Bull. Aust. Math. Soc. **112** (2025), 449–457
doi:10.1017/S0004972725000139

ON 5^k-REGULAR PARTITIONS MODULO POWERS OF 5

B. HEMANTHKUMAR and D. S. GIREESH✉

(Received 6 November 2024; accepted 18 January 2025; first published online 14 April 2025)

Abstract

In this work, we investigate the arithmetic properties of $b_{5^k}(n)$, which counts the partitions of n where no part is divisible by 5^k. By constructing generating functions for $b_{5^k}(n)$ across specific arithmetic progressions, we establish a set of Ramanujan-type congruences.

2020 *Mathematics subject classification*: primary 11P83; secondary 05A15, 05A17.

Keywords and phrases: partitions, ℓ-regular partitions, generating functions, congruences.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n, with the generating function

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{f_1}.$$

Here and throughout the paper,

$$f_r := (q^r;q^r)_\infty = \prod_{m=1}^{\infty}(1-q^{rm}).$$

Ramanujan [2] conjectured and Watson [5] proved that

$$p(5^k n + \delta_k) \equiv 0 \pmod{5^k}, \tag{1.1}$$

where $k \geq 1$ and δ_k is the reciprocal modulo 5^k of 24. Hirschhorn and Hunt [1] proved (1.1) by establishing generating functions

$$\sum_{n\geq 0} p(5^{2k-1}n + \delta_{2k-1})q^n = \sum_{j\geq 1} x_{2k-1,j}\, q^{j-1}\frac{f_5^{6j-1}}{f_1^{6j}} \tag{1.2}$$

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and

$$\sum_{n\geq 0} p(5^{2k}n + \delta_{2k})q^n = \sum_{j\geq 1} x_{2k,j}\, q^{j-1} \frac{f_5^{6j}}{f_1^{6j+1}},$$

where the coefficient vectors $\mathbf{x}_k = (x_{k,1}, x_{k,2}, \ldots)$ are given by

$$\mathbf{x}_1 = (x_{1,1}, x_{1,2}, \ldots) = (5, 0, 0, \ldots),$$

and for $k \geq 1$,

$$x_{k+1,i} = \begin{cases} \displaystyle\sum_{j\geq 1} x_{k,j}\, m_{6j,j+i} & \text{if } k \text{ is odd,} \\ \displaystyle\sum_{j\geq 1} x_{k,j}\, m_{6j+1,j+i} & \text{if } k \text{ is even,} \end{cases}$$

where the first five rows of $M = (m_{i,j})_{i,j\geq 1}$ are

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2\times 5 & 5^3 & 0 & 0 & 0 & 0 & \cdots \\ 9 & 3\times 5^3 & 5^5 & 0 & 0 & 0 & \cdots \\ 4 & 22\times 5^2 & 4\times 5^5 & 5^7 & 0 & 0 & \cdots \\ 1 & 4\times 5^3 & 8\times 5^5 & 5^8 & 5^9 & 0 & \cdots \end{bmatrix}$$

and, for $i \geq 6$, $m_{i,1} = 0$, and for $j \geq 2$,

$$m_{i,j} = 25m_{i-1,j-1} + 25m_{i-2,j-1} + 15m_{i-3,j-1} + 5m_{i-4,j-1} + m_{i-5,j-1}.$$

For any positive integer ℓ, a partition is called ℓ-regular if none of its parts is divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n; its generating function is given by

$$\sum_{n\geq 0} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{1.3}$$

In [4], Wang derived congruences for 5-regular partitions analogous to (1.1). He proved that, for each $n, \beta \geq 0$,

$$b_5\left(5^{2\beta+1}n + \frac{5^{2\beta+2}-1}{6}\right) \equiv 0 \pmod{5^{\beta+1}} \tag{1.4}$$

and

$$b_5\left(5^{2\beta+2}n + \frac{r\cdot 5^{2\beta+1}-1}{6}\right) \equiv 0 \pmod{5^{\beta+2}}, \quad r \in \{11, 17, 23, 29\}. \tag{1.5}$$

Ranganatha [3] extended these results to 25-regular partitions: for each $n, \beta \geq 0$,

$$b_{25}(5^{\beta+1}(n+1) - 1) \equiv 0 \pmod{5^{\beta+1}}. \tag{1.6}$$

We establish congruences for 5^k-regular partitions for all positive integers k. The congruences previously derived by Wang and Ranganatha can be seen as specific instances of these results. The main results are stated in the following theorem.

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THEOREM 1.1. *For each $n, \beta \geq 0$ and $k \geq 1$,*

$$b_{5^{2k-1}}\left(5^{2k+2\beta-1}n + \frac{4 \cdot 5^{2k+2\beta} - 5^{2k-1} + 1}{24}\right) \equiv 0 \pmod{5^{2k+\beta-1}}, \tag{1.7}$$

$$b_{5^{2k}}\left(5^{2k+\beta-1}(n+1) - \frac{5^{2k}-1}{24}\right) \equiv 0 \pmod{5^{2k+\beta-1}} \tag{1.8}$$

and

$$b_{5^{2k-1}}\left(5^{2k+2\beta}n + \frac{r \cdot 5^{2k+2\beta-1} - 5^{2k}}{6} + \delta_{2k-1}\right) \equiv 0 \pmod{5^{2k+\beta}}, \tag{1.9}$$

where $r \in \{11, 17, 23, 29\}$.

If we put $k = 1$ in (1.7) and (1.9), we obtain congruences (1.4) and (1.5), respectively, and putting $k = 1$ in (1.8) gives congruence (1.6).

2. Preliminaries

In this section, we state some lemmas which play a vital role in proving our results. Let H be the 'huffing' operator modulo 5, that is,

$$H\left(\sum a_n q^n\right) = \sum a_{5n} q^{5n}.$$

LEMMA 2.1 [1]. *If $G = f_5^6/q^4 f_1 f_{25}^5$ and $u = f_5^6/q^5 f_{25}^6$, then*

$$H(G^i) = \sum_{j=1}^{i} m_{i,j} u^{i-j}. \tag{2.1}$$

LEMMA 2.2. *For all $i \geq 1$,*

$$H\left(q^i \frac{f_5^{6i-5}}{f_1^{6i-5}}\right) = \sum_{j=1}^{5i-4} m_{6i-5,i+j-1} q^{5j} \frac{f_{25}^{6j-1}}{f_5^{6j-1}}, \tag{2.2}$$

$$H\left(q^{i-1} \frac{f_5^{6i-1}}{f_1^{6i-1}}\right) = \sum_{j=1}^{5i} m_{6i-1,i+j-1}\, q^{5j-5} \frac{f_{25}^{6j-5}}{f_5^{6j-5}}, \tag{2.3}$$

$$H\left(q^i \frac{f_5^{6i}}{f_1^{6i}}\right) = \sum_{j=1}^{5i} m_{6i,i+j}\, q^{5j} \frac{f_{25}^{6j}}{f_5^{6j}}, \tag{2.4}$$

$$H\left(q^{i+1} \frac{f_5^{6i+1}}{f_1^{6i+1}}\right) = \sum_{j=1}^{5j+1} m_{6i+1,i+j}\, q^{5j} \frac{f_{25}^{6j-1}}{f_5^{6j-1}}. \tag{2.5}$$

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PROOF. We can rewrite (2.1) as

$$H\left(q^i\frac{f_{25}^i}{f_1^i}\right)=\sum_{j=1}^{i}m_{i,j}\,q^{5j}\frac{f_{25}^{6j}}{f_5^{6j}}. \tag{2.6}$$

From (2.6) and the fact that $m_{6i-5,j}=0$ for $1\le j<i$,

$$H\left(\left(q\frac{f_{25}}{f_1}\right)^{6i-5}\right)=\sum_{j=i}^{6i-5}m_{6i-5,j}\,q^{5j}\frac{f_{25}^{6j}}{f_5^{6j}}=\sum_{j=1}^{5i-4}m_{6i-5,j+i-1}\,q^{5j+5i-5}\frac{f_{25}^{6j+6i-6}}{f_5^{6j+6i-6}},$$

which yields (2.2). Similarly, we can prove (2.3)–(2.5). □

3. Generating functions

In this section, we establish generating functions for $b_{5^\ell}(n)$ within specific arithmetic progressions.

THEOREM 3.1. *For each $\beta\ge 0$ and $k\ge 1$,*

$$\sum_{n\ge 0}b_{5^{2k-1}}\left(5^{2k+2\beta-1}n+\frac{4\cdot 5^{2k+2\beta}-5^{2k-1}+1}{24}\right)q^n=\sum_{i\ge 1}y_{2\beta+1,i}^{(2k-1)}\,q^{i-1}\frac{f_5^{6i-1}}{f_1^{6i-1}} \tag{3.1}$$

and

$$\sum_{n\ge 0}b_{5^{2k-1}}\left(5^{2k+2\beta}n+\frac{4\cdot 5^{2k+2\beta}-5^{2k-1}+1}{24}\right)q^n=\sum_{i\ge 1}y_{2\beta+2,i}^{(2k-1)}\,q^{i-1}\frac{f_5^{6i-5}}{f_1^{6i-5}}, \tag{3.2}$$

where the coefficient vectors are defined by

$$y_{1,j}^{(2k-1)}=x_{2k-1,j}$$

and

$$y_{\beta+1,j}^{(2k-1)}=\begin{cases}\displaystyle\sum_{i\ge 1}y_{\beta,i}^{(2k-1)}m_{6i-1,j+i-1} & \text{if } \beta \text{ is odd},\\ \displaystyle\sum_{i\ge 1}y_{\beta,i}^{(2k-1)}m_{6i-5,j+i-1} & \text{if } \beta \text{ is even},\end{cases}$$

for all $\beta,j\ge 1$.

PROOF. From (1.3),

$$\sum_{n\ge 0}b_{5^{2k-1}}(n)q^n=f_{5^{2k-1}}\sum_{n\ge 0}p(n)q^n. \tag{3.3}$$

Extracting the terms involving $q^{5^{2k-1}n+\delta_{2k-1}}$ on both sides of (3.3) and dividing throughout by $q^{\delta_{2k-1}}$,

$$\sum_{n\ge 0}b_{5^{2k-1}}(5^{2k-1}n+\delta_{2k-1})q^{5^{2k-1}n}=f_{5^{2k-1}}\sum_{n\ge 0}p(5^{2k-1}n+\delta_{2k-1})q^{5^{2k-1}n}.$$

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If we replace $q^{5^{2k-1}}$ by q and use (1.2),

$$\sum_{n\ge 0} b_{5^{2k-1}}(5^{2k-1}n+\delta_{2k-1})q^n = \sum_{j\ge 1} x_{2k-1,j}\, q^{j-1}\frac{f_5^{6j-1}}{f_1^{6j-1}},$$

which is the case $\beta = 0$ of (3.1).

We now assume that (3.1) is true for some integer $\beta \ge 0$. Applying the operator H to both sides, by (2.3),

$$\begin{aligned}\sum_{n\ge 0} b_{5^{2k-1}}\Big(5^{2k+2\beta}n + \frac{4\cdot 5^{2k+2\beta}-5^{2k-1}+1}{24}\Big)q^{5n} &= \sum_{i\ge 1} y^{(2k-1)}_{2\beta+1,i}\, H\Big(q^{i-1}\frac{f_5^{6i-1}}{f_1^{6i-1}}\Big)\\ &= \sum_{i\ge 1} y^{(2k-1)}_{2\beta+1,i}\sum_{j=1}^{5i} m_{6i-1,i+j-1}\, q^{5j-5}\frac{f_{25}^{6j-5}}{f_5^{6j-5}}\\ &= \sum_{j\ge 1}\Big(\sum_{i\ge 1} y^{(2k-1)}_{2\beta+1,i}\, m_{6i-1,i+j-1}\Big)q^{5j-5}\frac{f_{25}^{6j-5}}{f_5^{6j-5}} = \sum_{j\ge 1} y^{(2k-1)}_{2\beta+2,j}\, q^{5j-5}\frac{f_{25}^{6j-5}}{f_5^{6j-5}}.\end{aligned}$$

That is,

$$\sum_{n\ge 0} b_{5^{2k-1}}\Big(5^{2k+2\beta}n + \frac{4\cdot 5^{2k+2\beta}-5^{2k-1}+1}{24}\Big)q^{n} = \sum_{i\ge 1} y^{(2k-1)}_{2\beta+2,i}\, q^{i-1}\frac{f_5^{6i-5}}{f_1^{6i-5}}.$$

Hence, if (3.1) is true for some integer $\beta \ge 0$, then (3.2) is true for β.

Suppose that (3.2) is true for some integer $\beta \ge 0$. Applying the operating H to both sides, by (2.2),

$$\begin{aligned}\sum_{n\ge 0} b_{5^{2k-1}}\Big(5^{2k+2\beta+1}n + \frac{4\cdot 5^{2k+2\beta+2}-5^{2k-1}+1}{24}\Big)q^{5n+5} &= \sum_{i\ge 1} y^{(2k-1)}_{2\beta+2,i}\sum_{j=1}^{5i-4} m_{6i-5,i+j-1}q^{5j}\frac{f_{25}^{6j-1}}{f_5^{6j-1}}\\ &= \sum_{j\ge 1}\Big(\sum_{i\ge 1} y^{(2k-1)}_{2\beta+2,i}\, m_{6i-5,i+j-1}\Big)q^{5j}\frac{f_{25}^{6j-1}}{f_5^{6j-1}} = \sum_{j\ge 1} y^{(2k-1)}_{2\beta+3,j}\, q^{5j}\frac{f_{25}^{6j-1}}{f_5^{6j-1}},\end{aligned}$$

which yields

$$\sum_{n\ge 0} b_{5^{2k-1}}\Big(5^{2k+2\beta+1}n + \frac{4\cdot 5^{2k+2\beta+2}-5^{2k-1}+1}{24}\Big)q^{n} = \sum_{j\ge 1} y^{(2k-1)}_{2\beta+3,j}\, q^{j-1}\frac{f_5^{6j-1}}{f_1^{6j-1}}.$$

This is (3.1) with β replaced by $\beta+1$. This completes the proof. □

THEOREM 3.2. *For each $\beta \ge 0$ and $k \ge 1$,*

$$\sum_{n\ge 0} b_{5^{2k}}\Big(5^{2k+\beta-1}(n+1) - \frac{5^{2k}-1}{24}\Big)q^{n} = \sum_{i\ge 1} y^{(2k)}_{\beta+1,i}\, q^{i-1}\frac{f_5^{6i}}{f_1^{6i}}, \tag{3.4}$$

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where the coefficient vectors are defined by

$$y_{1,j}^{(2k)} = x_{2k-1,j}$$

and

$$y_{\beta+2,j}^{(2k)} = \sum_{i\geq 1} y_{\beta+1,i}^{(2k)} m_{6i,i+j}.$$

PROOF. Equation (1.3) can be written as

$$\sum_{n\geq 0} b_{5^{2k}}(n)q^n = f_{5^{2k}} \sum_{n\geq 0} p(n)q^n.$$

Extracting the terms involving $q^{5^{2k-1}n+\delta_{2k-1}}$ on both sides of the above equation and dividing throughout by $q^{\delta_{2k-1}}$,

$$\sum_{n\geq 0} b_{5^{2k}}(5^{2k-1}n + \delta_{2k-1})q^{5^{2k-1}n} = f_{5^{2k}} \sum_{n\geq 0} p(5^{2k-1}n + \delta_{2k-1})q^{5^{2k-1}n}.$$

If we replace $q^{5^{2k-1}}$ by q and use (1.2),

$$\sum_{n\geq 0} b_{5^{2k}}(5^{2k-1}n + \delta_{2k-1})q^n = \sum_{j\geq 1} x_{2k-1,j}\, q^{j-1} \frac{f_5^{6j}}{f_1^{6j}},$$

which is the case $\beta = 0$ of (3.4).

We now assume that (3.4) is true for some integer $\beta \geq 0$. Applying the operator H to both sides, by (2.4),

$$\sum_{n\geq 0} b_{5^{2k}}\left(5^{2k+\beta}(n+1) - \frac{5^{2k}-1}{24}\right)q^{5n+5}$$

$$= \sum_{i\geq 1} y_{\beta+1,i}^{(2k)} \sum_{j=1}^{5i} m_{6i,i+j}\, q^{5j} \frac{f_{25}^{6j}}{f_5^{6j}} = \sum_{j\geq 1} \left(\sum_{i\geq 1} y_{\beta+1,i}^{(2k)} m_{6i,i+j} \right) q^{5j} \frac{f_{25}^{6j}}{f_5^{6j}} = \sum_{j\geq 1} y_{\beta+2,j}^{(2k)}\, q^{5j} \frac{f_{25}^{6j}}{f_5^{6j}},$$

which implies that

$$\sum_{n\geq 0} b_{5^{2k}}\left(5^{2k+\beta}(n+1) - \frac{5^{2k}-1}{24}\right)q^n = \sum_{j\geq 1} y_{\beta+2,j}^{(2k)} q^{j-1} \frac{f_5^{6j}}{f_1^{6j}}.$$

So we obtain (3.4) with β replaced by $\beta + 1$. □

4. Proof of the congruences

For a positive integer n, let $\pi(n)$ be the highest power of 5 that divides n and define $\pi(0) = +\infty$.

LEMMA 4.1 [1, Lemma 4.1]. *For each $i, j \geq 1$,*

$$\pi(m_{i,j}) \geq \left\lceil \frac{5j - i - 1}{2} \right\rceil.$$

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LEMMA 4.2 [1, Lemma 4.3]. *For each $k, j \geq 1$,*

$$\pi(x_{2k-1,j}) \geq 2k - 1 + \left\lfloor \frac{5j-5}{2} \right\rfloor. \tag{4.1}$$

LEMMA 4.3. *For each $j, k \geq 1$ and $\beta \geq 0$,*

$$\pi\left(y_{2\beta+1,j}^{(2k-1)}\right) \geq 2k + \beta - 1 + \left\lfloor \frac{5j-5}{2} \right\rfloor \tag{4.2}$$

and

$$\pi\left(y_{2\beta+2,j}^{(2k-1)}\right) \geq 2k + \beta - 1 + \delta_{1,j} + \left\lfloor \frac{5j-6}{2} \right\rfloor, \tag{4.3}$$

where

$$\delta_{1,j} = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j \neq 1. \end{cases}$$

PROOF. In view of Theorem 3.1,

$$y_{1,j}^{(2k-1)} = x_{2k-1,j}.$$

From (4.1), we can see that the inequality (4.2) holds for $\beta = 0$.

We now assume that (4.2) is true for some $\beta \geq 0$. Then,

$$\begin{aligned} \pi\left(y_{2\beta+2,j}^{(2k-1)}\right) &\geq \min_{i\geq 1}\left\{\pi\left(y_{2\beta+1,i}^{(2k-1)}\right) + \pi(m_{6i-1,i+j-1})\right\} \\ &\geq \min_{i\geq 1}\left\{2k + \beta - 1 + \left\lfloor \frac{5i-5}{2} \right\rfloor + \left\lfloor \frac{5j-i-5}{2} \right\rfloor\right\} \\ &\geq 2k + \beta - 1 + \left\lfloor \frac{5j-6}{2} \right\rfloor + \delta_{1,j}, \end{aligned}$$

which is (4.3). In the last inequality, we have used the fact that $m_{5,1} = 1$.

Now, suppose (4.3) holds for some $\beta \geq 0$. Then,

$$\begin{aligned} \pi\left(y_{2\beta+3,j}^{(2k-1)}\right) &\geq \min_{i\geq 1}\left\{\pi\left(y_{2\beta+2,i}^{(2k-1)}\right) + \pi(m_{6i-5,i+j-1})\right\} \\ &\geq \min_{i\geq 1}\left\{2k + \beta - 1 + \left\lfloor \frac{5j-i-1}{2} \right\rfloor\right\} \\ &\geq 2k + \beta - 1 + \left\lfloor \frac{5j-2}{2} \right\rfloor \geq 2k + \beta + \left\lfloor \frac{5j-5}{2} \right\rfloor, \end{aligned}$$

which is (4.2) with $\beta + 1$ for β. This completes the proof. □

LEMMA 4.4. *For each $j, k \geq 1$ and $\beta \geq 0$,*

$$\pi\left(y_{\beta+1,j}^{(2k)}\right) \geq 2k + \beta - 1 + \left\lfloor \frac{5j-5}{2} \right\rfloor. \tag{4.4}$$

PROOF. In view of Theorem 3.2,

$$y_{1,j}^{(2k)} = x_{2k-1,j}.$$

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From (4.1), we can see that (4.4) holds for $\beta = 0$. Now, assume that (4.4) is true for some $\beta \geq 0$. Then,

$$\begin{aligned}\pi\left(y^{(2k)}_{\beta+2,j}\right) &\geq \min_{i\geq 1}\left\{\pi\left(y^{(2k)}_{\beta+1,i}\right) + \pi(m_{6i,i+j})\right\}\\ &\geq \min_{i\geq 1}\left\{2k+\beta-1+\left[\frac{5i-5}{2}\right]+\left[\frac{5j-i-1}{2}\right]\right\}\\ &\geq 2k+\beta-1+\left[\frac{5j-2}{2}\right]\\ &= 2k+\beta+\left[\frac{5j-4}{2}\right] \geq 2k+\beta+\left[\frac{5j-5}{2}\right],\end{aligned}$$

which is (4.4) with $\beta+1$ for β. This completes the proof. □

PROOF OF THEOREM 1.1. Congruence (1.7) follows from (4.2) together with (3.1), and congruence (1.8) follows from (4.4) and (3.4).

In view of (4.2), we can express (3.1) as

$$\sum_{n\geq 0} b_{5^{2k-1}}\left(5^{2k+2\beta-1}n + \frac{4\cdot 5^{2k+2\beta} - 5^{2k-1}+1}{24}\right)q^n \equiv y^{(2k-1)}_{2\beta+1,1}\frac{f_5^5}{f_1^5} \pmod{5^{2k+\beta+1}}.$$

From the binomial theorem,

$$\frac{f_5^5}{f_1^5} \equiv f_5^4 \pmod 5.$$

Thus,

$$\sum_{n\geq 0} b_{5^{2k-1}}\left(5^{2k+2\beta-1}n + \frac{4\cdot 5^{2k+2\beta} - 5^{2k-1}+1}{24}\right)q^n \equiv y^{(2k-1)}_{2\beta+1,1} f_5^4 \pmod{5^{2k+\beta}}.$$

Equating the coefficients of q^{5n+l}, $l \in \{1,2,3,4\}$, we arrive at (1.9). □

Acknowledgment

The authors would like to thank the anonymous referee for careful reading and helpful comments.

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https://doi.org/10.1017/S0004972725000139 Published online by Cambridge University Press

B. HEMANTHKUMAR, Department of Mathematics, RV College of Engineering, RV Vidyanikethan Post, Mysore Road, Bengaluru 560 059, Karnataka, India
e-mail: hemanthkumarb.30@gmail.com

D. S. GIREESH, Department of Mathematics, BMS College of Engineering, P.O. Box No. 1908, Bull Temple Road, Bengaluru 560 019, Karnataka, India
e-mail: gireeshdap@gmail.com

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