}^{k}\left(\sum_{j=1}^{n}|f(b_i(x_j)) - f(b_i(x_{j-1}))|^p\right)^{1/p} \\ &\le \left(\frac{1}{k}\sum_{j=1}^{n}\sum_{i=1}^{k}|f(b_i(x_j)) - f(b_i(x_{j-1}))|^p\right)^{1/p} \\ &\le \frac{1}{k^{1/p}} v_p(f, \underline{y}) \le \frac{1}{k^{1/p}}\,\mathrm{BV}_p(f). \end{aligned}$$

By taking the upper bound over $\underline{x}$, we obtain (2).

To apply Lemma 2.3, observe that a p-BV function f such that $\int f\,d\mu = 0$ for some probability measure μ must be non-negative at some point and non-positive at some point, so that $\|f\|_\infty \le \mathrm{BV}_p(f)$. The existence of an eigenprobability is ensured by the compactness of Ω and the fact that L_0^* preserves the set of probability measures. We obtain that L_0 has a spectral gap of size $(k^{1/p}-1)/(k^{1/p}+1)$ with constant 1. Since $\mathrm{BV}_p(\Omega)$ is uniformly dense in the space of continuous functions, it follows that there is in fact a unique probability measure fixed by L_0^*.

The same computation as in the proof of Theorem 3.3 shows that for all $\varphi \in \mathrm{BV}_p(\Omega)$, if there is a constant c such that

$$\|\varphi - c\|_{\mathrm{BV}_p} \le \log\left(1 + \frac{(k^{1/p}-1)^2}{16k^{1/p}(k^{1/p}+1)}\right)$$

then the transfer operator L_φ has a spectral gap. Taking $c = (\sup\varphi + \inf\varphi)/2$, we get as before $\|\varphi - c\|_{\mathrm{BV}_p} \le \frac{3}{2}\,\mathrm{BV}_p(\varphi)$, so that a sufficient condition to have a spectral gap is

$$\mathrm{BV}_p(\varphi) \le \frac{2}{3}\log\left(1 + \frac{(k^{1/p}-1)^2}{24k^{1/p}(k^{1/p}+1)}\right). \qquad \square$$

5. *Three shades of spectral gaps*

In this section we show that several possible definitions of spectral gap are equivalent. This will enable us to ensure that the spectral gaps proved above actually match the assumption needed in [**GKLMF15**].

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5.1. *Hypotheses on the function space.* We fix a compact metric space Ω and space of functions $\mathcal{X}(\Omega)$ endowed with a norm $\|\cdot\|$. As in [**GKLMF15**], we shall first assume the following hypothesis.

(H1) $\mathcal{X}(\Omega)$ is a Banach algebra of Borel measurable, bounded functions $\Omega \to \mathbb{R}$, which includes all constant functions, whose norm dominates the uniform norm (for some constant A we have $\|f\| \geq A\|f\|_\infty$) and such that for all positive functions $f \in \mathcal{X}(\Omega)$, bounded away from zero, the function $\log f$ also lies in $\mathcal{X}(\Omega)$.

The last condition of log stability could be lifted up to changing the formulation of Theorem 5.1 below (replacing $J \in \mathcal{X}$ by e^φ, where $\varphi \in \mathcal{X}$). The measurability of elements of $\mathcal{X}$ and the domination condition $\|\cdot\| \geq A\|\cdot\|_\infty$ ensure that every finite measure on Ω can be seen as a continuous linear form, i.e. we have a natural map $C^0(\Omega)^* \to \mathcal{X}(\Omega)^*$. We will also assume the following hypothesis.

(H1′) If a finite measure μ on Ω is such that $\int f\, d\mu > 0$ for all positive functions $f \in \mathcal{X}(\Omega)$ that are bounded away from 0, then μ must be a positive measure. Moreover, the set of finite positive measures is closed in $\mathcal{X}(\Omega)^*$ (by which we mean that the natural map $C^0(\Omega)^* \to \mathcal{X}(\Omega)^*$ has closed image).

This assumption means that $\mathcal{X}(\Omega)$ is large enough to detect positivity of measures, and not to have too exotic linear forms which can be approximated by positive measures. This notably avoids trivial cases such as $\mathcal{X}(\Omega) = \{\text{constants}\}$. To ensure one can identify finite measures with elements of $\mathcal{X}(\Omega)^*$, we would need to ensure that the natural map $C^0(\Omega)^* \to \mathcal{X}(\Omega)^*$ is injective. We will dispense with this, but in an abuse of terminology still say that a form $\phi \in \mathcal{X}(\Omega)^*$ 'is a finite (positive) measure' whenever it is the image of a finite (positive) measure by the natural maps, irrespective of the uniqueness of this measure.

We will prove at the end of the section that $\mathrm{Hol}_\alpha(\Omega)$ and $\mathrm{BV}_p([a, b])$ satisfy both hypotheses.

We only consider bounded linear operators $\mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$ and, from now on, use $\mathcal{X}$ as a shorthand for $\mathcal{X}(\Omega)$.

THEOREM 5.1. *Assume that $\mathcal{X}$ satisfies hypotheses (H1) and (H1′) and let* L *be an operator of the form*

$$\mathrm{L}f(x) = \int_\Omega J(y) f(y)\, dm_x(y)$$

for some positive function $J \in \mathcal{X}$ and transition kernel $(m_x)_{x\in\Omega}$. Assume that L *has positive eigendata λ, h. Then the three definitions (SG1), (SG2) and (SG3) are equivalent for* L *(with the same gap size δ, but possibly different constants C).*

Much of what follows is either classical or of classical flavor, but Theorem 5.1 seems insufficiently obvious for us to dispense with a proof, and we need some slightly non-standard versions of common concepts (e.g. we will use cones that need not be convex, but are open). Note that we work with real Banach spaces; the above definitions of a spectral gap (e.g. (SG1)) are nonetheless well known to be equivalent to the *complex* spectrum to be contained in $\mathbb{D}_{\lambda(1-\delta)} \cup \{\lambda\}$, where $\mathbb{D}_r$ is the closed disk in the complex plane, centered at 0 and of radius r.

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Proof. Since (SG1) is well known to be equivalent to (SG2), and (SG3) clearly implies (SG1) under our hypotheses (simply take $G = \ker \mu$), the part of Theorem 5.1 that needs a proof is that (SG1) implies (SG3). Assume that L satisfies (SG1).

The dual operator L^* acting on $\mathcal{X}^*$ by $\mathrm{L}^*\phi(u) := \phi(\mathrm{L}u)$ preserves the line $G^\perp := \{\phi \in \mathcal{X}^* \mid \ker \phi \supset G\}$ and thus has an eigenvector ϕ_0, which up to a multiplicative constant encodes the projection on $\langle u \rangle$ along G. Moreover, ϕ_0 has the same eigenvalue λ as u_0.

Similarly, L^* must preserve the hyperplane $u_0^\perp := \{\phi \in \mathcal{X}^* \mid \phi(u_0) = 0\}$, and one can easily check (SG1) for L^*, with $u_0^\perp$ in the role of G and the same spectral gap δ. In particular, λ is a simple isolated eigenvalue of L^*. Moreover, for all $\phi \in \mathcal{X}^*$ and all $u \in \mathcal{X}$, writing $u = au_0 + g$ for its decomposition along $\langle u \rangle \oplus G$, we have

$$\left(\frac{1}{\lambda^n}\mathrm{L}^{*n}\right)\phi(au_0 + g) = \phi\left(au_0 + \frac{1}{\lambda^n}\mathrm{L}^n g\right) \to a\phi(u_0).$$

Now, to apply this in our context we will want ϕ_0 to be a measure, even if $\mathcal{X}^*$ is larger than the space of measures (which often happens since $\mathcal{X}$ usually is rather smaller than the space of continuous functions). We will also need ϕ_0 to be a *positive* measure; let us thus use the additional structure of $\mathcal{X}$ as a space of real functions. We say that $P \subset \mathcal{X}$ is a *cone* if it is stable under multiplication by all positive scalars. We say it is a *positive cone* if it is a cone, is open in the subspace it generates, and $\overline{P} \cap (-P) = \varnothing$. The openness assumption is to be noted: it is used below, and somewhat restricts the scope of application of this method (e.g. in the space of L^1 real functions, the subset of positive functions is not open but generates the whole space).

Any positive cone has a dual $P^* := \{\phi \in \mathcal{X}^* \mid \forall u \in P,\ \phi(u) > 0\}$ which is a cone of $\mathcal{X}^*$ (but we do not claim it to be a positive cone) whose closure is, as soon as P^* is not empty, $\overline{P^*} = \{\phi \in \mathcal{X}^* \mid \forall u \in P,\ \phi(u) \geq 0\}$.

LEMMA 5.2. *If P is a positive cone of $\mathcal{X}$ which is preserved by* L*, then P^* is preserved by* L^*. *If additionally $u_0 \in P$, then $G \cap P = \varnothing$ and up to multiplying it by -1, $\phi_0 \in P^*$.*

If additionally $\mathcal{Y} \subset \mathcal{X}^*$ is a subspace preserved by L^* *such that $\overline{P^*} \cap \mathcal{Y}$ is closed in $\mathcal{X}^*$ then either $\mathcal{Y} \subset u_0^\perp$ or $\phi_0 \in P^* \cap \mathcal{Y}$.*

Proof. Let $\phi \in P^*$ and $u \in P$. Then $\mathrm{L}^*\phi(u) = \phi(Lu)$; since $Lu \in P$ this is a positive number. Thus $\mathrm{L}^*\phi \in P^*$, and L^* preserves P^*.

Assume now that $u_0 \in P$. If there exists some $g \in G \cap P$, then since P is open in the space it generates there is some $a < 0$ such that $au_0 + g \in P$. Then for all n we have $\lambda^{-n}\mathrm{L}^n(au_0 + g) \in P$, which converges to $au_0 \in -P$. But that would give an element of $\overline{P} \cap -P$, which is excluded by definition. Thus $G \cap P = \varnothing$.

Next we need to prove that for an appropriate choice of the dual eigenvector, for all $u \in P$ we have $\phi_0(u) > 0$. By construction, up to a multiplicative constant we can assume that $\phi_0(au_0 + g) = a$ for all $a \in \mathbb{R}$ and $g \in G$. If $u = au_0 + g$ is in P, then $a < 0$ is excluded, as otherwise $au_0 = \lim \lambda^{-n}\mathrm{L}^n u$ would be an element of $\overline{P} \cap -P$; and $a = 0$ is excluded by $G \cap P = \varnothing$. Therefore $a > 0$ and $\phi_0 \in P^*$.

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Finally, assume that there exists $\phi \in \mathcal{Y} \setminus u_0^\perp$ and write $\phi = b\phi_0 + \psi$ where $\psi \in u_0^\perp$. Then

$$\left(\frac{1}{\lambda^n}\mathrm{L}^{*n}\right)\phi = b\phi_0 + \left(\frac{1}{\lambda^n}\mathrm{L}^{*n}\right)\psi \to b\phi_0.$$

Since $\overline{P^*} \cap \mathcal{Y}$ is closed and b is non-zero (as $\phi \notin u_0^\perp$), we conclude that $\phi_0 \in \mathcal{Y}$. We already know it is in P^*, concluding the proof. □

Let us now finish the proof of Theorem 5.1. We consider the set P consisting of all positive functions $\Omega \to \mathbb{R}$ in $\mathcal{X}$ which are bounded away from 0. It is a cone, and, since the norm dominates the uniform norm, it is open in $\mathcal{X}$. An element of $\overline{P} \cap (-P)$ would be a non-negative and positive function, thus P is a positive cone. The dual cone P^* is the set of continuous linear forms of $\mathcal{X}$ which are positive on all elements of P; denoting by $\mathcal{Y}$ the subspace of $\mathcal{X}^*$ consisting of finite measures, assumption (H1′) ensures that $\overline{P^*} \cap \mathcal{Y}$ is closed in $\mathcal{X}^*$. The form of L ensures that $\mathcal{Y}$ is preserved by L^*. It is not true that $\mathcal{Y} \subset h^\perp$, therefore Lemma 5.2 ensures that L^* has an eigenvector in $P^* \cap \mathcal{Y}$, which by (H1′) must be a positive measure. It is not zero since $\phi_0(u_0) \neq 0$, and up to proper scaling we can make it a probability measure. Then it is easily checked that G must be the kernel of this eigenprobability, and (SG3) holds. □

PROPOSITION 5.3. *Let Ω be a compact metric space and $\mathcal{X}$ be a Banach space of functions $\Omega \to \mathbb{R}$ satisfying (H1). If $\mathcal{X}$ is dense (for the uniform norm) in the space $C^0(\Omega)$ of continuous functions, then $\mathcal{X}$ also satisfies (H1′).*

Proof. We prove a slightly stronger form of the first property. Let μ be a finite measure on Ω such that for all $f \in \mathcal{X}$ bounded away from 0, $\mu(f) := \int f \, d\mu \geq 0$, and let g be any continuous positive function. Let $f_n \in \mathcal{X}$ be such that $\|g - f_n\|_\infty \to 0$; since μ is finite, it follows that $\mu(f_n) \to \mu(g)$. By compactness $\inf g > 0$; thus for n large enough, f_n is bounded away from 0 and $\mu(f_n) \geq 0$. It follows that $\mu(g) \geq 0$, and μ is a positive measure.

Now let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of finite positive measures, and assume there is a continuous linear form $\phi \in \mathcal{X}^*$ such that $\|\mu_n - \phi\|^* \to 0$ (where μ_n are identified with elements of $\mathcal{X}^*$ and $\|\cdot\|^*$ is the dual norm on $\mathcal{X}^*$). We have to prove that ϕ is a finite positive measure.

First, $\lim \mu_n(\mathbf{1}) = \phi(\mathbf{1}) \in \mathbb{R}$ ensures that the total mass of μ_n is bounded by some $M > 0$, uniformly in n, i.e. for all bounded measurable f (in particular, $f \in \mathcal{X}$) we have $|\mu_n(f)| \leq M\|f\|_\infty$ (positivity of the μ_n is crucial here).

Consider any non-zero $f \in \mathcal{X}$. There exists an $n \in \mathbb{N}$ such that $\|\mu_n - \phi\|^* \leq \|f\|_\infty / \|f\|$, and then

$$\begin{aligned} |\phi(f)| &\leq |\phi(f) - \mu_n(f)| + |\mu_n(f)| \\ &\leq (1 + M)\|f\|_\infty \end{aligned}$$

so that ϕ is continuous in $\|\cdot\|_\infty^*$ norm on $\mathcal{X}$. It then admits an extension $\mu \in C^0(\Omega)^*$, which by the Riesz representation theorem is a finite measure. Since $\mu_n(f) \to \mu(f)$ for all $f \in \mathcal{X}$ and the μ_n are positive, we have $\mu(f) \geq 0$ for all non-negative $f \in \mathcal{X}$. In particular,

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we can apply the strengthened first property proved above to deduce that μ is positive. In other words, ϕ identifies with a finite positive measure, as needed. □

The following corollary follows easily (using the fact that log is locally Lipschitz and that $\mathrm{Hol}_\alpha(\Omega)$ is uniformly dense in the space of continuous functions [**Geo67**]).

COROLLARY 5.4. *For all compact metric spaces* Ω*, the Banach space* $\mathrm{Hol}_\alpha(\Omega)$ *satisfies (H1) and (H1′).*

For all bounded intervals $I \subset \mathbb{R}$*, the space* $\mathrm{BV}_p(I)$ *satisfies (H1) and (H1′).*

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