Ergod. Th. & Dynam. Sys. (2013), **33**, 1611–1643 © *Cambridge University Press, 2012*
doi:10.1017/etds.2012.102

Rational weak mixing in infinite measure spaces

JON AARONSON

School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel
(e-mail: aaro@post.tau.ac.il)

(*Received* 17 *January* 2012 *and accepted in revised form* 10 *May* 2012)

Abstract. Rational weak mixing is a measure theoretic version of Krickeberg's strong ratio mixing property for infinite measure preserving transformations. It requires 'density' ratio convergence for every pair of measurable sets in a dense hereditary ring. Rational weak mixing implies weak rational ergodicity and (spectral) weak mixing. It is enjoyed for example by Markov shifts with Orey's strong ratio limit property. The power, subsequence version of the property is generic.

1. *Introduction: Hopf's example*

Hopf gave an example in [**H**] of a transformation of the infinite strip, $\mathbb{R}_+ \times [0, 1]$, preserving Lebesgue measure m, which satisfies the ratio mixing property:

$$\frac{m(A \cap T^{-n}B)}{u_n} \xrightarrow[n\to\infty]{} m(A)m(B) \quad \text{for all } A,\ B \text{ bounded with } m(\partial A) = m(\partial B) = 0, \tag{✠}$$

where $u_n = \sqrt{2/\pi n}$.

Hopf mentioned that if (✠) could be established for every bounded measurable set, this would imply ergodicity of T. This latter property is also invariant under isomorphism.

The theory of weakly wandering sets as in [**HK**] shows that (✠) cannot hold for every pair of sets in any dense, hereditary collection (see below) in the absence of absolutely continuous, invariant probabilities (and this cannot be used to establish ergodicity of Hopf's example).

We show here that Hopf's example is rationally weakly mixing in a sense which implies that, for every pair of bounded measurable sets, (✠) is satisfied along a subsequence of full density.

1.1. *Organization of the paper.*

In §2, we give definitions and preliminary discussions. The main results are stated in §3. In §4, we study the modes of convergence involved in the rational weak mixing properties. Section 5 contains the proof of the 'basic' Proposition 0 and the 'density convergence' Theorem A. In §6, we establish sufficient conditions for rational weak mixing (Lemmas B and C). We collect in §7 some facts on the mean ergodic theorem with weighted averages for use in §8 to prove Theorem D which connects

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

subsequence rational weak mixing with other mixing properties. Markov shifts are treated in §9 (where there is some discussion of smoothness of renewal sequences) and Gibbs–Markov towers are studied in §10 via their local limit properties. We 'make categorical statements' (Theorem F) in §11 and some closing remarks in §12.

2. *Definitions and preliminaries*

2.1. *Notation and basics.* In this paper $(X, \mathcal{B}, m, T)$ denotes a measure preserving transformation T of a non-atomic, σ-finite, standard measure space $(X, \mathcal{B}, m)$.

Unless otherwise stated, the measure will be infinite ($m(X) = \infty$). Measure preserving transformations of finite measure spaces are referred to as probability preserving transformations.

A standing assumption on $(X, \mathcal{B}, m, T)$ is *conservativity*:

$$m\left(A \setminus \bigcup_{n=1}^{\infty} T^{-n}A\right) = 0 \quad \text{for all } A \in \mathcal{B}.$$

The collection of measurable subsets of X with finite measure is denoted $\mathcal{F} := \{A \in \mathcal{B} : m(A) < \infty\}$ and for any $\mathcal{C} \subset \mathcal{B}$ the 'positive elements' of $\mathcal{C}$ are denoted

$$\mathcal{C}_+ := \{A \in \mathcal{C} : m(A) > 0\}.$$

2.2. *Krickeberg mixing.* Krickeberg [**Kri1**] noted that Hopf's example is isomorphic to the Markov shift of the simple symmetric random walk on $\mathbb{N}$ with a reflecting barrier at 1 which has an irreducible, recurrent transition matrix [**KM**] and is therefore conservative and ergodic [**HR**]. He also formulated a concept of topological ratio mixing for transformations preserving infinite measures.

Let $(X, \mathcal{B}, m, T)$ be a measure preserving transformation and let $\alpha \subset \mathcal{B}$ be a countable partition, generating $\mathcal{B}$ under T in the sense that $\sigma(\bigcup_{n\in\mathbb{Z}} T^n\alpha) = \mathcal{B}$. The measure preserving transformation $(X, \mathcal{B}, m, T)$ is said to be *Krickeberg α-mixing* if there exists $u_n > 0$ $(n \geq 1)$ such that (✠) (as in §1) is satisfied for all $A, B \in \mathcal{C}_\alpha$, the collection of (α, T)-*cylinder sets* defined by

$$\mathcal{C}_\alpha = \mathcal{C}_\alpha(T) := \left\{[a_1, \ldots, a_N]_k := \bigcap_{j=1}^{N} T^{jk}a_j : N \in \mathbb{N}, k \in \mathbb{Z}, a_1, \ldots, a_N \in \alpha\right\};$$

hence [**Kri1**] for all A, B with $m(\partial A) = m(\partial B) = 0$ when X is considered equipped with the product topology from $\alpha^{\mathbb{Z}}$.

Markov shifts with the *strong ratio limit property* (SRLP) as in [**O**] (e.g. Hopf's example) are Krickeberg α-mixing with α the natural partition according to the state occupied at time 0 [**Kri1**]. Examples of Krickeberg α-mixing measure preserving transformations are also given in [**Fri, MT, Pap, T**]. Other definitions of mixing are discussed in [**L**].

It follows from Theorem 9.1 that Markov shifts whose associated renewal sequences have the strong ratio limit property are rationally weakly mixing.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

2.3. *Hereditary rings.* Let $(X, \mathcal{B}, m)$ be a σ-finite measure space. A collection $\mathcal{C} \subset \mathcal{B}$ is said to be *hereditary* if

$$\mathcal{H}(\mathcal{C}) := \{A \in \mathcal{B},\, A \subseteq B \in \mathcal{C}\} = \mathcal{C}.$$

A *hereditary ring* $\mathcal{H} \subset \mathcal{B}$ is a hereditary collection which is closed under finite union. It is *dense* if

$$\text{for all } A \in \mathcal{F} := \{F \in \mathcal{B} : m(F) < \infty\},\ \epsilon > 0 \text{ there exists } H \in \mathcal{H} \text{ such that } m(A \backslash H) < \epsilon.$$

For example, both $\mathcal{F}$ and the collection $\mathcal{R}_b$ of bounded measurable subsets of the infinite strip $\mathbb{R}_+ \times [0, 1]$ are dense hereditary rings. The collection of null sets is a hereditary ring which is not dense. We will denote the minimal hereditary ring containing the collection $\mathcal{C} \subset \mathcal{B}$ by $\mathcal{HR}(\mathcal{C})$.

Any two dense, hereditary rings in the same measure space intersect and thus many ergodic properties involving such are isomorphism invariant (e.g. rational weak mixing).

2.4. *Standard measure spaces.* We assume that all measure preserving transformations are defined on standard measure spaces. The σ-finite measure space $(X, \mathcal{B}, m)$ is *standard* if X is a Polish space, $\mathcal{B}$ is the collection of Borel sets and m is non-atomic. The standardness assumption is used as follows.

- If $(X, \mathcal{B}, m, T)$ is a conservative, ergodic, measure preserving transformation of a standard measure space then there exists a countable partition $\alpha \subset \mathcal{F} := \{A \in \mathcal{B} : m(A) < \infty\}$ which generates $\mathcal{B}$ under T and, up to isomorphism, T is the shift on $X = \alpha^{\mathbb{Z}}$ equipped with the product topology (a homeomorphism).

2.5. *Weights.* Our results involve averaging techniques using certain non-negative, bounded weight sequences. We call a bounded sequence $u = (u_0, u_1, \ldots)$ an *admissible weight sequence* (abbreviated to *weight*) if

$$u_n \geq 0 \text{ for all } n \geq 1 \quad \text{and} \quad a_u(n) := \sum_{k=1}^{n} u_k \longrightarrow \infty,$$

and we denote the collection of weights by $\mathfrak{W}$.

We will write, for (eventually) positive sequences $u = (u_0, u_1, \ldots)$ and $w = (w_0, w_1, \ldots)$:

- $u_n \sim w_n$ if $u_n / w_n \xrightarrow[n\to\infty]{} 1$.

Also, for non-negative sequences $u = (u_0, u_1, \ldots)$ and $w = (w_0, w_1, \ldots)$, we will write:

- $u_n \ll w_n$ if there exists $M > 0$ such that $u_n \leq M w_n$ for all $n \geq 0$;
- $u_n \asymp w_n$ if $u_n \ll w_n$ and $u_n \gg w_n$.

Given a subsequence $\mathfrak{K} \subset \mathbb{N}$, we call weights $u, w \in \mathfrak{W}$:

- $\mathfrak{K}$-*asymptotic* $(u \overset{\mathfrak{K}}{\approx} w)$ if $(1/a_u(n)) \sum_{k=1}^{n} |u_k - w_k| \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0$.

Evidently,

$$u \overset{\mathfrak{K}}{\approx} w \text{ implies } \frac{a_w(n)}{a_u(n)} \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 1.$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

The converse implication sometimes holds and will be discussed in the following.

We call $u \in \mathfrak{W}$:

- $\mathfrak{K}$-*smooth* if $(1/a_u(n)) \sum_{k=1}^{n} |u_k - u_{k+1}| \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0$ (equivalently $(u_0, u_1, \ldots) \overset{\mathfrak{K}}{\approx} (u_1, u_2, \ldots)$).

We will say that weights $u, w \in \mathfrak{W}$ are *asymptotic* ($u \approx w$) if $u \overset{\mathbb{N}}{\approx} w$, and that $u \in \mathfrak{W}$ is *smooth* if it is $\mathbb{N}$-smooth and *subsequence smooth* if it is $\mathfrak{K}$-smooth for some subsequence $\mathfrak{K} \subset \mathbb{N}$.

2.6. *Intrinsic weights.* For $(X, \mathcal{B}, m, T)$ a conservative, ergodic, measure preserving transformation and $E, F \in \mathcal{F}_+$, the *intrinsic weight* $u(E, F) \in \mathfrak{W}$ is defined by

$$u_n(E, F) := \frac{m(F \cap T^{-n}F)}{m(E)m(F)}.$$

We denote $a_n(E, F) = a_{u(E,F)}(n) := \sum_{k=0}^{n-1} u_k(E, F)$ and write $u(F) := u(F, F)$.

2.7. *Rational weak mixing.* Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence. We will call the measure preserving transformation $(X, \mathcal{B}, m, T)$ *rationally weakly mixing along* $\mathfrak{K}$ if there exists $F \in \mathcal{F}_+$ so that

$$m(A \cap T^{-n}B) \overset{\mathfrak{K}}{\approx} m(A)m(B)u_n(F) \quad \text{for all } A, B \in \mathcal{B} \cap F. \tag{$\bigstar_{\mathfrak{K}}$}$$

We call the measure preserving transformation $(X, \mathcal{B}, m, T)$:

- *rationally weakly mixing* if it is rationally weakly mixing along $\mathbb{N}$; and
- *subsequence rationally weakly mixing* if it is rationally weakly mixing along some $\mathfrak{K} \subset \mathbb{N}$.

2.8. *Weak rational ergodicity.* Again for $\mathfrak{K} \subset \mathbb{N}$ a subsequence, the measure preserving transformation $(X, \mathcal{B}, m, T)$ is said to be *weakly rationally ergodic along* $\mathfrak{K}$ if there exists $F \in \mathcal{F}_+$ so that

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} m(B \cap T^{-k}C) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} m(B)m(C) \quad \text{for all } B, C \in \mathcal{B} \cap F \tag{$\bigstar\!\!\bigstar_{\mathfrak{K}}$}$$

where $a_n(F) := (1/m(F)^2) \sum_{k=0}^{n-1} m(F \cap T^{-k}F)$. Weak rational ergodicity entails conservativity and ergodicity.

The proof of Theorem 3.3 in [**FL**] easily adapts to show that $F \in \mathcal{F}_+$ satisfies ($\bigstar\!\!\bigstar_{\mathfrak{K}}$) if and only if

$$\left\{ \frac{S_n^{(T)}(1_F)}{a_n(F)} : n \in \mathfrak{K} \right\} \quad \text{is uniformly integrable on } F.$$

A useful sufficient condition for this is

$$\sup_{n\in\mathfrak{K}} \frac{1}{a_n(F)^2} \int_F S_n(1_F)^2 \, dm < \infty,$$

and $(X, \mathcal{B}, m, T)$ is said to be *rationally ergodic along* $\mathfrak{K}$ if there exists $F \in \mathcal{F}_+$ with this property. See [**A, A1**] (for the special case $\mathfrak{K} = \mathbb{N}$).

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

In the case when T is weakly rationally ergodic along $\mathfrak{K}$, the following hold.

- The collection of sets $R_{\mathfrak{K}}(T)$ satisfying $(\bigstar_{\mathfrak{K}})$ is a T-invariant, dense, hereditary ring.
- There exists $a_n(T)$ (the *return sequence along* $\mathfrak{K}$) such that

$$\frac{a_n(A)}{a_n(T)} \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 1 \quad \text{for all } A \in R_{\mathfrak{K}}(T).$$

- For conservative, ergodic T, $R_{\mathfrak{K}}(T) = \mathcal{F}$ only when $m(X) < \infty$.

The proofs of these statements are analogous to those in [**A1, A**] (for the special case $\mathfrak{K} = \mathbb{N}$).

We will call a measure preserving transformation $(X, \mathcal{B}, m, T)$:

- *[weakly] rationally ergodic* if it is [weakly] rationally ergodic along $\mathbb{N}$; and
- *subsequence [weakly] rationally ergodic* if it is [weakly] rationally ergodic along some $\mathfrak{K} \subset \mathbb{N}$.

We set $R(T) := R_{\mathbb{N}}(T)$ (as in [**A, A1**]).

For example, conservative, ergodic Markov shifts are rationally ergodic. For further examples, see [**A**].

We will see that rational weak mixing along $\mathfrak{K}$ implies weak rational ergodicity along $\mathfrak{K}$ and that for T rationally weakly mixing along $\mathfrak{K}$,

$$\{F \in \mathcal{F}_+ : (\bigstar_{\mathfrak{K}}) \text{ holds}\} = R_{\mathfrak{K}}(T).$$

2.9. *Weak mixing.* For a measure preserving transformation $(X, \mathcal{B}, m, T)$ of a σ-finite measure space (as shown in [**ALW**]) the following conditions are equivalent:

(i) $f \in L^\infty$, $\lambda \in \mathbb{S}^1$, $f \circ T = \lambda f$ almost everywhere implies f is constant almost everywhere;

(ii) $T \times S$ is ergodic for all ergodic, probability preserving S;

(iii) $$\frac{1}{n}\sum_{k=0}^{n-1}\left|\int_X uf \circ T^k \, dm\right| \xrightarrow[n\to\infty]{} 0 \quad \text{for all } u \in L^1_0,\ f \in L^\infty.$$

We will call a measure preserving transformation satisfying (any one of) these conditions *spectrally weakly mixing*. This is in the interest of disambiguation. In [**ALW**] and elsewhere 'spectral weak mixing' is called 'weak mixing'.

We will see that subsequence rational weak mixing implies spectral weak mixing.

2.10. *Categorical statements.* Let $(X, \mathcal{B}, m)$ be a standard σ-finite, non-atomic, infinite measure space and consider $\mathrm{MPT}(X, \mathcal{B}, m)$, the collection of invertible measure preserving transformations of $(X, \mathcal{B}, m)$ equipped with the *weak operator topology* defined by $T_n \to T$ if

$$m(TA \Delta T_n A) + m(T^{-1}A \Delta T_n^{-1} A) \xrightarrow[n\to\infty]{} 0 \quad \text{for all } A \in \mathcal{F}.$$

It follows that $\mathrm{MPT}(X, \mathcal{B}, m)$ is a Polish group. A *categorical statement* is a statement concerning the Baire category of a subset of MPT. For a review of this subject, see [**CP**].

We will see that (the power version of) the subsequence rational weak mixing elements of MPT form a residual set in MPT.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

3. *Results*

PROPOSITION 0. (Basics) *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence and suppose that $(X, \mathcal{B}, m, T)$ is rationally weakly mixing along $\mathfrak{K}$. Then the following hold:*

(i) *T is weakly rationally ergodic along $\mathfrak{K}$;*

(ii) *$\{F \in \mathcal{F}_+ : (\bigstar_{\mathfrak{K}}) \text{ holds}\} = R_{\mathfrak{K}}(T)$;*

(iii) *$u(F, G)$ is $\mathfrak{K}$-smooth for all $F, G \in R_{\mathfrak{K}}(T)$, $m(F), m(G) > 0$;*

(iv) *for each $p \in \mathbb{Z}$, $p \neq 0$, T^p is rationally weakly mixing along $1/|p|\mathfrak{K} := \{\lfloor (k/|p|) \rfloor : k \in \mathfrak{K}\}$ and $R_{(1/|p|)\mathfrak{K}}(T^p) = R_{\mathfrak{K}}(T)$.*

3.1. *Corollary: isomorphism invariance.*

If $(X, \mathcal{B}, m, T)$ is rationally weakly mixing along $\mathfrak{K}$, and is isomorphic by measure preserving transformation to $(X', \mathcal{B}', m', T')$, then $(X', \mathcal{B}', m', T')$ is also rationally weakly mixing along $\mathfrak{K}$, and the intrinsic weights of T' are $\mathfrak{K}$-asymptotic to those of T.

THEOREM A. (Density convergence) *Suppose that $(X, \mathcal{B}, m, T)$ is rationally weakly mixing and that there exists $E \in R_{\mathfrak{K}}(T)$ such that $u(E) \approx v$ where $v \in \mathfrak{W}$ is regularly varying with index $s \in (-1, 0]$ ($v_{\lfloor \lambda n \rfloor}/v_n \xrightarrow[n\to\infty]{} \lambda^s$ for all $\lambda > 0$). Then, for $F \in R_{\mathfrak{K}}(T)_+$,*

$$\frac{m(A \cap T^{-n}B)}{u_n(F)} \xrightarrow[n\to\infty]{\text{density}} m(A)m(B) \quad \text{for all } A, B \in R_{\mathfrak{K}}(T).$$

Here $s_n \xrightarrow[n\to\infty]{\text{density}} L$ means $s_n \xrightarrow[n\to\infty, n\notin K]{} L$ where $K \subset \mathbb{N}$ has zero density in the sense that $\#(K \cap [1, n]) = o(n)$ as $n \to \infty$.

LEMMA B. (Sufficient conditions) *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence. Suppose that X is a Polish space and the following hold.*

(i) *$(X, \mathcal{B}, m, T)$ is an invertible, measure preserving transformation, weakly rationally ergodic along $\mathfrak{K}$.*

(ii) *There exists $\Omega \in R_{\mathfrak{K}}(T)$ open in X and a countable base $\mathcal{C}$ for the topology of Ω such that:*

  (a) *for all $\{C_i\}_{i=1}^k \subset \mathcal{C}$ there exists $\{D_j\}_{j=1}^\ell \subset \mathcal{C}$, $m(D_i \cap D_j) = 0$ for all $i \neq j$; $\bigcup_{i=1}^k C_i \overset{m}{=} \bigcup_{j=1}^\ell D_j$;*

  (b) *$m(A \cap T^{-n}B) \overset{\mathfrak{K}}{\approx} m(A)m(B)u_n(\Omega)$ for all $A, B \in \mathcal{C}$.*

Then $(X, \mathcal{B}, m, T)$ is rationally weakly mixing along $\mathfrak{K}$.

LEMMA C. (Sufficient conditions) *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence. Suppose that the following hold.*

(i) *$(X, \mathcal{B}, m, T)$ is an invertible, measure preserving transformation, weakly rationally ergodic along $\mathfrak{K}$.*

(ii) *There exists a countable generating partition $\alpha \subset R_{\mathfrak{K}}(T)$ and $\Omega \in \mathcal{C}_\alpha$ such that*

$$m(A \cap T^{-n}B) \overset{\mathfrak{K}}{\approx} m(A)m(B)u_n \quad \text{for all } A, B \in \mathcal{C}_\alpha,$$

where $u = u(\Omega)$.

Then $(X, \mathcal{B}, m, T)$ is rationally weakly mixing along $\mathfrak{K}$.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

THEOREM D. (Mixing properties) *Let* $\mathfrak{K} \subset \mathbb{N}$ *be a subsequence and suppose that* $(X, \mathcal{B}, m, T)$ *is rationally weakly mixing along* $\mathfrak{K}$. *Then the following hold:*

(i) $(X, \mathcal{B}, m, T)$ *is spectrally weakly mixing;*

(ii) $T \times S$ *is rationally weakly mixing along* $\mathfrak{K}$ *for all weakly mixing, probability preserving transformations* $(\Omega, \mathcal{F}, P, S)$.

Invertible rationally weakly mixing measure preserving transformations of infinite measure spaces are obtained via the following corollary.

COROLLARY E. *The natural extension of a measure preserving transformation, rationally weakly mixing along* $\mathfrak{K}$, *is also rationally weakly mixing along* $\mathfrak{K}$ *with* $\mathfrak{K}$*-asymptotic intrinsic weights.*

Let

$$\mathrm{RWM}(X) := \{T \in \mathrm{MPT}(X) : T \text{ is rationally weakly mixing}\},$$
$$\mathrm{SRWM}(X) := \{T \in \mathrm{MPT}(X) : T \text{ is subsequence rationally weakly mixing}\}.$$

For $T \in \mathrm{MPT}(X)$, $\Delta \geq 1$ and $(\kappa_1, \ldots, \kappa_\Delta) \in \mathbb{Z}^\Delta$, let

$$T^{(\kappa_1,\ldots,\kappa_\Delta)} := T^{\kappa_1} \times T^{\kappa_2} \times \cdots \times T^{\kappa_\Delta} \in \mathrm{MPT}(X^\Delta).$$

Call $T \in \mathrm{MPT}$ *power, subsequence rationally weakly mixing* if $T^{(\kappa_1,\ldots,\kappa_\Delta)}$ is subsequence rationally weakly mixing for all $\Delta \geq 1$ and $(\kappa_1, \ldots, \kappa_\Delta) \in (\mathbb{Z}\backslash\{0\})^\Delta$. Let

$$\mathrm{PSRWM}(X) := \{T \in \mathrm{MPT}(X) : T^{(\kappa_1,\ldots,\kappa_\Delta)} \in \mathrm{SRWM}(X^\Delta) \text{ for all } \Delta \geq 1, (\kappa_1, \ldots, \kappa_\Delta) \in (\mathbb{Z}\backslash\{0\})^\Delta\}.$$

THEOREM F. (Baire category)

(i) *The collection RWM is meagre in MPT.*

(ii) *The collection PSRWM*(X) *is residual in MPT*(X).

4. *Convergence*

In this section we study the modes of convergence involved in the rational weak mixing properties. Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence.

4.1. $(u, \mathfrak{K})$-*small sets.*

Let $u \in \mathfrak{W}$. We will say that the set $K \subset \mathbb{N}$ is:

- $(u, \mathfrak{K})$-*small* if $a_u(K, n)/a_u(n) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0$ where $a_u(K, n) := \sum_{k\in K\cap[1,n]} u_k$; and $(u, \mathfrak{K})$-*large* if K^c is $(u, \mathfrak{K})$-*small*.

We will call a set u-*small* if it is $(u, \mathbb{N})$-small.

Recall that the set $K \subset \mathbb{N}$:

- has *density* $d(K)$ if $(1/n)\#(K \cap [1, n]) \longrightarrow d(K)$ as $n \to \infty$; and
- has *zero density* if $d(K) = 0$ (equivalently: K is $\mathbb{1}$-small where $\mathbb{1} \in \mathfrak{W}$, $\mathbb{1}_n = 1$ for all n).

The following remark collects some elementary facts about $(u, \mathfrak{K})$-smallness.

Remark 4.1. Suppose that $u \in \mathfrak{W}$, then the following hold.

(o) If $K \subset \mathbb{N}$ is $(u, \mathfrak{K})$-small, it is also $(u, \mathfrak{K}')$-small whenever $\mathfrak{K}' \subset \mathfrak{K} + F$ for some finite set $F \subset \mathbb{Z}$. This is because if $n \in \mathfrak{K}'$ and $n = m + f$, where $m \in \mathfrak{K}$ and $f \in F$,

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

then

$$|a_u(K, n) - a_u(K, m)| \le \sup_k u_k \cdot \max_{j \in F} |j|.$$

(i) A finite union of $(u, \mathfrak{K})$-small sets is itself $(u, \mathfrak{K})$-small.

(ii) If $v \in \mathfrak{W}$ satisfies $u_n \asymp v_n$ then $K \subset \mathbb{N}$ is $(u, \mathfrak{K})$-small if and only if it is $(v, \mathfrak{K})$-small.

(iii) If $K_1 \subset K_2 \subset \cdots$ is an increasing sequence of $(u, \mathfrak{K})$-small sets, then there exists $N_1 < N_2 < \cdots$ so that $K_\infty := \bigcup_{j=1}^{\infty} K_j \cap [N_j + 1, N_{j+1}]$ is a $(u, \mathfrak{K})$-small set.

Proof. A suitable sequence is obtained by choosing $N_j \uparrow$ such that $a_u(K_j, n)/a_u(n) < 1/j$ for all $n \ge N_j$, $n \in \mathfrak{K}$. □

(iv) If $u, v \in \mathfrak{W}$ and $u \overset{\mathfrak{K}}{\approx} v$, then $K \subset \mathbb{N}$ is $(u, \mathfrak{K})$-small if and only if it is $(v, \mathfrak{K})$-small.

Proof. $|a_u(K, n) - a_v(K, n)| \le \sum_{k=0}^{n} |u_k - v_k| = o(a_u(n))$ along $\mathfrak{K}$. □

(v) If $u \in \mathfrak{W}$ is $\mathfrak{K}$-smooth, then $K \subset \mathbb{N}$ is $(u, \mathfrak{K})$-small if and only if $K + 1$ is $(u, \mathfrak{K})$-small.

(vi) If $u \in \mathfrak{W}$ is $\mathfrak{K}$-smooth, $p \ge 1$, and $u_n^{(p)} := u_{pn}$, then $a_{u^{(p)}}(n) \sim (1/p)a_u(pn)$ along $(1/p)\mathfrak{K}$ and $u^{(p)}$ is $(1/p)\mathfrak{K}$-smooth where $(1/p)\mathfrak{K} := \{\lfloor j/p \rfloor : j \in \mathfrak{K}\}$.

Proof. This follows from

$$\sum_{k=1}^{n} |u_k^{(p)} - u_{k+1}^{(p)}| \le p \sum_{k=1}^{pn} |u_k - u_{k+1}| = o(a_u(pn))$$

and

$$\begin{aligned} |pa_{u^{(p)}}(n) - a_u(pn)| &\le \sum_{r=1}^{p-1} \sum_{k=1}^{n} |u_{pk} - u_{pk+r}| \\ &\le p^2 \sum_{k=1}^{pn} |u_k - u_{k+1}| = o(a_u((p+1)n)). \end{aligned}$$

□

PROPOSITION 4.1. *Suppose that $u \in \mathfrak{W}$ and $u_n \asymp a_u(n)/n$ and $u_n \asymp v_n$ where $v \in \mathfrak{W}$ and $v_n \downarrow$, then a set is u-small if and only if it has zero density.*

Proof. By Remark 4.1(ii), there is no loss of generality in assuming $u_n \downarrow$.

Proof of $\Rightarrow$. Suppose that $u_n \ge \eta a_u(n)/n$. Since $u_n \downarrow$,

$$a_u(K, n) = \sum_{k \in [1,n] \cap K} u_k \ge u_n |K \cap [1, n]| \ge \eta \frac{a_u(n)}{n} |K \cap [1, n]|,$$

and hence

$$\frac{|K \cap [1, n]|}{n} \le \frac{a_u(K, n)}{\eta a_u(n)}.$$

□

Proof of $\Leftarrow$. We show first that

$$\overline{\lim_{n \to \infty, \epsilon \to 0}} \frac{a_u(\epsilon n)}{a_u(n)} = 0.$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

To do this, it suffices to show that

$$a_u(2n) \gtrsim (1 + \eta \log 2) a_u(n)$$

where $u_n \geq \eta a_u(n)/n$.

Indeed,

$$a_u(2n) - a_u(n) = \sum_{k=n+1}^{2n} u_k \geq \eta \sum_{k=n+1}^{2n} \frac{a_u(k)}{k}$$
$$\geq \eta a_u(n) \sum_{k=n+1}^{2n} \frac{1}{k} \gtrsim \eta \log 2 \cdot a_u(n). \qquad \square$$

Next, since $u_n \downarrow$, $u_n \leq M(a_u(n)/n)$, and

$$a_u(K, n) = \sum_{k \in [1,n] \cap K} u_k \leq a_u(n\epsilon) + \sum_{k \in [\epsilon n, n] \cap K} u_k$$
$$\leq a_u(n\epsilon) + u_{n\epsilon} |[1, n] \cap K|$$
$$\leq a_u(n\epsilon) + \frac{a_u(n\epsilon)}{n\epsilon} |[1, n] \cap K|,$$

and hence

$$\frac{a_u(K, n)}{a_u(n)} \leq \frac{a_u(n\epsilon)}{a_u(n)} + \frac{1}{\epsilon} \cdot \frac{|K \cap [1, n]|}{n}. \qquad \square$$

Remark 4.2.

(i) In the case when $u \in \mathfrak{W}$ and u_n is *regularly varying* with *index* $s \in (-1, 0)$ (i.e. $u_{\lfloor \lambda n \rfloor}/u_n \xrightarrow[n\to\infty]{} \lambda^s$), then (see e.g. [**BGT**]) $u_n \sim (1 + s) a_u(n)/n$ and there exists $v \in \mathfrak{W}$ such that $v_n \downarrow, v_n \sim u_n$. Thus, Proposition 4.1 applies.

(ii) The conclusion of Proposition 4.1 fails for $u_n = 1/(n + 1)$. The set $K := \bigcup_{k=1}^{\infty} [2^{k^2}, k2^{k^2}] \cap \mathbb{N}$ is u-small, but $\overline{\lim}_{n\to\infty}(|K \cap [1, n]|/n) = 1$.

4.2. *$(u, \mathfrak{K})$-density and $(u, \mathfrak{K})$-strong Cesaro convergence.* Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence and let $u \in \mathfrak{W}$.

We will say that a sequence s_n:

- *converges in $(u, \mathfrak{K})$-density to $L \in \mathbb{R}$* ($s_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} L$) if there exists $K \subset \mathbb{N}$ $(u, \mathfrak{K})$-small such that
$$s_n \xrightarrow[n\to\infty, n\notin K]{} L;$$
and
- *s_n converges $(u, \mathfrak{K})$-strongly Cesaro to $L \in \mathbb{R}$* ($s_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-s.C.}} L$) if
$$\frac{1}{a_u(n)} \sum_{k=0}^{n} u_k |s_k - L| \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0.$$

Remark 4.3. Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence and let $u \in \mathfrak{W}$, $s = (s_1, s_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ and $L \in \mathbb{R}$.

(i) The convergence $s_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} L$ holds if and only if
$$K_\epsilon := \{n \in \mathbb{N} : |s_n - L| > \epsilon\} \text{ is } (u, \mathfrak{K})\text{-small} \quad \text{for all } \epsilon > 0.$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Proof. Evidently $s_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} L \Rightarrow K_\epsilon$ is $(u, \mathfrak{K})$-small for all $\epsilon > 0$. To see the reverse implication: if K_ϵ is $(u, \mathfrak{K})$-small for all $\epsilon > 0$, then by Remark 4.1(iii) (on p. 1618), there exists $N_1 < N_2 < \cdots$ so that $K := \bigcup_{j=1}^{\infty} K_{1/j} \cap [N_j + 1, N_{j+1}]$ is a $(u, \mathfrak{K})$-small set and $s_n \xrightarrow[n\to\infty, n\notin K]{} 0$. □

(ii) If $v \in \mathfrak{W}$, $u \overset{\mathfrak{K}}{\approx} v$, then

$$s_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} L \quad \text{if and only if } s_n \xrightarrow[n\to\infty]{(v,\mathfrak{K})\text{-d.}} L.$$

(iii) If $s_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-s.C.}} L$ then $s_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} L$.

Proof. Suppose that $s_n \ge 0$, $L = 0$ and that $(1/a_u(n)) \sum_{k=1}^{n} u_k s_k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0$. By the Chebyshev–Markov inequality,

$$\frac{a_u(K_\epsilon, n)}{a_u(n)} = \frac{1}{a_u(n)} \sum_{k=1, k\in K_\epsilon}^{n} u_k \le \frac{1}{\epsilon a_u(n)} \sum_{k=1}^{n} u_k s_k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0. \qquad \square$$

We call a sequence $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ *one-sidedly bounded* if it is either bounded above or below (or both).

PROPOSITION 4.2. *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence and let $u \in \mathfrak{W}$. Suppose that $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ is one-sidedly bounded and $L \in \mathbb{R}$, then*

$$x_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-s.C.}} L \tag{‡}$$

if and only if

$$\frac{1}{a_u(n)} \sum_{k=0}^{n} u_k x_k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} L \quad \text{and} \quad x_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} L. \tag{†}$$

Proof of (†) ⇒ (‡). We assume (without loss of generality) that $x_n \ge 0$ for all $n \ge 1$ and $L \ge 0$. Fix $\epsilon > 0$ and set

$$K_{+,\epsilon} := \{n \ge 1, x_n > L + \epsilon\}, \quad K_{-,\epsilon} := \{n \ge 1, x_n < L - \epsilon\},$$
$$K_\epsilon := K_{+,\epsilon} \cup K_{-,\epsilon}.$$

By our assumptions there exists $N_\epsilon \ge 1$ such that

$$a_u(K_\epsilon, [1, n]) < \epsilon a_u(n) \quad \text{and} \quad L - \epsilon < \frac{1}{a_u(n)} \sum_{k=0}^{n} u_k x_k < L + \epsilon \quad \text{for all } n > N_\epsilon, n \in \mathfrak{K}.$$

For large enough $n > N_\epsilon$, $n \in \mathfrak{K}$,

$$\begin{aligned}
\sum_{k=1}^{n} u_k |x_k - L| &= \left(\sum_{k\in K_\epsilon^c \cap [1,n]} + \sum_{k\in K_{+,\epsilon}\cap[1,n]} + \sum_{k\in K_{-,\epsilon}\cap[1,n]} \right) u_k |x_k - L| \\
&\le \epsilon \sum_{k\in K_\epsilon^c\cap[1,n]} u_k + \sum_{k\in K_{+,\epsilon}\cap[1,n]} u_k (x_k - L) + \sum_{k\in K_{-,\epsilon}\cap[1,n]} u_k (L - x_k) \\
&\le \epsilon a_u(n) + L a_u(K_\epsilon, n) + \sum_{k\in K_\epsilon\cap[1,n]} u_k x_k \\
&< (1 + L)\epsilon a_u(n) + \sum_{k\in K_\epsilon\cap[1,n]} u_k x_k.
\end{aligned}$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Now (for large enough $n > N_\epsilon$, $n \in \mathfrak{K}$),

$$\begin{aligned}\sum_{k\in K_\epsilon\cap[1,n]} u_k x_k &= \sum_{k\in[1,n]} u_k x_k - \sum_{k\in K_\epsilon^c\cap[1,n]} u_k x_k\\ &< (L+\epsilon)a_u(n) - (L-\epsilon)a_u(K_\epsilon^c, n)\\ &= (L+\epsilon)a_u(K_\epsilon, n) + 2\epsilon a_u(K_\epsilon^c, n)\\ &< \epsilon(L+\epsilon)a_u(n) + 2\epsilon a_u(n)\\ &= (L+2+\epsilon)\epsilon a_u(n).\end{aligned}$$

Reassembling,

$$\sum_{k=1}^{n} u_k|x_k - L| < (2L+3+\epsilon)\epsilon a_u(n). \qquad \square$$

A version of the following proposition is implicit in [**GL**].

PROPOSITION 4.3. *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence and let $u \in \mathfrak{W}$. Suppose that $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ and that $L \in \mathbb{R}$.*

If $(1/a_u(n)) \sum_{k=0}^{n} u_k x_k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} L$ and either (i) or (ii) below holds, then $x_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-s.C.}} L$.

(i) *$x = (x_1, x_2, \ldots)$ is bounded below and there exists a $(u, \mathfrak{K})$-small set $K_0 \subset \mathbb{N}$ such that $\underline{\lim}_{n\to\infty, n\notin K_0} x_n \geq L$.*

(ii) *$x = (x_1, x_2, \ldots)$ is bounded above and there exists a $(u, \mathfrak{K})$-small set $K_0 \subset \mathbb{N}$ such that $\overline{\lim}_{n\to\infty, n\notin K_0} x_n \leq L$.*

Proof. By Proposition 4.2, it suffices to prove that $x_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} L$. By symmetry it suffices to prove the proposition under assumption (i). By possibly translating x with a constant sequence, we reduce to the case $x_n \geq 0$ for all $n \geq 1$ and $L \geq 0$.

For $\epsilon > 0$, set $K_\epsilon := \{n \notin K_0 : x_n > L + \epsilon\}$. It suffices to prove that K_ϵ is $(u, \mathfrak{K})$-small for all $0 < \epsilon < \frac{1}{2}$.

To see this, fix $0 < \epsilon < \frac{1}{2}$, and let N_ϵ be so that

$$x_n > L - \epsilon^2 \quad \text{for all } n \geq N_\epsilon,\ n \notin K_0.$$

Then for large $n \gg N_\epsilon$, $n \in \mathfrak{K}$,

$$\begin{aligned}(L+\epsilon^2)a_u(n) &> \sum_{k=N_\epsilon}^{n} x_k u_k\\ &\geq \sum_{k\in K_\epsilon\cap[N_\epsilon,n]} x_k u_k + \sum_{k\in K_\epsilon^c\cap K_0^c\cap[N_\epsilon,n]} x_k u_k\\ &> (L+\epsilon)a_u(K_\epsilon, n) + (L-\epsilon^2)a_u(K_\epsilon^c\cap K_0^c, n) - (2L+\epsilon-\epsilon^2)a_u(N_\epsilon)\\ &> (L+\epsilon)a_u(K_\epsilon, n) + (L-\epsilon^2)a_u(K_\epsilon^c, n)\\ &\quad - a_u(K_0, n) - (2L+1)a_u(N_\epsilon)\\ &= (L+\epsilon)a_u(K_\epsilon, n) + (L-\epsilon^2)a_u(K_\epsilon^c, n) - \mathcal{E}_n,\end{aligned}$$

where $\mathcal{E}_n := a_u(K_0, n) + (2L+1)a_u(N_\epsilon)$.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Writing

$$(L+\epsilon^2)a_u(n) = (L+\epsilon^2)a_u(K_\epsilon, n) + (L+\epsilon^2)a_u(K_\epsilon^c, n),$$

we see that

$$(\epsilon - \epsilon^2)a_u(K_\epsilon, n) \le 2\epsilon^2 a_u(K_\epsilon^c, n) + \mathcal{E}_n.$$

For large $n \in \mathfrak{K}$, $\mathcal{E}_n < \epsilon^2 a_u(n)$ and hence

$$a_u(K_\epsilon, n) \le \frac{3\epsilon}{1-\epsilon} \cdot a_u(n) < 6\epsilon a_u(n). \qquad \square$$

COROLLARY 4.4. *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence and let $u \in \mathfrak{W}$. If $u_{n+1}/u_n \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} 1$, then u is $\mathfrak{K}$-smooth.*

Proof. If $u_n > 0$ for all $n \ge 0$, the corollary follows from Proposition 4.2, since $\sum_{k=0}^n u_k |(u_{k+1}/u_k) - 1| = \sum_{k=0}^n |u_{k+1} - u_k|$.

If this is not the case, define $v \in \mathfrak{W}$ by

$$v_n = \begin{cases} u_n, & u_n > 0, \\ \dfrac{1}{2^n}, & u_n = 0. \end{cases}$$

Evidently $a_u(n) \le a_v(n) \le a_u(n) + 2$ so $a(n) := a_u(n) \sim a_v(n)$.

Moreover, if $K \subset \mathbb{N}$, then $a_u(K, n) \le a_v(K, n) \le a_u(K, n) + 2$ and so K is $(u, \mathfrak{K})$-small if and only if it is $(v, \mathfrak{K})$-small.

Next, if $1/2 < u_{n+1}/u_n < 2$, then $v_n = u_n$ and $v_{n+1} = u_{n+1}$, $v_{n+1}/v_n = u_{n+1}/u_n$. Thus $v_{n+1}/v_n \xrightarrow[n\to\infty]{v\text{-d.}} 1$ and, by Proposition 4.2 (as above),

$$\frac{1}{a(n)} \sum_{k=0}^n |v_{k+1} - v_k| \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0.$$

Finally,

$$\frac{1}{a(n)} \sum_{k=0}^n |u_{k+1} - u_k| \le \frac{1}{a(n)} \sum_{k=0}^n |v_{k+1} - v_k| + \frac{2}{a(n)} \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0. \qquad \square$$

5. *Proofs of Proposition 0 and Theorem A*

Proof of Proposition 0. Fix $F \in \mathcal{F}$ satisfying ($\bigstar_{\mathfrak{K}}$) (on p. 1614). Evidently, for A, $B \in \mathcal{B} \cap F$,

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} m(A \cap T^{-k} B) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} m(A)m(B) \quad \text{for all } A, B \in \mathcal{B} \cap F.$$

This shows that $F \in R_{\mathfrak{K}}(T)$ and that T is weakly rationally ergodic along $\mathfrak{K}$, thus proving (i).

To prove (ii), let $F \in \mathcal{F}$ satisfy ($\bigstar_{\mathfrak{K}}$). It suffices to show that

$$m(B \cap T^{-n} C) \overset{\mathfrak{K}}{\approx} m(B)m(C)u_n(F) \quad \text{for all } B, C \in R_{\mathfrak{K}}(T). \tag{5.1}$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Proof of (5.1). Fix $B, C \in R_{\mathfrak{K}}(T)$. Then $G := B \cup C \in R_{\mathfrak{K}}(T)$ and we claim the following.

¶1 There exists $K \subset \mathbb{N}$ $(u(F), \mathfrak{K})$-small such that

$$\varliminf_{n\to\infty, n\notin K} \frac{m(B \cap T^{-n}C)}{u_n(F)} \geq m(B)m(C).$$

Proof of ¶1. Let $\epsilon > 0$, then there exists $B_0, \ldots, B_N, C_0, \ldots, C_N \in \mathcal{B} \cap F$ so that

$$B' := \dot{\bigcup_{k=0}^{N}} T^{-k}B_k \subset B, \quad C' := \dot{\bigcup_{k=0}^{N}} T^{-k}C_k \subset C, \quad m(B\backslash B') < \epsilon, m(C\backslash C') < \epsilon.$$

Using $(\bigstar_{\mathfrak{K}})$,

$$\begin{aligned}
\frac{m(B \cap T^{-n}C)}{u_n(F)} &\geq \frac{m(B' \cap T^{-n}C')}{u_n(F)} \\
&= \sum_{k,\ell=0}^{N} \frac{m(T^{-k}B_k \cap T^{-n-\ell}C_\ell)}{u_n(F)} \\
&\xrightarrow[n\to\infty]{(u(F),\mathfrak{K})\text{-d.}} \sum_{k,\ell=0}^{N} m(T^{-k}B_k)m(T^{-\ell}C_\ell) \\
&= m(B')m(C') > (m(B) - \epsilon)(m(C) - \epsilon).
\end{aligned}$$

Choose $\epsilon_n \downarrow 0$. By the above there exists $K_1 \subset K_2 \subset \cdots \subset \mathbb{N}$, each K_ν being $(u(F), \mathfrak{K})$-small, such that

$$\varliminf_{n\to\infty, n\notin K_\nu} \frac{m(B \cap T^{-n}C)}{u_n(F)} \geq m(B)m(C) - \epsilon_\nu \quad \text{for all } \nu \geq 1.$$

By Remark 4.1(iii) there exists $K \subset \mathbb{N}$ realizing ¶1. □

By weak rational ergodicity along $\mathfrak{K}$,

$$\frac{1}{a_n(F)} \sum_{k=0}^{n} m(B \cap T^{-k}C) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} m(B)m(C),$$

so, by Proposition 4.3(i),

$$\frac{m(B \cap T^{-n}C)}{u_n(F)} \xrightarrow[n\to\infty]{(u(F),\mathfrak{K})\text{-s.C.}} m(B)m(C) \quad \text{for all } B, C \in \mathcal{B} \cap A;$$

equivalently, $m(B \cap T^{-n}C) \overset{\mathfrak{K}}{\approx} m(B)m(C)u_n(F)$. This proves (5.1). □

This proves (ii).

To see (iii) ($\mathfrak{K}$-smoothness of $u(F)$ for $F \in R_{\mathfrak{K}}(T)$), take $B = F$, $C = T^{-1}F$ in (5.1).

To prove (iv), fix $p \in \mathbb{N}$. To see rational weak mixing of T^p along $(1/p)\mathfrak{K}$, let $A \in R_{\mathfrak{K}}(T)$. By Remark 4.1(vi), $a_n^{(T^p)}(A) \sim (1/p)a_{np}^{(T)}(A)$ along $(1/p)\mathfrak{K}$. It also follows

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

that, for B, $C \in \mathcal{B} \cap A$,

$$\begin{aligned}\sum_{k=0}^{n} |m(B \cap T^{-kp}C) - m(B)m(C)u_k^{(p)}| &= \sum_{k=0}^{n} |m(B \cap T^{-kp}C) - m(B)m(C)u_{pk}| \\ &\le \sum_{k=0}^{pn} |m(B \cap T^{-k}C) - m(B)m(C)u_k| \\ &= o(a_{pn}(A)) \quad \text{along } \frac{1}{p}\mathfrak{K}.\end{aligned}$$

This shows that $R_{\mathfrak{K}}(T) \subset R_{(1/p)\mathfrak{K}}(T^p)$.

The other inclusion follows from results in [**FL**]. The proof of [**FL**, Theorem 3.3] shows that

$$A \in R_{\mathfrak{K}}(T) \Leftrightarrow \left\{\frac{S_n^{(T)}(1_A)}{a_n^{(T)}(A)} : n \in \mathfrak{K}\right\} \quad \text{is uniformly integrable on } A.$$

Now

$$S_{pn}^{(T)}(1_A) = \sum_{k=0}^{pn-1} 1_A \circ T^k = \sum_{\nu=0}^{p-1}\sum_{k=0}^{n-1} 1_A \circ T^{kp+\nu} = \sum_{\nu=0}^{p-1} S_n^{(T^p)}(1_A) \circ T^\nu,$$

and hence

$$\begin{aligned}A \in R_{(1/p)\mathfrak{K}}(T^p) \;&\Leftrightarrow\; \left\{\frac{S_n^{(T^p)}(1_A)}{a_n^{(T^p)}(A)} : n \in \frac{1}{p}\mathfrak{K}\right\} \quad \text{is uniformly integrable on } A \\ &\Longrightarrow \left\{\frac{S_{pn}^{(T)}(1_A)}{a_{pn}^{(T)}(A)} : n \in \frac{1}{p}\mathfrak{K}\right\} \quad \text{is uniformly integrable on } A \\ &\Longrightarrow \left\{\frac{S_n^{(T)}(1_A)}{a_n^{(T)}(A)} : n \in \mathfrak{K}\right\} \quad \text{is uniformly integrable on } A \\ &\Longrightarrow A \in R_{\mathfrak{K}}(T). \qquad \square\end{aligned}$$

Proof of Theorem A. This follows from Proposition 0(ii) via Proposition 4.1. $\square$

6. *Proof of Lemmas B and C and Corollary E*

Proof of Lemma B. Let $\mathcal{U}$ be the collection of finite unions of sets in $\mathcal{C}$. It follows from assumptions (ii)(a) and (b) that

$$\frac{m(A \cap T^{-k}B)}{u_k} \xrightarrow[k\to\infty]{(u,\mathfrak{K})\text{-d.}} m(A)m(B) \quad \text{for all } A,\, B \in \mathcal{U}. \tag{6.1}$$

Let

$$\mathcal{K} := \{K \subset \Omega : K \text{compact}\}.$$

We claim first that

$$\frac{m(A \cap T^{-k}B)}{u_k} \xrightarrow[k\to\infty]{(u,\mathfrak{K})\text{-d.}} m(A)m(B) \quad \text{for all } A,\, B \in \mathcal{K},\, A,\, B \subset \Omega. \tag{6.2}$$

Proof of claim. We show first that, for all A, $B \in \mathcal{K}$, there exists $K_1 \subset \mathbb{N}$ $(u, \mathfrak{K})$-large, so that

(1) $$\overline{\lim_{k\to\infty, k\in K_1}} \frac{m(A \cap T^{-k}B)}{u_k} \le m(A)m(B).$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

To see this, we show that, for all $\epsilon > 0$, there exists $U, V \in \mathcal{U}$ so that

$$A \subset U,\ B \subset V, \quad m(U \setminus A), m(V \setminus B) < \epsilon. \tag{2}$$

Given $\epsilon > 0$, the Borel property of the measure m ensures open sets U, V satisfying (2). Each of these is a countable union of members of $\mathcal{C}$. By compactness of A, B we can reduce to finite unions $U, V \in \mathcal{U}$.

By (5.1) there exists $K_\epsilon \subset \mathbb{N}$ $(u, \mathfrak{K})$-small, so that

$$\frac{m(U \cap T^{-k}V)}{u_k} \xrightarrow[k\to\infty, k\notin K_\epsilon]{} m(U)m(V)$$

and

$$\overline{\lim_{k\to\infty, k\in K_\epsilon}} \frac{m(A \cap T^{-k}B)}{u_k} < (m(A) + \epsilon)(m(B) + \epsilon).$$

Fix N_ν $(\nu \geq 1)$ so that $a_u(K_{1/\nu}, n) < (1/\nu)a_n(\Omega)$ for all $n \in \mathfrak{K}$, $n \geq \nu$ and

$$\frac{m(A \cap T^{-k}B)}{u_k} < \left(m(A) + \frac{1}{\nu}\right)\left(m(B) + \frac{1}{\nu}\right) \quad \text{for all } k \in K^c_{1/\nu} \cap [N_\nu, \infty).$$

The set

$$K_1 := \bigcup_{\nu\geq 1}^{\cdot} K_{1/\nu} \cap [N_\nu, N_{\nu+1})$$

is as required for (1).

Now fix for all $A, B \in \mathcal{K}$, $A, B \subset \Omega$. Since $\Omega \in R_{\mathfrak{K}}(T)$,

$$\sum_{k=0}^{n-1} m(A \cap T^{-k}B) \sim m(A)m(B)a_n(\Omega) \quad \text{as } n \to \infty,\ n \in \mathfrak{K},$$

and claim (6.2) follows from (1) and Proposition 4.3. □

To complete the proof that Ω satisfies $(\bigstar_{\mathfrak{K}})$, let $A, B \in \mathcal{B} \cap \Omega$. Then there exists $E_N, F_N \in \mathcal{K}$ such that, modulo m,

$$E_N \uparrow A \quad \text{and} \quad F_N \uparrow B,$$

and hence, by (6.2), there exists K_N $(u, \mathfrak{K})$-small so that, for all $N \geq 1$,

$$\underline{\lim_{n\to\infty, n\notin K_N}} \frac{m(A \cap T^{-n}B)}{u_n} \geq m(E_N)m(F_N).$$

As above, there exists $K \subset \mathbb{N}$ $(u, \mathfrak{K})$-small such that

$$\underline{\lim_{n\to\infty, n\notin K}} \frac{m(A \cap T^{-n}B)}{u_n} \geq m(A)m(B)$$

and $(\bigstar_{\mathfrak{K}})$ follows from Proposition 4.3. □

Proof of Lemma C. By standardness, up to isomorphism, $X = \alpha^{\mathbb{Z}}$, $T : X \to X$ is the shift and the collection $\mathcal{C}_\alpha$ of (α, T)-cylinder sets forms a base of clopen sets for the Polish topology on X. Thus $\mathcal{C} := \mathcal{C}_\alpha \cap \Omega$ satisfies assumption (ii) of Lemma B and Lemma C follows. □

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Proof of Corollary E. Let $(X, \mathcal{B}, m, T)$ be rationally weakly mixing along $\mathfrak{K}$, and let $\pi : (X', \mathcal{B}', m', T') \to (X, \mathcal{B}, m, T)$ be its natural extension; that is,

$$T' \text{ invertible}, \pi \circ T' = T \circ \pi, \pi^{-1}\mathcal{B} \subset \mathcal{B}', m' \circ \pi^{-1} = m \text{ and } \bigvee_{n\geq 0} T'^n \pi^{-1}\mathcal{B} = \mathcal{B}'.$$

It follows from uniform integrability considerations (as in [**FL**, Theorem 3.3]) that T' is weakly rationally ergodic along $\mathfrak{K}$ with $R_{\mathfrak{K}}(T') \supseteq \mathcal{HR}(\pi^{-1}R_{\mathfrak{K}}(T))$.

To see that T' is rationally weakly mixing along $\mathfrak{K}$, fix a countable, one-sided T-generator $\alpha \subset R_{\mathfrak{K}}(T)$. Then $\alpha' := \pi^{-1}\alpha \subset R_{\mathfrak{K}}(T')$ is a countable, two-sided T'-generator.

Fix $\Omega \in \alpha'$ and let $u := u(\Omega)$. By rational weak mixing of T along $\mathfrak{K}$,

$$\frac{m'(A \cap T'^{-n}B)}{u_n} \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} m'(A)m'(B) \quad \text{for all } A, B \in \mathcal{C}_{\alpha'} = \pi^{-1}\mathcal{C}_\alpha,$$

and hence, by Lemma C, $(X', \mathcal{B}', m', T')$ is rationally weakly mixing along $\mathfrak{K}$. □

7. *Mean ergodic theorem for weighted averages*

Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence. Call a weight $u \in \mathfrak{W}$ (good for the) *mean ergodic theorem along* $\mathfrak{K}$ (abbreviated to $\text{MET}_{\mathfrak{K}}$) if, for any ergodic, probability preserving transformation $(\Omega, \mathcal{A}, P, S)$, we have that

$$\frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k f \circ S^k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{\Lambda^2(P)} E(f) \quad \text{for all } f \in L^2(P).$$

We let $\text{MET} := \text{MET}_{\mathbb{N}}$.

Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence.

Using the spectral theorem for unitary operators, it can be shown (see [**Kre**]) that the following conditions are equivalent for $u \in \mathfrak{W}$:

- u is $\text{MET}_{\mathfrak{K}}$;
- $(1/a_u(n)) \sum_{k=0}^{n} u_k z^k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0$ for $z \in \mathbb{C}$, $|z| = 1$, $z \neq 1$;
- u is good for the *weak ergodic theorem along* $\mathfrak{K}$ (abbreviated to $\text{WET}_{\mathfrak{K}}$) in the sense that, for any ergodic, probability preserving transformation $(\Omega, \mathcal{A}, P, S)$,

$$\frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k f \circ S^k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} E(f) \text{ weakly in } L^2(P) \quad \text{for all } f \in L^2(P).$$

The recurrent renewal sequences form an important subclass of weights. A weight $u \in \mathfrak{W}$ is a *recurrent renewal sequence* if $u_0 = 1$ and there exists $f \in \mathcal{P}(\mathbb{N})$, called the associated *lifetime distribution*, satisfying the *renewal equation*

$$u_n = \sum_{k=1}^{n} f_k u_{n-k} \quad (n \geq 1).$$

The renewal sequence u is said to be *aperiodic* if $\langle\{n \in \mathbb{N} : u_n > 0\}\rangle = \mathbb{Z}$.

It follows from the renewal equation that any *aperiodic*, recurrent renewal sequence satisfies $|\sum_{k=0}^{n} u_k z^k| < \infty$ for $z \in \mathbb{C}$, $|z| = 1$, $z \neq 1$ and hence is MET. Proposition 7.2 (below) generalizes this.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence. Any $\mathfrak{K}$-smooth weight $u \in \mathfrak{W}$ is $\mathrm{MET}_{\mathfrak{K}}$ (see [**HP, Kre**] and the references therein). A weight $u \in \mathfrak{W}$ which is MET and not $\mathfrak{K}$-smooth for any subsequence $\mathfrak{K} \subset \mathbb{N}$ is exhibited in [**HP**].

We will need the following lemma.

LEMMA 7.1. *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence and suppose that $u \in \mathfrak{W}$ is $\mathrm{MET}_{\mathfrak{K}}$, and that $(\Omega, \mathcal{A}, P, S)$ is a weakly mixing probability preserving transformation. Then*

$$P(A \cap S^{-n}B) \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} P(A)P(B) \quad \textit{for all } A, B \in \mathcal{A}.$$

Proof. It follows from $(\mathrm{WET}_{\mathfrak{K}})$ for S and $A, B \in \mathcal{A}$ that

$$\frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k P(A \cap S^{-k}B) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} P(A)P(B), \tag{7.1}$$

and it follows from (7.1) for $S \times S$ (which is ergodic) and $A \times A$, $B \times B \in \mathcal{A} \otimes \mathcal{A}$ that

$$\begin{aligned} \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k P(A \cap S^{-k}B)^2 &= \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k P \times P(A \times A \cap (S \times S)^{-k} B \times B) \\ &\xrightarrow[n\to\infty, n\in\mathfrak{K}]{} P \times P(A \times A) \cdot P \times P(B \times B) \\ &= P(A)^2 P(B)^2. \end{aligned} \tag{7.2}$$

Using (7.1) and (7.2),

$$\begin{aligned} &\frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k (P(A \cap S^{-k}B) - P(A)P(B))^2 \\ &= \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k (P(A \cap S^{-k}B)^2 - 2P(A)P(B)P(A \cap S^{-k}B) + P(A)^2 P(B)^2) \\ &\xrightarrow[n\to\infty, n\in\mathfrak{K}]{} 0, \end{aligned}$$

and hence $P(A \cap S^{-n}B) \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}} P(A)P(B)$. □

PROPOSITION 7.2. *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence. Suppose that $(X, \mathcal{B}, m, T)$ is weakly rationally ergodic along $\mathfrak{K}$ and spectrally weakly mixing. Then $u(E, F)$ is $\mathrm{MET}_{\mathfrak{K}}$ for all $E, F \in R_{\mathfrak{K}}(T)_+$.*

Proof. Let $(\Omega, \mathcal{A}, P, S)$ be an ergodic, probability preserving transformation.

It follows from the assumptions that $T \times S$ is weakly rationally ergodic along $\mathfrak{K}$ and $R_{\mathfrak{K}}(T \times S) \supset R_{\mathfrak{K}}(T) \times \Omega$.

It suffices to show that, for $E, F \in R_{\mathfrak{K}}(T)_+$,

$$A_n f \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} E(f) \text{ weakly in } L^2(P) \quad \text{for all } f \in L^2(P), \tag{7.3}$$

where $A_n f := (1/a_n(E, F)) \sum_{k=0}^{n-1} u_k(E, F) f \circ S^k$.

Since $E \times \Omega$, $F \times \Omega \in R_{\mathfrak{K}}(T \times S)$,

$$\frac{1}{a_n} \sum_{k=0}^{n-1} u_k(E, F) P(C \cap S^{-k}D) \xrightarrow[n\to\infty]{} P(C)P(D) \quad \text{for all } C, D \in \mathcal{B}(\Omega).$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

This shows that (7.3) holds for indicators, and hence for simple functions f. By the triangle inequality $\|A_n f\|_2 \le \|f\|_2$ for all $f \in L^2(P)$ and (7.3) follows by approximation. □

8. *Proof of Theorem D*

We assume that T is invertible. By Corollary E, this involves no loss of generality.

Proof of Theorem D(i). We will prove spectral weak mixing of T by showing that $T \times S$ is weakly rationally ergodic along $\mathfrak{K}$ for any ergodic, probability preserving transformation $(\Omega, \mathcal{A}, P, S)$. To this end, let $(\Omega, \mathcal{A}, P, S)$ be an ergodic, probability preserving transformation. We claim first that

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} m(A \cap T^{-k}B)P(C \cap S^{-k}D) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} m(A)m(B)P(C)P(D) \quad \text{for all } A, B \in R_{\mathfrak{K}}(T),\ C, D \in \mathcal{A}. \tag{8.1}$$

Proof of (8.1). Fix $A, B \in R_{\mathfrak{K}}(T)_+$ and set $v = u(A, B)$. By Proposition 0(iii), v is smooth and therefore $\mathrm{MET}_{\mathfrak{K}}$; and (8.1) follows from $(\mathrm{WET}_{\mathfrak{K}})$ for $C, D \in \mathcal{A}$. □

Let $\mu := m \times P$, $\mathcal{C} := \mathcal{B} \otimes \mathcal{A}$ and $\tau := T \times S$.

We claim next that, for $F \in R_{\mathfrak{K}}(T)$, $A \in \mathcal{B} \cap F$, $B \in \mathcal{A}$, $C \in \mathcal{C} \cap (F \times \Omega)$,

$$\frac{1}{a_n} \sum_{k=0}^{n-1} \mu(C \cap \tau^{-k}(A \times B)) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} \mu(C)\mu(A \times B). \tag{8.2}$$

Proof of (8.2). By weak rational ergodicity along $\mathfrak{K}$ the collection $\{(1/a_n)\sum_{k=0}^{n-1} 1_A \circ T^k : n \in \mathfrak{K}\}$ is uniformly integrable on F. It follows that the collection $\{(1/a_n)\sum_{k=0}^{n-1} 1_{A\times B} \circ \tau^k : n \in \mathfrak{K}\}$ is uniformly integrable on $F \times \Omega$.

Let $\Phi \in \{(1/a_n)\sum_{k=0}^{n-1} 1_{A\times B} \circ \tau^k : n \in \mathfrak{K}\}'$ be a weak limit. Then, by (8.1),

$$\int_{C\times D} \Phi \, d\mu = \mu(A \times B)\mu(C \times D) \quad \text{for all } C \in \mathcal{B} \cap F,\ D \in \mathcal{A}.$$

It follows that $\Phi \equiv \mu(A \times B)$, and hence $(1/a_n)\sum_{k=0}^{n-1} 1_{A\times B} \circ \tau^k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} \mu(A \times B)$ weakly in $L^1(F \times \Omega)$ and (8.2) follows. □

Finally we complete the proof of Theorem D(i) by showing that

$$F \times \Omega \in R_{\mathfrak{K}}(\tau);$$

namely, for $F \in R_{\mathfrak{K}}(T)$, $C, D \in \mathcal{C} \cap (F \times \Omega)$,

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} \mu(C \cap \tau^{-k}D) \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} \mu(C)\mu(D). \tag{8.3}$$

Proof of (8.3). Since $D \subset F \times \Omega$, the collection $\{(1/a_n)\sum_{k=0}^{n-1} 1_D \circ \tau^k : n \in \mathfrak{K}\}$ is uniformly integrable on $F \times \Omega$. Let $\Psi \in \{(1/a_n)\sum_{k=0}^{n-1} 1_D \circ \tau^k : n \in \mathfrak{K}\}'$ be a weak limit. Then, by (8.2) for τ^{-1},

$$\int_{A\times B} \Phi \, d\mu = \mu(A \times B)\mu(D) \quad \text{for all } A \in \mathcal{B} \cap F,\ B \in \mathcal{A},$$

and hence $(1/a_n)\sum_{k=0}^{n-1} 1_D \circ \tau^k \xrightarrow[n\to\infty, n\in\mathfrak{K}]{} \mu(D)$ weakly in $L^1(F \times \Omega)$ and (8.3) follows. □

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Remark. Spectral weak mixing alone does not imply subsequence rational weak mixing. See [**ALV**] for squashable, spectrally weakly mixing, transformations. These are not even subsequence weakly rationally ergodic. We do not know whether weak rational ergodicity and spectral weak mixing together imply subsequence rational weak mixing.

Proof of Theorem D(ii). Fix a countable, $\mathcal{B}$-generating partition $\alpha \subset R_{\mathfrak{K}}(T)$. By standardness, up to isomorphism, $X = \alpha^{\mathbb{Z}}$ and $T : X \to X$ is the shift. The collection $\mathcal{C}_\alpha$ of (α, T)-cylinder sets forms a base of clopen sets for the T-invariant, measurable, Polish topology on X.

Let $(\Omega, \mathcal{A}, P, S)$ be a weakly mixing, probability preserving transformation. Fix a compact S-invariant, completely disconnected, measurable topology on Ω generating $\mathcal{A}$.

We must show that the measure preserving transformation

$$(Z, \mathcal{C}, \mu, \tau) := (X \times \Omega, \mathcal{B} \otimes \mathcal{A}, m \times P, T \times S)$$

is rationally weakly mixing along $\mathfrak{K}$.

For this, it suffices to show that, for $F \in \alpha$, $m(F) > 0$, $F \times \Omega$ satisfies ($\bigstar_{\mathfrak{K}}$) with respect to τ.

By Lemma B, it suffices to establish

$$\frac{m(A \cap T^{-n}B)P(C \cap S^{-n}D)}{u_n(F)} \xrightarrow[n\to\infty]{(u(F),\mathfrak{K})\text{-d}} m(A)m(B)P(C)P(D) \quad \text{for all } A, B \in \mathcal{B} \cap F,\ C, D \in \mathcal{A}. \tag{8.4}$$

Proof of (8.4). By Proposition 0(iii), $u(F)$ is $\mathfrak{K}$-smooth and hence $\mathrm{MET}_{\mathfrak{K}}$, and by Lemma 7.1,

$$P(C \cap S^{-n}D) \xrightarrow[n\to\infty]{(u(F),\mathfrak{K})\text{-d}} P(C)P(D) \quad \text{for all } C, D \in \mathcal{A}.$$

Thus, since F satisfies ($\bigstar_{\mathfrak{K}}$), for $A, B \in \mathcal{B} \cap F$,

$$\frac{m(A \cap T^{-n}B)}{u_n(F)} \xrightarrow[n\to\infty]{(u(F),\mathfrak{K})\text{-d}} m(A)m(B).$$

These two $(u(F), \mathfrak{K})$-density convergences imply (8.4) and thus, via Lemma B, Theorem D(ii). □

9. *Markov shift examples*

Let S be a countable set (the *state space*) and let $P : S \times S \to [0, 1]$ be a stochastic matrix (the *transition matrix*) on S ($\sum_{t\in S} p_{s,t} = 1$ for all $s \in S$) with an invariant distribution $\pi : S \to \mathbb{R}_+$ ($\sum_{u\in S} \pi_u p_{u,t} = \pi_t$).

The stationary, *two-sided Markov shift* of (P, π) is the quadruple

$$(S^{\mathbb{Z}}, \mathcal{B}, m, T),$$

where $T : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is the shift,

$$\mathcal{B} := \sigma(\{\text{cylinders}\}),$$

a *cylinder* being a set of the form

$$[s_1, \ldots, s_n]_k := \{x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in S^{\mathbb{Z}} : x_{j+k} = s_j \text{ for all } 1 \le j \le n\}$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

(for $s_1, \ldots, s_n \in S^n$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$), and the measure m is defined by

$$m([s_1, \ldots, s_n]_k) = \pi_{s_1} p_{s_1,s_2} \cdots p_{s_{n-1},s_n} \quad \text{for all } s_1, \ldots, s_n \in S^n,\ n \in \mathbb{N}.$$

The stationary Markov shift $(S^{\mathbb{Z}}, \mathcal{B}, m, T)$ is a measure preserving transformation.

As shown in [**HR**], T is:

- conservative if and only if P is *recurrent* ($\sum_{n=0}^{\infty} p_{s,s}^{(n)} = \infty$ for all $s \in S$) and, in this case, T is;
- ergodic if and only if P is *irreducible* (for all $s, t \in S$, there exists $n \in \mathbb{N} \ni p_{s,t}^{(n)} > 0$).

The (stationary) *one-sided, Markov shift* is $(S^{\mathbb{N}}, \mathcal{B}_+, m_+, \tau)$, where $\tau : S^{\mathbb{N}} \to S^{\mathbb{N}}$ is the shift,

$$\mathcal{B}_+ := \sigma(\{\text{one-sided cylinders}\}),$$

a *one-sided cylinder* being a set of the form

$$[s_1, \ldots, s_n] := \{x = (x_1, x_2, \ldots) \in S^{\mathbb{N}} : x_j = s_j \text{ for all } 1 \le j \le n\}$$

for $(s_1, \ldots, s_n) \in S^n$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, and the measure m_+ is defined by

$$m_+([s_1, \ldots, s_n]) = \pi_{s_1} p_{s_1,s_2} \cdots p_{s_{n-1},s_n} \quad \text{for all } s_1, \ldots, s_n \in S^n,\ n \in \mathbb{N}.$$

As shown in [**BF**], if the stochastic matrix P is irreducible, recurrent and *aperiodic* ($\gcd\{n \ge 1 : p_{s,s}^{(n)} > 0\} = 1$ for some and hence all $s \in S$), then T is a conservative K-automorphism (natural extension of an exact endomorphism), and hence (see [**ALW**]) spectrally weakly mixing.

As shown in [**A1**], a conservative, ergodic Markov shift $(S^{\mathbb{N}}, \mathcal{B}, m, T)$ is rationally ergodic with $R_{\mathfrak{K}}(T) \supset \mathcal{HR}(\mathcal{C}_\alpha)$, where $\alpha := \{[s]_0 : s \in S\}$, with $a_n(T) = a_n(P) \sim (1/\pi_s) \sum_{k=0}^{n-1} p_{s,s}^{(k)}$ (for all $s \in S$).

THEOREM 9.1. *Let $\mathfrak{K} \subset \mathbb{N}$ be a subsequence. The Markov shift $(S^{\mathbb{Z}}, \mathcal{B}, m, T)$ of the irreducible, recurrent, aperiodic transition matrix P on state space S is rationally weakly mixing along $\mathfrak{K}$ if and only if there exists $s \in S$ with $u([s]_0)$ $\mathfrak{K}$-smooth.*

Proof. By Proposition 0(iii), if T is rationally weakly mixing along $\mathfrak{K}$, then $u([s]_0)$ is $\mathfrak{K}$-smooth for all $s \in S$.

To prove the other implication, we will need the following lemma.

LEMMA 9.2. *Let S be a countable set and let $P : S \times S \to [0, 1]$ be an irreducible, recurrent, aperiodic stochastic matrix with the property that, for some $s \in S$, $u = u([s]_0)$ is $\mathfrak{K}$-smooth. Then*

$$\frac{p_{r,t}^{(n+\ell)}}{u_n} \xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-s.C.}} \pi_t \quad \textit{for all } r, t \in S,\ \ell \in \mathbb{Z}. \tag{9.1}$$

Lemma 9.2 is a $(u, \mathfrak{K})$-density version of [**O**, Lemma 1].

Proof of Lemma 9.2. Recall from [**Ch**] that the P-stationary distribution $\pi : S \to \mathbb{R}_+$ with $\pi_s = 1$ is given by

$$\pi_t = \sum_{n=1}^{\infty} {}_s p_{s,t}^{(n)}$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

where

$$ {}_sp_{s,t}^{(1)} := p_{s,t} \quad \text{and} \quad {}_sp_{s,t}^{(n+1)} := \sum_{r\in S\setminus\{s\}} {}_sp_{s,r}^{(n)} p_{r,t}. $$

As shown in [**Ch**], for all $r, t \in S$,

$$ \frac{1}{a_u(n)} \sum_{k=0}^{n-1} p_{r,t}^{(k)} \xrightarrow[n\to\infty]{} \pi_t. $$

In view of this, to show (9.1), it suffices by Proposition 4.3(i) to show that for all $r, t \in S$, $\ell \in \mathbb{Z}$, there exists $K_{r,t,\ell} \subset \mathbb{N}$ $(u, \mathfrak{K})$-small such that

$$ \varliminf_{n\to\infty, n\notin K_{r,t,\ell}} \frac{p_{r,t}^{(n+\ell)}}{u_n} \geq \pi_t. \tag{9.2} $$

Let $K_0 \subset \mathbb{N}$ be $(u, \mathfrak{K})$-small such that $u_{n+k}/u_n \xrightarrow[n\to\infty, n\notin K_0]{} 1$ for all $k \in \mathbb{Z}$.

To see (9.2) for $r = s$ and $t \in S$, $\ell \in \mathbb{Z}$,

$$ p_{s,t}^{(n+\ell)} = \sum_{k=0}^{n+\ell-1} u_{n+\ell-k}\, {}_sp_{s,t}^{(k)} \geq \sum_{k=0}^{N-1} u_{n+\ell-k}\, {}_sp_{s,t}^{(k)} \quad \text{for all } n+\ell > N \geq 1. $$

Thus

$$ \frac{p_{s,t}^{(n+\ell)}}{u_n} \geq \sum_{k=0}^{N-1} {}_sp_{s,t}^{(k)} \frac{u_{n+\ell-k}}{u_n} \xrightarrow[n\to\infty, n\notin K_0]{} \sum_{k=0}^{N-1} {}_sp_{s,t}^{(k)}, $$

and (9.2) holds with $K_{s,t} = K_0$. As mentioned above, we now have (9.1) with $r = s$.

To see (9.2) for general $r, t \in S$, fix first K_t $(u, \mathfrak{K})$-small such that

$$ \frac{p_{s,t}^{(n+k)}}{u_n} \xrightarrow[n\to\infty, n\notin K_t]{} \pi_t \quad \text{for all } k \in \mathbb{Z}. $$

Next,

$$ p_{r,t}^{(n+\ell)} = \sum_{k=0}^{n-1} {}_sp_{r,s}^{(k)}\, p_{s,t}^{(n+\ell-k)} \geq \sum_{k=0}^{N-1} {}_sp_{r,s}^{(k)}\, p_{s,t}^{(n+\ell-k)} \quad \text{for all } n+\ell > N \geq 1, $$

$$ \frac{p_{r,t}^{(n+\ell)}}{u_n} \geq \sum_{k=0}^{N-1} {}_sp_{r,s}^{(k)} \frac{p_{s,t}^{(n+\ell-k)}}{u_n} \xrightarrow[n\to\infty, n\notin K_t]{} \sum_{k=0}^{N-1} {}_sp_{r,s}^{(k)} \cdot \pi_t \xrightarrow[N\to\infty]{} \pi_t $$

(because $\sum_{k=0}^{\infty} {}_sp_{r,s}^{(k)} = 1$), and (9.2) holds with $K_{r,t} = K_t$. □

Proof of Theorem 9.1 continued. Suppose that $A = [a_1, \ldots, a_I]_k$ and $B = [b_1, \ldots, b_J]_\ell \in \mathcal{C}_\alpha$. Then, for $n \in \mathbb{Z}$,

$$ A \cap T^{-n}B = \{x \in S^{\mathbb{Z}} : x_{k+i} = a_i \text{ for all } 1 \leq i \leq I \text{ and } x_{n+\ell+j} = b_j \text{ for all } 1 \leq j \leq J\}, $$

and for $n > k + I - \ell$,

$$ \begin{aligned} m(A \cap T^{-n}B) &= \pi_{a_1} p_{a_1,a_2} \cdots p_{a_{I-1},a_I} p_{a_I,b_1}^{(n+\ell-I)} p_{b_1,b_2} \cdots p_{b_{J-1},b_J} \\ &= m(A)m(B) \cdot \frac{p_{a_I,b_1}^{(n+\ell-I)}}{\pi_{b_1}}. \end{aligned} $$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Hence

$$\frac{m(A\cap T^{-n}B)}{u_n}=\frac{m(A)m(B)}{\pi_{b_1}}\cdot\frac{p_{a_1,b_1}^{(n+\ell-1)}}{u_n}\xrightarrow[n\to\infty]{(u,\mathfrak{K})\text{-d.}}m(A)m(B).$$

Rational weak mixing follows from Lemma C. □

9.1. *Smoothness of renewal sequences.*

Remark 9.1. If u is a recurrent, *aperiodic* renewal sequence, whose associated lifetime distribution $f\in\mathcal{P}(\mathbb{N})$ has tails $f([n,\infty))$ which are $(-\gamma)$-regularly varying with $\gamma\in(0,1]$, then

$$\frac{nu_n}{a_n}\begin{cases}\xrightarrow[n\to\infty]{}1, & \gamma=1 \text{ by } [\mathbf{E}],\\ \xrightarrow[n\to\infty]{}\gamma, & \frac{1}{2}<\gamma<1 \text{ by } [\mathbf{GL}],\\ \xrightarrow[n\to\infty]{\text{density}}\gamma, & \gamma\le\frac{1}{2} \text{ by } [\mathbf{GL}].\end{cases}$$

By Proposition 4.1, the convergence in the third case (which follows from Lemma 10.2 below) is also in u-density. In all cases, u is smooth and any corresponding Markov chain is rationally weakly mixing by Theorem 9.1.

PROPOSITION 9.3. *Suppose that $u=(u_0,u_1,\ldots)$ is an aperiodic, recurrent renewal sequence with lifetime distribution $f\in\mathcal{P}(\mathbb{N})$. Let $L(n):=\sum_{k=1}^{n}f([k,\infty))$ and $V(t):=\sum_{1\le n\le t}n^2f_n$.*

(i) *If for some $N\ge1$, $\sum_{n=N}^{\infty}(1/V(n)^2)<\infty$, then $\sum_{n=1}^{\infty}(u_n-u_{n+1})^2<\infty$.*

(ii) *If, in addition, $L(n)/\sqrt{n}\xrightarrow[n\to\infty]{}0$, then u is smooth.*

Proof of (i). By Parseval's formula and the renewal equation,

$$\int_{-\pi}^{\pi}\frac{|\theta|^2\,d\theta}{|1-f(\theta)|^2}<\infty\iff\sum_{n=1}^{\infty}(u_n-u_{n+1})^2<\infty,$$

where $f(\theta):=\sum_{n=1}^{\infty}f_ne^{in\theta}$. By aperiodicity, $\sup_{\epsilon\le|\theta|\le\pi}|f(\theta)|<1$ for all $\epsilon>0$, and hence (using symmetry)

$$\int_{-\pi}^{\pi}\frac{|\theta|^2\,d\theta}{|1-f(\theta)|^2}<\infty\iff\int_{0}^{\epsilon}\frac{\theta^2\,d\theta}{|1-f(\theta)|^2}<\infty\quad\text{for some }\epsilon>0.$$

Next,

$$\begin{aligned}|1-f(\theta)|\ge Re\,(1-f(\theta))&=2\sum_{n=1}^{\infty}f_n\sin^2\left(\frac{n\theta}{2}\right)\\&\ge2\sum_{1\le n\le\pi/\theta}f_n\sin^2\left(\frac{n\theta}{2}\right)\ge\frac{2\theta^2}{\pi^2}\sum_{1\le n\le\pi/\theta}n^2f_n\\&=:C\theta^2V\left(\frac{\pi}{\theta}\right).\end{aligned}$$

For large N, $V(N)>0$ and

$$\int_0^{\pi/N}\frac{\theta^2\,d\theta}{|1-f(\theta)|^2}\le\sum_{n=N}^{\infty}\int_{\pi/(n+1)}^{\pi/n}\frac{d\theta}{(C\theta V(\pi/\theta))^2}\le C'\sum_{n=N}^{\infty}\frac{1}{V(n)^2}.$$

□

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Proof of (ii). It follows from the renewal equation (see [**A**, Lemma 3.8.5]) that $a_u(n) \asymp n/L(n)$. Hence

$$\frac{\sqrt{n}}{a_u(n)} \asymp \sqrt{n} \cdot \frac{L(n)}{n} = \frac{L(n)}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0,$$

and therefore, by (i),

$$\frac{1}{a_u(n)} \sum_{k=1}^{n} |u_k - u_{k+1}| \le \frac{\sqrt{n}}{a_u(n)} \sqrt{\sum_{n\ge 1} |u_n - u_{n+1}|^2} \xrightarrow[n\to\infty]{} 0. \qquad \square$$

For example, let $f \in \mathcal{P}(\mathbb{N})$ be the winnings distribution in the St Petersburg game:

$$f_k = \begin{cases} \dfrac{1}{2^{n+1}}, & k = 2^n \ (n \ge 0), \\ 0, & \text{else.} \end{cases}$$

The associated aperiodic, recurrent renewal sequence is smooth by Proposition 9.3 (Remark 9.1 above does not apply).

The following 'extends' Dyson's example [**Ch**, p. 55] of an aperiodic renewal sequence without the strong ratio limit property.

PROPOSITION 9.4. *There is a subsequence smooth, recurrent, renewal sequence which does not have the strong ratio limit property.*

Proof. We consider $\mathbb{P} := \{f \in \mathcal{P}(\mathbb{N}) : f_1 > 0\}$ metrized by

$$d(f, g) := \left| \frac{1}{f_1} - \frac{1}{g_1} \right| + \sum_{n\ge 1} |f_n - g_n|.$$

This space is Polish (complete and separable).

For $f \in \mathbb{P}$, let $u^{(f)}$ be the associated (aperiodic, recurrent) renewal sequence. Let

$$\mathbb{P}_{\text{SRLP}} := \{f \in \mathbb{P} : u^{(f)} \text{ has the strong ratio limit property}\}$$

and

$$\mathbb{P}_{\text{ss}} := \{f \in \mathbb{P} : \text{there exists } \mathfrak{K} \subset \mathbb{N},\ u^{(f)} \text{ is } \mathfrak{K}\text{-smooth}\}.$$

We show that

$$\mathbb{P}_{\text{ss}} \backslash \mathbb{P}_{\text{SRLP}} \quad \text{is residual in } \mathbb{P}$$

(and therefore not empty).

By Baire's theorem, it suffices to show residuality of $\mathbb{P}_{\text{ss}}$ and $\mathbb{P}\backslash\mathbb{P}_{\text{SRLP}}$.

Proof that $\mathbb{P}_{ss}$ is residual. For each $n \ge 1$, the function $f \mapsto u_n^{(f)}$ is continuous ($\mathbb{P} \to \mathbb{R}$), being a polynomial function of $(f_1, f_2, \ldots, f_n)$. Thus

$$\mathbb{P}_{\text{ss}} = \bigcap_{k=1}^{\infty} \bigcup_{N=k}^{\infty} \left\{ f \in \mathbb{P} : \sum_{j=1}^{N} |u_j^{(f)} - u_{j+1}^{(f)}| < \frac{1}{k} \sum_{j=1}^{N} u_j^{(f)} \right\}$$

is a G_δ set. By the renewal theorem $\mathbb{P}_{\text{ss}} \supset \mathbb{P}_+ := \{f \in \mathbb{P} : \sum_{n\ge 1} n f_n < \infty\}$, which is dense in $\mathbb{P}$. $\square$

Proof that $\mathbb{P}\backslash\mathbb{P}_{SRLP}$ is residual. For each $k \in \mathbb{N}$,

$$\Pi_k := \left\{ f \in \mathbb{P} : \text{there exists } N > k \text{ such that } u_{N-1}^{(f)} < \frac{u_N^{(f)}}{k} \right\}$$

is open.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

Since $\mathbb{P}\backslash\mathbb{P}_{\mathrm{SRLP}} \supseteq \bigcap_{k\geq 1} \Pi_k$, it suffices to prove that each Π_k is dense. To this end, fix $k \geq 1$, $f \in \mathbb{P}$ and $\epsilon > 0$. We will show that there exists $g \in \Pi_k$, $d(f, g) < 2\epsilon$. To this end note first that there exists $h \in \mathbb{P}$ so that $d(f, h) < \epsilon$ and so that the set $\{n \in \mathbb{N} : h_n > 0\}$ is infinite. Using this, find $\ell > k$ so that

$$0 < 1 - H := \sum_{j=\ell+1}^{\infty} h_j < \epsilon.$$

For $L > \ell$ define $g^{(L)} \in \mathbb{P}$ by

$$g_n^{(L)} := \begin{cases} h_n, & n \leq \ell, \\ \displaystyle\sum_{j=\ell+1}^{\infty} h_j, & n = L, \\ 0, & \text{else.} \end{cases}$$

We claim that, for all large L, $u_{L-1}^{(g^{(L)})} < (1 - H)/k \leq u_L^{(g^{(L)})}/k$, and hence $g^{(L)} \in \Pi_k$.

To see this, define the defective renewal sequence v by

$$v_0 = 1, \quad v_n := \sum_{k=1}^{n\wedge\ell} h_k v_{n-k}.$$

Then $u_j^{(g^{(L)})} = v_j$ for all $1 \leq j < L$.

Let $V_r := \max_{v\geq r\ell+1} v_v$. For $j \geq 1$,

$$v_{r\ell+j} = \sum_{i=1}^{\ell} h_i v_{r\ell+j-i} = \sum_{i=1}^{\ell} h_i v_{(r-1)\ell+j+\ell-i} \leq H V_{r-1},$$

and hence $V_r \leq H V_{r-1}$ and $v_{r\ell} \leq H^{r-1} v_\ell$. Now fix $L > \ell$ so that $v_{L-1} < (1 - H)/k$. It follows as above that $g^{(L)} \in \Pi_k$ and $d(f, g^{(L)}) < 2\epsilon$. □

□

10. *Examples with local limit sets*

In this section, we prove a generalization of part of [**GL**, Theorem 1.1] thereby establishing sufficient conditions for rational weak mixing. It is necessary to deal with essentially non-invertible transformations. By Corollary E, rational weak mixing passes to the natural extensions of these non-invertible transformations.

Suppose that $(X, \mathcal{B}, m, T)$ is a pointwise dual ergodic, measure preserving transformation (as in [**A**]) with γ-regularly varying return sequence $a(n) = a_n(T)$ $(0 < \gamma < 1)$. As shown in [**A**, Ch. 3], T is rationally ergodic, and T is not invertible.

By the Darling-Kac theorem ([**DK**], see also [**A**, Ch. 3]),

$$\frac{1}{a(n)} S_n(f) \xrightarrow{\mathfrak{d}} X_\gamma m(f) \quad \text{for all } f \in L_+^1$$

on $(X, \mathcal{B}, m)$ where X_γ is the Mittag-Leffler distribution of order γ normalized so that $E(X_\gamma) = 1$, $m(f) := \int_X f\, dm$, and $\xrightarrow{\mathfrak{d}}$ on $(X, \mathcal{B}, m)$ denotes convergence in distribution with respect to all m-absolutely continuous probabilities.

Let $\Omega \in \mathcal{B}$, $m(\Omega) = 1$ (the normalization $m(\Omega) = 1$ is not necessary, but convenient).

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

The *return time function* to Ω is $\varphi = \varphi_\Omega : \Omega \to \mathbb{N}$ defined by $\varphi(\omega) := \min\{n \geq 1 : T^n\omega \in \Omega\} < \infty$ almost surely, by conservativity. The *induced transformation* on Ω is $T_\Omega : \Omega \to \Omega$ defined by $T_\Omega(\omega) := T^{\varphi(\omega)}(\omega)$. As is well known, T_Ω is an ergodic, probability preserving transformation of $(\Omega, \mathcal{B}(\Omega), m_\Omega)$.

The return time process on Ω satisfies the stable limit theorem. Indeed, by [**A2**, Proposition 1],

$$\frac{1}{B(n)}\varphi_n \xrightarrow{\mathfrak{d}} Z_\gamma$$

on $(\Omega, \mathcal{B}(\Omega), m_\Omega)$, where $B(n) := a^{-1}(n)$, $Z_\gamma = Y_\gamma^{-1/\gamma}$ is the stable random variable of order γ and $\varphi_n := \sum_{k=0}^{n-1} \varphi \circ T_\Omega^k$.

The above is true for any $\Omega \in \mathcal{B}$, $m(\Omega) = 1$. By 'choosing' Ω carefully, it may be possible to obtain stronger properties.

Accordingly, in the above situation, we call $\Omega \in R(T)$, $m(\Omega) = 1$ a *local limit set* (LLT) if there exists a countable partition $\beta \subset \mathcal{B}(\Omega)$ generating $\mathcal{B}(\Omega)$ under T_Ω such that $\varphi_\Omega^{-1}\{n\} \in \sigma(\beta)$ for all $n \geq 1$ and such that, for all A, $B \in \mathcal{C}_\beta(T_\Omega)$,

$$B(n)m(A \cap T_\Omega^{-n} B \cap [\varphi_n = k_n]) \xrightarrow[n\to\infty, k_n/B(n)\to x]{} f_{Z_\gamma}(x)m(A)m(B)$$

uniformly in $x \in [c, d]$ whenever $0 < c < d < \infty$, where $f = f_{Z_\gamma}$ is the probability density function of Z_γ.

To be an LLT set, essentially, the return time stochastic process to Ω needs to satisfy the conditional, γ-stable, local limit theorem.

Example 10.1. If $(X, \mathcal{B}, m, T)$ is the tower over the a Gibbs Markov fibred system (as in [**AD**]), or an AFU fibred system (as in [**ADSZ**]) $(\Omega, \mathcal{A}, P, S, \alpha)$ with an α-measurable height function φ satisfying $E(\varphi \wedge t)$ regularly varying at infinity with index in $(0, 1)$, then $(X, \mathcal{B}, m, T)$ is pointwise dual ergodic, $\Omega \in R(T)$ with $a_n(T) = a_n(\Omega) \propto n/E(\varphi \wedge n)$ and the return time stochastic process to Ω satisfies the conditional, γ-stable, local limit theorem. See [**AD**] and [**ADSZ**] respectively. Thus, Ω is an LLT set.

THEOREM 10.1. *Suppose that $(X, \mathcal{B}, m, T)$ is pointwise dual ergodic with $a(n) = a_n(T)$ γ-regularly varying ($\gamma \in (0, 1)$) and which has an LLT set, then $(X, \mathcal{B}, m, T)$ is rationally weakly mixing.*

Proof. Let $\Omega \in R(T)$ be an LLT set with accompanying T_Ω-generating partition β. By standardness, up to isomorphism, $\Omega = \alpha^{\mathbb{N}}$, $T_\Omega : \Omega \to \Omega$ is the shift and the collection $\mathcal{C}_\beta(T_\Omega)$ of (β, T_Ω)-cylinder sets forms a base of clopen sets for the Polish topology on Ω. The proof is via Lemma C, whose use is enabled by the following Lemma 10.2, which is a version of the 'local limit' [**GL**, proof of Theorem 1.1]. Analogous results are established in [**MT**]. □

LEMMA 10.2. *Suppose that $(X, \mathcal{B}, m, T)$ is pointwise dual ergodic with return sequence $a(n) = a_n(T)$ which is γ-regularly varying ($\gamma \in (0, 1)$) and which has an LLT set $\Omega \in R(T)$, $m(\Omega) = 1$. Then*

$$\varliminf_{n\to\infty} \frac{m(A \cap T^{-n} B)}{u_n} \geq m(A)m(B) \quad \textit{for all } A, B \in \mathcal{C}_\beta(T_\Omega), \tag{GL}$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

where $u_n := \gamma a(n)/n$ and β is the accompanying T_Ω-generating partition.

Proof. (As in [**GL**]) Fix $A, B \in \mathcal{C}_\beta(T_\Omega)$ and $0 < c < d < \infty$. Writing $x_{k,n} := n/B(k)$ for $1 \le k \le n$ and using the LLT property of Ω, we have

$$\begin{aligned} m(A \cap T^{-n}B) &= \sum_{k=1}^{n} m(A \cap T_\Omega^{-k}B \cap [\varphi_k = n]) \\ &\ge \sum_{1\le k\le n, x_{k,n}\in[c,d]} m(A \cap T_\Omega^{-k}B \cap [\varphi_k = x_{k,n}B(k)]) \\ &\sim \sum_{1\le k\le n, x_{k,n}\in[c,d]} \frac{f(x_{k,n})}{B(k)} m(A)m(B) \end{aligned}$$

as $n \to \infty$ since Ω is an LLT set. We are going to show that the last sum is in fact a Riemann sum.

Now,

$$x_{k,n} - x_{k+1,n} = \frac{n}{B(k)} - \frac{n}{B(k+1)} \sim \frac{n}{\gamma k B(k)}$$

as $k, n \to \infty$, $x_{k,n} \in [c, d]$, since $B = a^{-1}$ is $1/\gamma$-regularly varying.

Also

$$a(n) = a(x_{k,n}B(k)) \sim x_{k,n}^{\gamma} a(B(k)) \sim x_{k,n}^{\gamma} k$$

as $k, n \to \infty$, $x_{k,n} \in [c, d]$ by the uniform convergence theorem for regularly varying functions, so

$$\frac{1}{B(k)} \sim \frac{\gamma k}{n} \cdot (x_{k,n} - x_{k+1,n}) \sim \frac{\gamma a(n)}{n} \cdot \frac{x_{k,n} - x_{k+1,n}}{x_{k,n}^{\gamma}},$$

and hence, as $n \to \infty$,

$$\begin{aligned} \sum_{1\le k\le n, x_{k,n}\in[c,d]} \frac{f(x_{k,n})}{B(k)} &\sim \frac{\gamma a(n)}{n} \sum_{1\le k\le n, x_{k,n}\in(c,d)} \frac{(x_{k,n} - x_{k+1,n})}{x_{k,n}^{\gamma}} f(x_{k,n}) \\ &\sim \frac{\gamma a(n)}{n} \int_{[c,d]} \frac{f(x)\,dx}{x^{\gamma}} = \frac{\gamma a(n)}{n} \mathbb{E}(1_{[c,d]}(Z_\gamma) Z_\gamma^{-\gamma}). \end{aligned}$$

Now

$$\mathbb{E}(1_{[c,d]}(Z_\gamma) Z_\gamma^{-\gamma}) = \mathbb{E}(1_{[c,d]}(X_\gamma^{-1/\gamma}) X_\gamma) \xrightarrow[c\to 0+, d\to\infty]{} \mathbb{E}(X_\gamma) = 1,$$

$$\text{therefore } m(A \cap T^{-n}B) \gtrsim \frac{\gamma a(n)}{n} m(A)m(B).$$

Theorem 10.1 now follows from Lemma C. □

Remark 10.1. In some cases, $\underline{\lim}$ in Lemma 10.2 is actually lim and the transformation has Krickeberg's mixing property. This occurs in:

(i) the Markov case when $\gamma \in (\frac{1}{2}, 1]$ (in Remark 9.1) (see [**GL**] for $\gamma \in (\frac{1}{2}, 1)$ and [**E**] for $\gamma = 1$; see also [**Fre**]);

(ii) Example 10.1 when $\gamma \in (\frac{1}{2}, 1)$ and sometimes when $\gamma = 1$ (in Theorem 10.1) (see [**MT**]).

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

11. *Proof of Theorem F*

Proof of (i). Recall from [**A2**] that for $T \in \mathrm{MPT}$, $d_k > 0$ and Y a random variable on $[0, \infty]$, $S_{n_k}^{(T)}/d_k \xrightarrow{\mathfrak{d}} Y$ if

$$g\left(\frac{S_{n_k}^{(T)}(f)}{d_k}\right) \xrightarrow[k\to\infty]{} E\left(g\left(Y\int_X f\,dm\right)\right) \text{ weak-* in } L^\infty \quad \text{for all } g \in C([0,\infty]).$$

The sequence $\{m_j\}$ is called a *loose sequence* for T if

$$n_k = m_{j_k} \to \infty,\ d_k > 0,\ \frac{S_{n_k}^{(T)}}{d_k} \xrightarrow{\mathfrak{d}} Y \Longrightarrow \mathrm{Prob}([Y \in (0,\infty)]) = 0.$$

As shown in [**A2**, Proof of Theorem 2], the collection

$$\{T \in \mathrm{MPT} : T \text{ has a loose sequence}\}$$

is residual in MPT. No weakly rationally ergodic transformation has a loose sequence and so the collection of these is meagre in MPT. Thus RWM is contained in a meagre collection. □

We commence the proof of (ii) by showing that subsequence, rational, weak mixing is residual. We will use the following lemma.

CONJUGACY LEMMA. (See e.g. [**A, Kri2, S**]) *For aperiodic $T \in MPT$,*

$$\{\psi^{-1} \circ T \circ \psi : \psi \in MPT\}$$

is dense in MPT.

By the isomorphism theorem, we may assume without loss of generality that $(X, \mathcal{B}, m)$ is as in Hopf's example:

$$X = \mathbb{R}_+ \times [0,1], \quad \mathcal{B} = \mathcal{B}(\mathbb{R}_+ \times [0,1]) \quad \text{and} \quad m = \mathrm{Leb}.$$

A *dyadic square* in X is a square $S = I \times J$ with I, J dyadic intervals in $\mathbb{R}$ (i.e. ∂I, $\partial J \in \mathbb{Q}_2$) of the same length. A *dyadic set* in X is a finite union of dyadic squares. Let $\mathcal{D} := \{\text{dyadic sets in } X\}$.

We will need the (standard) result that for $N \geq 2$ there is a measure space isomorphism $\Phi_N : X^N \to X$ so that

$$\Phi_N^{-1}(\mathcal{D}) = \{\text{finite unions of sets in } \mathcal{D} \underbrace{\otimes \cdots \otimes}_{N\text{-times}} \mathcal{D}\}.$$

11.1. *Permutations.* An automorphism $T \in \mathrm{MPT}(X, \mathcal{B}, m)$ is called a *permutation* if there exist finitely many disjoint dyadic squares $E_1, \ldots, E_N$ and a permutation $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$ so that the following hold:

- T maps each E_i onto $E_{\sigma(i)}$;
- $T(x) = x$ for all $x \notin \bigcup_{i=1}^N E_i$.

The proof of [**Kri2**, Satz 2] applies to show that the collection $\Pi :=$ {permutations in MPT} is dense in MPT. This immediately implies the following.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

PERMUTATION CONJUGACY LEMMA. *For aperiodic $T \in MPT$,*

$$\{\psi^{-1} \circ T \circ \psi : \psi \in \Pi\}$$

is dense in MPT.

Note that $\psi\mathcal{R}_b = \mathcal{R}_b$ (the collection of bounded measurable sets) for $\psi \in \Pi$, but not for arbitrary $\psi \in \mathrm{MPT}$.

11.2. *Markov shifts in MPT(X).* We show that any conservative, ergodic, stationary Markov shift with infinite stationary distribution is isomorphic to a piecewise affine transformation $T \in \mathrm{MPT}(X)$ with a Markov partition whose cylinder sets are bounded rectangles in X.

We consider (without loss of generality) only Markov chains with state space $\mathbb{N}$.

Let $P : \mathbb{N} \times \mathbb{N} \to [0, 1]$ be a stochastic matrix with infinite stationary distribution $\pi : \mathbb{N} \to \mathbb{R}_+$.

We show first that the one-sided shift of (P, π) is isomorphic to a measure preserving, piecewise affine map $\tau_{(P,\pi)} : \mathbb{R}_+ \to \mathbb{R}_+$. To this end, let:

- $\alpha = \{a_k : k \in \mathbb{N}\}$ be a partition modulo 0 of $\mathbb{R}_+$ into open intervals so that $\lambda(a_s) = \pi_s$ for all $s \in \mathbb{N}$ where λ denotes Lebesgue measure on $\mathbb{R}_+$; and
- for each $s \in \mathbb{N}$, $\{a_{s,t} : t \in \mathbb{N},\ p_{s,t} > 0\}$ be a partition modulo 0 of a_s into open intervals so that $\lambda(a_{s,t}) = \pi_s p_{s,t}$ for all $t \in \mathbb{N}$.

Now define $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tau(x) := \frac{\pi_t}{\pi_s p_{s,t}} \cdot x + \gamma_{s,t}, \quad x \in a_{s,t}\ (s, t \in \mathbb{N},\ p_{s,t} > 0),$$

where $\gamma_{s,t}$ is chosen so that $\tau a_{s,t} = a_t$.

It is standard to show that $\tau_{(P,\pi)}$ preserves λ and is isomorphic to the one-sided shift of (P, π).

To obtain the two-sided shift of (P, π), we represent the natural extension of τ on $\mathbb{R}_+ \times [0, 1]$.

For $s \in \mathbb{N}$, define $v_{a_s} : \tau a_s \to a_s$ by

$$v_{a_s}(y) := \frac{\pi_s p_{s,t}}{\pi_t} \cdot (y - \gamma_{s,t}), \quad y \in a_t \subseteq \tau(a_s).$$

Note that $v'_{a_s} = \sum_{t \in \mathbb{N},\, p_{s,t} > 0} (\pi_s p_{s,t}/\pi_t) 1_{a_t}$.

Define for $x \in \mathbb{R}_+$

$$q_0(x) := 0, \quad q_k(x) := \sum_{1 \le j \le k} 1_{\tau a_j}(x) v'_{a_j}(x) \quad (k \ge 1),$$

and let $F_{x,a_k} : [0, 1] \to [q_{k-1}(x), q_k(x)]$ be the increasing affine map

$$F_{x,a_k}(y) := 1_{\tau a_k}(x) v'_{a_k}(x) y + q_{k-1}(x).$$

Now define $T = T_{(P,\pi)} : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+ \times [0, 1]$ by

$$T(x, y) := (\tau(x), F_{\tau x, \alpha(x)}(y)), \quad \text{where } x \in \alpha(x) \in \alpha.$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

It is standard to show that $T_{(P,\pi)} \in \mathrm{MPT}(X)$ is a natural extension of τ, and hence isomorphic to the two-sided shift of (P, π). The partition $\beta := \alpha \times [0, 1]$ is a Markov partition whose cylinder sets are finite unions of bounded rectangles, and hence $\mathcal{HR}(\mathcal{C}_\beta) = \mathcal{R}_b$.

Let $P : S \times S \to [0, 1]$ be a stochastic matrix on the state space S with invariant distribution $\pi : S \to \mathbb{R}_+$, and let $T_{(P,\pi)}$ be a Markov shift in MPT isomorphic to the stationary Markov shift of (P, π). Fix $s \in S$ and let $u = u([s]_0)$, $a(n) := \sum_{k=0}^{n-1} u_k$.

Assume that u is smooth, then $T = T_{(P,\pi)}$ is rationally weakly mixing with $R(T) \supset \mathcal{HR}(\mathcal{C}_\beta) = \mathcal{R}_b$, and hence

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} |m(D \cap T^{-n} D') - u_n m(D) m(D')| \xrightarrow[n\to\infty]{} 0 \quad \text{for all } D, D' \in \mathcal{D}, \tag{11.1}$$

which implies

$$\frac{1}{a(n)} \int_D S_n^{(T)}(1_D)\, dm \xrightarrow[n\to\infty]{} m(D)^2 \quad \text{for all } D \in \mathcal{D}.$$

We claim that also

$$\varlimsup_{n\to\infty} \frac{1}{a(n)^2} \int_D S_n^{(T)}(1_D)^2\, dm \le 2m(D)^3 \quad \text{for all } D \in \mathcal{D}. \tag{11.2}$$

Proof of (11.2). Let $\beta = \alpha \times [0, 1]$, the Markov partition of T. We first show (11.2) for $A \in \mathcal{U}_\beta$. Let $\tau = \tau_{(P,\pi)} : \mathbb{R}_+ \to \mathbb{R}_+$ be as above (isomorphic to the one-sided Markov shift of (P, π)). It is pointwise dual ergodic in the sense that

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{\tau}^k 1_A \xrightarrow[n\to\infty]{} \lambda(A) \quad \text{for all } A \in \mathcal{F}, \tag{a}$$

where $\widehat{\tau} : L^1(\lambda) \to L^1(\lambda)$ is the transfer operator defined by

$$\int_{\mathbb{R}_+} \widehat{\tau} f \cdot g\, d\lambda = \int_{\mathbb{R}_+} f \cdot g \circ \tau\, d\lambda$$

(see [**A**, §3.7]).

Now

$$\sup_{\mathbb{R}_+} \sum_{k=0}^{n-1} \widehat{\tau}^k 1_A = \sup_A \sum_{k=0}^{n-1} \widehat{\tau}^k 1_A.$$

For $A \in \mathcal{C}_\alpha$, the convergence (a) is uniform on A, and hence

$$\frac{1}{a(n)} \sup_{\mathbb{R}_+} \sum_{k=0}^{n-1} \widehat{\tau}^k 1_A = \frac{1}{a(n)} \sup_A \sum_{k=0}^{n-1} \widehat{\tau}^k 1_A \xrightarrow[n\to\infty]{} m(A) \quad \text{for all } A \in \mathcal{C}_\alpha. \tag{b}$$

From (b) we see that

$$\varlimsup_{n\to\infty} \frac{1}{a(n)} \sup_{\mathbb{R}_+} \sum_{k=0}^{n-1} \widehat{\tau}^k 1_A \le m(A) \quad \text{for all } A \in \mathcal{C}_\alpha. \tag{b$'$}$$

The statement (b$'$) holds

$$\text{for all } A \in \mathcal{U}_\alpha := \left\{ \bigcup_{k=1}^{N} C_k : N \ge 1,\, C_1, \ldots, C_N \in \mathcal{C}_\alpha \right\},$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

and it follows that, for $A \in \mathcal{U}_\alpha$ and n large so that $\sum_{k=0}^{n-1} \widehat{\tau}^k 1_A \le 2m(A)a(n)$,

$$\begin{aligned}\int_A S_n(1_A)^2\,d\lambda &\le 2\sum_{0\le i\le j\le n-1} \lambda(A\cap \tau^{-i}A\cap \tau^{-j}A)\\ &= 2\int_A \sum_{0\le i\le n-1} \widehat{\tau}^i 1_A S_{n-i}(1_A)\circ\tau^i\,d\lambda\\ &\le \int_A S_n(1_A)\sum_{k=0}^{n-1}\widehat{\tau}^k 1_A\,d\lambda\\ &\le 4m(A)a(n)\int_A S_n(1_A)\,d\lambda\\ &\le 8m(A)^3a(n)^2. \qquad \text{(c)}\end{aligned}$$

If $\psi : X = \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ is the projection $\psi(x,y) = x$, then

$$\mathcal{U}_\beta = \bigcup_{n\ge 1} T^n\psi^{-1}\mathcal{U}_\alpha$$

and (11.2) follows for $A \in \mathcal{U}_\beta$. Dyadic sets can be monotonically approximated by sets in $\mathcal{U}_\beta$, and (11.2) follows. □

Now enumerate $\mathcal{D} := \{D_i : i \in \mathbb{N}\}$, and define

$$\mathcal{G} := \bigcap_{k\ge 1}\bigcup_{N\ge k}\bigcap_{1\le i,j\le k} U(N,k,i,j),$$

where

$$U(N,k,i,j) := \Big\{T \in \mathrm{MPT} : \sum_{\nu=0}^{N-1} |m(D_i\cap T^{-\nu}D_j) - m(D_i)m(D_j)u_\nu| < \frac{a(N)}{k},$$
$$\text{and } \int_{D_i} S_N(1_{D_i})^2\,dm < 8m(D_i)^3a(N)^2\Big\}.$$

Evidently each $U(N,k,i,j)$ is open in MPT, and hence the set $\mathcal{G}$ is a G_δ set. We will complete the proof of residuality of SRWM by showing that:

- $\mathcal{G}$ is dense in MPT;
- Each $T \in \mathcal{G}$ is subsequence rationally weakly mixing.

Proof of density of $\mathcal{G}$. By (11.1) and (11.2), $T_{(P,\pi)} \in \mathcal{G}$. Since $\psi\mathcal{R}_b = \mathcal{R}_b$ for all $\psi \in \Pi$,

$$\{\psi^{-1}\circ T_{(P,\pi)}\circ\psi : \psi \in \Pi\} \subset \mathcal{G}.$$

Since $T_{(P,\pi)}$ is ergodic, by the permutation conjugacy lemma,

$$\overline{\mathcal{G}} \supset \overline{\{\psi^{-1}\circ T_{(P,\pi)}\circ\psi : \psi\in\Pi\}} = \mathrm{MPT}.$$ □

Proof of the subsequence rational weak mixing of elements of $\mathcal{G}$. Let $T \in \mathcal{G}$. Then there exists a subsequence $\mathfrak{K} \subset \mathbb{N}$ such that:

(a) $$\frac{1}{a(N)}\sum_{\nu=0}^{N-1} |m(D\cap T^{-\nu}D') - m(D)m(D')u_\nu| \xrightarrow[N\to\infty, N\in\mathfrak{K}]{} 0 \text{ for all } D, D' \in \mathcal{D}; \text{ and}$$

(b) $$\int_D S_N(1_D)^2\,dm < 8m(D)^3a(N)^2 \quad \text{for all } D\in\mathcal{D},\ N\in\mathfrak{K}.$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

It follows from (a) that

$$\frac{a_N^{(T)}(D)}{a(N)} \xrightarrow[N\to\infty, N\in\mathfrak{K}]{} m(D)^2 \quad \text{for all } D\in\mathcal{D},$$

and hence, by (b), T is weakly rationally ergodic along $\mathfrak{K}$ with return sequence $a(n)$ along $\mathfrak{K}$ and $\mathcal{D}\subset R_{\mathfrak{K}}(T)$. This enables use of (a) and Lemma C to show that T is rationally weakly mixing along $\mathfrak{K}$. □

Proof of Theorem F(ii). For $\underline{\kappa}=(\kappa_1,\dots,\kappa_\Delta)\in(\mathbb{Z}\setminus\{0\})^\Delta$, define $\psi_{\underline{\kappa}}:\mathrm{MPT}(X)\to\mathrm{MPT}(X)$ by

$$\psi_{\underline{\kappa}}(T):=\phi_\Delta\circ T^{(\kappa_1,\dots,\kappa_\Delta)}\circ\phi_\Delta^{-1}\in\mathrm{MPT}(X),$$

where, as above, $\Phi_\Delta: X^\Delta\to X$ so that

$$\Phi_\Delta^{-1}(\mathcal{D})=\{\text{finite unions of sets in } \mathcal{D}\underbrace{\otimes\cdots\otimes}_{\Delta\text{-times}}\mathcal{D}\}.$$

If $\psi_{\underline{\kappa}}(T)\in\mathcal{G}$, then $T^{(\kappa_1,\dots,\kappa_\Delta)}\in\mathrm{SRWM}$. Thus it suffices to show that

$$\mathcal{G}_{\mathrm{power}}:=\bigcap_{\Delta=1}^{\infty}\bigcap_{(\kappa_1,\dots,\kappa_\Delta)\in(\mathbb{Z}\setminus\{0\})^\Delta}\psi_{\underline{\kappa}}^{-1}\mathcal{G}$$

is residual.

It is not hard to see that:

- each $\psi_{\underline{\kappa}}:\mathrm{MPT}(X)\to\mathrm{MPT}(X)$ is a continuous homomorphism, and hence $\mathcal{G}_{\mathrm{power}}$ is a G_δ set in $\mathrm{MPT}(X)$;
- $\psi_{\underline{\kappa}}(\Pi)=\Pi$, and hence $\psi^{-1}\circ T\circ\psi\in\mathcal{G}_{\mathrm{power}}$ for all $T\in\mathcal{G}_{\mathrm{power}}$, $\psi\in\Pi$, because for $T\in\mathcal{G}_{\mathrm{power}}$ and $\pi\in\Pi$, $\psi_{\underline{\kappa}}(\pi)\mathcal{D}=\mathcal{D}$ and

$$\psi_{\underline{\kappa}}(\pi^{-1}\circ T\circ\pi)=\psi_{\underline{\kappa}}(\pi)^{-1}\circ\psi_{\underline{\kappa}}(T)\circ\psi_{\underline{\kappa}}(\pi)\in\mathcal{G}.$$

To prove density of $\mathcal{G}_{\mathrm{power}}$ (and thus complete the proof of Theorem F(ii)) it suffices to exhibit $T\in\mathcal{G}_{\mathrm{power}}$ for then T is ergodic and $\overline{\mathcal{G}_{\mathrm{power}}}\supset\overline{\{\pi^{-1}\circ T\circ\pi:\pi\in\Pi\}}=\mathrm{MPT}$ by the permutation conjugacy lemma.

Renewal shifts. Let u be a recurrent renewal sequence with lifetime distribution $f\in\mathcal{P}(\mathbb{N})$. Define (as in [**Ch**]) a stochastic matrix $P=P_u$ on $\mathbb{N}$ by

$$P_{1,n}:=f_n \quad\text{and}\quad P_{n+1,n}=1 \text{ for all } n\in\mathbb{N}.$$

This has stationary distribution $\pi=\pi_u$ defined by $\pi_n:=\sum_{k=n}^\infty f_k$ and $P_{1,1}^{(n)}=u_n$. The Markov shift of (P,π) is called the *renewal shift* of u. Let $T_u:=T_{(P,\pi)}\in\mathrm{MPT}$.

If u is smooth, then $T_u\in\mathrm{RWM}$.

Now suppose that $\Delta\ge1$, $\underline{\kappa}=(\kappa_1,\dots,\kappa_\Delta)\in(\mathbb{Z}\setminus\{0\})^\Delta$. Then, as evidently $T_u^{-1}\cong T_u$,

$$T_u^{\kappa_1}\times\cdots\times T_u^{\kappa_\Delta}\cong T_u^{|\kappa_1|}\times\cdots\times T_u^{|\kappa_\Delta|},$$

and we may assume without loss of generality that $\underline{\kappa}\in\mathbb{N}^\Delta$. Now $T_u^{\kappa_1}\times\cdots\times T_u^{\kappa_\Delta}$ is also the Markov shift of an irreducible, aperiodic, stochastic matrix with renewal sequence $u^{(\underline{\kappa})}$ defined by

$$u_n^{(\underline{\kappa})}:=\prod_{j=1}^{\Delta}u_{\kappa_j n}.$$

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

If u is smooth and $u^{(\underline{\kappa})}$ is recurrent, then $u^{(\underline{\kappa})}$ is also smooth, $T_u^{\kappa_1} \times \cdots \times T_u^{\kappa_\Delta}$ is rationally weakly mixing and $\psi_{\underline{\kappa}}(T_u) \in \mathrm{RWM}$.

Now let u be the sequence defined by

$$u_n := \frac{1}{\log(n+e)} \quad (n \geq 0).$$

Then u is a *Kaluza sequence* in the sense that $u_0 = 1$ and $u_{n+1}/u_n \uparrow 1$, and hence a smooth, recurrent renewal sequence; as can be easily checked, so is $u^{(\underline{\kappa})}$ for all $\Delta \geq 1$, $\underline{\kappa} = (\kappa_1, \ldots, \kappa_\Delta) \in \mathbb{N}^\Delta$. It follows that $T_u \in \mathcal{G}_{\mathrm{power}}$. □

12. *Closing remarks*

All infinite, rationally weakly mixing examples in this paper are of the form $T \times S$ where T is an infinite K-automorphism and S is a weakly mixing probability preserving transformation.

Their Koopman operators all have countable Lebesgue spectrum. This is shown in [**Par**] for K-automorphisms, and a simple argument shows that multiplying by a weakly mixing probability preserving transformation does not change this.

The transformation $T \in \mathrm{MPT}$ is said to be *rigid* if there exists $\mathfrak{L} \subset \mathbb{N}$ so that

$$m(A \Delta T^{-n} A) \xrightarrow[n\to\infty, n\in\mathfrak{L}]{} 0 \quad \text{for all } A \in \mathcal{F}.$$

The spectrum of a rigid transformation is Dirichlet, and hence singular.

As shown in [**AS**], the collection RIGID of rigid transformations in MPT is residual; by Theorem F, so is PSRWM ∩ RIGID, and so there is a rigid, power, subsequence, rationally weak mixing, measure preserving transformation with a singular spectrum.

Acknowledgement. The research was supported by Israel Science Foundation grant No. 1114/08.

References

[A] J. Aaronson. *An Introduction to Infinite Ergodic Theory (Mathematical Surveys and Monographs, 50)*. American Mathematical Society, Providence, RI, 1997.

[A1] J. Aaronson. Rational ergodicity and a metric invariant for Markov shifts. *Israel J. Math.* **27**(2) (1977), 93–123.

[A2] J. Aaronson. The asymptotic distributional behaviour of transformations preserving infinite measures. *J. Anal. Math.* **39** (1981), 203–234.

[AD] J. Aaronson and M. Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs–Markov maps. *Stoch. Dyn.* **1**(2) (2001), 193–237.

[ADSZ] J. Aaronson, M. Denker, O. Sarig and R. Zweimüller. Aperiodicity of cocycles and conditional local limit theorems. *Stoch. Dyn.* **4**(1) (2004), 31–62.

[ALV] J. Aaronson, M. Lemańczyk and D. Volný. A cut salad of cocycles. *Fund. Math.* **157** (1998), 99–119.

[ALW] J. Aaronson, M. Lin and B. Weiss. Mixing properties of Markov operators and ergodic transformations, and ergodicity of Cartesian products. *A Collection of Invited Papers on Ergodic Theory*. *Israel J. Math.* **33**(3–4) (1979), 198–224 (1980).

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press

[AS] O. N. Ageev and C. E. Silva. Genericity of rigid and multiply recurrent infinite measure-preserving and nonsingular transformations. *Proceedings of the 16th Summer Conference on General Topology and its Applications (New York, NY, 2001)*. *Topology Proc.* **26**(2) (2002), 357–365.

[BF] D. Blackwell and D. Freedman. The tail σ-field of a Markov chain and a theorem of Orey. *Ann. Math. Stat.* **35** (1964), 1291–1295.

[BGT] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation*. Cambridge University Press, Cambridge, 1989.

[Ch] K. L. Chung. *Markov Chains with Stationary Transition Probabilities (Grundlehren der Mathematischen Wissenshaften, 104)*. Springer, Berlin, 1960.

[CP] J. R. Choksi and V. S. Prasad. Approximation and Baire category theorems in ergodic theory. *Measure Theory and its Applications (Proceedings of Conference, Sherbrooke, 1982) (Lecture Notes in Mathematics, 1033)*. Springer, Berlin, 1983, pp. 94–113.

[DK] D. A. Darling and M. Kac. On occupation times for Markoff processes. *Trans. Amer. Math. Soc.* **84** (1957), 444–458.

[E] K. B. Erickson. Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151** (1970), 263–291.

[FL] S. R. Foguel and M. Lin. Some ratio limit theorems for Markov operators. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **23** (1972), 55–66.

[Fre] J. B. G. Frenk. The behavior of the renewal sequence in case the tail of the waiting time distribution is regularly varying with index -1. *Adv. in Appl. Probab.* **14** (1982), 870–884.

[Fri] N. Friedman. Mixing transformations in an infinite measure space. *Studies in Probability and Ergodic Theory (Advances in Mathematics: Supplementary Studies, 2)*. Academic Press, NY, New York–London, 1978, pp. 167–184.

[GL] A. Garsia and J. Lamperti. A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* **37** (1963), 221–234.

[HK] A. Hajian and S. Kakutani. Weakly wandering sets and invariant measures. *Trans. Amer. Math. Soc.* **110** (1964), 136–151.

[HP] D. L. Hanson and G. Pledger. On the mean ergodic theorem for weighted averages. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **13** (1969), 141–149.

[HR] T. E. Harris and H. Robbins. Ergodic theory of Markov chains admitting an infinite invariant measure. *Proc. Natl. Acad. Sci. USA* **39** (1953), 860–864.

[H] E. Hopf. *Ergodentheorie (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3, 5)*. Springer, Berlin, 1937.

[KM] S. Karlin and J. McGregor. Random walks. *Illinois J. Math.* **3** (1959), 66–81.

[Kre] U. Krengel. *Ergodic Theorems*. Walter de Gruyter, Berlin, 1985.

[Kri1] K. Krickeberg. Strong mixing properties of Markov chains with infinite invariant measure. *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability (Berkeley, CA, 1965/66),* Vol. II, *Part 2*. University of California Press, Berkeley, CA, 1967, pp. 431–446.

[Kri2] K. Krickeberg. Mischende Transformationen auf Mannigfaltigkeiten unendlichen Masses. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **7** (1967), 235–247.

[L] M. Lenci. On infinite-volume mixing. *Comm. Math. Phys.* **298**(2) (2010), 485–514.

[MT] I. Melbourne and D. Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.* **189** (2012), 61–110.

[O] S. Orey. Strong ratio limit property. *Bull. Amer. Math. Soc.* **67** (1961), 571–574.

[Pap] F. Papangelou. Strong ratio limits, R-recurrence and mixing properties of discrete parameter Markov processes. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **8** (1967), 259–297.

[Par] W. Parry. Ergodic and spectral analysis of certain infinite measure preserving transformations. *Proc. Amer. Math. Soc.* **16** (1965), 960–966.

[S] U. Sachdeva. On category of mixing in infnite measure spaces. *Math. Syst. Theory* **5** (1971), 319–330.

[T] M. Thaler. The asymptotics of the Perron–Frobenius operator of a class of interval maps preserving infinite measures. *Studia Math.* **143**(2) (2000), 103119.

https://doi.org/10.1017/etds.2012.102 Published online by Cambridge University Press