THE LAMINAR BOUNDARY-LAYER EQUATION: A METHOD OF SOLUTION BY MEANS OF AN AUTOMATIC COMPUTER

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ABSTRACT. A method, very suitable for use with an automatic computer, of solving the Hartree-Womersley approximation to the incompressible boundary-layer equation is developed. It is based on an iterative process and the Choleski method of solving a simultaneous set of linear algebraic equations. The programming of this method for an automatic computer is discussed. Tables of a solution of the boundary-layer equation in a region upstream of the separation point are given. In the upstream neighbourhood of separation this solution is compared with Goldstein's asymptotic solution and the agreement is good.

1. *Introduction.* The equation for steady laminar boundary-layer flow of an incompressible fluid is, in reduced dimensionless variables,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

and the boundary conditions are

$$\left.\begin{aligned} u=0, \quad v=0 \quad &\text{at} \quad y=0, \\ u\to U(x) \quad &\text{as} \quad y\to\infty. \end{aligned}\right\} \tag{3}$$

$U(x)$ is the mainstream velocity, and we are interested in a solution for

$$U(x)=1-\tfrac{1}{8}x. \tag{4}$$

The pressure gradient $\partial p/\partial x$ is given by the relation $\partial p/\partial x=-U(dU/dx)$, so that $\partial p/\partial x$ is positive, and under this condition the flow separates from the solid boundary at a point a certain distance downstream from the beginning of the solid boundary.

Howarth (5), Hartree (3), and others have obtained solutions of the boundary-layer equation for the mainstream velocity of (4). Hartree, in a hand computation, used an approximation introduced by Hartree and Womersley (2) for solving an equation of heat conduction. The resulting ordinary differential equation was then solved by a method using step-by-step integration which in that context was very laborious and is unsuitable for automatic computation.

Hartree (3) and Jones (6) compared, near separation, Hartree's numerical solution with the theoretical solution of Goldstein (1) and found that the comparison was not completely satisfactory.

The results of the present paper are the outcome of work undertaken with the following purposes in mind:

(i) to investigate numerical methods of solving the boundary-layer equation by means of an automatic computer;

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(ii) to obtain a more accurate solution of the boundary-layer equation for the case (4), particularly in the neighbourhood of the separation point;

(iii) to compare this numerical solution in the upstream neighbourhood of the separation point with the theoretical asymptotic solution based on the assumption of a singularity at separation.

2. *Method of solution.* In the Hartree-Womersley approximation to the boundary-layer equation, derivatives in the x-direction are replaced by finite differences while other quantities are replaced by averages. We suppose that the solution u at x_1 is known and is denoted by u_1, and that we wish to find the solution u_2 at x_2. Then replacing v in (1) by $-\int(\partial u/\partial x)\,dy$ from (2) and making the necessary finite-difference approximations in (1) we have

$$\tfrac{1}{2}(u_2+u_1)\frac{u_2-u_1}{x_2-x_1}-\int_0^y \frac{u_2-u_1}{x_2-x_1}\,dy\,\tfrac{1}{2}(u_2'+u_1') = \frac{1}{2}\left\{\left(U\frac{dU}{dx}\right)_2+\left(U\frac{dU}{dx}\right)_1\right\}+\tfrac{1}{2}(u_2''+u_1''), \qquad (5)$$

where dashes denote differentiation with respect to y.

2·1. Equation (5) is an ordinary non-linear differential equation with two-point boundary conditions for u_2 if u_1 and its derivatives in y are known. Hartree used a step-by-step method of integration which requires covering the range of integration several times until the third boundary condition $(u_2')_0$, the value of u_2' at $y=0$, is found such that the boundary condition at $y=\infty$ is satisfied. The formula used for each step of the integration was

$$\delta f=\int_{y_0}^{y_0+\delta y} f'\,dy=\frac{\delta y}{2}\{f_0'+f_1'-\tfrac{1}{12}(\delta^2 f_0'+\delta^2 f_1')\}, \qquad (6)$$

where the subscripts 0 and 1 indicate values at the ends of the interval δy. Formula (6) is suitable for the solution of (5) by means of hand computation because it is simple and easily applied in advancing the integration through one step in y. However, the integration of (5) in the y-direction by a step-by-step method has the disadvantage that the solution is very sensitive to small changes in the initial conditions at $y=0$. Hartree found in his computation that over the range in y in which we are interested the difference between two trial solutions may increase by a factor of 10^{15}. It was therefore necessary to interpolate at some intermediate y between the solutions calculated with different initial values of $(u_2')_0$, and to start a new integration from the interpolated values. This interpolation process had to be repeated as often as 10 or 20 times before trial solutions could be obtained over the whole range and near enough together for the solution satisfying the conditions at infinity to be interpolated between them. This sensitivity of the step-by-step solution becomes greater the smaller the values of (x_2-x_1). The more accurate we try to make the solution by reducing the truncation error of the finite-difference approximation in the x-direction, the more difficult becomes the solution of the resulting ordinary differential equation by step-by-step methods.

A step-by-step method of integration with the difficulty of this very sensitive behaviour is unsuitable for use with an automatic computer. A program constructed to carry out this procedure completely automatically would be cumbersome and

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inefficient since it would contain a large proportion of instructions of an administrative or organizational character. When considering the solution of a problem by means of an automatic computer it is not a serious disadvantage if the computational procedure is complicated, but the prime consideration is that the application of the computational procedure be straightforward or repetitive and involve little or no judgement.

2·2. Rearranging (5) and putting $w = u_2 + u_1$ we have

$$w'' + \frac{1}{(x_2 - x_1)}\left(\int_0^y w\,dy - 2\int_0^y u_1\,dy\right)w' - \frac{1}{(x_2 - x_1)}(w - 2u_1)\,w = 2P, \tag{7}$$

where

$$2P = \left\{\left(U\frac{dU}{dx}\right)_2 + \left(U\frac{dU}{dx}\right)_1\right\}.$$

This is a non-linear third-order differential equation with boundary conditions

$$\left.\begin{aligned} w &= 0 \quad \text{at} \quad y = 0, \\ w &= U_2 + U_1 \quad \text{at} \quad y = \infty. \end{aligned}\right\} \tag{8}$$

(5) is put in the form of (7) so that in solving for u_2 we need only u_1 and $\int u_1\,dy$, and no derivatives of u_1. When w has been determined by the solution of (7), u_2 is given by $u_2 = w - u_1$.

Now if w_m is an approximation to the solution of (7), let a function w_{m+1} be determined by the linear second-order differential equation

$$w''_{m+1} + \frac{1}{(x_2 - x_1)}\left\{\int_0^y (w_m - 2u_1)\,dy\right\}w'_{m+1} - \frac{1}{(x_2 - x_1)}(w_m - 2u_1)\,w_{m+1} = 2P. \tag{9}$$

Equation (9) defines an iterative process for finding the solution w of (7) if $\lim_{m\to\infty} w_m$ exists.

We now make appropriate finite-difference substitutions for the derivatives and integrals in (9) at N mesh points in the range of y. This gives a set of N simultaneous linear algebraic equations for w_{mj} $(j = 1, 2, \ldots, N)$, the values of w_m at mesh points. These equations were solved by the Choleski matrix method which will be described in greater detail later. The procedure was programmed for the EDSAC*, but successive iterations did not converge. On examining the process of iteration determined by (9) we can see why this process is not likely to converge. A small percentage error in w_m results in a large percentage error in the coefficients of this linear differential equation. For example, if we take $2u_1$ as an initial estimate of w_m, equation (9) becomes

$$w''_{m+1} = 2P.$$

Then w_{m+1} is a parabola through the origin and the positive value of velocity at 'infinity', and has positive curvature, and is therefore negative over a large part of the range. Hence, although $2u_1$ as an estimate of $(u_2 + u_1)$ may be in error by only a few per cent, the result of the first iteration does not even have the right sign over part of the range.

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2·3. The following iteration procedure was tried next:

$$w''_{m+1}-\frac{w_m}{(x_2-x_1)}w_{m+1}+\frac{w'_m}{(x_2-x_1)}\int_0^y w_{m+1}dy = 2P-\frac{1}{(x_2-x_1)}\left\{2u_1w_m-\int_0^y 2u_1dy\,w'_m\right\}. \quad (10)$$

(10) is effectively a third-order linear differential equation if w_m is a known approximation to the solution w of (7).

We make the following finite-difference replacements (where the second subscript j indicates the value at the jth mesh point):

$$\left.\begin{aligned} w''_{m+1,j} &= \frac{1}{h^2}(w_{m+1,j+1}-2w_{m+1,j}+w_{m+1,j-1}),\\ \int_0^{j\delta y} w_{m+1}dy &= h(w_{m+1,1}+w_{m+1,2}+\ldots+w_{m+1,j-1}+\tfrac{1}{2}w_{m+1,j}),\\ w'_{mj} &= \frac{1}{2h}(w_{m,j+1}-w_{m,j-1}) = \frac{\alpha_{mj}}{h},\\ 2u_{1j} &= \delta_j,\\ \int_0^{j\delta y} 2u_1dy &= h(\delta_1+\delta_2+\ldots+\delta_{j-1}+\tfrac{1}{2}\delta_j) = h\gamma_j. \end{aligned}\right\} \quad (11)$$

Multiplying (10) through by (x_2-x_1), putting $\beta=(x_2-x_1)/h^2$, and substituting the expressions of (11) in (10), after rearranging we have for the equation at the jth point

$$\alpha_{mj}w_{m+1,1}+\alpha_{mj}w_{m+1,2}+\ldots+\alpha_{mj}w_{m+1,j-2}+A_{mj}w_{m+1,j-1}+B_{mj}w_{m+1,j}+\beta w_{m+1,j+1}=C_{mj}, \quad (12)$$

where

$$\begin{aligned} A_{mj} &= \alpha_{mj}+\beta,\\ B_{mj} &= \tfrac{1}{2}\alpha_{mj}-2\beta w_{mj},\\ C_{mj} &= 2P(x_2-x_1)-\delta_j w_{mj}+\gamma_j\alpha_{mj}. \end{aligned}$$

The equations at the ends of the range at $j=1$ and $j=N$ are, respectively,

$$\left.\begin{aligned} B_{m1}w_{m+1,1}+\beta w_{m+1,2} &= C_{m1},\\ \alpha_{mN}w_{m+1,1}+\alpha_{mN}w_{m+1,2}+\ldots+\alpha_{mN}w_{m+1,N-2}+A_{mN}w_{m+1,N-1}&\\ +B_{mN}w_{m+1,N} &= C_{m+1,N}-\beta(U_2+U_1). \end{aligned}\right\} \quad (13)$$

Equations (12) and (13) are a set of N simultaneous linear algebraic equations for the values of the $(m+1)$th iteration w_{m+1} of equation (10) at the N mesh points. This process was found to converge.

This matrix method of solving (7) has one very important advantage over a step-by-step method of solving (7). Hartree found in using step-by-step integration in the y-direction that in order to get greater accuracy in the solution of the boundary-layer equation by reducing the step in the x-direction, the process of integration in the y-direction became more difficult. However, in the case of the matrix method of integration the reverse is true: when (x_2-x_1) is made smaller, the convergence of the iterative process is more rapid. Decreasing (x_2-x_1) decreases β and results in an increase of the relative magnitude of the diagonal elements of the matrix of coefficients of equations (12) and (13). Generally it is observed that iterative processes involving matrices converge more rapidly the larger the diagonal elements.

2·4. There is a truncation error arising from the use of finite intervals in the y-direction. By performing two integrations w_h, w_{2h} of (7) at intervals h and $2h$ in y

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we may remove the leading term of the truncation error by the h^2-extrapolation process of Richardson (7). The h^2-extrapolated solution is $w_h + \frac{1}{3}(w_h - w_{2h})$. The validity of the procedure was tested by doing three integrations at different intervals, and it was found that the differences between the solutions were in the correct ratio.

In Hartree's solution (3) of the boundary-layer equation it was assumed that the process of h^2-extrapolation could be applied in the x-direction, thereby reducing the truncation error considerably. However, in the recent calculations on the EDSAC it did not appear that the largest term of the truncation error could be removed by h^2-extrapolation. The interval $x = 0{\cdot}80$ to $x = 0{\cdot}88$ was covered in 1 step, 2 steps, 4 steps and 8 steps. The solutions at $x = 0{\cdot}88$ indicate that h^2-extrapolation cannot be applied. The theoretical analysis of the h^2-extrapolation applied to the Hartree-Womersley approximation showed that this process should remove the leading term in the truncation error; however the coefficients of the higher order terms may be very large (particularly in the neighbourhood of the singularity). The analysis followed that of Hartree and Womersley (2) for the heat conduction equation. Therefore no extrapolation procedure was used to attempt to reduce the truncation error due to finite intervals in the x-direction.

3. *Programming.* 3·1. We shall now discuss the programming of the solution of (7) by the iterative procedure of (10). For each iteration we must solve the set of simultaneous linear algebraic equations (12) and (13). We write the set of equations (12) and (13) as a matrix equation

$$\mathbf{A}_m \mathbf{w}_{m+1} = \mathbf{C}_m, \tag{14}$$

where $\mathbf{A}_m$ is the matrix

$$\begin{pmatrix} B_{m1} & \beta & 0 & 0 & 0 & 0 & . \\ A_{m2} & B_{m2} & \beta & 0 & 0 & 0 & . \\ \alpha_{m3} & A_{m3} & B_{m3} & \beta & 0 & 0 & . \\ \alpha_{m4} & \alpha_{m4} & A_{m4} & B_{m4} & \beta & 0 & . \\ . & . & . & . & . & . & . \\ \alpha_{m,N-1} & . & . & \alpha_{m,N-1} & A_{m,N-1} & B_{m,N-1} & \beta \\ \alpha_{mN} & . & . & . & \alpha_{mN} & A_{mN} & B_{mN} \end{pmatrix}$$

and $\mathbf{w}_{m+1}$ is the column vector

$$\mathbf{w}_{m+1} = \{w_{m+1,1}, w_{m+1,2}, \ldots, w_{m+1,N}\},$$

and $\mathbf{C}_m$ is the column vector

$$\mathbf{C}_m = \{C_{m1}, C_{m2}, \ldots, C_{m,N-1}, [C_{mN} - \beta(U_2 + U_1)]\}.$$

The matrix equation (14) was solved by the Choleski method (Hartree (4), p. 164). The matrix $\mathbf{A}_m$ is factorized into the product of a lower triangular matrix $\mathbf{L}$ times an upper triangular matrix $\mathbf{U}$. Then the matrix equation $\mathbf{L}\mathbf{y} = \mathbf{C}_m$ is solved for the intermediate variable $\mathbf{y}$ and then $\mathbf{U}\mathbf{w}_{m+1} = \mathbf{y}$ is solved for the solution $\mathbf{w}_{m+1}$. $\mathbf{L}$ has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . \\ l_{21} & 1 & 0 & 0 & . \\ l_{31} & l_{32} & 1 & 0 & . \\ . & . & . & . & . \end{pmatrix}$$

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and **U** is given by

$$\begin{pmatrix} u_1 & \beta & 0 & 0 & . \\ 0 & u_2 & \beta & 0 & . \\ 0 & 0 & u_3 & \beta & . \\ . & . & . & . & . \end{pmatrix}.$$

The y's are given by

$$\left.\begin{aligned} y_1 &= C_{m1}, \\ y_2 &= C_{m2} - l_{21} y_1, \\ y_3 &= C_{m3} - l_{31} y_1 - l_{32} y_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}\right\} \tag{15}$$

and the w_{m+1}'s are given by

$$\left.\begin{aligned} w_{m+1,N} &= y_N / u_N, \\ w_{m+1,N-1} &= (y_{N-1} - \beta w_{m+1,N})/u_{N-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ w_{m+1,1} &= (y_1 - \beta w_{m+1,2})/u_1. \end{aligned}\right\} \tag{16}$$

The number of intervals taken in the y-direction was as much as eighty, and hence in programming this process for an automatic computer one has to keep the total number of storage locations simultaneously in use as small as is practicable. If there are N intervals, then there are more than $\frac{1}{2}N^2$ elements of **L**, but we do not need storage for any of these, since when $l_{j,i-1}$ is calculated it can be used immediately in the calculation of l_{ji} and y_j in (15), and then it is no longer required. The amount of storage required can be considerably reduced by calculating the elements of $\mathbf{A}_m$ as they are required in the factorization process. We require N locations for the velocity distribution u_{1j} at x_1 which must be preserved for the succeeding iterations. We also require $2N$ locations for u_j, y_j which are required for (16). Therefore the minimum storage needed if α_{mj}, A_{mj}, B_{mj}, C_{mj} are computed from u_{1j} and w_{mj} as they are required is a total of $3N$ locations. This also results in a saving of time since the machine is required to perform fewer copying operations.

3·2. In this method of solving the boundary-layer equation a few small auxiliary programs are needed in conjunction with the relatively large program for performing the process expressed by (10). The programs for the whole process of taking the solution from x_1 to x_2 are:

(i) solution of (10) for $\delta y = 0{\cdot}2$, initial estimate $w_0 = 2u_1$;

(ii) interpolation of results of (i) at intervals of $\delta y = 0{\cdot}1$;

(iii) solution of (10) for $\delta y = 0{\cdot}1$, initial estimate using interpolation of (ii);

(iv) h^2-extrapolation in y-direction on results of (i) and (iii), and calculation of $u_2 = w - u_1$.

Forty points in the range of $y = 0$ to 8 were taken in (i) and eighty in (iii). The number of operations in solving (10) by the matrix method is approximately proportional to N^2 since the matrix **A** contains elements in the entire lower triangle. Therefore (iii) takes about 4 times as long as (ii), and so it is worth while to have a fairly

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accurate initial estimate for (iii). Each iteration of (i) takes about $2\frac{1}{2}$ min. including output punching of results and each iteration of (iii) takes about 8 min. At about $x = 0{\cdot}8$, (i) required six to eight iterations for eight decimals (scaled) and (iii) took from three to four iterations. Near separation many more iterations were required: ten to fifteeen for (i) and about eight for (iii). Note that this behaviour does not contradict the statement made near the end of § 2·3 that for smaller β fewer iterations are required; for near separation the solution is becoming singular.

4. *Solution in the upstream neighbourhood of the separation point.* The solution of the laminar boundary-layer equation by means of the Hartree-Womersley approximation was carried out from $x = 0{\cdot}8$ to separation. The velocity distribution at $x = 0{\cdot}8$ was taken from Hartree's results* (3). The integration was carried out from $x = 0{\cdot}80$ to $x = 0{\cdot}94$ in steps of 0·01, and then from 0·94 to 0·956 at the points 0·940, 0·945, 0·950, 0·954, 0·956. The following steps were then taken:

(i) 0·956 to 0·958,
(ii) 0·956 to 0·957,
(iii) 0·957 to 0·958,
(iv) 0·956 to 0·9584,
(v) 0·957 to 0·9584.

Tables of the velocity distribution u at the points $x = 0{\cdot}956$, 0·957, 0·958, and 0·9584, appear in Table 1. Not all of the results have been tabulated.

4·1. We shall discuss briefly now the accuracy of the solution. First it may be noted that near separation the value of the boundary condition at $y = \infty$ has very little effect on the solution nearer $y = 0$: for example, a change of 10 % in the solution at $y = 8{\cdot}0$ does not effect the solution in the sixth decimal place at $y = 6{\cdot}0$.

From the magnitude of the h^2-extrapolation correction in the y-direction we may conclude that as far as integration in the y-direction is concerned, the velocity distribution is determined to six decimal places, except possibly to only five from $x = 0{\cdot}956$ to separation.

The truncation error due to finite intervals in the x-direction has a maximum error of 5 in the fifth decimal place, and near $y = 0$ there are nearly six decimal places correct, particularly upstream from separation.

In claiming this degree of accuracy in the solution we have said nothing as yet about the accuracy of Hartree's solution (3) at $x = 0{\cdot}8$, which was used as a starting velocity distribution. The value of $8u$ is given to four decimal places (nearly five decimal places in u), but since the validity of the h^2-extrapolation process is in question these results may only represent the actual velocity distribution in some places to about three decimal places in u. However, Hartree's figures can be said to represent some velocity distribution to five decimal places at $x = 0{\cdot}8$. The essential nature of the solution in the neighbourhood of the separation point should not be altered by slightly different velocity distributions at $x = 0{\cdot}8$.

It may be asked: How can we claim to have the solution correct to six decimal places near $y = 0$ when the original velocity distribution at $x = 0{\cdot}80$ is only given to

* There are two typographical errors in Hartree's results at $x = 0{\cdot}80$. At $y = 2{\cdot}9$, 5·0442 should read 5·0447 and at $y = 3{\cdot}1$, 5·3932 should read 5·3832.

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five decimal places? After several steps in the integration the irregularities in the sixth place are removed everywhere in y. This is not due to the method of integration, but to the fact that the boundary layer equation tends to smooth out perturbations in the flow as one proceeds downstream.

Table 1. *Tangential velocity* **u**

y \ x	0·9560	0·9570	0·9580	0·9584
	0·	0·	0·	0·
0·0	000000	000000	000000	000000
0·1	001513	001279	000971	000760
0·2	004124	003659	003040	002618
0·3	007832	007136	006205	005572
0·4	012630	011705	010462	009618
0·5	018514	017358	015806	014752
0·6	025473	024088	022227	020964
0·7	033496	031882	029715	028244
0·8	042568	040729	038257	036582
0·9	052675	050612	047840	045961
1·0	063797	061512	058444	056365
1·1	075912	073411	070050	067775
1·2	088997	086282	082636	080169
1·3	103023	100100	096175	093521
1·4	117960	114835	110639	107802
1·5	133774	130453	125994	122982
1·6	150428	146918	142207	139025
1·7	167882	164191	159237	155894
1·8	186089	182226	177043	173546
1·9	205004	200978	195578	191936
2·0	224573	220396	214793	211016
2·2	265455	261008	255047	251032
2·4	308254	303591	297340	293134
2·6	352442	347618	341156	336810
2·8	397448	392528	385937	381507
3·0	442676	437725	431092	426637
3·2	487517	482601	476017	471598
3·4	531367	526554	520108	515783
3·6	573653	569005	562781	558608
3·8	613849	609424	603501	599531
4·0	651498	647348	641792	638070
4·4	717790	714297	709623	706497
4·8	770821	768055	764357	761889
5·2	810566	808505	805754	803926
5·6	838419	836968	835042	833770
6·0	856649	855677	854398	853564
6·4	867787	867157	866342	865823
6·8	874136	873730	873221	872909
7·2	877511	877240	876915	876729
7·6	879193	878996	878773	878655
8·0	880500	880375	880250	880200

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At the same time, the process of integration tends to increase small irregularities in the solution near $y = 0$. It is desirable to remove these irregularities before they become too large, because the values of u near $y = 0$ are required to be accurate in order to determine the skin friction $(\partial u/\partial y)_0$ accurately. Smoothing without using any external information in the process is difficult to justify on the grounds of increasing the accuracy of a solution. However, in this instance, we have the fact that

$$\left(\frac{\partial^3 u}{\partial y^3}\right)_0 = 0$$

(by differentiating the boundary-layer equation and putting $u = 0$ at $y = 0$). The values of u near $y = 0$ were adjusted to satisfy this condition. The smoothing resulted in a change in the values of u of at most two in the sixth decimal place. The process was carried out at $x = 0{\cdot}88$, $x = 0{\cdot}92$, $x = 0{\cdot}954$ and $x = 0{\cdot}958$.

4·2. The value of the skin friction $(\partial u/\partial y)_0$ is required in order to compare the numerical solution with the theoretical solution. In the step-by-step method of integrating equation (5), $(\partial u/\partial y)_0$ must be specified in order to carry out an integration and, indeed, $\partial u/\partial y$ over the whole range of y is determined along with u. However, in the method of solution of this paper only u is found directly. Consequently $(\partial u/\partial y)_0 = u_0'$ was found in the following manner. The Taylor series for u expanded about $y = 0$ is

$$u = u_0 + u_0' y + \frac{u_0''}{2!}y^2 + \frac{u_0'''}{3!}y^3 + \frac{u_0^{(\mathrm{iv})}}{4!}y^4 + \ldots.$$

Now $u_0 = 0$, and from the boundary-layer equation $u_0'' = \frac{1}{8}(1 - \frac{1}{8}x)$, and by differentiating the boundary-layer equation we have that $u_0''' = 0$. Then considering up to terms in y^4 and rearranging we have

$$\frac{1}{y}\{u - \tfrac{1}{16}(1 - \tfrac{1}{8}x)\, y^2\} = u_0' + \tfrac{1}{4}u_0^{(\mathrm{iv})} y^3. \tag{17}$$

The left-hand side of equation (17) was plotted against y^3 for $y = 0{\cdot}1\,(0{\cdot}1)\,1{\cdot}0$. The points $y = 0{\cdot}4\,(0{\cdot}1)\,1{\cdot}0$ lie on a straight line as nearly as can be plotted; while in going from about 0·4 to zero the plotted points gradually depart slightly from this straight line. For y near zero u is small and for the same absolute error in u the percentage error in the left-hand side of (17) for small y is large. The point at which the straight line intersects the vertical axis gives the value of u_0' and this could be determined to within about one unit in the sixth decimal place. In Table 2 the values of $(\partial u/\partial y)_0$ determined by the above method are given along with Hartree's values.

Table 2. *Skin friction* $(\partial u/\partial y)_0$

x	Leigh	Hartree
0·940	0·028917	—
0·945	0·024164	—
0·949	—	0·0201
0·950	0·018668	—
0·954	0·013157	—
0·956	0·009618	0·0097
0·957	0·007307	—
0·958	0·004200	0·0050
0·9584	0·002088	—

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4·3. The present calculations verify previous findings that near separation point x_s,

$$\left(\frac{\partial u}{\partial y}\right)_0 \propto (x_s - x)^q, \tag{18}$$

where $0 < q < 1$. By plotting $\log (\partial u/\partial y)_0$ against $\log (x_s - x)$ for different values of x_s, or by plotting $\{(\partial u/\partial y)_0\}^{1/q}$ against x for different values of q, Hartree (3) found that his results indicated a value of q definitely greater than $\frac{1}{2}$ and less than $\frac{3}{4}$, and that the values of $(\partial u/\partial y)_0$ from $x = 0{\cdot}8$ onwards were represented very closely by (18) with $q = 0{\cdot}6$. The same analysis for the present results indicates a value of q closer to $\frac{1}{2}$, but even with more values of $(\partial u/\partial y)_0$ than Hartree had, and a value at $x = 0{\cdot}9584$, it is difficult by these methods to establish the limiting behaviour of $(\partial u/\partial y)_0$ with any certainty.

Goldstein (1) made an analytical examination of the flow in the upstream neighbourhood of the separation point as a result of Hartree's calculations, and Jones (6) carried out certain numerical investigations in connexion with the theoretical work of Goldstein and compared the numerical and analytic solutions.

Goldstein showed that q must be $\frac{1}{2}$ and that the velocity near and upstream of separation is given by

$$u_1 = 2\xi^2[f_0'(\eta) + (\alpha_1 \xi) f_1'(\eta) + (\alpha_1 \xi)^2 f_2'(\eta) + \ldots], \tag{19}$$

where $\xi = x_1^{\frac{1}{2}}$ and $\eta = y_1/2^{\frac{1}{2}} x_1^{\frac{1}{2}}$, and quantities with the subscript 1 are non-dimensional in terms of the mainstream velocity at separation. x_1 is the non-dimensional distance from separation, reckoned positive upstream. The f's are solutions of ordinary differential equations in η. Hence the solution is determined apart from the parameter α_1 which Goldstein suggests is probably determined uniquely by the condition $u_1 \to 1$ as $y_1 \to \infty$.

For a given mainstream velocity and hence a given pressure distribution, the pressure distribution may or may not be independent of the position of separation x_s. When separation takes place, the non-dimensional velocity distribution at and near separation is independent of the initial distribution of u for the former pressure distribution, and for the others it is the same for all initial distributions of u for which separation takes place at the same value of x. This suggests that what has been found is an *asymptotic* solution at and near separation, and that all solutions of the boundary-layer equation that have a separation point behave asymptotically in the same way near separation.

Jones (6) has found the first few terms in the expansion for $(\partial u/\partial y)_0$ in a suitable form to be

$$\frac{100\,(\partial u/\partial y)_0}{\{100(x_s - x)\}^{\frac{1}{2}}} = 3{\cdot}3169\,[\alpha_1 + 0{\cdot}34526\,\{100(x_s - x)\}^{\frac{1}{2}}\alpha_1^2 + 0{\cdot}12478\,\{100(x_s - x)\}^{\frac{1}{2}}\alpha_1^3 + 0{\cdot}0596\,\{100(x_s - x)\}^{\frac{3}{2}}\alpha_1^4 + \ldots]. \tag{20}$$

He has also tabulated f_0, f_1, f_2, f_3, f_4, and f_5 and their first derivatives, and $u_1/2\xi^2$ as a function of η for several values of $\alpha_1 \xi$.

4·4. Our numerical solution can be compared with the asymptotic solution in the upstream neighbourhood of the separation point by fitting equation (20) to the values of $(\partial u/\partial y)_0$ and by comparing velocity profiles.

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In Fig. 1 $100(\partial u/\partial y)_0/\{100(x_s-x)\}^{\frac{1}{2}}$ has been plotted against $\{100(x_s-x)\}^{\frac{1}{2}}$ for different values of x_s. The quantity $100(\partial u/\partial y)_0/\{100(x_s-x)\}^{\frac{1}{2}}$ is very sensitive to changes in x_s for small values of (x_s-x). The broken lines are curves plotted from the right-hand side of (20) for various values of α_1. In order that the numerical solution should join up with the asymptotic solution, x_s should be between about 0·958540 and 0·958545 and α_1 between about 0·485 and 0·495.

For the moment we will consider Hartree's results. His velocity distributions show little disagreement with the present results except at $x = 0{\cdot}958$ where the difference is about 15 %, and this difference is seen in the values of $(\partial u/\partial y)_0$ of Table 2. Values of

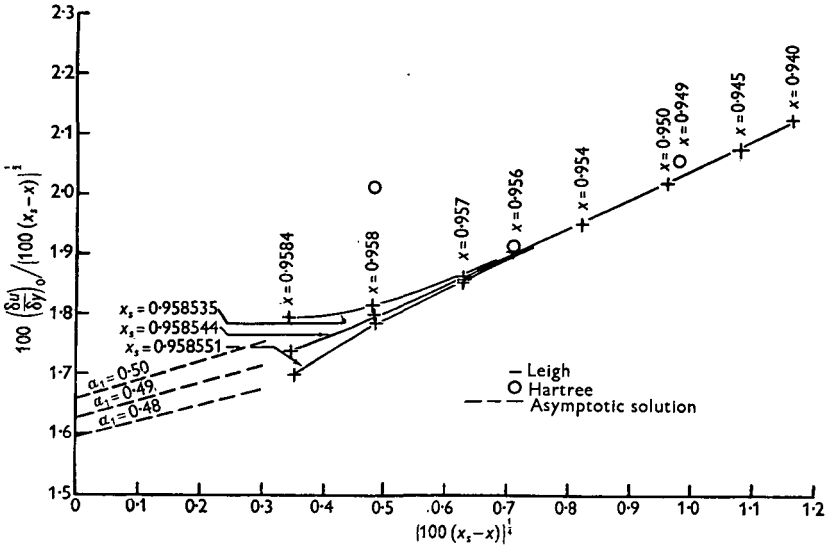


Fig. 1. Skin friction curves.

$100(\partial u/\partial y)_0/\{100(x_s-x)\}^{\frac{1}{2}}$ using Hartree's results for $(\partial u/\partial y)_0$ at $x = 0{\cdot}949$, 0·956, and 0·958 have been plotted in Fig. 1, for $x_s = 0{\cdot}958544$. The point for $x = 0{\cdot}958$ lies far from the curve for $x_s = 0{\cdot}958544$ of the present calculations, while the points for $x = 0{\cdot}956$ and 0·949 lie nearly on this curve as would the remainder of Hartree's results for $x < 0{\cdot}94$. Hence the 'bump' at about $x = 0{\cdot}95$ in the curve of

$$100\{2(\partial u/\partial y)_0\}/\{100(x_s-x)\}^{\frac{1}{2}} \quad \text{against} \quad \{100(x_s-x)\}^{\frac{1}{2}}$$

on p. 19 of his report (3) is due to the velocity distribution at $x = 0{\cdot}958$ being greatly in error leading to an error in the determination of x_s.

Jones ((6), Table 4) compared velocity profiles upstream of separation at $x = 0{\cdot}956$ for an $x_s = 0{\cdot}959$, using Hartree's velocity profile, and found a value of α_1 of about 0·47. However, the results of the present calculation differ by a unit in the fourth decimal place from Hartree's results at $x = 0{\cdot}956$ quoted by Jones, and also the value of $x_s = 0{\cdot}959$ could not be determined very accurately. Hence the value of α_1 so derived cannot be expected to be very good.

In the present calculations we are almost certain that the velocity profiles are correct to five decimal places near $y = 0$ and with a good value of x_s we are in a position

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to make a definitive comparison. It appears that the most profitable comparison can be made at $x = 0{\cdot}958$. Velocity profiles were compared at $x = 0{\cdot}956$ for various α_1 and by varying x_s, but the asymptotic solution could not be made to coincide with the true solution; it must be concluded that $\alpha_1 \xi$ is too large for the asymptotic solution to be approximated by the first few terms of the series. Profiles were also compared at $x = 0{\cdot}9584$ and here agreement was very good to five decimals; but $\alpha_1 \xi$ is small and the asymptotic solution here is not very sensitive to changes in α_1, and furthermore x_s could not be determined any more accurately than by the skin friction curve of Fig. 1.

Table 3. *Comparison of velocity profiles at* $x = 0{\cdot}958$

y	True solution	Asymptotic solution			
		$x_s = 0{\cdot}958544$			$x_s = 0{\cdot}958540$
		$\alpha_1 = 0{\cdot}490$	$\alpha_1 = 0{\cdot}492$	$\alpha_1 = 0{\cdot}495$	$\alpha_1 = 0{\cdot}492$
0	0	0	0	0	0
0·1	0·00097	0·00100	0·00100	0·00100	0·00099
0·2	0·00304	0·00306	0·00306	0·00307	0·00306
0·3	0·00620	0·00619	0·00619	0·00620	0·00620
0·4	0·01046	0·01046	0·01047	0·01048	0·01047
0·5	0·01580	0·01581	0·01582	0·01583	0·01582
0·6	0·02223	0·02218	0·02219	0·02221	0·02220
0·7	0·02971	0·02969	0·02971	0·02972	0·02971
0·8	0·03826	0·0382	0·0382	0·0382	0·0383
0·9	0·04784	0·0479	0·0479	0·0479	0·0479
1·0	0·05844	0·0584	0·0584	0·0584	0·0585
1·1	0·07005	0·0700	0·0700	0·0700	0·0701
1·2	0·08264	0·0826	0·0826	0·0826	0·0828
1·3	0·09618	0·0962	0·0962	0·0962	0·0962
1·4	0·11064	0·1106	0·1106	0·1106	0·1109

Velocity profiles at $x = 0{\cdot}958$ are tabulated in Table 3 for different values of α_1 and x_s. The fifth decimal of the asymptotic solution may be in error by a unit. The agreement is about as good as one can expect and leads to the conclusion that

$$\alpha_1 = 0{\cdot}492$$

with a possible error of $\pm 0{\cdot}002$. Considering this value of α_1 and Fig. 1, we may conclude that the velocity distribution of this computation indicates a value of x_s of 0·958542 with a possible error of ± 2 in the last decimal place. However, since the integration was started from $x = 0{\cdot}8$, using Hartree's velocity distribution which is correct to three and at most four decimal places, we can only be certain of the first four decimal places in the actual value of x_s. Hence we have that the separation point of the solution of the boundary-layer equation is

$$x_s = 0{\cdot}9585.$$

4·5. Evidence of the existence of a singularity at the point of separation is derived not only from the numerical solution in the immediate neighbourhood, but also from the breakdown of the process of integration when one attempts to carry it through the

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separation point. Hartree attempted to carry his integration process through the separation point and also to start from the separation point and work downstream. Using his step-by-step method of integration in the y-direction he found that in neither case was it possible to get any solution at all satisfying the boundary condition at $y = \infty$, for any starting value of $(\partial u_2/\partial y)_0$. The writer experienced similar difficulties in his matrix method of integration. An integration was attempted from $x_1 = 0{\cdot}958$ to $x_2 = 0{\cdot}959$ for $\delta y = 0{\cdot}2$: the iterations of the process indicated by equation (10) did not converge but went through an extremely wide range of variations (successive iterations did not usually agree even to the first figure: the iterations appeared to be cyclic, every seventh agreeing to almost three figures). An integration from $x_1 = 0{\cdot}958$ to $x_2 = 0{\cdot}9585$ was tried and did not meet with much success, even though the separation point x_s is greater than $0{\cdot}9585$. With $\delta y = 0{\cdot}2$ successive iterations did not converge to more than the third decimal place and the remaining figures cycled in the above manner. For $\delta y = 0{\cdot}1$ a solution could not be obtained even to the first figure. An integration from $x_1 = 0{\cdot}958$ to $x_2 = 0{\cdot}9584$ displayed the same behaviour as that from $0{\cdot}958$ to $0{\cdot}9585$. However, for $x_1 = 0{\cdot}957$ and $x_2 = 0{\cdot}9584$ the solutions converged for both $\delta y = 0{\cdot}1$ and $\delta y = 0{\cdot}2$. The evidence appears to substantiate the existence of a singularity, for if there were not a singularity one would not expect any difficulty in integrating through the separation point.

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