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High frequency diffraction by a circular disc

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Abstract. The problem considered is that of transforming the integral equations obtained by Jones(2, 3), for diffraction by a circular disc so that they can be solved by iteration for any general incident field. A method of achieving this transformation is presented for the case of axi-symmetric incident fields. The generalization of the method to non-axi-symmetric problems is also examined.

1. *Introduction.* Jones(2, 3) has shown that the problem of the scattering of an acoustic plane wave normally incident on a circular disc can be reduced to the solution of a Fredholm integral equation which can be solved by an indirect iterative method for high frequencies. The Jones formal analysis is applicable to an arbitrary axi-symmetric incident field but the development of a suitable iterative scheme for this more general incident field is not obvious. The principal difficulty is the determination of a suitable first iterate as Jones's method of obtaining this iterate relies heavily on the simple form of the incident field.

Newby(4) has recently presented a different derivation of the Jones results in an attempt to obtain a systematic approach which could be generalized to other incident fields. Newby's analysis is somewhat elaborate but he succeeds in establishing a method which can be generalized to set up an iterative scheme for the case of an obliquely incident plane wave. Unfortunately, however, Newby's analysis relies on special properties associated with the incident field and its generalization to more general cases does not seem possible. The particular form of the equations derived by Newby (which differ only slightly from Jones), however, is such that that they suggest a simple method of developing an iterative scheme suitable for high frequencies and this paper is concerned with the development of this method.

The two separate cases of soft and rigid discs are considered and in both instances it is shown that an appropriate iterative scheme may be established, the first iterate in both cases being expressed comparatively simply in terms of the solution to the problem of the reflexion of the incident wave by an infinite plane. A detailed examination of particular problems is not presented but the general method described here has been applied by Miss R. M. Hamson to various electromagnetic and acoustic problems involving point sources and also to the problem of the radiation from a piston in a concentric finite rigid baffle.

Non-axially symmetric solutions of Helmholtz's equation can, by Fourier superposition, be expressed in terms of derivatives of axi-symmetric solutions of this equation and the application of this technique to scattering of an obliquely incident plane wave by a rigid disc is discussed briefly.

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2. *Diffraction by a soft disc.* The problem of the scattering of an axi-symmetric time harmonic (the time variation being proportional to $e^{i\omega t}$) acoustic wave by a soft circular disc of unit radius reduces to the solution of the integral equation

$$u_0(r) = \frac{1}{4\pi}\int_0^1\int_0^{2\pi} \frac{r_1 f(r_1)\, e^{-i\alpha(r_1^2+r^2-2rr_1\cos\theta)^{\frac{1}{2}}}}{(r_1^2+r^2-2rr_1\cos\theta)^{\frac{1}{2}}}\, d\theta\, dr_1 \quad (0<r<1). \tag{1}$$

α is the wave number, u_0 denotes the value of the incident velocity potential in the plane of the disc and $f(r)$ is the discontinuity in the velocity across the disc.

Jones (2) has shown that equation (1) may be transformed into

$$\int_0^1 wf(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v}+\frac{e^{-i\alpha(w-v)}}{w-v}\right\}dw = F^{(0)}(v), \tag{2}$$

where
$$F^{(0)}(v) = 2\frac{d}{dv}\int_0^v z^{\frac{1}{2}}\frac{\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{(v^2-z)^{\frac{1}{2}}}\frac{d}{dz}\int_0^z \frac{u_0(y^{\frac{1}{2}})\cosh\alpha(z-y)^{\frac{1}{2}}}{(z-y)^{\frac{1}{2}}}\,dy\,dz, \tag{3}$$

for a plane wave normally incident on the disc $u_0 = 1$.

It has been shown by Newby (4) that the non-axially symmetric problem of scattering by an obliquely incident wave can also be reduced to the solution of equations of the general form of equation (2) with $F^{(0)}$ defined by equation (3) and u_0 now being given by

$$u_0(r) = \frac{1}{(i\alpha\sin\beta)^n}\left\{J_0(\alpha r\sin\beta)+\sum_{m=0}^{n-1} b_m r^{2m}\right\}. \tag{4}$$

n is any positive integer and for $n = 0$ the series in equation (4) is zero. For $n = \beta = 0$ u_0 defined by equation (4) reduces to the form appropriate to a normally incident plane wave.

Jones has also shown that equation (2) may be further transformed to give

$$F(v) = F^{(0)}(v) - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{F(t)}{t+v}e^{-i\alpha t}\,dt, \tag{5}$$

where
$$F(v) = \int_0^1 \frac{wf(w)\, e^{-i\alpha(w-v)}}{w-v}\,dw. \tag{6}$$

Though equation (5) is an integral equation of the second kind it is not in a form which is directly suitable for solution by iteration for large α. For the particular case of a normally incident plane wave Jones has shown that it is possible to transform equation (5) into an equation which can be solved by iteration for large α. A similar transformation for the case $u_0 = r^2$ has been presented by Thomas (5) in investigating the problem of the torsional oscillations set up in an elastic half space by a disc attached to its boundary. Thomas's analysis is a slight extension of Jones's and both methods rely heavily on the extremely simple forms of u_0 which are involved and give no insight into the possibility of obtaining an appropriate transformation of equation (5) for arbitrary u_0. The problem of obtaining a systematic method of deriving the appropriate transformation for equation (6) for a normally incident plane wave has also been examined by Newby and he does set up a procedure for this special case which he succeeds in generalizing to cover the case of u_0 defined by equation (4). Newby's analysis is still somewhat lengthy and elaborate, particularly for the more general

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form of u_0, and still relies heavily on a knowledge of the particular form of u_0. It is not obvious how the analysis could be applied to general forms of u_0 but the final form of the integral equation obtained by Newby is extremely useful in that, from this equation, it is possible to develop a simple method of transforming equation (5) into one which can be solved fairly easily by iteration for large α.

Newby shows that by writing F as $\mathscr{F} + Q(v)$, where Q is an appropriately chosen known function, it is possible to re-write equation (5) as

$$\mathscr{F} = g(v) - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v}\mathscr{F}(t)\,dt, \tag{7}$$

where

$$g(v) = \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v} i\int_1^{1-i\infty} \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v}\gamma(t)\,dt. \tag{8}$$

The function γ is defined by Newby in terms of a contour integral; for $u_0 = 1$ it reduces to $2\pi\alpha t$ and for u_0 defined by equation (6),

$$\gamma = \frac{2\pi}{(i\alpha\sin\beta)^n} \times \left\{\alpha t\cos\beta J_0(\alpha t\sin\beta) - \sum_{m=0}^{n-1}(-)^m (m!)^2 b_m \alpha^{-2m}\sum_{s=0}^{m}\frac{(-)^s (t\alpha)^{2s+1}(2m-2s)!}{(s!)^2\{(m-s)!\}^2(2m-2s-1)}\right\}. \tag{9}$$

The particular form of γ derived by Newby is somewhat more complicated than that given in equation (9) and the demonstration of the equivalence of Newby's result and equation (9) is sketched in the Appendix. It should be noted that for the diffraction problem the precise forms of the coefficients of t^{2r} are not required as the b_m are themselves arbitrary constants but for completeness it seems worth including the explicit forms of the coefficients.

The analysis which led to equations (7) and (8) is in no way relevant to our present purpose as it is the final form of the equations which provide the clue for developing an appropriate transformation of equation (5). For the case $u_0 = 1$ equations (7) and (8) were first derived by Jones(2); the form obtained by Jones differs only extremely slightly from Newby's but this slight difference seems to have been sufficient to obscure the general nature of the transformation from equation (5) to equation (7).

In order to present a general method of calculating γ it is necessary to examine the first step in the iteration scheme proposed by Jones for solving equation (7). If $\mathscr{F}_2$ is defined by

$$\mathscr{F} = \int_0^\infty \frac{\Psi_1(t)\,e^{-i\alpha(t-v)}}{t-v}dt + \mathscr{F}_2 \tag{10}$$

then Jones has shown that

$$\mathscr{F}_2 = \left(\frac{v}{1+v}\right)^{\frac{1}{2}}\int_1^\infty \left(\frac{w-1}{w}\right)^{\frac{1}{2}}\frac{e^{-i\alpha(w+v)}}{w+v}\Psi_1(w)\,dw - \frac{1}{\pi}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v}\int_0^1\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{e^{-i\alpha t}}{t+v}\mathscr{F}_2(t)\,dt, \tag{11}$$

provided that

$$\int_0^\infty \Psi_1(w)\left[\frac{e^{-i\alpha(w-v)}}{w-v} + \frac{e^{-i\alpha(w+v)}}{w+v}\right]dw = g(v). \tag{12}$$

The contour in equation (8) may be deformed to the real axis from 1 to ∞ if γ is algebraic and $o(1)$ as $t \to \infty$ and even if γ has a polynomial behaviour at infinity g can

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still be defined by equation (8) with the path of integration from $t = 1$ to $t = \infty$ provided that the integral is interpreted in the sense of generalized functions. Whenever it is necessary to do so integrals will be interpreted in this latter sense.

It now follows from inspection of equations (10) to (12) that equation (5) can be transformed directly to the form of equation (7) by setting

$$F = \mathscr{F} + \frac{i}{\pi}\int_0^\infty \frac{\gamma(t)\, e^{-i\alpha(t-v)}}{t-v}\, dt$$

where

$$\int_0^\infty \gamma(w)\left\{\frac{e^{-i\alpha(w-v)}}{w-v} + \frac{e^{-i\alpha(w+v)}}{w+v}\right\} dw = -\pi i F^{(0)}(v). \tag{13}$$

Equation (13) is one of a class of integral equations for which Jones(1) has obtained a general method of solution; the solution obtained by Jones is in terms of infinite integrals involving Bessel functions and as $F^{(0)}$ is itself somewhat complicated this approach does not provide a simple method of determining γ. The general forms of equations (2) and (13) are however very similar and it will now be shown that this similarity enables a simple method of calculating γ to be presented.

The problem of obtaining a solution u of

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + \alpha^2 u = 0, \tag{14}$$

in $z \geqslant 0$, satisfying the appropriate radiation condition and such that $u = u_0$ on $z = 0$, can by methods almost identical to those used in obtaining equation (1), be reduced to solving

$$u_0(r) = -\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} r_1 h(r_1) \frac{\exp -i\alpha(r_1^2 + r^2 - 2rr_1\cos\theta)^{\frac{1}{2}}\, d\theta\, dr_1}{(r_1^2 + r^2 - 2rr_1\cos\theta)^{\frac{1}{2}}}, \tag{15}$$

where

$$h(r_1) = (\partial u/\partial z)_{z=0}.$$

It now follows immediately from equations (1), (2), (13) and (15) that

$$\gamma(t) = 2\pi i t h(t). \tag{16}$$

The boundary-value problem for u is trivial if u_0 is the value on $z = 0$ of a solution $u_0(r, z)$ of equation (14), produced by sources in $z > 0$ and satisfying the radiation condition at infinity. In this case

$$u = -u_0(r, -z). \tag{17}$$

The problem when $u_0(r) = 1$ is a particular case of this general class with $u_0 = e^{-i\alpha z}$ and thus $\gamma(t) = 2\pi\alpha t$ in agreement with Newby's result. It is not possible to use this simple approach to treat u_0 defined by the series in equation (4) but for u_0 defined by the first term of this equation the appropriate solution u of equation (14) is

$$J_0(\alpha r \sin\beta)\, e^{-i\alpha z\cos\beta}/(i\alpha\sin\beta)^n$$

and the contribution of this term to γ is

$$\frac{2\pi\alpha t\cos\beta J_0(\alpha t\sin\beta)}{(i\alpha\sin\beta)^n},$$

this is in agreement with the first term of equation (9).

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The boundary-value problem for equation (14) with $u = r^{2m}$ on $z = 0$ is of a novel type but can be solved by considering

$$u = \frac{A}{2\pi i}\int_C \frac{e^{-(\lambda^2-\alpha^2)^{\frac{1}{2}}z}}{\lambda^{2m+1}} J_0(\lambda r)\,d\lambda, \tag{18}$$

where C is a circle centre $\lambda = 0$ and of radius less than α. Clearly all the conditions imposed on u will be satisfied if $A = (-)^m 2^{2m}(m!)^2$ and hence by evaluating $\partial u/\partial z$ at $z = 0$ it is deduced that the contribution of the second term in equation (4) to γ is

$$-\frac{2\pi\alpha}{(i\alpha\sin\beta)^n}\sum_{m=0}^{n-1}(-)^m(m!)^2 b_m \sum_{s=0}^{m}\frac{(-)^s t^{2s+1}\alpha^{2(s-m)}(2m-2s)!}{(s!)^2\{(m-s)!\}^2(2m-2s-1)}$$

in agreement with the second term of equation (9).

In view of the simple form of equation (17) the problem of obtaining the appropriate integral equation for large α for axi-symmetric problems such as a dipole on the axis of a circular disc becomes comparatively easy and the general method developed here has been applied by Miss R. M. Hamson to problems of this type.

3. *Diffraction by a rigid disc.* The problem of scattering of an axi-symmetric acoustic wave by a rigid circular disc of unit radius reduces to the solution of the differential integral equation

$$-\left(\frac{\partial u_0}{\partial z}\right)_{z=0} = \frac{1}{4\pi}\left(\frac{\partial}{\partial z}\right)_{z=0}\int_0^1\int_0^{2\pi} r_1 g(r_1)\left(\frac{\partial}{\partial z_1}\right)_{z_1=0} \times \frac{e^{-i\alpha[r^2+r_1^2-2r_1 r\cos\theta+(z-z_1)^2]^{\frac{1}{2}}}}{[r^2+r_1^2-2r_1 r\cos\theta+(z-z_1)^2]^{\frac{1}{2}}}\,d\theta\,dr_1 \quad (0<r<1). \tag{19}$$

The disc is taken to be in the plane $z = 0$, u_0 denotes the incident velocity potential and g the increase in the velocity potential in crossing the disc in the positive z direction.

Jones(3) has shown that g satisfies the equation

$$\int_0^1 g(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} - \frac{e^{-i\alpha(w-v)}}{w-v}\right\}dw = G^{(0)}(v), \tag{20}$$

where
$$G^{(0)}(v) = \int_0^{v^{\frac{1}{2}}} \frac{\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{z^{\frac{1}{2}}(v^2-z)^{\frac{1}{2}}}\int_0^z \frac{\cosh\alpha(z-t)^{\frac{1}{2}}}{(z-t)^{\frac{1}{2}}} g^{(0)}(t^{\frac{1}{2}})\,dt\,dz$$

and
$$g^{(0)}(r) = \left\{\frac{\partial}{\partial z}u_0(r,z)\right\}_{z=0}.$$

Jones has also shown that equation (20) may be transformed to give

$$G(v) = -G^{(0)}(v) - \left(\frac{1+v}{v}\right)^{\frac{1}{2}}\frac{e^{-i\alpha v}}{\pi}\int_0^1\left(\frac{t}{1-t}\right)^{\frac{1}{2}} G(t)\frac{e^{-i\alpha t}}{t+v}\,dt \tag{21}$$

where
$$G(v) = \int_0^1 \frac{e^{-i\alpha(w-v)}}{w-v} g(w)\,dw.$$

In order to determine g from G it is necessary to use the edge condition $g(1) = 1$.

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Equation (21) is not in a form which is suitable for solution for large values of α and we thus write

$$G = \int_0^\infty \frac{L_0(t)\, e^{-i\alpha(t-v)}}{t-v}\, dt + G_1, \tag{22}$$

and it may be deduced from the analysis on p. 196 of Jones (3) that

$$G_1(v) = -\left(\frac{1+v}{v}\right)^{\frac{1}{2}} \int_1^\infty \left(\frac{t}{t-1}\right)^{\frac{1}{2}} \frac{L_0\, e^{-i\alpha(t+v)}}{t+v}\, dt - \frac{e^{-i\alpha v}}{\pi} \left(\frac{1+v}{v}\right)^{\frac{1}{2}} \int_0^1 \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \frac{e^{-i\alpha w}\, G_1(w)}{w+v}\, dw, \tag{23}$$

provided that

$$\int_0^\infty L_0(t) \left\{\frac{e^{-i\alpha(t-v)}}{t-v} - \frac{e^{-i\alpha(t+v)}}{t+v}\right\} dt = -G^{(0)}(v). \tag{24}$$

Equation (23) is in the form suitable for solution by iteration for large α and it now only remains to calculate the function L_0. L_0 can be obtained almost immediately by inspection of equations (20), (23) and (24) and using arguments similar to those employed in calculating γ. If we consider the problem of determining a solution $u(r, z)$ of equation (14) such that $\partial u/\partial z = \partial u_0/\partial z$ then it follows immediately that $L_0(r) = 2u(r, 0)$ and furthermore, if the only sources of u_0 are in $z \geqslant 0$, $u = -u_0(r, -z)$ and hence

$$L_0(r) = -2u_0(r, 0).$$

For the particular case considered by Jones of a plane wave normally incident on the disc $L_0 = -2$ and equation (22) with $L_0 = -2$ is in agreement with equation (9) of Jones (3).

The general approach described above for axi-symmetric problems may also be used to solve non-axi-symmetric problems. Boundary-value problems for the latter problems can, by Fourier superposition, be reduced to problems for

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{n^2}{r^2} v + \frac{\partial^2 v}{\partial z^2} + \alpha^2 v = 0, \tag{25}$$

with $\partial u/\partial z = v_n(r)$ on $z = 0$ and u satisfying an appropriate radiation condition at infinity. Direct differentiation shows that a solution of equation (25) is given by

$$v = r^n \left(\frac{1}{r}\frac{d}{dr}\right)^n u, \tag{26}$$

where u is a solution of equation (14). Thus the boundary-value problem for equation (25) reduces to one for equation (14) with

$$\frac{\partial u}{\partial z} = 2^{1-n} \int_0^r v_n(y)\, y^{1-n} (r^2 - y^2)^{n-1}\, dy + \sum_{m=0}^{n-1} b_m r^{2m}, \tag{27}$$

where the b_m's are arbitrary constants and have to be found by requiring that $v = 0$ on $z = 0$ when $r = 1$. An exactly similar situation occurs in the problem of scattering by a soft disc examined by Newby and the above method of relating equations (25) and (14) can also be used to obtain a more direct formulation of Newby's problem.

For the case when u_0 is a plane wave incident at an angle β to the z axis

$$v_n = (-i)^{n+1} \cos\beta J_n(\alpha r \sin\beta)$$

and equation (27) then becomes

$$\frac{\partial u}{\partial z} = \frac{(-i)^{n+1} \cos\beta}{(\lambda \sin\beta)^n} J_0(\alpha r \sin\beta) + \sum_{m=0}^{n-1} b_m r^{2m}. \tag{28}$$

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For $\partial u/\partial z$ defined by the right-hand side of equation (28),

$$L_0(t) = -2\left\{\frac{J_0(\alpha t \sin\beta)}{(i\lambda\sin\beta)^n} + \sum_{m=0}^{n-1} b_m K_m(t)\right\}$$

where
$$K_m(t) = -\frac{1}{2\pi i}\int_C \frac{(-)^m 2^{2m}(m!)^2 J_0(\lambda t)\,d\lambda}{\lambda^{2m+1}(\lambda^2-\alpha^2)^{\frac{1}{2}}}, \tag{29}$$

where C is the contour of equation (18). The integral in equation (29) may be evaluated to give

$$K_m(t) = (-)^m (m!)^2 i \sum_{s=0}^{m} \frac{(-)^s (2m-2s)!\,\alpha^{2s-2m-1} t^{2s}}{(s!)^2\{(m-s)!\}^2}. \tag{30}$$

APPENDIX

If the contribution of the second term in equation (4) is written as

$$\left(\sum_{m=0}^{n-1} b_m \gamma_m\right)(i\alpha\sin\beta)^n$$

then Newby has shown that

$$\gamma_m = \frac{(-)^{m+1} m!\,2\pi i t}{\alpha^m} \sum_{r=0}^{m-1}{}' \frac{(2m-2-r)!}{(m-1-r)!\,r!(2\alpha)^{m-r-1}(r+1)!}\left(\frac{i}{2}\right)^{1+r}\left\{\frac{d^{r+2}(y^2-t^2)^{r+1}e^{-i\alpha y}}{dy^{r+2}}\right\}_{y=0}, \tag{A 1}$$

where the dash implies that, for $m = 0$, the summation reduces to the single term given by $r = 0$.

The differentiation in (A 1) can be carried out by using Leibnitz' product rule and employing the binomial expansion to evaluate the s'th derivative of

$$(y^2-t^2)^{r+1} \quad \text{at} \quad y = 0.$$

The resulting double series can be re-arranged as a power series in t^2 and the final result is

$$\gamma_m = \frac{(-)^m 2\pi m!}{\alpha^{2m}} \sum_{k=0}^{m} \frac{(\alpha t)^{2k+1}(-)^k}{k!} \sum_{r=0}^{k+1} \frac{(-)^r (2m-1-k-r)!\,(k+r)(k+r+1)}{(m-k-r)!\,(k+1-r)!\,r!}. \tag{A 2}$$

If the inner-series in equation (A 2) is denoted by S then we have that

$$\begin{aligned} S &= (m-1)!\ \text{Coefficient of } x^{m-1} \text{ in } \sum_{r=0}^{k+1} \frac{(-)^r (1+x)^{2m-1-k-r}(k+r)(k+r+1)}{r!(k+1-r)!} \\ &= (m-1)!\ \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{2m+1}\frac{d^2}{dx^2}\sum_{r=0}^{k+1}\frac{(-)^r(1+x)^{-k-r}}{r!(k+1-r)!} \\ &= \frac{(m-1)!}{(k+1)!}\ \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{2m+1}\frac{d^2}{dx^2}\left[\frac{x^{k+1}}{(1+x)^{2k+1}}\right] \\ &= \frac{-m!(2m-2k)!}{k![(m-k)!]^2(2m-2k-1)}. \end{aligned}$$

Hence
$$\gamma_m = \frac{(-)^{m+1}2\pi(m!)^2}{\alpha^{2m}} \sum_{k=0}^{m} \frac{(-)^k(\alpha t)^{2k+1}}{(k!)^2} \frac{(2m-2k)!}{[(m-k)!]^2(2m-2k-1)}$$

and thus the equivalence of equations (9) and (A 1) is established.

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