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AUTOMORPHISM GROUPS OF BICOSET DIGRAPHS

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Abstract

We examine bicoset digraphs and their natural properties from the point of view of symmetry. We then consider connected bicoset digraphs that are X-joins with collections of empty graphs, and show that their automorphism groups can be obtained from their natural irreducible quotients. We further show that such digraphs can be recognised from their connection sets.

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1. Introduction

The oldest and most popular construction of vertex-transitive graphs is that of a Cayley graph. The basic idea is that, given a group G and a subset of G, one constructs a graph with a large amount of symmetry. There are now a variety of graphs and digraphs that have similar constructions, in that they also use a group G and a subset S of G, and produce a graph or digraph with a large amount of symmetry. The differences between the constructions are usually in how the group acts. Today, there are Cayley digraphs, coset digraphs, Haar graphs, bicoset graphs and digraphs that are all constructed in this manner. The work in this paper is the third in an unplanned series of papers that investigate what is in hindsight an obvious question. Given a group G and subset S, what symmetry information can one obtain from a Cayley digraph about the symmetries of the other graphs and digraphs constructed with the different actions of G? As the literature contains only a few actions of a group G for which there are vertex-transitive graphs and digraphs whose automorphism group contains

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the subgroup induced by the action, one should focus on these actions and the graphs and digraphs mentioned above.

The first paper in this sequence, motivated by an isomorphism problem (the so-called BCI problem), is [4], where a relationship was given between the automorphism groups of the Cayley digraphs of abelian groups and their corresponding Haar graphs. The second paper [1], motivated by the lack of a definition of a 'generalised wreath product' for Cayley graphs of nonabelian groups, gave a recognition theorem for Cayley and coset digraphs that can be written as a wreath product from the set S used to construct them. (This concerns a very important class of vertex-transitive digraphs which exist, for example, for Cayley digraphs of every abelian group A of every order other than prime order, and make one of three families of automorphism groups of Cayley digraphs of cyclic groups; see [3].) Fortuitously, the recognition theorem is exactly what is needed in order to give the relationship between the automorphism groups of Cayley digraphs and coset digraphs. It turns out that, while different, the automorphism group of one can be obtained from the automorphism group of the other [1, Theorem 3.3]. Every vertex-transitive digraph is isomorphic to a coset digraph, and so this result reduces the problem of finding the automorphism group of vertex-transitive digraphs to the problem of finding the automorphism groups of Cayley digraphs.

In this, the third paper in the series, we prove in Theorem 5.4 an analogue of the recognition theorem mentioned in the previous paragraph for Haar graphs and digraphs and for bicoset graphs and digraphs, the bipartite analogues of Cayley graphs and digraphs and of coset graphs and digraphs. For these graphs, the action of G that is used need not be transitive, and so a generalisation of the wreath product, the X-join, will play the role that the wreath product played in [1]. Although Hemminger [8] determined a considerable amount of information on the automorphism group of the X-join, he only considered graphs (and we need that information for digraphs as well). It is easier to derive the information we need about the X-join independently (see Section 3). We will also need the automorphism groups of bicoset digraphs that can be written as X-joins with empty digraphs. We prove the results we need in Section 4, giving the automorphism groups in Theorem 4.8. Section 5 gives the main recognition theorem (Theorem 5.4), together with several corollaries and some examples which highlight some of the ways in which X-joins of bicoset digraphs with empty digraphs differ from the coset digraphs which can be written as wreath products.

In a subsequent paper [2], we will give applications of the recognition theorem for X-joins, which are approximate analogues of the applications of the recognition theorem for wreath products. We will show that the automorphism groups of bicoset digraphs can be obtained from the automorphism groups of their corresponding Haar graphs, and a subgroup of the automorphism group of a Haar graph of index at most 2 can be obtained from the automorphism group of their corresponding bicoset digraph. Additionally, we will show that two bicoset digraphs of a group G are isomorphic if and only if the corresponding Haar graphs of G are isomorphic.

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2. Old and new families of digraphs

We now define the graphs and digraphs studied in this paper, and discuss some of their symmetry properties.

DEFINITION 2.1. Let G be a group and $S \subseteq G$. Define a *Cayley digraph of* G, denoted by $\mathrm{Cay}(G, S)$, to be the digraph with vertex set $V(\mathrm{Cay}(G, S)) = G$ and arc set $A(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call S the *connection set of* $\mathrm{Cay}(G, S)$.

DEFINITION 2.2. Let G be a group and $g \in G$. Define $g_L : G \to G$ by $g_L(x) = gx$. The map g_L is a *left translation of* G. The *left regular representation of* G, denoted by G_L, is $G_L = \{g_L : g \in G\}$. That is, G_L is the group of all left translations of G. It is straightforward to verify that G_L is a group isomorphic to G.

DEFINITION 2.3. Let G be a group and $H, K < G$. For each $g \in G$, the set $HgK = \{hgk : h \in H, k \in K\}$ is called the (H, K)-*double coset of* g *in* G. If $H = K$, then the (H, K)-double cosets are referred to as the *double cosets of* H *in* G.

The $(\{1\}, K)$-double cosets are the left cosets of K in G and the $(H, \{1\})$-double cosets are the right cosets of H in G. In general, HgK is a union of right cosets of H as well as the union of left cosets of K. We use G/H to denote the set of all left cosets of H in G, which is not necessarily a quotient group as we do not assume H is normal in G.

DEFINITION 2.4. Let G be a group, $H \leq G$ and $S \subset G$ such that $S \cap H = \emptyset$ and $HSH = S$. Define a digraph $\mathrm{Cos}(G, H, S)$ with vertex set $V(\mathrm{Cos}(G, H, S)) = G/H$ the set of left cosets of H in G, and arc set $A(\mathrm{Cos}(G, H, S)) = \{(gH, gsH) : g \in G \text{ and } s \in S\}$. The digraph $\mathrm{Cos}(G, H, S)$ is called the *coset digraph of* G with *connection set* S (or HSH).

The condition that an ordered pair (gH, gsH) is an arc of $\mathrm{Cos}(G, H, S)$ is often written in the equivalent form $(xH, yH) \in A(\mathrm{Cos}(G, H, S))$ if and only if $x^{-1}y \in S$. A Cayley digraph of G is canonically isomorphic to a coset digraph of G when $H = 1$. Similar to Cayley digraphs, for $g \in G$, the map $xH \mapsto gxH$ is an automorphism of $\mathrm{Cos}(G, H, S)$. The group of all such maps has no standard name; we denote it by (G, H).

We now give the natural bipartite analogues of Cayley and coset digraphs. It turns out that there are two different ways one can do this.

DEFINITION 2.5. Let G be a group and $S \subseteq G$. Define the *Haar graph* with *connection set* S, denoted by $\mathrm{Haar}(G, S)$, to be the graph with vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(0, g)(1, gs) : g \in G \text{ and } s \in S\}$.

Some authors refer to Haar graphs as bi-Cayley graphs, and denote them accordingly. As with Cayley graphs, note that $(0, x)(1, y) \in E(\mathrm{Haar}(G, S))$ if and only if $x^{-1}y \in S$. Haar graphs are the natural bipartite analogues of Cayley digraphs and have bipartition $\{\{i\} \times G : i \in \mathbb{Z}_2\}$. Haar graphs were introduced in [9].

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DEFINITION 2.6. For $g \in G$, define $\widehat{g_L} : \mathbb{Z}_2 \times G \to \mathbb{Z}_2 \times G$ by $\widehat{g_L}(i,j) = (i, g_L(j))$. It is straightforward to verify that $\widehat{g_L}$ is a bijection. Define $\widehat{G_L} = \{\widehat{g_L} : g \in G\}$.

It is straightforward to verify that $\widehat{G_L}$ is a group and $\widehat{G_L} \leq \mathrm{Aut}(\mathrm{Haar}(G,S))$ for every $S \subseteq G$. Also, $\widehat{G_L}$ has two orbits and its induced action on each orbit is regular and isomorphic to G_L.

DEFINITION 2.7. Let G be a group, let H_0 and H_1 be subgroups of G, and let $S \subseteq G$ be such that $S = H_0 S H_1$. The *bicoset graph* with respect to H_0, H_1, with *connection set* S, denoted by $\Gamma = B(G, H_0, H_1, S)$, is the graph with vertex set $(G/H_0) \cup (G/H_1)$ and edge set $E(\Gamma) = \{\{gH_0, gsH_1\} : g \in G, s \in S\}$.

Bicoset graphs are bipartite with bipartition $\{\{G/H_i\} : i \in \mathbb{Z}_2\}$. Bicoset graphs were introduced in [7]. Our definition is more general than in [7], where it is assumed that the largest normal subgroup of G contained in $H_0 \cap H_1$ (the *core* of $H_0 \cap H_1$ in G) is trivial. This condition ensures that the action of G on $(G/H_0) \cup (G/H_1)$ is faithful. In [7, Lemma 2.3] it was shown that the action of G by left multiplication on $V(\Gamma)$ is a subgroup of $\mathrm{Aut}(B(G, H_0, H_1; S))$ with two orbits. We denote this subgroup by $\widehat{G}$.

In order to define bipartite digraphs that can be thought of as analogues of Cayley and coset digraphs, we will need two connection sets, which give the arcs between the bipartition sets.

DEFINITION 2.8. Let G be a group and $S_0, S_1 \subseteq G$. The *Haar digraph* of G with *connection sets* S_0, S_1, denoted by $\mathrm{Haar}(G, S_0, S_1)$, is the digraph with vertex set $\mathbb{Z}_2 \times G$ and arc set $\{((0,g),(1,gs_0)) : g \in G,\ s_0 \in S_0\} \cup \{((1,g),(0,gs_1)) : g \in G,\ s_1 \in S_1\}$.

Haar digraphs were first defined in [6], and they are directed bipartite analogues of Cayley digraphs. Like Haar graphs, their automorphism groups contain $\widehat{G_L}$. We next introduce directed bicoset graphs.

DEFINITION 2.9. Let G be a group, let H_0 and H_1 be subgroups of G, and let $S_0, S_1 \subseteq G$ such that $S_0 = H_0 S_0 H_1$ and $S_1 = H_1 S_1 H_0$. The *bicoset digraph* of G with respect to H_0, H_1 with *connection sets* S_0 and S_1, denoted by $B(G, H_0, H_1; S_0, S_1)$, is the digraph with vertex set $(G/H_0) \cup (G/H_1)$ and arc set $\{(gH_0, gs_0H_1) : g \in G, s_0 \in S_0\} \cup \{(gH_1, gs_1H_0) : g \in G, s_1 \in S_1\}$.

It is easy to verify that the permutation representation $\widehat{G}$ of the action of G on $V(B(G, H_0, H_1; S_0, S_1))$ by left multiplication is contained in $\mathrm{Aut}(B(G, H_0, H_1; S_0, S_1))$ and is semiregular on G/H_0 and G/H_1. That is, $\widehat{G} = \{\widehat{g} : g \in G\}$ where $\widehat{g} : (G/H_0) \cup (G/H_1) \to (G/H_0) \cup (G/H_1)$ is defined by $\widehat{g}(kH_i) = (gk)H_i$, with $k \in G$ and $i \in \mathbb{Z}_2$. It is possible that $|H_0| \neq |H_1|$, in which case the cells of the natural bipartition have different orders.

EXAMPLE 2.10. Let $G = \mathbb{Z}_{15}$, $H_0 = \langle 3 \rangle$ and $H_1 = \langle 5 \rangle$. One can easily see that $|B_0| = |G|/|H_0| = 3$, while $|B_1| = |G|/|H_1| = 5$. So any bicoset digraph $B(G, H_0, H_1, S_0, S_1)$ will be a bicoset digraph in which the cells of the natural bipartition have different orders, where $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$, $i \in \mathbb{Z}_2$. In particular, if $S_0 = \mathbb{Z}_{15}$

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and $S_1 = \emptyset$, then $B(G, H_0, H_1, S_0, S_1)$ is the complete graph $K_{3,5}$ with every edge replaced with an arc from the cell of the bipartition of size 3 to the cell of the bipartition of size 5.

When $H_0 = H_1 = \{1\}$ in the definition of a bicoset digraph, the sets S_0, S_1 are just subsets of G, and the bicoset digraph with respect to H_0, H_1 with connection sets S_0 and S_1 is simply $\mathrm{Haar}(G, S_0, S_1)$. We now verify some basic facts about bicoset digraphs which are analogues of properties of coset digraphs. The next result is basically identical to showing that the arc set of a coset digraph $\mathrm{Cos}(G, H, S)$ is well defined if and only if its connection set S is a union of (H, H)-double cosets (see [5, Lemma 1.3.2], for example). Its proof is included for completeness.

LEMMA 2.11. *Let G be a group with $H_i \leq G$ and $S_i \subseteq G$ for $i \in \mathbb{Z}_2$. The arc set of $B(G, H_0, H_1; S_0, S_1)$ is well defined if and only if S_i is a union of (H_i, H_{i+1})-double cosets, $i \in \mathbb{Z}_2$.*

PROOF. Set $\Gamma = B(G, H_0, H_1; S_0, S_1)$. For $g \in G$, $i \in \mathbb{Z}_2$ and $s_i \in S_i$,

$$\begin{aligned}
&(gH_i, gs_iH_{i+1}) \in A(\Gamma)\\
&\quad \text{if and only if} \quad (gh_iH_i, gh_is_iH_{i+1}) \in A(\Gamma) \text{ for all } h_i \in H_i\\
&\quad \text{if and only if} \quad (gh_iH_i, gh_is_ih_{i+1}H_{i+1}) \in A(\Gamma) \text{ for all } h_i \in H_i, h_{i+1} \in H_{i+1}\\
&\quad \text{if and only if} \quad (gH_i, gh_is_ih_{i+1}H_{i+1}) \in A(\Gamma) \text{ for all } h_i \in H_i, h_{i+1} \in H_{i+1}\\
&\quad \text{if and only if} \quad h_is_ih_{i+1} \in S_i \text{ for all } h_i \in H_i, h_{i+1} \in H_{i+1}.
\end{aligned}$$

Thus $(gH_i, gs_iH_{i+1}) \in A(\Gamma)$ if and only if S_i is a union of (H_i, H_{i+1})-double cosets. □

We adopt the following standard notation for the rest of the paper. For a bipartite digraph Γ, denote its (natural) bipartition by $\mathcal{B} = \{B_0, B_1\}$. For a partition $\mathcal{P}$ of $V(\Gamma)$ that refines $\mathcal{B}$ (that is, every element of $\mathcal{P}$ is a subset of some element of $\mathcal{B}$), we denote by $\mathcal{P}_i$ those sets of $\mathcal{P}$ that are contained in B_i, $i \in \mathbb{Z}_2$.

DEFINITION 2.12. Let $G \leq S_n$ with orbit $\mathcal{O}$, and $g \in G$. Then g induces a permutation on $\mathcal{O}$ by restricting the domain of g to $\mathcal{O}$. We denote the resulting permutation in $S_{\mathcal{O}}$ by $g^{\mathcal{O}}$. The group $G^{\mathcal{O}} = \{g^{\mathcal{O}} : g \in G\}$ is the *transitive constituent* of G on $\mathcal{O}$.

Let G be a group and $H \leq G$. Then G acts by left multiplication on G/H, and this action is faithful if and only if H contains no proper nontrivial normal subgroups of G. That is, H is *core-free* in G. Many authors (but not we) insist, when defining a coset digraph $\mathrm{Cos}(G, H, S)$, that H be core-free in G, as if H has a normal subgroup $N \triangleleft H$, then $\mathrm{Cos}(G/N, H/N, T) \cong \mathrm{Cos}(G, H, S)$ where T is the set of cosets of N in S. The next result shows that a similar result holds for bicoset digraphs with $\mathrm{core}(H_0) \cap \mathrm{core}(H_1)$ playing the role of $\mathrm{core}(H)$.

LEMMA 2.13. *Let G be a group, $i \in \mathbb{Z}_2$, $H_i \leq G$, $S_i \subseteq G$ such that $S_i = H_iS_iH_{i+1}$, and $\Gamma = B(G, H_0, H_1; S_0, S_1)$. Let $N = \mathrm{core}_G(H_0) \cap \mathrm{core}_G(H_1)$. Then*

$$B(G, H_0, H_1; S_0, S_1) \cong B(G/N, H_0/N, H_1/N; \{s_0N : s_0 \in S_0\}, \{s_1N : s_1 \in S_1\}).$$

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PROOF. Define $\gamma : (G/H_0) \cup (G/H_1) \to ((G/N)/(H_0/N)) \cup ((G/N)/(H_1/N))$ by $\gamma(gH_i) = gN(H_i/N)$. Suppose $\gamma(gH_i) = \gamma(g'H_i)$. Then $gN(H_i/N) = g'N(H_i/N)$ or $(g^{-1}g'N)H_i/N = H_i/N$, that is, $g^{-1}g'N \in H_i/N$. Thus $gH_i/N = g'H_i/N$ and γ is a well-defined injection. As $|G/H_i| = |(G/N)/(H_i/N)|$ and G is finite, γ is a bijection. Let $T_i = \{s_iN : s_i \in S_i\}$. As $S_i = H_iS_iH_{i+1}$ and $N \trianglelefteq H_i$, we see that $(H_i/N)T_i(H_{i+1}/N) = T_i$ as $h_iNs_iNh_{i+1}N = h_is_ih_{i+1}N \in T$ for every $h_i \in H_i$ and $h_{i+1} \in H_{i+1}$. Thus $B(G/N, H_0/N, H_1/N; T_0, T_1)$ is a well-defined digraph. Let $(gH_i, gs_iH_{i+1}) \in A(B(G, H_0, H_1; S_0, S_1))$, so $s_i \in S_i$. Then

$$\gamma(gH_i, gs_iH_{i+1}) = (gN(H_i/N), gs_iN(H_{i+1}/N)) \in B(G/N, H_0/N, H_1/N; T_0, T_1)$$

as $s_iN \in T_i$ as $s_i \in S_i$. The result follows. □

3. X-joins with empty digraphs

The role of the wreath product in [1] for coset digraphs will be played by the X-join with a collection of empty digraphs (that is, digraphs with no arcs) for bicoset digraphs. This is because instead of replacing each vertex with an empty digraph of the same order for coset digraphs, we may use empty digraphs of different orders in each cell of the bipartition $\mathcal{B}$ of a bicoset digraph. In this section, we define the X-join, and prove the results that we will need, culminating in Corollary 3.12 which determines the automorphism groups of X-joins.

DEFINITION 3.1. Let X be a digraph, and $Y = \{Y_x : x \in V(X)\}$ a collection of digraphs indexed by $V(X)$. The *X-join* of Y is the digraph $Z = \bigvee(X, Y)$ with vertex set $V(Z) = \{(x, y) : x \in X, y \in Y_x\}$ and arc set

$$A(Z) = \{((x, y), (x', y')) : (x, x') \in A(X) \text{ or } x = x' \text{ and } (y, y') \in A(Y_x)\}.$$

In other words, the digraph Z is obtained by replacing each vertex of X by the digraph Y_x and inserting either all or none of the possible arcs between the vertices of Y_u and Y_v depending on whether or not there is an arc between u and v in X. If the Y_x are all isomorphic, then the X-join of $\{Y_x : x \in X\}$ is the wreath product $X \wr Y$, where $Y \cong Y_x$ for all $x \in X$.

Much of the work on automorphism groups of wreath products and X-joins has been based on two equivalence relations, one of which (R_Γ) we now define.

DEFINITION 3.2. Let Γ be a digraph. Define the equivalence relations R_Γ and D_Γ on $V(\Gamma)$ as follows:

(1) $u \, R_\Gamma \, v$ if and only if $N^+_\Gamma(u) = N^+_\Gamma(v)$ and $N^-_\Gamma(u) = N^-_\Gamma(v)$, and
(2) $u \, D_\Gamma \, v$ if and only if $u = v$.

Γ is called *irreducible* if $R_\Gamma = D_\Gamma$, otherwise it is called *reducible*. We call the set of equivalence classes of R_Γ the *unworthy* partition of $V(\Gamma)$. D_Γ is called the diagonal (or identity) relation. We will abbreviate the condition in (1) by saying $v \, R_\Gamma \, u$ if $N^\pm_\Gamma(v) = N^\pm_\Gamma(u)$.

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We note that R_Γ is a G-congruence (see [5, Definition 2.3.1]) if $G \leq \mathrm{Aut}(\Gamma)$ is transitive. Some people say u and v are *twins* if $u\, R_\Gamma\, v$.

Our terminology here is taken from a wide variety of sources, as the equivalence relation R_Γ has been discovered many times by many people in many contexts. The term 'irreducible' is due to Sabidussi [10] and 'unworthy' is due to Wilson [11]. We will need the following result [10, Lemma 1(ii)].

LEMMA 3.3. *If Γ is a reducible digraph with unworthy partition $\mathcal{P}$, then $\Gamma/\mathcal{P}$ is irreducible.*

Bipartite digraphs which can be written as X-joins are easy to recognise.

DEFINITION 3.4. Let X be a digraph and Y a collection of digraphs indexed by $V(X)$. The partition $\{\{(x, y) : y \in V(Y_x)\} : x \in V(X)\}$ is the *join partition of* $\bigvee(X, Y)$.

Hemminger [8, Theorem 2.8] gave necessary and sufficient conditions for the automorphism group of an X-join of Y to consist of what he calls the 'natural' automorphisms.

DEFINITION 3.5. Let Z be an X-join of $\{Y_x\}_{x\in X}$. Then a digraph automorphism φ of Z is called *natural* if for each $x_1 \in X$ there is an $x_2 \in X$ such that $\varphi(\{(x_1, y) : y \in V(Y_{x_1})\}) = \{(x_2, y) : y \in V(Y_{x_2})\}$. Otherwise, it is *unnatural*.

A natural automorphism of Z is an automorphism that permutes the elements of $\{Y_x\}_{x\in X}$, while an unnatural element is one that maps vertices in one $Y_x \in \{Y_x\}_{x\in X}$ to vertices in at least two different elements of $\{Y_x\}_{x\in X}$. It is important to note that if φ is a natural automorphism of Z, then φ induces an automorphism φ^* of X where $\varphi^*(x_1) = x_2$ if $\varphi(\{(x_1, y) : y \in V(Y_{x_1})\}) = \{(x_2, y) : y \in V(Y_{x_2})\}$. Similarly, if ϕ is an automorphism of X such that $Y_x \cong Y_{\phi(x)}$ for all $x \in X$, then ϕ induces a set of natural automorphisms of Z, where if ϕ' is one of these natural automorphisms, then $(\phi')^* = \phi$.

DEFINITION 3.6. Let X be a digraph, Y a collection of digraphs indexed by $V(X)$, and $Z = \bigvee(X, Y)$. We say that the set of natural automorphisms is *complete* if for each automorphism σ of X there exists a natural automorphism μ of Z such that $\mu^* = \sigma$.

The next result is well known, and shows that if $R_\Gamma \neq D_\Gamma$, then $\mathrm{Aut}(\Gamma)$ has automorphisms of a particular form. The proof is provided here for completeness.

LEMMA 3.7. *Let Γ be a digraph and E be an equivalence class of R_Γ. Let $\sigma \in S_E$. Define a permutation $\bar{\sigma} \in S_{V(\Gamma)}$ by $\bar{\sigma}(v) = \sigma(v)$ if $v \in E$ and $\bar{\sigma}(v) = v$ otherwise. Then $\bar{\sigma} \in \mathrm{Aut}(\Gamma)$.*

PROOF. Since for $v, v' \in E$ the neighbours of v and v' are the same, we see that $\bar{\sigma}$ preserves the arcs of Γ incident with a vertex of E. As $\bar{\sigma}^{V(\Gamma)-E} = 1$, $\bar{\sigma}$ preserves the arcs of Γ that are not incident with a vertex of E. Thus $\bar{\sigma}$ preserves the arcs of Γ, and so $\bar{\sigma} \in \mathrm{Aut}(\Gamma)$. □

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THEOREM 3.8. *Let X be a digraph and $Y = \{Y_x : x \in V(X)\}$ where each $Y_x \in Y$ is an empty digraph. If $\Gamma \cong \bigvee(X, Y)$, then the following statements are equivalent:*

(1) *X is irreducible;*
(2) *the join partition of Γ is the unworthy partition of Γ;*
(3) *every automorphism of Γ is natural.*

PROOF. (1) $\implies$ (2). Suppose X is irreducible. As Y_x is empty for all $x \in V(X)$, $u \, R_\Gamma \, v$ for all $u, v \in \{x\} \times V(Y_x)$. This implies that the unworthy partition of Γ is refined by the join partition of Γ. As X is irreducible, for any pair of distinct vertices $x_1, x_2 \in V(X)$, there exists a vertex $x_3 \in V(X)$ such that $(x_1, x_3) \in A(X)$ but $(x_2, x_3) \notin A(X)$ or such that $(x_3, x_1) \in A(X)$ but $(x_3, x_2) \notin A(X)$.

As there exists a vertex $x_3 \in V(X)$ such that x_1 is in-adjacent (respectively, out-adjacent) to x_3 but x_2 is not in-adjacent (respectively, out-adjacent) to x_3, no vertex of the set $\{(x_2, y) : y \in V(Y_{x_2})\}$ is in-adjacent (respectively, out-adjacent) to any vertex of $\{(x_3, y) : y \in V(Y_{x_3})\}$, while every vertex of $\{(x_1, y) : y \in V(Y_{x_1})\}$ is in-adjacent (respectively, out-adjacent) to every vertex of $\{(x_3, y) : y \in V(Y_{x_3})\}$. Thus, the cell of R_Γ containing the vertices $\{(x_1, y) : y \in V(Y_{x_1})\}$ must contain every element of the form $\{(x_1, y) : y \in V(Y_{x_1})\}$, and, as x_1, x_2 are arbitrary, the cell of R_Γ containing $\{(x_1, y) : y \in V(Y_{x_1})\}$ does not contain any element of the form $\{(x_2, y) : y \in V(Y_{x_i})\}$. As x_1 and x_2 are arbitrary, the cell of R_Γ containing $\{(x_1, y) : y \in V(Y_{x_1})\}$ is $\{(x_1, y) : y \in V(Y_{x_1})\}$.

(2) $\implies$ (3). Suppose that the join partition of Γ is the unworthy partition of Γ. Let $\gamma \in \mathrm{Aut}(\Gamma)$ and $u, v \in V(\Gamma)$ such that $u \, R_\Gamma \, v$. Then $\gamma(u) \, R_\Gamma \, \gamma(v)$. Thus γ maps the unworthy partition of Γ to the unworthy partition of Γ, and so maps the join partition of Γ to the join partition of Γ. This implies that γ is a natural automorphism.

(3) $\implies$ (1). Suppose every automorphism of Γ is natural. Towards a contradiction, suppose that X is reducible. Then there exist vertices $x_1, x_2 \in V(X)$ such that $N_X^{\pm}(x_1) = N_X^{\pm}(x_2)$. This implies, by definition, that every vertex of $\{(x_1, y) : y \in V(Y_{x_1})\}$ and every vertex of $\{(x_2, y) : y \in V(Y_{x_2})\}$ has the same out- and in-neighbours in Γ. The permutation $((x_1, y_1), (x_2, y_2))$, for all $y_1 \in V(Y_{x_1})$ and $y_2 \in V(Y_{x_2})$ is an automorphism of Γ and is an unnatural automorphism, a contradiction. □

The terminology and notation given above were introduced by Hemminger [8], where he studied the automorphism group of $Z = \bigvee(X, Y)$. In general, even if every automorphism of $\mathrm{Aut}(Z)$ is natural, $\mathrm{Aut}(Z)$ can be difficult to describe. In fact, Hemminger gave necessary and sufficient conditions for every automorphism of Z to be natural [8, Theorem 2.8], but did not write an explicit description of what this group looks like (as it depends upon which of the Y_x are isomorphic). However, in the introduction, he gave the following description of what a natural automorphism is (taken more or less verbatim from [8], with the only differences in notation). If $\mu \in \mathrm{Aut}(X)$, such that $Y_x \cong Y_w$ whenever $\mu(x) = w$ then μ induces a set of 'natural' graph automorphisms of Z, namely, the automorphisms that are obtained by first permuting the Y_x depending on how μ permutes the subscripts of the Y_x and then performing an arbitrary automorphism of each Y_x.

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In the case of bicoset digraphs Z, there are only two possibilities for the set Y of empty graphs (this is explicitly proven below). Either $\mathrm{Aut}(X)$ is transitive and all the empty graphs in Y have the same number of elements (and consequently $\mathrm{Aut}(Z) = \mathrm{Aut}(X) \wr S_m$, where $m = |Y_x|$, $Y_x \in Y$), or $\mathrm{Aut}(X)$ has two orbits, and if X is weakly connected, any two elements of Y contained in $B_i \in \mathcal{B}$, $i \in \mathbb{Z}_2$, have the same order. A complete set of natural automorphisms need not be a group, but it is easy to see that the set of all natural automorphisms of Z is a group.

DEFINITION 3.9. Let X be a digraph, Y a set of digraphs indexed by X and $Z = \bigvee(X, Y)$. We will call the set of all natural automorphisms of Z the *group of natural automorphisms of Z*. Additionally, if the group of natural automorphisms is also a complete set, we will call it the *complete group of natural automorphisms*.

For a bicoset digraph that is not vertex-transitive, the group of natural automorphisms will be generated by elements of $\mathrm{Aut}(X)$ permuting the elements of Y whose orders are the same, as such an element permutes the subscripts, together with the subgroups that are the symmetric groups on Y_x and the identity elsewhere, for every $Y_x \in Y$. The following result then gives the automorphism group of bicoset digraphs that can be written as a join that we will need.

DEFINITION 3.10. Let X be a set and $G \leq S_X$ be transitive. A subset $B \subseteq X$ is a *block* of G if whenever $g \in G$, then $g(B) \cap B = \emptyset$ or B.

A block B of a permutation group $G \leq S_X$ is a subset of X which is either fixed setwise by $g \in G$, or is moved 'away' from B. Note that if B is a block of G, then $g(B)$ is also a block of B for every $g \in G$ and is called a *conjugate block of B*.

DEFINITION 3.11. The set of all blocks conjugate to B is a partition of X, called a *block system of G*.

COROLLARY 3.12. *Let X be an irreducible bicoset digraph that is weakly connected when $\mathrm{Aut}(X)$ is transitive and $Y = \{Y_x : x \in V(X)\}$, where each $Y_x \in Y$ is an empty digraph. Let $\Gamma \cong \bigvee(X, Y)$. The following statements are equivalent:*

(1) $\mathrm{Aut}(\Gamma)$ *is the complete group of natural automorphisms;*
(2) *either X is vertex-transitive and every element of Y has the same order, or any two elements of Y contained in $B_i \in \mathcal{B}$, $i \in \mathbb{Z}_2$, have the same order.*

PROOF. As X is weakly connected when $\mathrm{Aut}(X)$ is transitive, Γ is weakly connected, and so $\mathcal{B}$ is a block system of $\mathrm{Aut}(\Gamma)$ when Γ is vertex-transitive [4, Lemma 1.10], while $\mathcal{B}$ is the set of orbits of $\mathrm{Aut}(\Gamma)$ otherwise. As X is irreducible, by Theorem 3.8 we see that every automorphism of Γ is natural.

Suppose that $\mathrm{Aut}(\Gamma)$ is the complete group of natural automorphisms. If X is vertex-transitive, then for every $x_1, x_2 \in V(X)$ there is $\sigma \in \mathrm{Aut}(X)$ (which depends on x_1, x_2) such that $\sigma(x_1) = x_2$. Then there exists $\delta \in \mathrm{Aut}(\Gamma)$ such that $\delta^* = \sigma$. Hence $\delta(Y_{x_1}) = Y_{x_2}$. Thus $|Y_{x_1}| = |Y_{x_2}|$. As x_1 and x_2 are arbitrary, every element of Y has the same order. An analogous argument shows that if X is not vertex-transitive, then

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$\mathrm{Aut}(X)$ is transitive on $B_i \in \mathcal{B}$, and any two elements of Y contained in $B_i \in \mathcal{B}$ have the same order.

Suppose that X is vertex-transitive and every element of Y has the same order. Let T be a set of size $|Y_x|$, $Y_x \in Y$, and assume without loss of generality that for every $x \in X$, we have $Y_x = \{(x,t) : t \in T\}$. Then $V(\Gamma) = V(X) \times T$. Let $\sigma \in \mathrm{Aut}(X)$. Define $\delta : V(X) \times T \to V(X) \times T$ by $\delta(x,s) = (\sigma(x), \omega_x(s))$, where $\omega_x \in S_T$ for every $x \in V(X)$. Let $((x_1,t_1),(x_2,t_2)) \in A(\Gamma)$. Then $(x_1,x_2) \in A(X)$, and so $\sigma(x_1,x_2) \in A(X)$. Hence $((\sigma(x_1),t_3),(\sigma(x_2),t_4)) \in A(\Gamma)$ for every $t_3,t_4 \in T$, and $\delta((x_1,t_1),(x_2,t_2)) \in A(\Gamma)$. Thus $\delta \in \mathrm{Aut}(\Gamma)$. The result follows in this case. When X is not vertex-transitive but X is a bicoset digraph, the argument is analogous, although instead of having one set T, we will need two sets T_1 and T_2, one for each $B_i \in \mathcal{B}$. □

4. Automorphism groups of reducible bicoset digraphs

We will need quotient digraphs. As bicoset digraphs need not be vertex-transitive [7], the natural partition by which to quotient them has two parts (one for each cell of the natural bipartition), and so is slightly more complicated. We first give this partition, which is a special case of Definition 4.3.

DEFINITION 4.1. Let G be a group, $i \in \mathbb{Z}_2$, $H_i \le K_i \le G$ and $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$. Let $\Gamma = B(G, H_0, H_1; S_0, S_1)$ and let $\mathcal{P}_i$ be the partition of B_i where each cell is the set of left cosets of H_i in G contained in a left coset of K_i in G. Define the *join partition of* $V(\Gamma)$ *with respect to* H_i *and* K_i, denoted by $\mathcal{P}(H_i,K_i)$, of the vertices of Γ as $\mathcal{P}(H_i,K_i) = \mathcal{P}_0 \cup \mathcal{P}_1$. That is, $\mathcal{P}_i = \{\{k_iH_i : k_i \in gK_i\} : g \in G, i \in \mathbb{Z}_2\}$.

Note that $\mathcal{P}_i$ is a block system of $\widehat{G}^{B_i}$, as H_i partitions G and $H_i \le K_i$, but the join partition $\mathcal{P}(H_i,K_i)$ of $V(\Gamma)$ is not necessarily a block system of $\mathrm{Aut}(\Gamma)$, as Γ may not be vertex-transitive. For example, if $|K_0/H_0| \ne |K_1/H_1|$ then Γ is never vertex-transitive.

DEFINITION 4.2. Let X be a set, and $G \le S_X$ (we note that G need not be transitive). A partition $\mathcal{P}$ of X will be called a *G-invariant partition of X* if $g(P) \in \mathcal{P}$ for every $P \in \mathcal{P}$.

We observe that if G is transitive, a G-invariant partition is simply a block system of G. However, we will only use this terminology when the group G is intransitive. The next result shows that for us, the join partition $\mathcal{P}(H_i,K_i)$ is natural in our context.

LEMMA 4.3. *Let G be a group and $i \in \mathbb{Z}_2$. Let $\mathcal{P}$ be a partition of $\mathbb{Z}_2 \times G$ that refines $\mathcal{B}$. Then $\mathcal{P}$ is a $\widehat{G}$-invariant partition of $\mathbb{Z}_2 \times G$ if and only if there exist $H_i \le K_i \le G$, such that $\mathcal{P} = \mathcal{P}(H_i,K_i)$.*

PROOF. It is clear that $\mathcal{P}(H_i,K_i)$ is a refinement of $\mathcal{B}$ and invariant under $\widehat{G}$ as left multiplication permutes left cosets of any subgroup of G, where $H_i \le K_i \le G$. Conversely, suppose that $\mathcal{P}$ refines $\mathcal{B}$ and is $\widehat{G}$-invariant. Let $\mathcal{P}_i$ consist of those subsets of $\mathcal{P}$ that are contained in B_i. As $\mathcal{P}$ refines $\mathcal{B}$, $\mathcal{P}_i$ is a partition of B_i. Furthermore, $\widehat{G}^{B_i}$ is transitive on B_i, so $\mathcal{P}_i$ is a block system of $\widehat{G}^{B_i}$. By [5, Exercise 2.3.16], $\mathcal{P}_i$ is the set of left cosets of some $H_i \le K_i \le G$. So $\mathcal{P} = \mathcal{P}(H_i,K_i)$. □

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LEMMA 4.4. *Let G be a group, $H_i \leq G$, and $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$. Let $\Gamma = B(G, H_0, H_1; S_0, S_1)$. The unworthy partition of Γ is $\widehat{G}$-invariant.*

PROOF. Let $u, v \in V(\Gamma)$, such that $N^{\pm}_{\Gamma}(u) = N^{\pm}_{\Gamma}(v)$. For any in- or out-neighbour y of u and v and $g \in G$, we see that $\widehat{g}(y)$ is an in- or out-neighbour of $\widehat{g}(u)$ and $\widehat{g}(v)$. Thus $N^{\pm}_{\Gamma}(\widehat{g}(u)) = N^{\pm}_{\Gamma}(\widehat{g}(v))$. □

LEMMA 4.5. *Let Γ be a bipartite digraph without isolated vertices with bipartition $\mathcal{B} = \{B_0, B_1\}$. The unworthy partition $\mathcal{P}$ of Γ refines $\mathcal{B}$.*

PROOF. Suppose towards a contradiction that there exists some $P \in \mathcal{P}$ such that $P \cap B_0 \neq \emptyset \neq P \cap B_1$. Let $x_i \in V(\Gamma)$ be such that $x_i \in P \cap B_i$, $i \in \mathbb{Z}_2$. As Γ has no isolated vertices, x_0 has some in- or out-neighbour y which, as Γ is bipartite, is contained in B_1. But, as the in- and out-neighbourhoods of x_0 and x_1 are the same, either $(x_1, y) \in A(\Gamma)$ or $(y, x_1) \in A(\Gamma)$. In either case, this implies that $\mathcal{B}$ is not a bipartition of Γ, a contradiction. □

LEMMA 4.6. *Let G be a group, $i \in \mathbb{Z}_2$, $H_i \leq G$, and $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$. Let $\Gamma = B(G, H_0, H_1; S_0, S_1)$. Let $\mathcal{P}$ be the unworthy partition of $V(\Gamma)$, and $\mathcal{P}_i$ consist of those elements of $\mathcal{P}$ that are contained in B_i. Then $\mathcal{P}_i$ is the set of orbits of the kernel of the induced action of F on B_{i+1}, where F is the setwise stabiliser of each $B_i \in \mathcal{B}$, and $\mathcal{P} = \mathcal{P}(H_i, K_i)$ for some $H_i \leq K_i \leq G$.*

PROOF. By Lemma 4.5, we see that $\mathcal{P}_i$ is a partition of B_i, $i \in \mathbb{Z}_2$. Let Ker_i be the kernel of the action of F on B_{i+1} and O be an orbit of Ker_i. Fix $v \in O$ and let $P_i \in \mathcal{P}_i$ such that $v \in P_i$. By definition, Ker_i fixes every vertex of B_{i+1}. Note that for every $y \in N^{\pm}(v)$, $y \in B_{i+1}$ and Ker_i^O acts transitively on O. Thus if $x \in N^+(v)$ then $u \in N^-(x)$ for all $u \in O$, and similarly if $x \in N^-(v)$ then $u \in N^+(x)$ for all $u \in O$. We conclude that if x is an in- and out-neighbour of v, then x is an in- and out-neighbour of every element of O. Thus every element of O has the same neighbourhood and $O \subseteq P_i$. Conversely, let $P_i \in \mathcal{P}_i$ and, for each $\sigma \in S_{P_i}$, define $\overline{\sigma} \in S_{V(\Gamma)}$ by $\overline{\sigma}(v) = \sigma(v)$ if $v \in P_i$ and $\overline{\sigma}(v) = v$ otherwise. By Lemma 3.7, $\overline{\sigma} \in F$ and so $P_i \subseteq O$. Thus $P_i = O$ and so $\mathcal{P}_i$ is the set of orbits of the kernel of the action of F on B_{i+1}. By Lemma 4.4, $\mathcal{P}$ is $\widehat{G}$-invariant, so by Lemma 4.3, there exist K_i satisfying $H_i \leq K_i \leq G$ and $\mathcal{P}_i = \mathcal{P}(H_i, K_i)$. □

We now give the automorphism group of every bicoset digraph that can be written as an X-join of a set of empty digraphs Y such that the partition $\mathcal{P} = \{V(Y_x) : x \in X\}$ refines the natural bipartition $\mathcal{B}$. But first, with the natural partition in hand, we now define quotient digraphs.

DEFINITION 4.7. Let Ω be a set and $\mathcal{P}$ be a partition of Ω. Let Γ be a digraph with vertex set Ω. Define the *quotient digraph* of Γ with respect to $\mathcal{P}$, denoted $\Gamma/\mathcal{P}$, by $V(\Gamma/\mathcal{P}) = \mathcal{P}$ and $(P_1, P_2) \in A(\Gamma/\mathcal{P})$ if and only if $(p_1, p_2) \in A(\Gamma)$ for some $p_1 \in P_1$ and $p_2 \in P_2$.

THEOREM 4.8. *Let G be a group, $i \in \mathbb{Z}_2$, $H_i \leq G$ and $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$. Let $\Gamma = B(G, H_0, H_1; S_0, S_1)$ be a bicoset digraph such that Γ is weakly connected when*

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$\mathrm{Aut}(\Gamma)$ *is transitive and* $\Gamma = \bigvee(X, Y)$ *for some digraph* X *and* Y *a collection of empty graphs that refines the natural bipartition* $\mathcal{B}$ *of* Γ. *Then there exists* $H_i \leq K_i \leq G$ *such that the unworthy partition* $\mathcal{P}$ *of* $V(\Gamma)$ *is the join partition* $\mathcal{P}(H_i, K_i)$, *and if* Y' *is the collection of empty graphs on the cells of* $\mathcal{P}$, *then* $\Gamma \cong \bigvee(\Gamma/\mathcal{P}, Y')$. *In addition,* $\mathrm{Aut}(\bigvee(\Gamma/\mathcal{P}, Y'))$ *is the complete group of natural automorphisms of* $\bigvee(\Gamma/\mathcal{P}, Y')$. *Also, if every empty digraph in* Y' *has the same order* n, *then* $\mathrm{Aut}(B(G, H_0, H_1; S_0, S_1)) = \mathrm{Aut}(\Gamma/\mathcal{P}) \wr S_n$.

PROOF. Let F be the setwise stabiliser of each block of $B_i \in \mathcal{B}$ and $\mathcal{P}_i$ consist of the elements of $\mathcal{P}$ contained in B_i. By Lemma 4.6, $\mathcal{P}_i$ is the set of orbits of the kernel of the induced action of F on B_{i+1} and $\mathcal{P} = \mathcal{P}(H_i, K_i)$ for some $H_i \leq K_i \leq G$. Thus the elements of $\mathcal{P}_i$ have the same order, and if Γ is vertex-transitive then all the elements of $\mathcal{P}$ have the same order. Let L be the subgroup of $\mathrm{Aut}(\Gamma/\mathcal{P})$ that fixes each cell of the bipartition of $\Gamma/\mathcal{P}$, and call it $\mathcal{B}/\mathcal{P}$, setwise.

Suppose towards a contradiction that the induced action of L on the left cosets of K_{i+1} in B_{i+1} is unfaithful. The kernel of the action fixes each left coset of K_{i+1} in B_{i+1} and is nontrivial on the left cosets of K_i in B_i. There exists $\delta_i \in \mathrm{Aut}(\Gamma/\mathcal{P})$ that is trivial on the left cosets of K_{i+1} in B_{i+1} but $\delta_i(K_i) = g_i K_i \neq K_i$ for some $g_i \in G$. Then K_i and $g_i K_i$ have the same neighbourhood in the quotient $\Gamma/\mathcal{P}$. If Γ has no arcs, then $\mathrm{Aut}(\Gamma)$ is a symmetric group and transitive. By hypothesis, Γ is weakly connected, a contradiction. Thus Γ has arcs. Then some left coset $k_i H_i$ of K_i is in- or out-adjacent to some left coset $\ell_{i+1} H_{i+1}$ in $\ell_{i+1} K_{i+1}$, while some left coset $g_i H_i$ in $g_i K_i$ is in- or out-adjacent to $\ell'_{i+1} H_{i+1}$ that is also contained in $\ell_{i+1} K_{i+1}$. As $\mathcal{P}$ is the unworthy partition of Γ, this implies that every left coset of H_i in K_i and $g_i K_i$ is in- and out-adjacent to every left coset of H_{i+1} in $\ell_{i+1} K_{i+1}$. As K_i and $g_i K_i$ have the same neighbours in $\Gamma/\mathcal{P}$, this implies that H_i and $g_i H_i$ have the same neighbours in Γ and so $H_i \, R_\Gamma \, g_i H_i$. Then K_i contains $g_i K_i$, a contradiction.

Thus the action of L on $\mathcal{B}/\mathcal{P}$ is faithful. So, $R_{\Gamma/\mathcal{P}} = D_{\Gamma/\mathcal{P}}$ and $\Gamma/\mathcal{P}$ is irreducible. As Γ is weakly connected when $\mathrm{Aut}(\Gamma)$ is transitive, by Corollary 3.12, $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\bigvee(\Gamma/\mathcal{P}, Y'))$ is the complete group of natural automorphisms of $\bigvee(\Gamma/\mathcal{P}, Y')$. Additionally, if every empty digraph in Y' has the same order n, then $\Gamma \cong (\Gamma/\mathcal{P}) \wr \overline{K}_n$ and it follows that $\mathrm{Aut}(B(G, H_0, H_1; S_0, S_1)) = \mathrm{Aut}(\Gamma/\mathcal{P}) \wr S_n$. □

COROLLARY 4.9. *Let* G *be a group,* $i \in \mathbb{Z}_2$, *and* $S_i \subseteq G$. *Let* $\Gamma = \mathrm{Haar}(G, S_0, S_1)$ *be a Haar digraph such that* Γ *is weakly connected when* $\mathrm{Aut}(\Gamma)$ *is transitive, and* $\Gamma = \bigvee(X, Y)$ *for some digraph* X *and* Y *a collection of empty graphs that refines the natural bipartition* $\mathcal{B}$ *of* Γ. *Then there exists* $K_i \leq G$, *such that the unworthy partition* $\mathcal{P}$ *of* $V(\Gamma)$ *is the join partition* $\mathcal{P}(1, K_i)$ *and if* Y' *is the collection of empty digraphs on the cells of* $\mathcal{P}$, *then* $\Gamma \cong \bigvee(\Gamma/\mathcal{P}, Y')$. *Also,* $\mathrm{Aut}(\bigvee(\Gamma/\mathcal{P}, Y'))$ *is the complete group of natural automorphisms of* $\bigvee(\Gamma/\mathcal{P}, Y')$. *Additionally, if every empty digraph in* Y' *has the same order* n, *then* $\mathrm{Aut}(\mathrm{Haar}(G, S)) = \mathrm{Aut}(\Gamma/\mathcal{P}) \wr S_n$.

PROOF. When $H_0 = H_1 = \{1_G\}$, $\Gamma = B(G, H_0, H_1; S_0, S_1) = \mathrm{Haar}(G, S_0, S_1)$. The result follows from Corollary 4.8. □

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5. Recognition of bicoset digraphs which are X-joins

Originally, our main goal for this section was to determine when a Haar graph is isomorphic to a wreath product of a graph and an empty graph by inspection of its connection set. However, just as quotients of Cayley digraphs need not be Cayley digraphs, quotients of Haar digraphs need not be Haar digraphs. So the proper context in which to proceed is in the class of digraphs which contains all bipartite digraphs and has a group acting transitively on each of the bipartition classes, as Haar digraphs have this property. These digraphs are the bicoset digraphs, as previously stated.

In the previous section, we determined the automorphism group of a bicoset digraph when it is isomorphic to a nontrivial X-join. In this section, we determine when a bicoset digraph is isomorphic to a nontrivial X-join by inspecting the connection sets S_0 and S_1.

We first prove a variation of a recognition theorem for wreath products (see, for example, [5, Theorem 4.2.15]) adapted for bicoset digraphs. We remind the reader that bicoset digraphs need not be vertex-transitive, and we do not assume this here. We first need a definition.

DEFINITION 5.1. A group $G \leq S_n$ is *semitransitive* if G has exactly two orbits.

In [7, Lemma 2.4] it was shown that every graph whose automorphism group has a semitransitive subgroup is isomorphic to a bicoset graph. A similar proof also works for digraphs. The next result generalises [5, Theorem 4.2.15] from wreath products to X joins, and can be thought of as a digraph-theoretic version of the recognition theorem.

THEOREM 5.2. *Let Γ be a bipartite digraph whose bipartition $\mathcal{C} = \{C_i : i \in \mathbb{Z}_2\}$ is the orbits of a semitransitive group G. Let $i \in \mathbb{Z}_2$. Let $\mathcal{P} \preceq \mathcal{C}$ be a G-invariant partition and $\mathcal{P}_i$ be the sets of $\mathcal{P}$ contained in C_i. Let $X = \Gamma/\mathcal{P}$ and Y be the set of all empty digraphs whose vertex sets are elements of $\mathcal{P}$. Then $\Gamma = \bigvee(X, Y)$ if and only if whenever $P_i \in \mathcal{P}_i$ and $P_{i+1} \in \mathcal{P}_{i+1}$, there is an arc (x_i, x_{i+1}) from a vertex $x_i \in P_i$ to a vertex $x_{i+1} \in P_{i+1}$ if and only if every arc of the form (x_i, x_{i+1}) with $x_i \in P_i$ and $x_{i+1} \in P_{i+1}$ is contained in $A(\Gamma)$.*

PROOF. Set $\mathcal{P} = \mathcal{P}(H_i, K_i)$. Suppose $\Gamma = \bigvee(X, Y)$. Let $x_i \in P_i \in \mathcal{P}_i$, $i \in \mathbb{Z}_2$. Then

$$\begin{aligned}(x_i, x_{i+1}) \in A(\Gamma) \quad &\text{if and only if} \quad (P_i, P_{i+1}) \in A(\Gamma/\mathcal{P})\\ &\text{if and only if} \quad (x_i, x_{i+1}) \in A(\Gamma) \text{ for all } x_i \in P_i \text{ and } x_{i+1} \in P_{i+1},\end{aligned}$$

by the definition of the X-join of Y.

For the converse, first observe that $V(\bigvee(X, Y)) = V(\Gamma)$. Let $i \in \mathbb{Z}_2$, $x \in P_i$ and $y \in P_{i+1}$. As $\mathcal{P} \prec \mathcal{C}$ and $\mathcal{C}$ is a bipartition of Γ, there exist $x \in P_x \in \mathcal{P}_i$ and $y \in P_y \in \mathcal{P}_{i+1}$. Suppose $(x, y) \in A(\Gamma)$. Then $(P_x, P_y) \in A(\Gamma/\mathcal{P})$ and $(x, y) \in A(\bigvee(X, Y))$. Thus $A(\Gamma) \subseteq A(\bigvee(X, Y))$. Now suppose $(x, y) \notin A(\Gamma)$. Then there is no arc from any vertex of P_i to any vertex of P_{i+1} and so $(P_x, P_y) \notin A(\bigvee(X, Y))$. Thus $A(\Gamma) = A(\bigvee(X, Y))$, and the result follows. □

The next result is the version of the previous result for bicoset digraphs.

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THEOREM 5.3. *Let G be a group, $i \in \mathbb{Z}_2$, $H_i \leq K_i \leq G$ and $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$. Let $\Gamma = B(G, H_0, H_1; S_0, S_1)$ and $X = \Gamma/\mathcal{P}(H_i, K_i)$. For $g \in G$, let $Y_{i,g}$ be the empty digraph on the left cosets of H_i contained in gK_i and $Y = \{Y_{i,g} : i \in \mathbb{Z}_2, g \in G\}$. Let $\mathcal{P}_i$ be the sets in $\mathcal{P}(H_i, K_i)$ contained in B_i. Then $\Gamma \cong \bigvee(X, Y)$ if and only if whenever $P_i \in \mathcal{P}_i$ and $P_{i+1} \in \mathcal{P}_{i+1}$ there is an arc (x_i, x_{i+1}) from a vertex $x_i \in P_i$ to a vertex $x_{i+1} \in P_{i+1}$ if and only if every arc of the form (x_i, x_{i+1}) with $x_i \in P_i$ and $x_{i+1} \in P_{i+1}$ is contained in $A(\Gamma)$.*

We are now ready to prove our main theorem regarding recognition of bicoset digraphs that are X-joins using their connection set. This is the analogue of [1, Theorem 3.3] for bicoset digraphs.

THEOREM 5.4. *Let G be a group, $i \in \mathbb{Z}_2$, $H_i \leq K_i \leq G$ and $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$. Let $\Gamma = B(G, H_0, H_1; S_0, S_1)$ have arcs and let $X = \Gamma/\mathcal{P}(H_i, K_i)$. For $g \in G$, let $Y_{i,g}$ be the empty digraph on the left cosets of H_i contained in gK_i, and $Y = \{Y_{i,g} : i \in \mathbb{Z}_2, g \in G\}$. Then $\Gamma = \bigvee(X, Y)$ if and only if $S_i = K_i S_i K_{i+1}$. If $S_i = K_i S_i K_{i+1}$, then $X = B(G, K_0, K_1; S_0, S_1)$. Also, if K_0 and K_1 are chosen to be as large as possible, then X is irreducible. In addition, if X is weakly connected when* $\mathrm{Aut}(X)$ *is transitive, then* $\mathrm{Aut}(\Gamma)$ *is the complete group of natural automorphisms of* $\bigvee(X, Y)$.

PROOF. Set $\mathcal{P} = \mathcal{P}(H_i, K_i)$. Suppose $\Gamma \cong \bigvee(X, Y)$. Let $s_i \in S_i$. Then H_i is out-adjacent to $s_i H_{i+1} \subseteq s_i K_{i+1}$. As $\Gamma \cong \bigvee(X, Y)$, by Theorem 5.3, every vertex of K_i is out-adjacent to every vertex of $s_i K_{i+1}$. Let $k_i^{-1} \in K_i$ and $k_{i+1} \in K_{i+1}$. Then $(k_i^{-1} H_i, s_i k_{i+1} H_{i+1}) \in A(\Gamma)$ as $k_i^{-1} H_i$ is a vertex of K_i and $k_{i+1} H_{i+1}$ is a vertex in K_{i+1} and so $k_i s_i k_{i+1} \in S_i$. As k_i and k_{i+1} are arbitrary, $K_i s_i K_{i+1} \subseteq S_i$, and as s_i is arbitrary, S_i is a union of (K_i, K_{i+1})-double cosets.

Conversely, suppose $H_i \leq K_i \leq G$ such that $S_i = K_i S_i K_{i+1}$. Let $(xH_i, yH_{i+1}) \in A(\Gamma)$, where $x, y \in G$. Then $x^{-1}y \in S_i$, and as $S_i = K_i S_i K_{i+1}$, we see that $k_i^{-1} x^{-1} y k_{i+1} \in S_i$ for every $k_i \in K_i$ and $k_{i+1} \in K_{i+1}$. Then $(xk_i H_i, yk_{i+1} H_{i+1}) \in A(\Gamma)$ for all $k_i \in K_i$ and $k_{i+1} \in K_{i+1}$. So every vertex contained in xK_i is out-adjacent in Γ to every vertex in yK_{i+1} (here K_i and K_{i+1} are viewed as unions of cosets of H_i and H_{i+1}, respectively) for every $(xH_i, yH_{i+1}) \in A(\Gamma)$. As $\Gamma[\{i\} \times gK_i]$ has no arcs, by Theorem 5.3, $\Gamma = \bigvee(\Gamma/\mathcal{P}, Y)$.

Suppose that $K_i S_i K_{i+1} = S_i$. We now show that $X = B(G, K_0, K_1; S_0, S_1)$. First, $B(G, K_0, K_1; S_0, S_1)$ exists and is well defined by supposition. By the definition of $B(G, K_0, K_1; S_0, S_1)$, we have $V(X) = \mathcal{P} = V(B(G, K_0, K_1; S_0, S_1))$. For $a, b \in G$,

$$\begin{aligned}(aK_i, bK_{i+1}) \in A(B(G, K_0, K_1; S_0, S_1)) \\ \text{if and only if} \quad & a^{-1}b \in S_i \\ \text{if and only if} \quad & (aH_i, bH_{i+1}) \in A(B(G, H_0, H_1; S_0, S_1)) \\ \text{if and only if} \quad & (aK_i, bK_{i+1}) \in A(X).\end{aligned}$$

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This shows that $X = B(G, K_0, K_1; S_0, S_1)$. To see that $\Gamma = \bigvee(X, Y)$, observe that $\bigvee(X, Y)$ has vertex set $V(B(G, H_0, H_1; S_0, S_1))$ and arc set

$$\{(ak_iH_i, bk_{i+1}H_{i+1}) : (aK_i, bK_{i+1}) \in A(X) \text{ and } k_iH_i \in K_i/H_i, k_{i+1}H_{i+1} \in K_{i+1}/H_{i+1}\}.$$

As $(aK_i, bK_{i+1}) \in A(B(G, K_0, K_1; S_0, S_1))$, we see that $b = as_i$ for some $s_i \in S_i$, or $s_i = a^{-1}b$. The arc $(ak_iH_i, bk_{i+1}H_{i+1})$ is an arc of Γ if and only if $k_{i+1}^{-1}a^{-1}bk_{i+1} \in S_i$, which, as $S_i = K_iS_iK_{i+1}$, is true. Thus $\Gamma = \bigvee(X, Y)$.

Now additionally assume that K_0 and K_1 are chosen to be as large as possible. Let $\mathcal{E}$ be the unworthy partition of Γ. By Lemma 4.5, $\mathcal{E} \prec \mathcal{B}$. Let $\mathcal{E}_i$ be the set of elements of $\mathcal{E}$ contained in B_i. By Lemma 4.6, $\mathcal{E}_i$ is the set of orbits of the kernel of the action of F on B_{i+1}, where F is the setwise stabiliser of each block of $\mathcal{B}$, and is the set of left cosets of some subgroup $H_i \leq K_i \leq G$. By Lemma 3.3, $\Gamma/\mathcal{E}$ is irreducible. Also, as $K_iS_iK_{i+1} = S_i$, by the first part of this theorem, $\Gamma = \bigvee(\Gamma/\mathcal{P}, Y)$. We then have $\mathcal{P} \leq \mathcal{E}$ as $uR_\Gamma v$ for any $u, v \in Y_x$, $x \in X$. However, if $\mathcal{P} \neq \mathcal{E}$, then $\Gamma/\mathcal{P}$ is not irreducible, and so by the first part of this theorem, there is $K_i < K_i'$ such that $K_i'S_iK_{i+1}' = S_i$ and K_i is not the largest subgroup for which $K_iS_iK_{i+1} = S_{i+1}$, a contradiction. Thus $\mathcal{P} = \mathcal{E}$, and $X = \Gamma/\mathcal{P}$ is irreducible by Theorem 3.8.

If, in addition to K_0 and K_1 being chosen to be as large as possible, so that the unworthy partition of $V(\Gamma)$ is the join partition $\mathcal{P}(H_i, K_i)$, the bicoset digraph Γ is weakly connected when $\mathrm{Aut}(\Gamma)$ is transitive, then by Theorem 4.8,

$$\mathrm{Aut}(\Gamma/\mathcal{P}(H_i, K_i)) = \mathrm{Aut}\left(\bigvee(X, Y)\right) = \mathrm{Aut}(\Gamma)$$

is the complete group of natural automorphisms of $\bigvee(X, Y)$. □

Some observations are in order. First, it is possible that a bicoset digraph $\Gamma = B(G, H_0, H_1; S_0, S_1)$ is isomorphic to an X-join of Y, where $X = B(G', H_0', H_1'; S_0, S_1)$ is a bicoset digraph and Y is a set of empty digraphs, but there is no relationship at all between G and G', as the next example shows. So the result above does not capture *all* the ways a bicoset digraph can be isomorphic to an X-join of a set of empty digraphs, but rather only those where X is a quotient of Γ using a partition of $V(\Gamma)$ which is a $\mathcal{P}(H_i, K_i)$-join partition of $V(\Gamma)$.

EXAMPLE 5.5. The bicoset graph $\Gamma = B(\mathbb{Z}_5, 1, \mathbb{Z}_5; \{0\}, \{0\})$ is a star on six vertices (that is, it has six vertices with one vertex of degree 5 and five vertices of degree 1), and is isomorphic to the K_2-join of $\{K_1, \bar{K}_5\}$, where $\bar{K}_5$ is the complement of K_5. Let $X = B(\mathbb{Z}_2, 1, \mathbb{Z}_2; \{0\}, \{0\})$ (so X is the star on three vertices), and label the vertices of X with elements of $\{x, y, z\}$ such that the unique vertex of X of degree 2 is x. Set $Y_x = K_1$, $Y_y = \bar{K}_2$, $Y_z = \bar{K}_3$, and $Y = \{Y_x, Y_y, Y_z\}$. Then Γ is isomorphic to the X-join of Y, and is also a star on six vertices.

PROBLEM 5.6. Determine necessary and sufficient conditions for a bicoset digraph to be isomorphic to an X-join with a collection of empty digraphs.

The next result shows when a bicoset digraph is a wreath product.

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COROLLARY 5.7. *Let G be a group, $i \in \mathbb{Z}_2$, $H_i \leq K_i \leq G$ and $S_i \subseteq G$ such that $S_i = H_i S_i H_{i+1}$. Let $\Gamma = B(G, H_0, H_1; S_0, S_1)$ have arcs and let $X = \Gamma/\mathcal{P}(H_i, K_i)$. For $g \in G$, let $Y_{i,g}$ be the empty digraph on the left cosets of H_i contained in gK_i and $Y = \{Y_{i,g} : i \in \mathbb{Z}_2, g \in G\}$. Then $\Gamma \cong \Gamma/\mathcal{P}(H_i, K_i) \wr \bar{K}_m$ if and only if $S_i = K_i S_i K_{i+1}$ and $m = [K_0 : H_0] = [K_1 : H_1]$. Also, if K_0 and K_1 are chosen to be as large as possible, then X is irreducible. In addition, if X is weakly connected when* Aut(X) *is transitive, then* Aut$(\Gamma) \cong$ Aut$(X) \wr S_m$.

PROOF. By Theorem 5.4, $\Gamma \cong \bigvee(X, Y)$. The digraph $\bigvee(X, Y) \cong \Gamma/\mathcal{P} \wr \bar{K}_m$ if and only if each $Y_{i,j}$ is isomorphic, which occurs if and only if $m = [K_0 : H_0] = [K_1 : H_1]$. □

It is possible for a bicoset digraph Γ to be simultaneously isomorphic to a wreath product and an X-join of empty digraphs that is not written as a wreath product.

EXAMPLE 5.8. Let $n \geq 3$ and G be a group of order n. Then $K_{n,n} \cong B(G, 1, 1; G, G) \cong K_2 \wr \bar{K}_n$. Let $\{S_1, S_2\}$ be a partition of G into two subsets of different sizes, $P_1 = \{(0, p_1) : p_1 \in S_1\}$, $P_2 = \{(0, p_2) : p_2 \in S_2\}$ and $P_3 = \{(1, g) : g \in G\}$. Set $\mathcal{P}_0 = \{P_1, P_2\}$ and $\mathcal{P}_1 = \{P_3\}$. Then $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ is a partition of $V(B(G, 1, 1; G, G))$. Let X be the digraph with $V(X) = \mathcal{P}$ and arc set $\{(P_1, P_3), (P_3, P_1), (P_2, P_3), (P_3, P_2)\}$ and let Y_{P_i} be the digraph with vertex set P_i and no arcs, $i = 1, 2, 3$. Then the vertex set of $B(G, 1, 1; G, G) = \bigvee(X, Y)$ is not the vertex set of a wreath product of two graphs, and so is not a wreath product as $|S_1| \neq |S_2|$ (it is though, of course, *isomorphic* to a wreath product).

When $H_0 = H_1 = \{1_G\}$, $B(G, H_0, H_1, S) \cong$ Haar(G, S), and we have a special case of Theorem 5.4 for Haar digraphs.

COROLLARY 5.9. *Let G be a group, $i \in \mathbb{Z}_2$, $K_i \leq G$ and $S_i \subseteq G$. Let $\Gamma =$* Haar(G, S_0, S_1) *have arcs and let $X = \Gamma/\mathcal{P}(1, K_i)$. For $g \in G$, let $Y_{i,g}$ be the empty digraph on the left cosets of gK_i and $Y = \{Y_{i,g} : i \in \mathbb{Z}_2, g \in G\}$. Then $\Gamma = \bigvee(X, Y)$ if and only if $S_i = K_i S_i K_{i+1}$. If $S_i = K_i S_i K_{i+1}$, then $X = B(G, K_0, K_1; S_0, S_1)$. Also, if the K_i are chosen to be as large as possible, then X is irreducible. In addition, if X is weakly connected when* Aut(Γ) *is transitive, then* Aut(Γ) *is the group of natural automorphisms of $\bigvee(X, Y)$.*

The next corollary shows when a Haar digraph is isomorphic to a wreath product.

COROLLARY 5.10. *Let G be a group, $i \in \mathbb{Z}_2$, $K_i \leq G$ and $S_i \subseteq G$. Let $\Gamma =$* Haar(G, S_0, S_1) *have arcs and let $X = \Gamma/\mathcal{P}(1, K_i)$. For $g \in G$, let $Y_{i,g}$ be the empty digraph on the left cosets of gK_i and $Y = \{Y_{i,g} : i \in \mathbb{Z}_2, g \in G\}$. Then $\Gamma = \Gamma/\mathcal{P} \wr \overline{K}_m$ if and only if $S_i = K_i S_i K_{i+1}$ and $m = |K_0| = |K_1|$. If $S_i = K_i S_i K_{i+1}$ and $|K_0| = |K_1|$, then $X = B(G, K_0, K_1; S_0, S_1)$. Also, if the K_i are chosen to be as large as possible, then X is irreducible. In addition, if X is weakly connected when* Aut(Γ) *is transitive, then* Aut$(\Gamma) \cong$ Aut$(X) \wr S_m$.

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