Philosophy of Science (2026), **00**, 1–16
doi:10.1017/psa.2026.10245

PSA PHILOSOPHY OF SCIENCE ASSOCIATION
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SYMPOSIA PAPER

A Deflationary Account of Quantum Theory and Its Implications for the Complex Numbers

Jacob A. Barandes

Departments of Philosophy and Physics, Harvard University, USA
Email: jacob_barandes@harvard.edu

(Received 15 April 2025 UTC; revised 12 June 2026 UTC; accepted 12 June 2026 UTC)

Abstract

Why does quantum theory need the complex numbers? With a view toward answering this question, I argue that the usual Hilbert-space formalism is a special case of the general method of Markovian embeddings. I then describe the "indivisible interpretation" of quantum theory, according to which a quantum system can be regarded as an "indivisible" stochastic process unfolding in an old-fashioned configuration space, with wave functions and other exotic Hilbert-space ingredients demoted from having an ontological status. The complex numbers end up being necessary to ensure that the Hilbert-space formalism is indeed a Markovian embedding.

1. Introduction

One of the most mysterious features of quantum theory is its apparent reliance on the complex numbers, which are included explicitly in the textbook theory's Dirac–von Neumann axioms (Dirac 1930; von Neumann 1932). There are arguments in the research literature that aim to establish the necessity of the complex numbers in quantum theory through various measurement scenarios (Renou et al. 2021). In this paper, I take a different and more foundational approach.

2. The complex numbers

An algebra is a vector space, with its usual rules for addition and multiplication by scalars, as well as an additive identity 0, together with a multiplication rule that takes two elements of the algebra and returns another element of the algebra. A normed division algebra has a norm $|q| \geq 0$ and includes for every nonzero element $q \neq 0$ a multiplicative reciprocal $1/q$, so that division is defined for all nonzero divisors.

A key example is the set of real numbers $\mathbb{R}$, which form a one-dimensional normed division algebra, with norm $|x| \equiv \sqrt{x^2}$.

As follows from the Cayley–Dickson construction (Dickson 1919) together with the Hurwitz theorem (Hurwitz 1922), there exists a unique two-dimensional normed division algebra, called the complex numbers $\mathbb{C}$. Elements of $\mathbb{C}$ can be identified with

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ordered pairs (a, b) of real numbers, a and b, together with the following arithmetic rules:

- Addition: $(a, b) + (c, d) \equiv (a + c, b + d)$.
- Multiplication by scalars: $c(a, b) \equiv (ca, cb)$ for c any real number.
- Multiplication: $(a, b)(c, d) \equiv (ac - bd, ad + bc)$.

Notice that addition and multiplication by scalars are carried out entry-wise, whereas multiplication of complex numbers with each other takes a much less obvious form. To see why entry-wise multiplication fails to give a division algebra, observe that $(a, 0)$ is not the additive identity $(0, 0)$, and yet, under entry-wise multiplication, it lacks a multiplicative reciprocal, because then $(1, 1)$ is the multiplicative identity, and there are no ordered pairs (c, d) such that $(a, 0)(c, d) = (1, 1)$ under entry-wise multiplication.

The complex numbers also come with several other important operations:

- Complex conjugation: $(a, b)^* \equiv \overline{(a, b)} \equiv (a, -b)$.
- Norm or modulus: $|(a, b)| \equiv \sqrt{(a, b)^*(a, b)} = \sqrt{a^2 + b^2}$.
- Real part: $Re\,(a, b) \equiv a$.
- Imaginary part: $Im\,(a, b) \equiv b$.

Any set of mathematical symbols that are in a one-to-one correspondence with these ordered pairs and exhibit the same properties give a representation of the complex numbers.

For example, denote:

$$\text{“}1\text{”} \equiv (1, 0), \quad \text{“}i\text{”} \equiv (0, 1). \tag{1}$$

Then:

$$z \equiv x + iy \equiv 1x + iy = (1, 0)x + (0, 1)y = \big(x, y\big). \tag{2}$$

One has

$$i^2 = -1, \tag{3}$$

so one can formally write down the mystical-looking formula

$$\text{“}i = \sqrt{-1}.\text{”} \tag{4}$$

However, there is nothing particularly mysterious going on here. Under the hood, there are just ordered pairs of real numbers with a special multiplication rule.

From the Taylor series for the exponential function e^x, one sees that

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{6}(i\theta)^3 + \cdots \\ &= 1 + i\theta - \frac{1}{2}\theta^2 - i\frac{1}{6}\theta^3 \pm \cdots \\ &= \left(1 - \frac{1}{2}\theta^2 \pm \cdots\right) + i\left(\theta - \frac{1}{6}\theta^3 \pm \cdots\right). \end{aligned}$$

Identifying the two quantities in parentheses with the respective Taylor series for the cosine and sine functions, one ends up with the Euler formula:

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$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{5}$$

As a special case, for $\theta = \pi$, this result leads to the famous Euler identity,

$$e^{i\pi} + 1 = 0, \tag{6}$$

which unites the fundamental constants 0, 1, e, i, and π together with the basic mathematical operations of addition $+$ and equality $=$.

From trigonometry, for a right triangle with opening angle θ, adjacent side of length x, opposite side of length y, and hypotenuse of length r, one has the basic relationships

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ r &= \sqrt{x^2 + y^2}. \end{aligned} \tag{7}$$

These formulas give the polar form of a complex number z:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{8}$$

Here, $r = |z|$ is the norm or modulus of the complex number, and $e^{i\theta}$ is called its phase factor, or phase for short. Phases have unit modulus:

$$|e^{i\theta}| = 1. \tag{9}$$

Taking the modulus of a general complex number $z = re^{i\theta}$ erases its phase information $e^{i\theta}$:

$$|z| = |re^{i\theta}| = r. \tag{10}$$

There exist other representations of the complex numbers $\mathbb{C}$. One interesting example is the "real" 2×2 representation (see, e.g., Myrheim 1999), in which one introduces the following pair of 2×2 matrices, both of which have only real-valued entries:

$$\mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

These matrices satisfy

$$\mathbf{I}^2 = -\mathbf{1}. \tag{12}$$

It follows that any linear combination of the two matrices $\mathbf{1}$ and $\mathbf{I}$ with real-valued coefficients x and y defines a 2×2 matrix of the specific form

$$\mathbf{Z} \equiv \mathbf{1}x + \mathbf{I}y = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}. \tag{13}$$

One can check that matrices of this form have the same arithmetic behavior as complex numbers, so they indeed provide an alternative representation of $\mathbb{C}$. Moreover, taking the transpose of $\mathbf{Z}$ implements complex conjugation,

$$\mathbf{Z}^{\mathrm{T}} = \mathbf{1}x - \mathbf{I}y, \tag{14}$$

taking the trace gives twice the real part,

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$$\mathrm{tr}(\mathbf{Z}) = 2x, \tag{15}$$

and taking the determinant yields the modulus-squared,

$$\det(\mathbf{Z}) = x^2 + y^2. \tag{16}$$

Additionally, recall that a 2×2 matrix that implements rotations by an angle θ is given by

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{17}$$

and observe that one can decompose the right-hand side as

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\theta + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin\theta.$$

That is,

$$\mathrm{Rot}(\theta) = \mathbf{1}\cos\theta + \mathbf{I}\sin\theta = e^{\mathbf{I}\theta}, \tag{18}$$

where the exponential of a matrix is defined by its Taylor expansion. This result is just the "real" 2×2 representation of the Euler formula (5).

The 2×2 matrix I here is called a linear complex structure. Notice that it does not involve any appearances of a "literal" i, but only real numbers. However, the complex numbers, as an algebraic structure, are still present. The complex numbers are fundamentally an algebraic structure, so any algebraic structure isomorphic to $\mathbb{C}$ is really just $\mathbb{C}$ itself.[1]

Returning to a more abstract representation of complex numbers as expressions of the form $z = x + iy$, it will turn out to be useful to introduce an operator K that implements complex conjugation:

$$Kz \equiv \bar{z}K. \tag{19}$$

By construction, K anticommutes with i:

$$Ki = -iK. \tag{20}$$

It follows that the symbols i, K, and iK generate a Clifford algebra known as the pseudo-quaternions (Stueckelberg 1960):

$$-i^2 = K^2 = (iK)^2 = (i)(K)(iK) = 1. \tag{21}$$

In the "real" 2×2 matrix representation (13), K is represented by a 2×2 matrix with real-valued entries that happens to coincide with the first Pauli sigma matrix σ_x:

[1] By contrast, Renou et al. (2021) argue that replacing i with the 2×2 matrix $\mathbf{I}$ qualifies as removing the complex numbers from quantum theory and results in a truly real theory. This argument is not correct—their theory still involves the complex numbers because their theory still involves factors of $\mathbf{I}$, with all its usual algebraic properties. Moreover, when replacing factors of i with $\mathbf{I}$, it follows from straightforward mathematical identities that one must give up certain assumptions about tensor-factorizability in order to obtain an equivalent formulation of standard quantum theory. Renou et al. (2021) decide not to give up those assumptions, and, rather unsurprisingly, they ultimately find that their still-complex theory disagrees with some of the predictions of standard quantum theory. It is not clear how significant it is to demonstrate the inequivalence between standard quantum theory and an arbitrarily defined new theory that still involves the complex numbers.

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$$\mathbf{K} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x. \tag{22}$$

3. Markovian embeddings

A Markov process is a dynamical system for which the laws that pick out future configurations, either deterministically or probabilistically, depend only on the system's present configuration and not on prior configurations in the past. It will be fruitful to examine the ways in which a non-Markovian system can be made to look Markovian by a suitable change of representation.

As a simple first example, consider a *discrete* system, whose kinematics consists of configurations

$$x = 1, 2, 3, \ldots, N \tag{23}$$

that make up a configuration space $\mathcal{C}$. At the level of dynamics, suppose that the system has a second-order deterministic law

$$x(t+1) = F(x(t), x(t-1)), \tag{24}$$

where F is some given function with two arguments. This dynamical law is said to be non-Markovian of second order because picking out the system's configuration at a later time $t+1$ requires specifying the system's configuration not merely at the present time t, but also at the previous time $t-1$. This model as a whole can be understood as a discrete version of a simple system of one degree of freedom in Newtonian mechanics.

One can then always make the following nomology–ontology trade-off: Simplify the model's dynamical law, or nomology, to be first order, or Markovian, at the cost of complicating the model's configuration space, meaning its ontology. According to the new kinematics, the "states" are now ordered pairs (x, y) making up a "state space" $\mathcal{C} \times \mathcal{C}$, and the new dynamics is given by the *pair* of *first-order* equations

$$\begin{aligned} x(t+1) &= F\big(x(t), y(t)\big), \\ y(t+1) &= x(t). \end{aligned} \tag{25}$$

One therefore ends up with a Markovian embedding of the originally second-order, non-Markovian model.

This Markovian embedding also features an emergent set of transformations that are allowed to mix up the variables x and y for convenience as long as the transformations do not entail losing any information. For example, one can define $X \equiv x + y$ and $Y \equiv x - y$, which would then satisfy the equations

$$\begin{aligned} X(t+1) &= F\left(\frac{X(t)+Y(t)}{2}, \frac{X(t)-Y(t)}{2}\right) + \frac{X(t)+Y(t)}{2}, \\ Y(t+1) &= F\left(\frac{X(t)+Y(t)}{2}, \frac{X(t)-Y(t)}{2}\right) - \frac{X(t)+Y(t)}{2}. \end{aligned} \tag{26}$$

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After solving these alternative first-order equations, one could then immediately write down the solutions to the original equations as $x(t) = (X(t) + Y(t))/2$ and $y(t) = (X(t) - Y(t))/2$.

As a second example, consider a *continuous* system whose kinematics consists of configurations x making up a configuration space $\mathcal{C}$, with dynamics given by a second-order deterministic law

$$\ddot{x}(t) = F(x(t), \dot{x}(t)), \tag{27}$$

using Newton's fluxion notation in which dots denote time derivatives. Again, one can make a nomology–ontology trade-off, with the new kinematics consisting of "states" (x, y) making up a "state space" $\mathcal{C} \times \mathcal{C}$ and the new dynamics consisting of a Markovian embedding given by the pair of first-order equations

$$\begin{aligned} \dot{x}(t) &= y(t), \\ \dot{y}(t) &= F\big(x(t), y(t)\big). \end{aligned} \tag{28}$$

As before, this Markovian embedding features an emergent set of transformations that can freely mix up x and y as long as they do not lose track of any information. This example is a template for the Hamiltonian formulation of Newtonian mechanics, with x playing the role of a canonical coordinate, y playing the role of a canonical momentum, and the set of transformations mixing up x and y containing the set of canonical transformations.

For t_0 any choice of origin for the time coordinate, one can define a time-reversal transformation by the replacement

$$t_0 + t \mapsto t_0 - t, \tag{29}$$

so that translations forward in time from t_0 become translations backward in time from t_0, and vice versa. For simplicity, the origin for the time coordinate will be taken to be $t_0 = 0$. Then it follows from the chain rule that for the original second-order formulation of the continuous model, time-reversal transformations act on the degree of freedom $x(t)$ and its time derivatives according to the rules

$$\begin{aligned} x(t) &\mapsto +x(-t), \\ \dot{x}(t) &\mapsto -\dot{x}(-t), \\ \ddot{x}(t) &\mapsto +\ddot{x}(-t). \end{aligned} \tag{30}$$

The original second-order law (27) is therefore time-reversal invariant precisely if

$$F(x, -\dot{x}) = F(x, \dot{x}). \tag{31}$$

For the equivalent Markovian embedding, a time-reversal transformation must instead be required to take the form

$$\begin{aligned} x(t) &\mapsto +x(-t), \\ \dot{x}(t) &\mapsto -\dot{x}(-t), \\ y(t) &\mapsto -y(-t), \\ \dot{y}(t) &\mapsto +\dot{y}(-t), \end{aligned} \tag{32}$$

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where $y(t)$ and $\dot{y}(t)$ transform with opposite signs as compared with $x(t)$ and $\dot{x}(t)$. The first-order Markovian laws then transform as

$$\begin{aligned} \dot{x}(t) = y(t) &\mapsto -\dot{x}(t) = -y(t), \\ \dot{y}(t) = F\big(x(t), y(t)\big) &\mapsto \dot{y}(t) = F\big(x(t), -y(t)\big). \end{aligned} \tag{33}$$

The dynamical laws are time-reversal-invariant if

$$F\big(x, -y\big) = F\big(x, y\big). \tag{34}$$

Many Markovian embeddings are naturally expressible using complex numbers. For the present example, one can define

$$z(t) \equiv x(t) + iy(t) \tag{35}$$

and

$$\mathcal{F}(z(t), z^*(t)) \equiv y(t) + iF\big(x(t), y(t)\big). \tag{36}$$

Then one ends up with a *single* first-order Markovian law:

$$\dot{z}(t) = \mathcal{F}(z(t), z^*(t)). \tag{37}$$

So one has made the dynamical law look even simpler, at the cost of making the ontology murkier and introducing the purportedly exotic complex numbers into the story.

This complex form of the dynamics makes it particularly easy to implement time-reversal transformations:

$$\begin{aligned} z(t) &\mapsto Kz(-t), \\ \dot{z}(t) &\mapsto -K\dot{z}(-t). \end{aligned} \tag{38}$$

Notice, in particular, how the transformation law for $z(t)$ entails *both* a change in its argument from t to $-t$ *together* with an application of the complex-conjugation operator K, as originally introduced in (19). Meanwhile, the first-order Markovian law (37) transforms according to

$$-K\dot{z}(-t) = \mathcal{F}(z^*(-t), z(-t)). \tag{39}$$

The dynamical law is therefore time-reversal-invariant if

$$-K\mathcal{F}(z^*, z) = \mathcal{F}(z, z^*). \tag{40}$$

This trick of embedding a non-Markovian model into a Markovian model with a more complicated state space generalizes to systems with

- arbitrarily many degrees of freedom,
- non-Markovian dynamics of arbitrarily high order, and
- stochastic dynamical laws.

The embedded-Markovian formulation may have a murky ontology; a large class of strange-looking transformations of its state-space variables; and exotic mathematical structures, such as the complex numbers. If one were handed such an embedded-Markovian formulation at the outset, one could easily imagine endless debates over its correct physical interpretation.

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4. The Strocchi–Heslot formulation

A closed quantum system with state vector $|\Psi(t)\rangle$ evolving according to the *Schrödinger equation*

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = H|\Psi(t)\rangle \tag{41}$$

(conventionally written with a partial time derivative $\partial/\partial t$) is a Markov process with a murky ontology, features a large class of strange (unitary change-of-basis) transformations, and involves the complex numbers. Reversing the logic of the previous section, one might then naturally ask whether there is some non-Markovian system with a more transparent ontology for which this Hilbert-space formulation is merely its Markovian embedding. Remarkably, the answer turns out to be in the affirmative. One can find hints in this direction from works like that of Glick and Adami (2020) that demonstrate clear instances of non-Markovian behavior for reasonably straightforward, *closed* quantum systems. (For *open* systems, non-Markovianity has long been known to be ubiquitous, in both the classical and quantum cases, as a result of prosaic feedback effects from the environment.)

Moreover, there exists an intriguing but little-known reformulation of quantum theory as a *classical* Hamiltonian system of coupled harmonic oscillators. This reformulation comes from two beautiful papers, the first by Franco Strocchi (1966) and the second by André Heslot (1985).[2]

To start, consider a quantum system with an N-dimensional Hilbert space $\mathcal{H}$, a state vector $|\Psi(t)\rangle$, and a Hamiltonian $H = H^\dagger$. Again, the Schrödinger equation reads

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = H|\Psi(t)\rangle.$$

Next, pick an arbitrary orthonormal basis—a choice to be revisited shortly—so that the state vector $|\Psi(t)\rangle$ is represented by the column matrix

$$\begin{pmatrix} \Psi_1(t) \\ \vdots \\ \Psi_N(t) \end{pmatrix}, \tag{42}$$

and the Hamiltonian H is represented by the square matrix

$$\begin{pmatrix} H_{11} & H_{12} & & \\ H_{21} & \ddots & & \\ & & & H_{NN} \end{pmatrix}. \tag{43}$$

Writing out the Schrödinger equation in terms of individual entries, one has

$$i\frac{d\Psi_i(t)}{dt} = \sum_{j=1}^{N} H_{ij}\Psi_j(t). \tag{44}$$

[2] For the generalization to infinite-dimensional Hilbert spaces, see the work of Ashtekar and Schilling (1999).

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Then decompose all the individual entries of $|\Psi(t)\rangle$ and H into their real and imaginary parts:[3]

$$\begin{aligned}\Psi_i(t) &= \frac{1}{\sqrt{2}}\big(q_i(t) + ip_i(t)\big),\\ H_{ij} &= A_{ij} + iB_{ij},\end{aligned} \tag{45}$$

where $H = H^\dagger$ implies the symmetry conditions

$$\begin{aligned}A_{ij} &= A_{ji},\\ B_{ij} &= -B_{ji}.\end{aligned} \tag{46}$$

The Schrödinger equation then becomes

$$i\frac{d\big(q_i(t) + ip_i(t)\big)}{dt} = \sum_{j=1}^{N}\big(A_{ij} + iB_{ij}\big)\big(q_j(t) + ip_j(t)\big). \tag{47}$$

Breaking up this equation into its real and imaginary parts, and simplifying fully, one ends up with a system of $2N$ classical Hamiltonian equations of motion:

$$\begin{aligned}\dot{q}_i(t) &= \frac{\partial H_{\mathrm{SH}}}{\partial p_i},\\ \dot{p}_i(t) &= -\frac{\partial H_{\mathrm{SH}}}{\partial q_i}.\end{aligned} \tag{48}$$

Here, H_{SH} is the Strocchi–Heslot Hamiltonian, which is just a classical Hamiltonian that ends up with a manifestly bilinear form as a result of the linearity of the Schrödinger equation:

$$H_{\mathrm{SH}} \equiv \sum_{k,l}\left(\frac{1}{2}A_{kl}p_kp_l + B_{kl}q_kp_l + \frac{1}{2}A_{kl}q_kq_l\right). \tag{49}$$

Schrödinger time evolution is therefore isomorphic, on the Strocchi–Heslot formulation, to the dynamics of a system of *classical* coupled harmonic oscillators. Changing orthonormal basis on the Hilbert-space side is then equivalent to carrying out a linear canonical transformation on the Strocchi–Heslot side.

Moreover, under the most straightforward notion of a time-reversal transformation, one has $q_i(t) \mapsto q_i(-t)$ and $p_i \mapsto -p_i(-t)$, with an extra minus sign for $p_i(t)$, precisely as in (32). Hence, when $q_i(t)$ and $p_i(t)$ are recombined into the components $\Psi_i(t) = (1/\sqrt{2})\big(q_i(t) + ip_i(t)\big)$ of the system's state vector, one sees in analogy with the transformation law for $z(t)$ in (38) that $\Psi_i(t)$ transforms according to the rule

$$\Psi_i(t) \mapsto K\Psi_i(-t). \tag{50}$$

That is, $\Psi_i(t)$ experiences *both* a reversal of the time argument from t to $-t$ *together* with an application of the complex-conjugation operator K. The Strocchi–Heslot

[3] The $1/\sqrt{2}$ normalization is convenient for rewriting Poisson brackets in terms of complex variables.

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formulation therefore gives a nice way of seeing why state vectors obey this odd-looking transformation rule.[4]

As a particularly interesting choice of orthonormal basis, picking the energy eigenbasis corresponds to carrying out a linear canonical transformation to the set of normal modes, for which the oscillators are uncoupled and independent, with individual frequencies proportional to the eigenvalues of the original quantum Hamiltonian H. Mathematically speaking, this canonical frame is arguably the simplest description of the evolution for any quantum system—after all, there are few systems simpler than a set of uncoupled, classical harmonic oscillators.

Some interpretations of quantum theory attempt to reify its usual Markovian formulation. For such interpretations, it is not immediately clear why the Hilbert-space formulation should be favored, metaphysically speaking, over the Strocchi–Heslot formulation. If, however, the Strocchi–Heslot formulation is reified, then is the implication that the universe is just a collection of harmonic oscillators? In that case, why should having zero amplitude—say, for a "branch" or "world" in some versions of Everettian quantum theory—be taken to mean a lack of physical existence? After all, a harmonic oscillator that is not oscillating still physically exists. One therefore sees the dangers of trying to build an ontology directly out of a purely mathematical structure, simply because mathematical structures have many different-looking formulations, and what fixes the formulation to be one over another?

Notice that for a quantum system with an N-dimensional Hilbert space, for N finite, the corresponding Strocchi–Heslot formulation features a continuously infinite state space of dimension $2N$. In particular, the Strocchi–Heslot state space is continuously infinite even for the simplest possible quantum system: a qubit, which has $N = 2$. Hence, if the Strocchi–Heslot state space can be understood as the state space of a Markovian embedding of some underlying non-Markovian system with a discrete configuration space consisting of just N elements, then the dynamics of that underlying system is presumably going to be extremely non-Markovian, far more so than second order.

5. Indivisible stochastic processes

An indivisible stochastic process is a simple generalization of a non-Markovian stochastic process. The new axioms, which are considerably simpler than the Dirac–von Neumann axioms and do not involve Hilbert spaces or even the complex numbers, are a combination of fixed and contingent ingredients. The fixed ingredients, which are the same in every real-world instantiation or run of the given indivisible model, are as follows:

- a kinematics consisting of a configuration space $\mathcal{C}$, and
- a dynamics consisting of transition maps $\Gamma(t \leftarrow t_0) : \mathcal{C} \times \mathcal{C} \rightarrow [0, 1]$ for pairs of times t, t_0 taken from respective index sets $\mathcal{T}$, called the target times, and $\mathcal{T}_0 \subset \mathcal{T}$, called the conditioning times, where each transition map $\Gamma(t \leftarrow t_0)$ encodes first-order conditional transition probabilities connecting t_0 to t.

[4] If one were to allow for a linear canonical transformation as part of the time-reversal transformation, then $\Psi_i(t)$ would instead transform as $\Psi_i(t) \mapsto VK\Psi_i(-t)$, where V is some unitary matrix. This more generic transformation law is, in fact, the most general textbook rule for the time reversal of a state vector.

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The contingent ingredient, which will generically differ between real-world instantiations or runs of the model, is

- a standalone probability distribution $p(t) : \mathcal{C} \to [0, 1]$ for times t in the set of target times $\mathcal{T}$.

Labeling distinct configurations in $\mathcal{C}$ with an index i or j, the standard rules of ordinary probability theory include the usual law of total probability, which implies that there is a *linear* marginalization rule connecting standalone probabilities $p_j(t_0)$ at conditioning times t_0 and standalone probabilities $p_i(t)$ at target times t:

$$p_i(t) = \sum_j \Gamma_{ij}(t \leftarrow t_0) p_j(t_0). \tag{51}$$

Here, the two-indexed quantities $\Gamma_{ij}(t \leftarrow t_0)$ define a transition matrix, which is then a (column) stochastic matrix, meaning a matrix whose entries are nonnegative real numbers and each of whose columns sum to 1.

Indivisibility is a simple and remarkably new idea. One can think of it as "a failure of iterativeness." The conceptual foundations of indivisibility originated in the theory of quantum channels (Wolf and Cirac 2008), but they were only first applied to ordinary probability theory and stochastic processes within the last few years (Milz and Modi 2021).

In short, for an indivisible stochastic process, not all target times are valid conditioning times at the level of the transition matrices. That is, even if the transition matrices $\Gamma(t \leftarrow t_0)$ and $\Gamma(t' \leftarrow t_0)$ exist for times $t > t' > t_0$, where t' is intermediate between t_0 and t, it may be the case that

$$\Gamma(t \leftarrow t_0) \neq \Gamma(t \leftarrow t')\Gamma(t' \leftarrow t_0) \tag{52}$$

for all matrices $\Gamma(t \leftarrow t')$ that satisfy the conditions of being stochastic and that are properly interpreted by the theory as describing first-order conditional transition probabilities connecting t' to t. The inequality (52) is called indivisibility.

As emphasized, for instance, in work by Gillespie (1998, 2000), a traditional non-Markovian stochastic process features an infinite hierarchy or Kolmogorov tower of conditional probabilities of first order, second order, third order, and higher orders:

$$\begin{aligned} &p\big(i, t|j_1, t_1\big), \\ &p\big(i, t|j_1, t_1; j_2, t_2\big), \\ &p\big(i, t|j_1, t_1; j_2, t_2; j_3, t_3\big). \end{aligned} \tag{53}$$

From this Kolmogorov tower of conditional probabilities, together with standalone probabilities $p(i, t)$, one can go on to construct all possible two-way joint probabilities, three-way joint probabilities, and so on.[5]

In a sense, an indivisible stochastic process represents an entire *equivalence class* of non-Markovian stochastic processes of arbitrary order that agree on the sparse set of first-order conditional probabilities encoded in the transition maps $\Gamma(t \leftarrow t_0)$. Each

[5] For pedagogical treatments of the theory of stochastic processes, see Rosenblatt (1962), Parzen (1962), Doob (1990), or Ross (1995).

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such non-Markovian stochastic process in this equivalence class is compatible with the laws of the given indivisible stochastic process and is called a *non-Markovian realizer*.[6] By contrast, a Markov process entails making the approximation that all second- and higher-order conditional probabilities $p(i, t|j_1, t_1; j_2, t_2; \ldots)$ are fixed and numerically equal to first-order conditional probabilities, conditioned on the time closest to the target time t.

To establish the equivalence between indivisible stochastic processes and quantum systems, called the stochastic–quantum correspondence, one needs to define the stochastic-to-quantum direction of the equivalence, and then the quantum-to-stochastic direction.

For the stochastic-to-quantum direction, one starts by considering any indivisible stochastic process with N configurations and then picking a conditioning time, taken to be 0 without loss of generality, as well as a target time t. Then, given the transition matrix $\Gamma(t \leftarrow 0)$, one proceeds to "solve" the nonnegativity condition $\Gamma_{ij}(t \leftarrow 0) \geq 0$ by introducing a nonunique "potential" matrix $\Theta(t \leftarrow 0)$ whose entries are allowed to be complex-valued, according to the entry-by-entry equality

$$\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2. \tag{54}$$

If it happens that the transition matrix $\Gamma(t \leftarrow 0)$ is unistochastic (Stueckelberg 1952; Horn 1954; Thompson 1989; Nylen et al. 1993), then $\Theta(t \leftarrow 0)$ can be assumed to be a unitary matrix $U(t \leftarrow 0)$, satisfying

$$U^\dagger(t \leftarrow 0) = U^{-1}(t \leftarrow 0) \tag{55}$$

and called the system's time-evolution operator.

Otherwise, one can define a set of $N \times N$ matrices $K_\beta(t \leftarrow 0)$, with $\beta = 1, \ldots, N$, that each take one column from $\Theta(t \leftarrow 0)$ and are otherwise filled with 0s. It is a straightforward exercise to show that the matrices $K_\beta(t \leftarrow 0)$ are Kraus operators, meaning that they satisfy the Kraus identity

$$\sum_\beta K_\beta^\dagger(t \leftarrow 0) K_\beta(t \leftarrow 0) = \mathbb{1}, \tag{56}$$

where $\mathbb{1}$ is the identity matrix, and also that the transition matrix $\Gamma(t \leftarrow 0)$ has the following Kraus decomposition:

$$\Gamma_{ij}(t \leftarrow 0) = \sum_{\beta=1}^{N} |K_{\beta,ij}(t \leftarrow 0)|^2. \tag{57}$$

It follows from an application of the Stinespring dilation theorem (Stinespring 1955; Barandes 2025a, 2025b, 2026) that by increasing N to N', which is bounded from above by N^3, one can replace the $N \times N$ matrix $\Theta(t \leftarrow 0)$ with an $N' \times N'$ unitary matrix $U(t \leftarrow 0)$ that can serve as the system's time-evolution operator. This argument gives a novel and compelling first-principles explanation of why unitary time evolution plays such a special role in quantum theory.

Crucially, for $N > 2$, an $N \times N$ unistochastic matrix will not generally be orthostochastic. An orthostochastic matrix is a unistochastic matrix for which the

[6] The author thanks Alexander Meehan for suggesting this terminology.

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underlying unitary matrix can be taken to have only real-valued entries and therefore to form an orthogonal matrix. Hence, to exploit the stochastic-to-quantum side of the correspondence, with unitary time evolution, it will be necessary to introduce the complex numbers, or an algebraic construct isomorphic to them. Of course, from the standpoint of the stochastic–quantum correspondence, Hilbert spaces are mere mathematical fictions anyway, so introducing the complex numbers to define the Hilbert-space picture is a harmless move. The complex numbers also provide important benefits and resources, like the spectral theorem, symmetry generators, Hamiltonians, energy eigenvalues, stationary states, the Schrödinger equation, the uncertainty principle, spinors, and more.

As it turns out, to implement time-reversal transformations, one again needs to introduce the complex-conjugation operator K, as defined in (19), which combines with i and iK to form the pseudo-quaternions, as in (21). In a sense, then, the Hilbert spaces of quantum systems are actually defined over the pseudo-quaternions, rather than just over the complex numbers, although observables are typically constructed out of only the complex numbers.

After encoding the initial probabilities $p_i(0)$ as the diagonal entries of an initial density matrix $\rho(0)$ that is otherwise filled with 0s, one can go on to define a time-evolving density matrix $\rho(t) \equiv U(t \leftarrow 0)\rho(0)U^\dagger(t \leftarrow 0)$, as well as the rest of the Hilbert-space picture, as detailed in other work (Barandes 2025a). If the density matrix is rank-one, then there exists an $N \times 1$ state vector or wave function $|\Psi(t)\rangle = U(t \leftarrow 0)|\Psi(0)\rangle$ that gives a simple outer-product factorization

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|, \tag{58}$$

where $\langle\Psi(t)| \equiv |\Psi(t)\rangle^\dagger$. Assuming that the unitary time evolution is sufficiently smooth as a function of t, the wave function then satisfies the Schrödinger equation (41),

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = H(t)|\Psi(t)\rangle,$$

for a self-adjoint Hamiltonian $H(t) = H^\dagger(t)$ defined in keeping with Stone's theorem (Stone 1930) in terms of the time-evolution operator $U(t \leftarrow 0)$ according to

$$H(t) \equiv i\frac{\partial U(t \leftarrow 0)}{\partial t}U^\dagger(t \leftarrow 0). \tag{59}$$

The unitarily-time-evolving Hilbert-space formalism therefore yields a "divisible" Markovian embedding of the original indivisible stochastic process, with a first-order differential equation for the time evolution. Moreover, the linearity of the Schrödinger equation now has a clear explanation: As one can check, it ultimately descends from the linearity of the law of total probability (51) connecting standalone probability distributions via the transition matrix.

From the perspective of this formulation, one sees that wave functions and the Schrödinger equation are secondary pieces of derived mathematics and not the primary ontological furniture of quantum systems. In particular, wave functions, like the luminiferous aether or Magritte's famous pipe (Magritte 1929) are not physical or ontic. Wave functions are not even entirely epistemic, but encode a blend of epistemic and

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nomological information, so they are not well-classified according to the Harrigan–Spekkens "ontological models" framework (Harrigan and Spekkens 2010).[7]

To establish the quantum-to-stochastic direction of the stochastic–quantum correspondence, one begins with a unitarily evolving quantum system, with an N-dimensional Hilbert space $\mathcal{H}$. After picking a convenient orthonormal basis (which is *also* a step used to define the path-integral formulation), one proceeds to define an indivisible stochastic process via introducing a manifestly unistochastic transition matrix $\Gamma(t \leftarrow 0)$ according to the rule

$$\Gamma_{ij} \equiv |U_{ij}(t \leftarrow 0)|^2. \tag{60}$$

One might initially be concerned that this formula erases phase information from $U(t \leftarrow 0)$, as in (10). However, this phase information does not matter as long as one remembers that all empirical results come from measurement processes and one is careful to include measuring devices and environments in models of quantum systems. If, say, a measuring device is properly regarded as a subsystem of the overall indivisible stochastic process, and the chosen orthonormal basis for the overall system's Hilbert space properly captures the measuring device's pointer variables, then, by construction, the dynamics of the overarching indivisible stochastic process will deliver the measuring device into its correct read-out configurations with the correct measurement-outcome probabilities, in accord with the Born rule. Keeping track of phase information in $U(t \leftarrow 0)$ is only necessary if one wishes to *exclude* the measuring device from the system, for reasons of mathematical simplicity and convenience, in keeping with the usual Dirac–von Neumann axioms. A detailed treatment of the measurement process within this framework is covered in other work (Barandes 2025a).

One therefore arrives at a comprehensive stochastic–quantum correspondence, according to which the Hilbert-space formalism is a kind of a Markovian embedding, serving as a form of "analytical mechanics" for highly non-Markovian stochastic systems, giving rise to an effective first-order, "divisible" differential equation. Like any form of analytical mechanics, the Hilbert-space formalism provides a powerful set of mathematical tools for specifying microphysical laws in a systematic manner, for studying dynamical symmetries, and for calculating predictions.

6. Conclusion

To summarize the arguments in this article, one can regard a Hilbert-space quantum system as a Markovian embedding for an indivisible stochastic process, which essentially represents an equivalence class of non-Markovian stochastic processes of arbitrarily high order. From this perspective, the complex numbers are only necessary as part of reformulating an indivisible stochastic process in the Hilbert-space picture with unitary time evolution.

The claim that every quantum system is really an indivisible stochastic process in disguise leads to a new interpretation of quantum theory, naturally called "indivisible quantum theory," which is based on simpler axioms than textbook quantum theory,

[7] Indeed, Harrigan and Spekkens expressly acknowledge that their ontic–epistemic dichotomy "is presumably a category mistake" in such cases.

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does away with treating superposition as a literal fact of physical objects being in multiple configurations at once, arguably does not suffer from a measurement problem, demotes wave functions from having a directly physical or ontological status, and deflates a lot of the exotic talk about quantum phenomena more generally. Finally, as argued elsewhere (Barandes 2024), one can introduce a plausible microphysical theory of causal influences according to which indivisible quantum theory can be regarded as a causally local theory.

Acknowledgments. The author would like to acknowledge Jill North, Craig Callender, and Noel Swanson for many helpful conversations and feedback. This article is based on a talk delivered at the Philosophy of Science Association's meeting in November 2024.

Funding. None to declare.

Statements and Declarations. None to declare.

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Cite this article: Barandes, Jacob A. 2026. "A Deflationary Account of Quantum Theory and Its Implications for the Complex Numbers." *Philosophy of Science*. https://doi.org/10.1017/psa.2026.10245

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