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Functions on a crosscap

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1. *Introduction*

The study of the differential geometry of surfaces in 3-space has a long and celebrated history. Over the last 20 years a new approach using techniques from singularity theory has yielded some interesting results (see, for example [**3**, **5**, **19**] for surveys).

Of course surfaces arise in a number of ways: they are often defined explicitly as the image of a mapping $f\colon \mathbf{R}^2 \to \mathbf{R}^3$. Since the subject is *differential* geometry one normally asks that these defining mappings are smooth, that is infinitely differentiable, however it is not true, in any sense, that most such parametrisations will yield manifolds. For such mappings have self-intersections, and more significantly they may have crosscaps (also known as Whitney umbrellas). Moreover if we perturb the maps these singularities will persist; that is they are *stable*. (See [**14**, **20**] for details.)

Consequently when studying the differential geometry of surfaces in 3-space there are good reasons for studying surfaces with crosscaps.

In this paper we carry out a classification of mappings from 3-space to lines, up to changes of co-ordinates in the source preserving a crosscap. We can apply our results to the geometry of generic crosscap points. In [**10**] we computed geometric normal forms for the crosscap and used them to study the dual of the crosscap. We shall see that the approach here yields more information than that obtained in [**10**], although the latter has the advantage of being more explicit (in particular various aspects of the geometry can be compared using the normal forms).

We refer the reader to [**16**, **18**] for background material concerning singularity theory.

2. *The crosscap*

The Whitney umbrella is the variety defined by the equation $v^2 - u^2 w = 0$ and is depicted in Fig. 1.

Let $f\colon \mathbf{R}^2, 0 \to \mathbf{R}^3, 0$ be the map germ defined by $f(x, y) = (x, xy, y^2)$ in (u, v, w)-space. (Of course the same formula defines a complex analytic map germ $\mathbf{C}^2, 0 \to \mathbf{C}^3, 0$.) The image of f is the Whitney umbrella without its 'handle' (the

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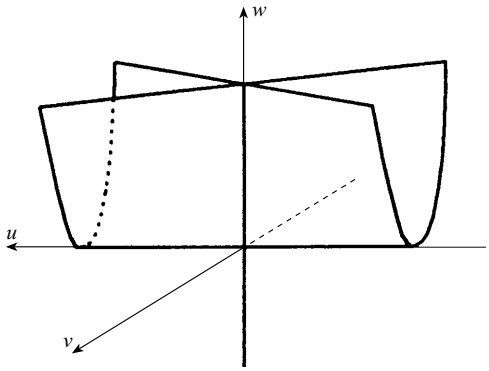


Fig. 1.

half line $u = v = 0$, $w < 0$). It can be shown that f is 2-$\mathscr{A}$-determined. The germ f is singular at the origin and the image of the derivative of f at this point is the u-axis. The mapping f also has a double point curve, the image of the y-axis, which ends at the crosscap point. By a *crosscap* we mean the image of any germ g that is $\mathscr{A}$-equivalent to f. We refer to the crosscap parametrised by f as the *standard crosscap*. Whitney proved that such singularities are stable [**20**], and that a stable mapping from a surface to a three manifold only admits local singularities of this type.

In what follows we shall need the following result which shows that two map germs which parametrise the same crosscap are $\mathscr{R}$-equivalent.

Proposition 2·1. *Let* $g\colon \mathbf{C}^2, 0 \longrightarrow \mathbf{C}^3, 0$ *be an immersion away from the origin, with image set the standard crosscap parametrised by* $f\colon \mathbf{C}^2, 0 \longrightarrow \mathbf{C}^3, 0, (x, y) \longmapsto (x, xy, y^2)$. *Then* g *is* $\mathscr{R}$*-equivalent to* f. *Moreover, the right-equivalence preserves the double point set.*

Proof. It is enough to check that f and g are normalisations of $f(\mathbf{C}^2)$ and hence $\mathscr{R}$-equivalent to any other normalisation (see [**15**], p. 34). Alternatively, we can use the fact that f is an immersion to see that there is a well defined map from the complement of the double point set of f to that of g which is a local diffeomorphism at each point. Using the fact that away from the pinch points we have transversely intersecting sheets along the double point curve, we can extend this map to the complement of the origin. By Hartog's theorem (see [**13**], p. 53) this extends to a smooth map between sources, which is invertible (we can use the same construction to get the inverse map). □

If X is a set in n-dimensional affine space $\mathbf{K}^n$ (where $\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$) and $x \in X$, then the tangent cone to X at x, denoted T_xX, is the limiting position of the secants that pass through X. Shafarevich ([**17**], p. 79) gives a full definition and shows that if X is an irreducible affine variety in $\mathbf{K}^n$ defined by the ideal I then the equations of T_xX are the lowest degree terms of the polynomials in I. So the tangent cone to the Whitney umbrella is the repeated plane $v^2 = 0$. Clearly the double point set of the standard crosscap is the non-negative w-axis. In the real or complex case the tangent cone is also the limit of the tangent planes to either sheet of the crosscap as we approach along the double point curve.

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Since tangent cones are preserved by diffeomorphisms in an obvious natural way we can deduce the following

PROPOSITION 2·2. *The tangent cone to the crosscap at the crosscap point is the plane which is the limit of the tangent planes to either sheet of the crosscap as we approach along the double point curve.*

Let $\mathcal{O}_n = \mathcal{O}(x_1, \ldots, x_n)$ denote the set of smooth function germs $\mathbf{K}^n, 0 \to \mathbf{K}$. We shall need to determine the $\mathcal{O}_3$-module of vector fields tangent to the crosscap.

Let h be an irreducible polynomial in $\mathbf{K}[x_1, \ldots, x_n]$ defining a hypersurface $X \subset \mathbf{K}^n$. If ξ is an analytic vector field on $\mathbf{K}^n$ that is tangent to X, then at any smooth point $x \in X$,

$$\xi h(x) = dh_x(\xi(x)) = 0.$$

Thus ξh vanishes at all smooth points of X. Since the set of smooth points of X is dense in X, ξh vanishes on all of X. So if ξ is tangent to X then $\xi h \in I(V < h >)$, (where as usual $I(W)$ is the ideal of functions vanishing on W, $V(J)$ the set of common zeros of elements of J). Conversely if $\xi h \in I(V < h >)$ then ξh vanishes on X. Hence ξh vanishes at the regular points of X and so ξ is tangent to X. In the complex case the Nullstellensatz yields $I(V < h >) = \sqrt{< h >} =< h >$, and so $\xi h = \lambda h$ for some function λ. In the case of the Whitney Umbrella, where $h = v^2 - u^2 w$ then this is also true over the reals.

LEMMA 2·3. *Let h be an irreducible polynomial in $\mathbf{R}[u, v, w]$. Denote the real and complex varieties determined by h by $V(h)_{\mathbf{R}}$ and $V(h)_{\mathbf{C}}$ respectively. If $h = v^2 - u^2 w$ then the ideal*

$$I(V(h)_{\mathbf{R}}) \subset \mathbf{R}[u, v, w]$$

is generated by h.

Proof. If ϕ: $\mathbf{R}^2 \to \mathbf{R}^3$ is the map defined by $f(x, y) = (x, xy, y^2)$, then $\phi(\mathbf{R}^2) \subset V(h)_{\mathbf{R}}$. Thus if g is a polynomial with real coefficients that vanishes on $V(h)_{\mathbf{R}}$ then the function $g \circ \phi$: $\mathbf{R}^2 \to \mathbf{R}$ is identically zero. Hence $g \circ \phi$: $\mathbf{C}^2 \to \mathbf{C}$ is identically zero. So g vanishes on $\phi(\mathbf{C}^2)$. Consequently g vanishes on the Zariski closure of $\phi(\mathbf{C}^2)$ which is $V(h)_{\mathbf{C}}$. So $g \in I(V(h)_{\mathbf{C}})$. Hence by the Nullstellensatz $g = \alpha h$, $\alpha \in \mathbf{C}[u, v, w]$ and since g and h are real,

$$g = \frac{(\alpha + \overline{\alpha})}{2} h \in \mathbf{R}[u, v, w] < h > . \qquad \square$$

We also need the following result.

LEMMA 2·4. *Consider the linear map*

$$L\colon \mathcal{O}_n^n \to \mathcal{O}_n$$

defined by

$$L(\alpha) = L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=0}^{n} \alpha_i x_i,$$

where $\alpha_1, \ldots, \alpha_n$ are the components of α. Then $\ker L$ *is spanned as an $\mathcal{O}_n$-module by*

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the set of maps

$$\{\gamma_{ij}\colon \gamma_{ij}(x_1,\ldots,x_n) = x_i e_j - x_j e_i, \quad 1 \leqslant i < j \leqslant n\},$$

where e_k is the vector in $\mathbf{K}^n$ with a 1 in the kth position and zeros elsewhere.

Proof. It is clear that the γ_{ij} lie in $\ker L$. We need to show that they generate $\ker L$; this is straightforward by induction on n. □

Proposition 2·5. *The $\mathcal{O}_3$-module of germs at the origin of vector fields on $\mathbf{C}^3$ that are tangent to the Whitney Umbrella is generated by*

$$\xi_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad \xi_2 = v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w}, \quad \xi_3 = u^2\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial w} \quad \textit{and} \quad \xi_4 = v\frac{\partial}{\partial u} + uw\frac{\partial}{\partial v}.$$

Proof. Let $h(u,v,w) = v^2 - u^2 w$ be the defining equation of the Whitney Umbrella. Then

$$\xi_1 h = 2h, \quad \xi_2 h = 2h, \quad \xi_3 h = 0 \quad \text{and} \quad \xi_4 h = 0,$$

so ξ_1, ξ_2, ξ_3 and ξ_4 are certainly tangent to the Whitney Umbrella. But if η is a germ at the origin of a vector field on $\mathbf{C}^3$ that is tangent to the Whitney Umbrella, then $\eta h = \lambda h$ for some polynomial λ. So

$$(\eta - \tfrac{1}{2}\lambda\xi_1)h = \eta h - \tfrac{1}{2}\lambda\xi_1 h = \eta h - \lambda h = 0,$$

and we need only check that ξ_1, ξ_2, ξ_3 and ξ_4 generate all germs of vector fields η such that $\eta h = 0$. Writing $\eta = \eta_1(\partial/\partial u) + \eta_2(\partial/\partial v) + \eta_3(\partial/\partial w)$, we have $\eta h = 0$ if and only if

$$(-2uw)\eta_1 + (2v)\eta_2 + (-u^2)\eta_3 = 0.$$

Clearly $\eta_2 = u\eta_2'$ for some η_2', so $(\eta_1, \eta_2', \eta_3)$ lies in the kernel of the linear map $M\colon \mathcal{O}_3^3 \to \mathcal{O}_3$ defined by

$$M(f_1, f_2, f_3) = (-f_3)u + (2f_2)v + (-2f_1)w.$$

Now Lemma 2·4 implies that $\ker M$ is spanned by the maps $(v, w, 0), (0, u, 2v)$ and $(u, 0, -2w)$. Substituting for $(\eta_1, \eta_2', \eta_3)$ yields the result.

3. *Functions on the crosscap*

One approach to studying the geometry of the crosscap is to classify function germs $\mathbf{R}^3, 0 \to \mathbf{R}, 0$ allowing only those diffeomorphisms in the source which preserve the Whitney Umbrella. We call such function germs *functions on the crosscap*. Functions on analytic varieties have been considered in [**9**] (see also [**1**]). We begin with some technical results.

Let $X, 0 \subset \mathbf{K}^p, 0$ be the germ of a reduced analytic subvariety of $\mathbf{K}^p$ at 0. We consider analytic function germs $\mathbf{K}^p, 0 \longrightarrow \mathbf{K}, 0$ and say two germs are equivalent if one can be obtained from the other by source coordinate changes which preserve X. As usual the theory works well when $\mathbf{K} = \mathbf{C}$.

Definition 3·1. Let $\mathcal{I}$ denote the ideal in $\mathcal{O}_p$ of germs of functions that vanish on X. A diffeomorphism $\phi\colon \mathbf{K}^p, 0 \longrightarrow \mathbf{K}^p, 0$ *preserves* X if $\phi(X)$ and X are equal as germs at 0, i.e. $\phi(X), 0 = X, 0$. Over $\mathbf{C}$ this is equivalent to the assertion that the induced isomorphism $\phi^*\colon \mathcal{O}_p \longrightarrow \mathcal{O}_p$ satisfies $\phi^*(\mathcal{I}) = \mathcal{I}$. The group of such diffeomorphisms is a subgroup of $\mathcal{R}$, and is denoted $\mathcal{R}(X)$.

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Two function germs $f, g \in \mathcal{O}_p$ are said to be $\mathcal{R}(X)$-equivalent if there exists $\phi \in \mathcal{R}(X)$ such that $g \circ \phi = f$.

Let δ be the germ of an analytic vector field on $\mathbf{K}^p$ at 0. Then δ is said to be *logarithmic* for $X, 0$ if, when considered as a derivation $\delta\colon \mathcal{O}_p \longrightarrow \mathcal{O}_p$, $f \mapsto \delta \cdot f$, we have $\delta \cdot f \in \mathcal{I}$ for all $f \in \mathcal{I}$, that is, $\delta \cdot f$ vanishes on X. The $\mathcal{O}_p$-module of logarithmic vector fields is denoted $\Theta(X)$.

Intuitively, an infinitesimal approach is given by integrating vector fields tangent to X to yield diffeomorphisms which preserve X. This is confirmed by the following proposition, which shows that the logarithmic vector fields are precisely those vector fields tangent to X.

Proposition 3·2. *When* $\mathbf{K} = \mathbf{C}$, *the germ at* 0 *of a vector field lies in* $\Theta(X)$ *if and only if at each smooth point* x (*sufficiently close to* 0) *of each irreducible component* X_i *of* X *the vector field* δ *is tangent to* X_i *at* x.

Now if $\mathbf{K} = \mathbf{C}$ *or* $\mathbf{R}$ *suppose that* $\delta \in \Theta(X)$ *vanishes at* 0. *Then the flow* ϕ_t *generated by* δ *preserves* $X, 0$. *Thus* $\phi_t \in \mathcal{R}(X)$ *for all* t.

Proof. See [**9**], section 1. □

The basic tools for classifying analytic function germs $\mathbf{K}^p, 0 \longrightarrow \mathbf{K}$ up to $\mathcal{R}(X)$-equivalence are generalisations of the standard results concerning functions under $\mathcal{R}$-equivalence, for which $X = \emptyset$. The group $\mathcal{R}(X)$ is one of Damon's 'geometric subgroups' of $\mathcal{A}$ in [**11**], so his version of the unfolding and determinacy theorems apply.

Definition 3·3. A function germ $f\colon \mathbf{K}^p, 0 \longrightarrow \mathbf{K}, 0$ is k-$\mathcal{R}(X)$-determined if all function germs with the same k-jet as f are $\mathcal{R}(X)$-equivalent to f.

Theorem 3·4. *A function germ* $f\colon \mathbf{K}^p, 0 \longrightarrow \mathbf{K}, 0$ *is finitely* $\mathcal{R}(X)$*-determined if the ideal*

$$J_X(f) = \{\delta \cdot f\colon \delta \in \Theta(X)\}$$

in $\mathcal{O}_p$ *contains some power of the maximal ideal* $\mathcal{M}_p$. *Moreover if* $J_X(f) \supset \mathcal{M}_p^k$ *then* f *is* $(k+1) - \mathcal{R}(X)$*-determined.*

Proof. See [**11**]. □

The definition and fundamental theorem of $\mathcal{R}(X)$-versality for unfoldings are analogous to those of $\mathcal{R}$-versality; this time the diffeomorphisms must preserve the variety X.

Definition 3·5. Let $f \in \mathcal{O}_n$ be a germ of a function. A *p-parameter unfolding* of f is a germ F of a smooth function at the origin of $\mathbf{K}^n \times \mathbf{K}^p$ (given coordinates (x, t)) such that

$$F_0(x) = F(x, 0) = f(x).$$

We define the *extended pseudo-group of diffeomorphisms preserving* X, denoted $\mathcal{R}_e(X)$ to be the pseudo-group obtained by integrating all vector fields $\delta \in \Theta(X)$ as in Proposition 3·2, but dropping the condition that $\delta(0) = 0$. Note that when X is the Whitney Umbrella, all the vector fields do vanish at the origin, so that in this case $\mathcal{R}_e(X) = \mathcal{R}(X)$.

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Two p-parameter unfoldings, F and G of the same germ $f \in \mathcal{O}_n$ are *isomorphic* if there exists a local diffeomorphism $\phi\colon \mathbf{K}^n \times \mathbf{K}^p, 0 \to \mathbf{K}^n \times \mathbf{K}^p, 0$ at the origin such that

(i) $\phi(x,t) = (\psi(x,t),t)$ and $\psi(x,0) = x$, i.e. ϕ is a p-parameter unfolding of the identity map on $\mathbf{K}^n$, this time with the property that the mapping $\psi_t\colon \mathbf{K}^n \to \mathbf{K}^n$ defined by $\psi_t(x) = \psi(x,t)$ is in $\mathscr{R}_e(X)$ for t near 0 in $\mathbf{K}^p$;

(ii) $G = F \circ \phi$.

An unfolding is called *trivial* if it is isomorphic to the constant unfolding $(x,t) \mapsto f(x)$.

As before we say that two p-parameter unfoldings F and G of a function germ f are *equivalent* if there exists a diffeomorphism $h\colon \mathbf{K}^p, 0 \longrightarrow \mathbf{K}^p, 0$ such that G is isomorphic to h^*F, the pullback of F by h. If H is now some q-parameter unfolding of f, we say that H can be *induced* from F if there exists a smooth map germ $h\colon \mathbf{K}^q, 0 \longrightarrow \mathbf{K}^p, 0$ such that H is isomorphic to h^*F.

We say that an unfolding F of $f \in \mathcal{O}_n$ is a *versal unfolding* of f if any other unfolding G of f can be induced from f.

Theorem 3·6. *A p parameter unfolding $F(x,u)$ of f is $\mathscr{R}(X)$-versal if and only if the initial speeds*

$$\frac{\partial F}{\partial u_1}(x,0), \ldots, \frac{\partial F}{\partial u_p}(x,0)$$

span $\mathcal{O}_n/J_X(f)$.

Proof. See [**11**]. □

We shall be considering the geometry of functions on the Whitney Umbrella, so we need to use the notion of $\mathscr{K}(X)$-equivalence. The subgroup $\mathscr{K}(X)$ of $\mathscr{K}$ is defined by replacing $\mathscr{R}$ by $\mathscr{R}(X)$ in the definition of $\mathscr{K}$. Thus $\mathscr{K}(X)$ can be thought of as the semi-direct product of $\mathscr{R}(X)$ and $\mathscr{C}$, and $\mathscr{K}(X)$ acts on $\mathscr{M}_n \cdot \mathcal{O}(n,p)$ as a subgroup of $\mathscr{K}$. If f is a smooth function germ, then the tangent space to the $\mathscr{K}(X)$-orbit of f is given by

$$L\mathscr{K}(X)\cdot f = L\mathscr{C}\cdot f + L\mathscr{R}(X)\cdot f =< f > + J_X(f).$$

We can define $\mathscr{K}(X)$-versal unfoldings in a similar way to $\mathscr{R}(X)$-versal unfoldings. The group $\mathscr{K}(X)$ is another of Damon's 'geometric subgroups' of $\mathscr{K}$, and so we have the following theorem.

Theorem 3·7. *A p parameter unfolding $F(x,u)$ of f is $\mathscr{K}(X)$-versal if and only if the initial speeds*

$$\frac{\partial F}{\partial u_1}(x,0), \ldots, \frac{\partial F}{\partial u_r}(x,0)$$

span $\mathcal{O}_n/L\mathscr{K}(X)\cdot f$.

Proof. See ([**11**]). □

3·1. *$\mathscr{R}(X)$-triviality and complete transversals*

Here we establish a complete transversal method for classifying functions on a variety, generalising the approach given in Chapter 11 of [**8**]. For an alternative approach see [**7**].

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In what follows we shall be dealing with the germ of an analytic variety $X, 0$ in $\mathbf{R}^n$. We shall suppose that we have a module of vector fields $\Theta(X)$ which are tangent to X. Note that the module $\Theta(X)$ is not necessarily the collection of all vector fields tangent to X. We wish to classify functions $f\colon \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ up to diffeomorphisms in the source preserving X. We shall as usual obtain these diffeomorphisms by integrating vector fields in $\Theta(X)$.

Definition 3·8. Let $F\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$ be a germ of a smooth 1-parameter family of functions with $F(0,t) = 0$ for small t, and let k be an integer $\geqslant 1$. We say that F is $\mathscr{R}(X)$-*trivial* if we can find a germ of a 1-parameter family of diffeomorphisms $H\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}^n, 0$ preserving X with $H(x,0) = x,\ H(0,t) = 0$ (for small t) and $F(H(x,t),t) = F(x,0)$.

We say that F is k-$\mathscr{R}(X)$-*trivial* if we can find a germ of a 1-parameter family of diffeomorphisms $H : \mathbf{R}^n \times \mathbf{R}, (0,0) \to \mathbf{R}^n, 0$ preserving X with $H(x,0) = x,\ H(0,t) = 0$ (for small t) and $F(H(x,t),t) = F(x,0) + \phi(x,t)$ for some $\phi \in \mathscr{M}_n^{k+1} \subset \mathscr{E}_{n+1}$.

The latter definition says that F is k-$\mathscr{R}(X)$-trivial if F is $\mathscr{R}(X)$-trivial up to and including terms of degree k. Obviously an $\mathscr{R}(X)$-trivial family is k-$\mathscr{R}(X)$-trivial for all k. We need criteria for a family to be $\mathscr{R}(X)$-trivial and k-$\mathscr{R}(X)$-trivial.

Proposition 3·9. (i) *Let*

$$F\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$$

be a smooth function with $F(0,t) = 0$ *for* t *small. Let* $\xi_1, \ldots, \xi_p$ *be vector fields in* $\Theta(X)$ *vanishing at* $0 \in \mathbf{R}^n$. *Then the family* F *is* $\mathscr{R}(X)$*-trivial if there is a smooth germ* $\alpha\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}^p$ *satisfying* $\sum_{i=1}^p \alpha_i \xi_i F + \partial F/\partial t \equiv 0$, *or equivalently*

$$\frac{\partial F}{\partial t} \in \langle \xi_1 F, \ldots, \xi_p F \rangle \subset \mathscr{E}_{n+1}. \tag{1}$$

(ii) *The family*

$$F\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$$

defined as above is k-$\mathscr{R}(X)$*-trivial if*

$$\frac{\partial F}{\partial t} \in \langle \xi_1 F, \ldots, \xi_p F \rangle + \mathscr{M}_n^{k+1} \subset \mathscr{E}_{n+1}. \tag{2}$$

Proof. (i) The proof here follows familiar lines. Suppose that α exists as in the statement of the Proposition. By the fundamental theorem on the existence of solutions of ordinary differential equations, if

$$\eta = \sum_{i=1}^p \alpha_i \xi_i = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x_i},$$

then the differential equation

$$\frac{\partial H}{\partial t}(x,t) = \eta(H(x,t),t),\ H(x,0) = x \tag{3}$$

has a solution defined on some neighbourhood of $(0,0) \in \mathbf{R}^n \times \mathbf{R}$. That is we can find $H\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \to \mathbf{R}^n, 0$ satisfying ((3)). The vector field η is tangent to X, so by Proposition 3·2, we obtain a diffeomorphism $H_t\colon \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ preserving X for each

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small t. Now define a new family $G\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \to \mathbf{R}, 0$ by $G(x,t) = F(H(x,t),t)$. Differentiating with respect to t we obtain

$$\begin{aligned}\frac{\partial G}{\partial t}(x,t) &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(H(x,t),t)\frac{\partial H_i}{\partial t}(x,t) + \frac{\partial F}{\partial t}(H(x,t),t)\\ &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(H(x,t),t)\eta_i(H(x,t),t) + \frac{\partial F}{\partial t}(H(x,t),t)\\ &= \left(\sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t}\right)(H(x,t),t) \equiv 0.\end{aligned}$$

Fixing x we see that $G(x,t)$ is constant, i.e. $G(x,t) = G(x,0)$ for all t and x. In other words $F(H(x,t),t) = F(H(x,0),0) = F(x,0)$, as required. Note that $\partial H/\partial t(0,t) = \eta(H(0,t),t)$ has the unique solution $H(0,t) \equiv 0$, since the vector fields ξ_i all vanish at the origin, so H does have the required properties.

(ii) Suppose that the germ α exists as in the statement of the Proposition, and $H\colon \mathbf{R}^n \times \mathbf{R}, (0,0) \to \mathbf{R}^n, 0$ solves the differential equation

$$\frac{\partial H}{\partial t}(x,t) = \eta(H(x,t),t), H(x,0) = x$$

where $\eta = \sum_{i=1}^{p} \alpha_i \xi_i$. As before, since the vector fields ξ are tangent to X, we know that H_t is a 1-parameter family of diffeomorphisms preserving X. Consider $G(x,t) = F(H(x,t),t)$ and differentiate with respect to t. We find that

$$\begin{aligned}\frac{\partial G}{\partial t}(x,t) &= \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(H(x,t),t)\frac{\partial H_i}{\partial t}(x,t) + \frac{\partial F}{\partial t}(H(x,t),t)\\ &= \left(\sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t}\right)(H(x,t),t).\end{aligned}$$

The term

$$\sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t}$$

lies in $\mathscr{M}_n^{k+1}$, and hence so does $\partial G/\partial t(x,t)$. In particular we can write $\partial G/\partial t(x,t)$ as a sum $\sum G_I(x,t)x^I$, where I is a multi-index with $|I| = k+1$. So

$$G(x,t) - G(x,0) = \int_0^t \frac{\partial G}{\partial u}(x,u)du = \sum \left(\int_0^t \frac{\partial G_I}{\partial u}(x,u)du\right) x^I$$

also lies in $\mathscr{M}_n^{k+1}$, and since $G(x,0) = F(x,0)$ the result follows. □

We shall use Proposition 3·9 to prove a complete transversal result for $\mathscr{R}(X)$-equivalence. Before doing this we make the following definition.

Definition 3·10. If $\Theta(X)$ is the set of vector fields tangent to the variety X then we write $\Theta_1(X)$ for those fields with 1-jet zero. This is a module over the ring $\mathscr{E}_n$.

Theorem 3·11. *Let $f\colon \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be a smooth germ (usually a polynomial of degree k) and let $\{h_1, \ldots, h_r\}$ be a collection of homogeneous polynomials of degree $k+1$ with*

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the property that

$$\Theta_1(X)\cdot f + \mathrm{sp}\,\{h_1,\ldots,h_r\} + \mathscr{M}_n^{k+2} \supset \mathscr{M}_n^{k+1}.$$

Then any germ $g\colon \mathbf{R}^n,0 \to \mathbf{R},0$ *with* $j^k g(0) = j^k f(0)$ *is* $\mathscr{R}(X)$*-equivalent to a germ of the form*

$$f(x) + \sum_{i=1}^{r} u_i h_i(x) + \phi(x)$$

where $\phi(x) \in \mathscr{M}_n^{k+2}$. *Moreover this* $\mathscr{R}(X)$*-equivalence has* 1*-jet the identity.*

Proof. Given a function germ $g\colon \mathbf{R}^n,0 \to \mathbf{R},0$ with $j^k g(0) = j^k f(0)$ consider the difference $g - f \in \mathscr{M}_n^{k+1}$. We write this as

$$\eta(f) + \sum_{i=1}^{r} u_i h_i + \phi$$

where the vector field $\eta = \sum \eta_i \partial/\partial x_i$ lies in $\Theta_1(X)$, the u_i are real numbers and ϕ lies in $\mathscr{M}_n^{k+2}$. Now set

$$F_t(x) = F(x,t) = f(x) + (1-t)(g-f)(x) + t\sum_{i=1}^{r} u_i G_i(x).$$

Note that $F_0 = g$ and $F_1 = f + \sum u_i G_i$ so that it is enough to prove that for each t_0 the family $F(x, t+t_0)$ is k-$\mathscr{R}(X)$-trivial. To see this first note that

$$\frac{\partial F}{\partial t}(x,t) = (f-g)(x) + \sum u_i G_i,$$

while

$$\eta F = \eta f + \psi(x,t), \quad \text{where} \quad \psi \in \mathscr{M}_n^{k+2}.$$

So we have

$$\begin{aligned}\eta F + \frac{\partial F}{\partial t} &= \sum \eta_i \frac{\partial f}{\partial x_i} + (f-g) + \sum u_i G_i + \psi \\ &= -\phi + \psi \in \mathscr{M}_n^{k+2}.\end{aligned}$$

It follows from Proposition 3·9 that the family $F(x,t)$ (based at $t_0 = 0$) is k-$\mathscr{R}(X)$-trivial.

If we set

$$f' = f + (1-t_0)(g-f) + t_0 \sum_{i=1}^{r} G_i(x),$$

then it can easily be checked that

$$\Theta_1(X) f' + \mathscr{M}_n^{k+2} = \Theta_1(X) f + \mathscr{M}_n^{k+2},$$

so that each hypothesis of the theorem applies if we replace f by f'. Consequently

$$F(x, t+t_0)\colon \mathbf{R}^n \times \mathbf{R},0 \to \mathbf{R},0$$

is k-$\mathscr{R}(X)$-trivial for each t_0, and the result follows by the usual compactness arguments. Since each η lies in $\Theta_1(X)$ the diffeomorphisms H_t all have 1-jet the identity. □

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Corollary 3·12. (*a*) *If*

$$\Theta_1(X) + \mathscr{M}_n^{k+2} \supset \mathscr{M}_n^{k+1},$$

then f *is* $k - \mathscr{R}(X)$*-determined.*

(*b*) *In particular, if every vector field in* $\Theta(X)$ *vanishes at the origin, then if*

$$\Theta(X) + \mathscr{M}_n^{k+2} \supset \mathscr{M}_n^{k+1},$$

f *is* $(k+1) - \mathscr{R}(X)$*-determined.*

Proof. (*a*) By Nakayama's Lemma $\Theta_1(X) \supset \mathscr{M}_n^{k+1}$ and the previous theorem establishes the result.

(*b*) If every vector field in $\Theta(X)$ vanishes at the origin then $\Theta_1(X) \supset \mathscr{M}_n \cdot \Theta(X)$. So if we have $\Theta(X) + \mathscr{M}_n^{k+2} \supset \mathscr{M}_n^{k+1}$, then $\mathscr{M}_n \cdot \Theta(X) + \mathscr{M}_n^{k+3} \supset \mathscr{M}_n^{k+2}$, and f is $(k+1) - \mathscr{R}(X)$-determined. □

When X is the Whitney Umbrella, every vector field in $\Theta(X)$ does vanish at the origin, and so we use the determinacy result of Corollary 3·12(*b*) for the classification that follows.

3·2. *Classifying functions on the crosscap*

We apply the results of the previous section to the case where X is the Whitney Umbrella. In this case $\Theta(X)$ is the $\mathscr{O}_3$ module of germs at the origin of vector fields on $\mathbf{C}^3$ generated by the vector fields listed in Proposition 2·5.

To perform the classification, we need to determine at least the 1-jets of the coordinate changes in $\mathscr{R}(X)$. Consider the linear parts of the vector fields generating $\Theta(X)$. These are

$$u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad v\frac{\partial}{\partial v} + 2w\frac{\partial}{\partial w}, \quad 2v\frac{\partial}{\partial w} \quad \text{and} \quad v\frac{\partial}{\partial u}.$$

Integrating these, we get the 1-jets of coordinate changes $(u,v,w) \mapsto (e^{\lambda}u, e^{\lambda}v, w)$, $(u,v,w) \mapsto (u, e^{\mu}v, e^{2\mu}w)$, $(u,v,w) \mapsto (u,v,w+\alpha v)$ and $(u,v,w) \mapsto (u+\beta v, v, w)$.

Theorem 3·13. *Any germ* $f\colon \mathbf{R}^3, X, 0 \to \mathbf{R}, 0$ *which is a submersion on the ambient space is* $\mathscr{R}(X)$*-equivalent to one of the germs in the following three families.*

(i) $UW_{k+1}^{\pm}$: $u \pm w^{k+1}$, $k \geqslant 0$,
(ii) $WU_{k+1}^{\pm}$: $w \pm u^{k+1}$, $k \geqslant 1$,
(iii) $VW_{k+1}^{\pm}$: $v \pm w^{k+1}$, $k \geqslant 1$.

Each of these germs is $(k+1) - \mathscr{R}(X)$*-determined.*

Proof. We may suppose that we are starting with a 1-jet $au + bv + cw$, with one of a, b, c non-zero. Now if $ac \neq 0$, then using the linear changes of coordinates given above, we may reduce to (*a*) $u \pm w$.

If on the other hand $ac = 0$, then using the linear changes given above, we may reduce to one of the following 1-jets (b) u; (c) w; (d) v.

(*a*) Suppose that $f(u,v,w) = u \pm w$. Then we have $\xi_1 f = u, \xi_2 f = \pm 2w, \xi_3 f = 2v$ and $\xi_4 f = v$. So $J_X(f) \supset \mathscr{M}_3$. Thus f satisfies the criteria of Corollary 3·12(*b*) and is 1-$\mathscr{R}(X)$-determined.

(*b*) Suppose that $f(u,v,w) = u$. Then we have $\xi_1 f = u, \xi_2 f = 0, \xi_3 f = 0$ and $\xi_4 f = v$. Thus if we have a germ with k-jet u, then a complete transversal is given by $u + \lambda w^{k+1}$. If $\lambda \neq 0$, we can apply the coordinate changes described above to fix $\lambda = \pm 1$. Now let

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$f(u,v,w) = u \pm w^{k+1}$. Then we have $\xi_1 f = u, \xi_2 f = \pm 2(k+1)w^{k+1}, \xi_3 f = \pm 2(k+1)vw^k$ and $\xi_4 f = v$. So $J_X(f) \supset \mathscr{M}_3^{k+1}$ and we can see that f is $(k+1) - \mathscr{R}(X)$-determined.

(*c*) If $f(u,v,w) = w$, then we have $\xi_1 f = 0, \xi_2 f = 2w, \xi_3 f = 2v$ and $\xi_4 f = 0$. Thus if we have a germ with k-jet w, then a complete transversal is given by $w + \lambda u^{k+1}$. If $\lambda \neq 0$, we can apply the coordinate changes described above to fix $\lambda = \pm 1$. Now let $f(u,v,w) = w \pm u^{k+1}$. Then $\xi_1 f = \pm(k+1)u^{k+1}, \xi_2 f = 2w, \xi_3 f = 2v$ and $\xi_4 f = \pm(k+1)vu^k$, and again it is clear that f is $(k+1) - \mathscr{R}(X)$-determined.

(*d*) Finally we consider the case where $f(u,v,w) = v$. Then we have $\xi_1 f = v, \xi_2 f = v, \xi_3 f = u^2$ and $\xi_4 f = uw$. So a complete 2-transversal is $v + \lambda u^2 + \mu uw + \nu w^2$. We think of this as a 1 parameter family F_λ. We show that this is $\mathscr{R}(X)$-trivial using Proposition 3·9. Now we have $\xi_1 F = v + 2\lambda u^2 + \mu uw, \xi_2 F = v + 2\mu uw + 4\nu w^2, \xi_3 F = u^2 + 2\mu uv + 4\nu vw$ and $\xi_4 F = uw + 2\lambda uv + \mu vw$. and $\partial F/\partial \lambda = u^2$. It is clear that

$$\partial F/\partial \lambda \in \Theta(X).F + \mathscr{M}_n^3,$$

so that by Proposition 3·9, the family F_λ is $2 - \mathscr{R}(X)$-trivial. Similarly we can show that the family $F_\mu = v + \mu uw + \nu w^2$ is $2 - \mathscr{R}(X)$-trivial. We can then scale ν to get the normal forms v and $v \pm w^2$.

If $f(u,v,w) = v$ is a k-jet, then a complete $(k+1)$-transversal is given by $g(u,v,w) = v + \lambda w^{k+1}$. Then we have $\xi_1 g = v, \xi_2 g = v \pm 2(k+1)w^{k+1}, \xi_3 g = u^2 \pm 2(k+1)vw^k$ and $\xi_4 g = uw$. Thus it is clear that g is $(k+1) - \mathscr{R}(X)$-determined, for all $k \geqslant 1$. □

We need to distinguish the $\mathscr{R}(X)$-types of the functions on the crosscap. Proposition 2·1 shows that if $H\colon \mathbf{R}^3, 0 \longrightarrow \mathbf{R}, 0$ is a function on the crosscap, then we can associate with it two germs which are well defined up to $\mathscr{R}$-equivalence. First we can choose any parametrisation of the crosscap $g\colon \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^3, 0$ and compose it with H to obtain a function germ $H \circ g\colon \mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$. Secondly, we can compose H with the parametrisation of the double point set $\alpha\colon \mathbf{R}, 0 \longrightarrow \mathbf{R}^3, 0$ to obtain the germ $H \circ \alpha\colon \mathbf{R}, 0 \longrightarrow \mathbf{R}, 0$. For the standard crosscap parametrised by $(x,y) \mapsto (x, xy, y^2)$, the parametrisation of the double point set is given by $\alpha(y) = (0, 0, y^2)$, so that $\alpha(y) = \alpha(-y)$. Since any crosscap may be obtained from the standard crosscap by a diffeomorphism in the target, we may always suppose that the parametrisation of the double point set of a general crosscap, $\beta\colon \mathbf{R}, 0 \to \mathbf{R}^3, 0$ satisfies $\beta(y) = \beta(-y)$. So if $H\colon \mathbf{R}^3, 0 \longrightarrow \mathbf{R}, 0$ is any function germ, then the composite $H \circ \beta\colon \mathbf{R}, 0 \longrightarrow \mathbf{R}, 0$ is $\mathbf{Z}_2$-equivariant with respect to the involution $y \mapsto -y$ on $\mathbf{R}, 0$.

THEOREM 3·14. *The classes $UW_{k+1}^{\pm}$, $WU_{k+1}^{\pm}$ and $VW_{k+1}^{\pm}$ of Theorem* 3·13 *are all $\mathscr{R}(X)$-inequivalent. (There is some redundancy with signs however: so for example for k even it is easy to see that a germ of type WU_{k+1}^{+} is $\mathscr{R}(X)$-equivalent to one of type WU_{k+1}^{-}.)*

Proof. In each case we compose the germ with the parametrisation of the standard crosscap $((x,y) \mapsto (x, xy, y^2))$ and the parametrisation of the double point curve $(y \mapsto (0, 0, y^2))$ to obtain

(i) $UW_{k+1}^{\pm}$: the germs $x \pm y^{2k+2}$ and $\pm y^{2k+2}$, i.e. a submersion and a singularity of type $A_{2k+1}^{\pm}$;
(ii) $WU_{k+1}^{\pm}$: the germs $y^2 \pm x^{k+1}$ and y^2, i.e. an $A_k^{\pm}$ singularity and an A_1 singularity;

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(iii) $VW_{k+1}^{\pm}$: the germs $xy \pm y^{2k+2}$ and $\pm y^{2k+2}$, i.e. an A_1 singularity and an $A_{2k+1}^{\pm}$ singularity.

The result then follows from the discussion above. □

Now we consider $\mathscr{R}(X)$-versal unfoldings of these germs.

Theorem 3·15. *The following are $\mathscr{R}(X)$-versal unfoldings of the germs of Theorem 3·13 and are of minimal dimension. They are also $\mathscr{K}(X)$-versal of minimal dimension. Note also that a germ of type UW_{k+1}^{+} is $\mathscr{K}(X)$-equivalent to one of type UW_{k+1}^{-}, and the same applies to the types $VW_{k+1}^{\pm}$. In particular their discriminants will be diffeomorphic.*

(i) $UW_{k+1}^{\pm}$: $u \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i$,

(ii) $WU_{k+1}^{\pm}$: $w \pm u^{k+1} + \sum_{i=0}^{k} a_i u^i$,

(iii) $VW_{k+1}^{\pm}$: $v \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i + bu$.

Proof. By Theorem 3·6, we need to find a complement to $J_X(f)$ in $\mathscr{O}_3$ for each of the function germs.

(i) Let $f(u,v,w) = u \pm w^{k+1}$. Then we have $\xi_1 f = u, \xi_2 f = \pm 2(k+1)w^{k+1}, \xi_3 f = \pm 2(k+1)vw^k$ and $\xi_4 f = v$. So a complement to $J_X(f)$ is given by $1, w, w^2, \cdots, w^k$, giving us the unfolding

$$F(u,v,w,a) = u \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i.$$

(ii) When $f(u,v,w) = w \pm u^{k+1}$, we have $J_X(f) =< u^{k+1}, v, w >$, so we do indeed have the unfolding

$$F(u,v,w,a) = w \pm u^{k+1} + \sum_{i=0}^{k} a_i u^i.$$

(iii) When $f = v \pm w^{k+1}$ we have $J_X(f) =< v, w^{k+1}, u^2, uw >$ with complement spanned by $1, w, w^2, \ldots, w^k, u$, giving the unfolding

$$F(u,v,w,a) = v \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i + bu.$$

Note that in each case $f \in J_X(f)$ so the resulting unfoldings are $\mathscr{K}(X)$-versal, and of minimal dimension. □

3·3. *The geometry of the functions on the crosscap*

Now we analyse the discriminants of the singularities $UW_{k+1}^{\pm}$, $WU_{k+1}^{\pm}$ and $VW_{k+1}^{\pm}$. If $f\colon \mathbf{R}^3, X, 0 \to \mathbf{R}, 0$ is a germ and $F\colon \mathbf{R}^3 \times \mathbf{R}^p, 0 \to \mathbf{R}, 0$ is an unfolding of f, we can identify a number of interesting subsets of the unfolding space $\mathbf{R}^p$.

Definition 3·16. (i) The set $\mathscr{D}_1(F)$ is defined by

$$\mathscr{D}_1(F) = \left\{ a\colon F = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = \frac{\partial F}{\partial w} = 0 \quad \text{at some point} \quad (u,v,w,a) \right\}.$$

Now set $G = F(x, xy, y^2, a)$.

(ii) The set $\mathscr{D}_2(F)$ is defined by

$$\mathscr{D}_2(F) = \left\{ a\colon G = \frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = 0 \quad \text{at some point} \quad (x,y,a) \right\}.$$

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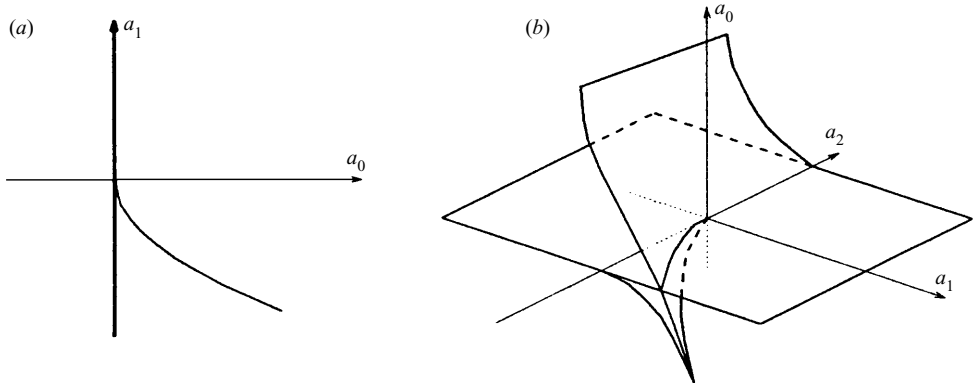


Fig. 2. (*a*) $\mathscr{D}_3(F)$ for UW_2^+. (*b*) $\mathscr{D}_3(F)$ for UW_3^+.

Set $H = F(0, 0, y^2, a)$

(iii) The set $\mathscr{D}_3(F)$ is defined by

$$\mathscr{D}_3(F) = \left\{ a\colon H = \frac{\partial H}{\partial y} = 0 \quad \text{at some point} \quad (y, a) \right\}.$$

(iv) The set $\mathscr{D}_4(F)$ is defined by

$$\mathscr{D}_4(F) = \{a\colon F(0, 0, 0, a) = 0\}.$$

So $\mathscr{D}_1(F)$ is the usual discriminant of F as a function on $\mathbf{R}^3$, $\mathscr{D}_2(F)$ is the discriminant of F when composed with the parametrisation of the standard crosscap, $\mathscr{D}_3(F)$ is the discriminant of F restricted to the double point set and $\mathscr{D}_4(F)$ corresponds to the fibres of F that go through the origin. We now identify these discriminants for the families $UW_{k+1}^{\pm}$, $WU_{k+1}^{\pm}$ and $VW_{k+1}^{\pm}$. Since each of the germs are submersions on the ambient space, $\mathscr{D}_1(F)$ is empty in all cases. Also it is clear that $\mathscr{D}_4(F)$ is given by the equation $a_0 = 0$. It remains to analyse the discriminants $\mathscr{D}_2(F)$ and $\mathscr{D}_3(F)$ for each of the unfoldings.

(i) $UW_{k+1}^{\pm}$: $F(u, v, w, a) = u \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i$.

We associate to F the unfoldings

$$G(x, y, a) = x \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i} \quad \text{and} \quad \mathrm{H}(y, a) = \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i}.$$

The deformation G unfolds the germ $x \pm y^{2k+2}$, which is a submersion at $(0, 0)$, so $\mathscr{D}_2(F)$ is empty in this case.

Now H is a versal unfolding of a B_{k+1} singularity in the terminology of [**2**], and thus $\mathscr{D}_3(F)$ is the discriminant of such a singularity. For the cases $k = 1$ and $k = 2$ the pictures are well known, see, for example, [**4**], and given in Fig. 2(*a*, *b*).

(ii) $WU_{k+1}^{\pm}$: $F(u, v, w) = w \pm u^{k+1} + \sum_{i=0}^{k} a_i u^i$.

We associate to F the unfoldings

$$G(x, y, a) = y^2 \pm x^{k+1} + \sum_{i=0}^{k} a_i x^i \quad \text{and} \quad H(y, a) = \pm y^2 + a_0.$$

We see immediately that $\mathscr{D}_3(F) = \mathscr{D}_4(F)$ is the hyperplane $a_0 = 0$. The deformation G is an $\mathscr{R}$-versal unfolding of an A_k singularity. The discriminant $\mathscr{D}_2(F)$ is the standard

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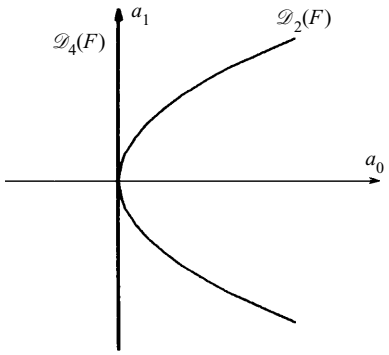


Fig. 3. $\mathscr{D}_2(F)$ and $\mathscr{D}_4(F)$ for WU_2^+.

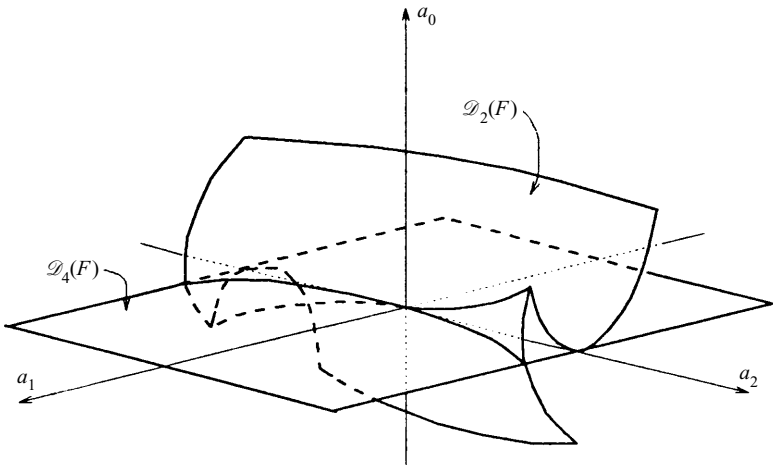


Fig. 4. $\mathscr{D}_2(F)$ and $\mathscr{D}_4(F)$ for WU_3^+.

discriminant of an A_k singularity. The sets $\mathscr{D}_2(F)$ and $\mathscr{D}_4(F)$ are shown in Fig. 3 for WU_2^+.

If $k = 2$ then $\mathscr{D}_2(F)$ consists of a cuspidal surface. The discriminant is defined by the equations

$$G(x, y, a) = y^2 \pm x^3 + a_0 + a_1 x + a_2 x^2 = \frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = 0.$$

The cuspidal edge is defined by $G = \partial G/\partial x = \partial^2 G/\partial x^2 = 0$ and is parametrised as $(a_0, a_1, a_2) = (-x^3, 3x^2, -3x)$ in the case of WU_3^+, and $(x^3, -3x^2, 3x)$ in the case of WU_3^-. The cuspidal surface meets the plane $a_0 = 0$ along the a_2-axis; looking at sections of the surface along this axis, we see that the plane $a_0 = 0$ is tangential here. The cuspidal surface also meets the plane $a_0 = 0$ transversely along the curve parametrised by $(a_1, a_2) = (x^2, -2x)$ in the case of WU_3^+, and along the curve parametrised by $(a_1, a_2) = (-x^2, 2x)$ in the case of WU_3^-. The sets $\mathscr{D}_2(F)$ and $\mathscr{D}_4(F)$ are shown in Fig. 4 for WU_3^+.

(iii) $VW_{k+1}^{\pm}$: $F(u, v, w, a, b) = v \pm w^{k+1} + \sum_{i=0}^{k} a_i w^i + bu$.

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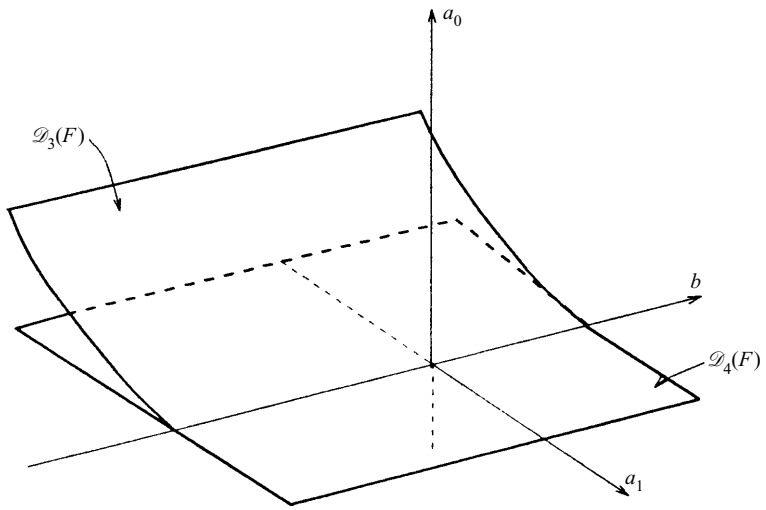


Fig. 5. The discriminants $\mathscr{D}_3(F)$ and $\mathscr{D}_4(F)$ for VW_2^+.

We have

$$G(x, y, a, b) = xy \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i} + bx.$$

This is an $\mathscr{R}$-versal unfolding of an A_1 singularity, so $\mathscr{D}_2(F)$ is diffeomorphic to a hyperplane in (a, b)-space. Indeed by considering the equations $G = \partial G/\partial x = \partial G/\partial y = 0$, we see that $\mathscr{D}_2(F)$ is defined by the equation

$$\pm b^{2k+2} + \sum_{i=0}^{k} a_i b^{2i} = 0.$$

As usual the discriminant $\mathscr{D}_4(F)$ is the hyperplane with equation $a_0 = 0$. The discriminant $\mathscr{D}_2(F)$ meets $\mathscr{D}_4(F)$ in the set defined by

$$b^2 \left(\pm b^{2k} + \sum_{i=1}^{k} a_i b^{2i-2} \right) = 0.$$

The unfolding H is given by

$$H(y, a, b) = \pm y^{2k+2} + \sum_{i=0}^{k} a_i y^{2i}.$$

As in the case of the family UW_{k+1}, H is a versal unfolding of a B_{k+1} singularity.

Now we consider the case where $k = 1$ in more detail. Then we have

$$F(u, v, w, a, b) = v \pm w^2 + a_0 + a_1 w + bu,$$
$$G(x, y, a, b) = xy \pm y^4 + a_0 + a_1 y^2 + bx$$

and

$$H(y, a) = \pm y^4 + a_0 + a_1 y^2.$$

The discriminant $\mathscr{D}_3(F)$ is parametrised by $(a_0, a_1, b) = (\pm y^4, \mp 2y^2, b)$. For VW_2^+ the configuration of $\mathscr{D}_3(F)$ and $\mathscr{D}_4(F)$ (the plane $a_0 = 0$) is given in Fig. 5.

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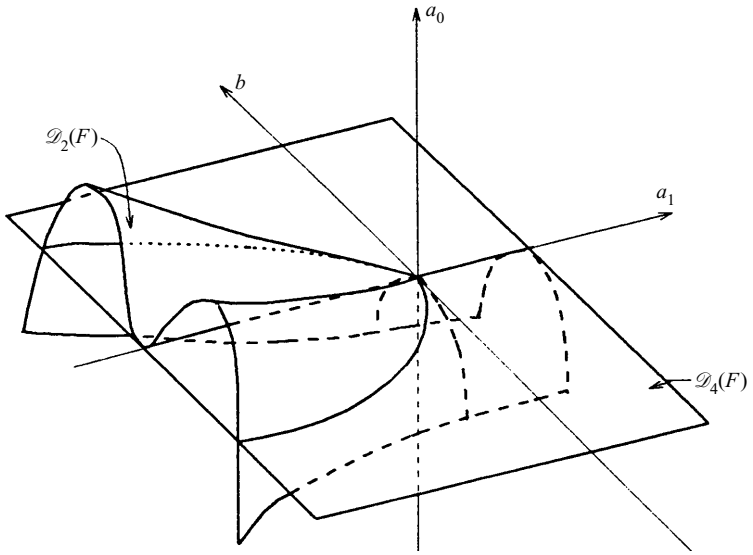


Fig. 6. The discriminants $\mathscr{D}_2(F)$ and $\mathscr{D}_4(F)$ for VW_2^+.

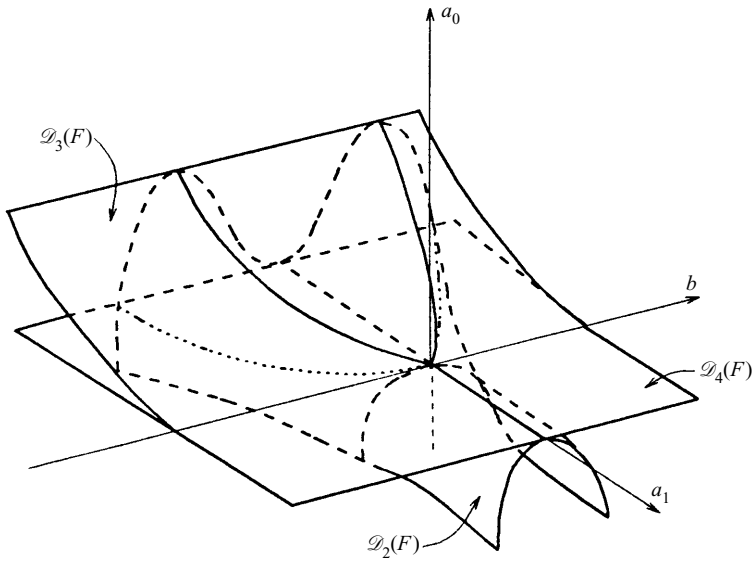


Fig. 7. The discriminants $\mathscr{D}_2(F)$, $\mathscr{D}_3(F)$ and $\mathscr{D}_4(F)$ for VW_2^+.

In the case $k = 1$, $\mathscr{D}_4(F)$ meets $\mathscr{D}_2(F)$ in the curve with equation

$$b^2(\pm b^2 + a_1) = 0,$$

and $\mathscr{D}_3(F)$ meets $\mathscr{D}_2(F)$ in the curve parametrised by

$$(a_0, a_1, b) = (\pm y^4, \mp 2y^2, -y)$$

Figures 6 and 7 illustrate this.

3·4. *The flat geometry of the crosscap*

Now we discuss one motivation for classifying functions on the Whitney Umbrella. In a previous paper we considered height functions on crosscaps. Our approach has

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been to compose the parametrisation of the crosscap with the height functions to obtain a germ $\mathbf{R}^2, 0 \longrightarrow \mathbf{R}, 0$. Proposition 2·1 shows that the result is well defined up to $\mathscr{R}$-equivalence.

The approach in this paper is to consider functions on the crosscap itself. More precisely, we know that the parametrisation of the crosscap g is $\mathscr{A}$-equivalent to the standard parametrisation $f(x, y) = (x, xy, y^2)$. In other words, there are germs of diffeomorphisms $\phi\colon \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$ and $\psi\colon \mathbf{R}^3, 0 \longrightarrow \mathbf{R}^3, 0$ such that $\psi \circ f = g \circ \phi$. So the height function, H say, defined on the image of g can be thought of as a function on the Whitney Umbrella by replacing f by $f \circ \psi$. The composite $H \circ \psi$ is well defined up to $\mathscr{R}(X)$-equivalence, for given any other diffeomorphisms $\phi_1\colon \mathbf{R}^2, 0 \longrightarrow \mathbf{R}^2, 0$ and $\psi_1\colon \mathbf{R}^3, 0 \longrightarrow \mathbf{R}^3, 0$ such that $\psi_1 \circ f = g \circ \phi_1$, we have $H \circ \psi_1 = H \circ \psi \circ (\psi^{-1} \circ \psi_1)$ and the diffeomorphism $\psi^{-1} \circ \psi_1$ preserves the Whitney Umbrella.

We can now apply the results on the classification of functions on the crosscap. This approach requires a transversality result and an understanding of the stratification of the jet-space given by our classification of germs. These results can be found in Appendix (A). So let $X, 0$ denote the standard crosscap in $\mathbf{R}^3$, and let $\phi\colon \mathbf{R}^3, 0 \to \mathbf{R}^3, 0$ be the germ of a diffeomorphism taking X to the geometric crosscap $\phi(X)$, with $f\colon \mathbf{R}^3 \times Z \to \mathbf{R}$ a family of functions parametrised by some manifold Z which we wish to apply to the geometric crosscap. Then we can consider the family

$$\begin{aligned} f_\phi\colon \mathbf{R}^3 \times Z &\to \mathbf{R} \\ (x, z) &\mapsto f(\phi(x), z) = f_{\phi z}(x). \end{aligned}$$

By studying this family of mappings on the standard crosscap we can describe the interaction between the geometric crosscap and the fibres of the family of functions $f_z\colon \mathbf{R}^3 \to \mathbf{R}$. We are interested in the family of height functions parametrised by the 2-sphere S^2 and given by

$$\begin{aligned} h\colon \mathbf{R}^3 \times S^2 &\to \mathbf{R} \\ (x, u) &\mapsto x \cdot u. \end{aligned}$$

Proposition 3·17. *Let $f\colon \mathbf{R}^3 \times Z \to \mathbf{R}$ be a family of functions parametrised by some manifold Z. Suppose that the function f_z is a submersion at the origin for each $z \in Z$ and the dimension of Z is at most 2. We define the family of functions f_ϕ by*

$$\begin{aligned} f_\phi\colon \mathbf{R}^3 \times Z &\to \mathbf{R} \\ (x, z) &\mapsto f(\phi(x), z) = f_{\phi z}(x). \end{aligned}$$

Then generically the function $f_{\phi z}$ at the origin must be $\mathscr{R}(X)$-equivalent to a germ of one of the following types, and each is $\mathscr{R}(X)$-versally unfolded by the family f_ϕ.

(i) UW_{k+1}, $k = 0, 1, 2$,
(ii) WU_{k+1}, $k = 1, 2$,
(iii) VW_2.

Proof. See Appendix (A), Proposition A·2. □

Now we can make deductions about the dual of the crosscap, and of the double point curve.

Proposition 3·18. *The singularities listed in Proposition* 3·17 *have the following geometric interpretations.*

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(i) UW_1: *the corresponding plane is transverse to both the crosscap surface and its double point curve.*

(ii) UW_2: *the plane is transverse to the surface and is in the pencil of planes obtained as limiting tangents to the double point curve.*

(iii) UW_3: *the plane is transverse to the surface and is the limiting osculating plane to the double point curve.*

(iv) WU_2: *the plane is tangent to the surface and transverse to the double point curve.*

(v) WU_3: *the plane has parabolic tangency with the surface and is transverse to the double point curve.*

(vi) VW_2: *the plane is the tangent cone, i.e. is tangent to both the surface and the double point curve.*

Proof. The geometrical results follow from composing the listed functions with the parametrisation of the standard crosscap. Note that we have changed coordinates so the fibres of these functions are no longer planes.

For the UW_{k+1} series, we get $x \pm y^{2k}$ and $\pm y^{2k}$ when composing the function with the parametrisations of the standard crosscap and its double point curve respectively. It is clear that for $k = 0, 1, 2$, the function UW_{k+1} corresponds to a plane that is transverse to the surface. We remark that the self intersection curve is 'doubly parametrised' and is thus highly degenerate. Nevertheless we can consider contact of planes with this curve. If F is an unfolding of f, then the discriminant $\mathscr{D}_3(F)$ discussed in Section 3·3 will give us a model for the dual of the double point curve. The plane $a_0 = 0$ corresponds to the set of planes that pass through the crosscap point (the discriminant $\mathscr{D}_4(F)$). So the dual to the double point curve is obtained by removing the plane $a_0 = 0$ that appears as part of the discriminants $\mathscr{D}_3(F)$. Note that in general an osculating plane has A_2 contact with a smooth space curve, and so the dual of the space curve will have a cuspidal edge. Considering the discriminant $\mathscr{D}_3(F)$ for UW_2 and UW_3 completes the proof of results (i)–(iii).

For the WU_{k+1}-series we get $y^2 \pm x^k$ and y^2. It follows that the corresponding planes are transverse to the double point curve and have A_1 (for $k = 1$) and A_2 (for $k = 2$) contact with the surface at the crosscap point. This gives results (iv) and (v).

Finally for VW_2 the composites are $xy \pm y^4$ and y^4. Here we can see that the corresponding plane is tangent to the double point line (set $x = 0$) and is tangent to the surface. This corresponds to the tangent cone plane by Lemma 2·2. □

We have seen that if F is an unfolding of one of the functions listed above then the discriminant $\mathscr{D}_3(F)$ corresponds to the dual of the double point curve, so that this approach gives additional information on the double point curve. Similarly the discriminant $\mathscr{D}_2(F)$ discussed in Section 3·3 will give us a model for the dual of the crosscap. Hence the discussion of these discriminants in Section 3·3 can be used to give an alternative treatment of structure of the dual, but with additional information about the dual of the curve of self-intersection. In the paper [**10**] we obtained a model for that part of the dual about the pencil of planes tangent at the cross-cap point. The relevant singularities here are (iv), (v) and (vi) above, and the discriminants described in (iv) and (v) give an alternative derivation of the corresponding parts of the dual. In (vi) above we have a model for the dual in a neighbourhood of the tangent cone which refines that given in [**10**]. Note that in the corresponding discriminant the sets $\mathscr{D}_2(F)$ and $\mathscr{D}_3(F)$ are tangent along a curve. It is clear from

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general properties of duals (as well as direct calculation) that this corresponds to the tangent planes to the surface along the double point curve.

Appendix A. Transversality result

This section contains the results needed to prove Proposition 3·17.

Let $X, 0$ denote the standard crosscap in $\mathbf{R}^3$, and let $\phi\colon \mathbf{R}^3, 0 \to \mathbf{R}^3, 0$ be the germ of a diffeomorphism taking X to the geometric crosscap $\phi(X)$. Let $f\colon \mathbf{R}^3 \times Z \to \mathbf{R}$ be a family of functions parametrised by some manifold Z which we wish to apply to the geometric crosscap. Then we can consider the family

$$\begin{aligned} f_\phi\colon \mathbf{R}^3 \times Z &\to \mathbf{R} \\ (x, z) &\mapsto f(\phi(x), z) = f_{\phi z}. \end{aligned}$$

By studying this family of mappings on the standard crosscap we can describe the interaction between the geometric crosscap and the fibres of the family of functions $f_z\colon \mathbf{R}^3 \to \mathbf{R}$.

We wish to prove a result concerning the generic crosscap, so we need a suitably large family of crosscaps to consider. So we let P_k denote the set of polynomial mappings $\mathbf{R}^3 \to \mathbf{R}^3$ of degree at most k, with no constant terms, so that the elements of P_k preserve the origin. Let V_k denote a neighbourhood of the identity which maps the unit ball at the origin diffeomorphically onto its image. Consider the family

$$\begin{aligned} F\colon \mathbf{R}^3 \times Z \times V_k &\to \mathbf{R}, \\ (u, z, p) &\mapsto f_z(p \circ \phi(u)). \end{aligned}$$

As before we shall use the coordinates (x, y) in the parametrising space $\mathbf{R}^2$ and the coordinates (u, v, w) in the ambient space $\mathbf{R}^3$.

Proposition A·1. *If the germs f_z are all submersions then the jet-extension map*

$$j_1^k F\colon Z \times V_k \to J^k(3, 1)$$

obtained by evaluating the k-jet of F at $(0, 0, 0)$ is a submersion.

Proof. Recall that $J^k(3, 1)$ is the vector space of polynomial functions with zero constant term of degree at most k. We shall see that we may as well suppose that the diffeomorphism ϕ is the identity. We consider for $p \in P_k$ the limit

$$\lim_{s \to 0} \frac{j_1^k(f_z(x + sp(x)) - f_z(x))}{s} = j^k \left(\frac{\partial f_z}{\partial u_1} p_1 + \frac{\partial f_z}{\partial u_2} p_2 + \frac{\partial f_z}{\partial u_3} p_3 \right)(0).$$

As f_z is a submersion then we obtain the result as one of the $\partial f_z / \partial u_i \neq 0$. □

We now wish to use the above result, and so we need to stratify the jet space. We do this using the above classification. In what follows we shall actually assume that each function in the family f_z is a submersion. In fact we are really interested in the family of height functions, so the f_z are indeed submersions and we have a 2-parameter family.

Proposition A·2. *We can stratify $J^3(3, 1)$ into strata consisting of the singular jets, submersions of type UW_{k+1}, $k = 0, 1, 2$; WU_{k+1}, $k = 1, 2$ and VW_2, and the orbits of u, v and w.*

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Let $f\colon \mathbf{R}^3\times Z \to \mathbf{R}$ be a family of functions parametrised by some manifold Z. Suppose that the function f_z is a submersion at the origin for each $z \in Z$ and the dimension of Z is at most 2. We define the family of functions f_ϕ by

$$f_\phi\colon \mathbf{R}^3 \times Z \to \mathbf{R}$$
$$(x, z) \mapsto f(\phi(x), z) = f_{\phi z}.$$

Then generically, the function $f_{\phi z}$ at the origin must be $\mathscr{R}(X)$-equivalent to a germ of one of the following types:

(i) UW_{k+1}, $k = 0, 1, 2$,
(ii) WU_{k+1}, $k = 1, 2$,
(iii) VW_2.

Generically, the functions $f_{\phi z}$ are $\mathscr{R}(X)$-versally unfolded by the family f_ϕ.

Proof. The group $\mathscr{R}(X)$ has well defined jet groups $J^k\mathscr{R}(X)$. For the orbits we consider those of the k-jets we have classified. We are only interested in those of codimension at most 2 since there are no problems with moduli.

The relevant germs are UW_{k+1}, $k = 0, 1, 2$; WU_{k+1}, $k = 1, 2$ and VW_2. The $\mathscr{R}(X)$-codimension is k in each case. Now stratify $J^3(3, 1)$ by the corresponding orbits and their complement, which consists of the singular germs (of codimension 3) and the orbits of u, v and w. We now ask that the jet-extension map is transverse to these strata (it will be for a generic choice of polynomial mapping, in other words for a generic choice of crosscap).

The result about the versality follows because of the infinitesimal criterion we gave before. Note that our jet-extension map does not have a copy of the source space $\mathbf{R}^3$ present as is usual in these situations. On the other hand the groups $\mathscr{R}(X)$ and $\mathscr{R}_e(X)$ coincide, so the transversality criterion for versality does not involve tangent vectors from the source space.

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