This function again is of lower degree than the original quartic and we can calculate the difference area G using integration rules that we derived earlier in the article. We get

$$G = -H^4\left(H - \frac{H}{2}\right) + 4H^3\left(\frac{H^2 - \left(\frac{H}{2}\right)^2}{2}\right) - 6H^2\left(\frac{H^3 - \left(\frac{H}{2}\right)^3}{3}\right) + 4H\left(\frac{H^4 - \left(\frac{H}{2}\right)^4}{4}\right) = \frac{3H^5}{16}.$$

By reflection symmetry, we find again $A = G + 2a$. Using what we know about G and a we get this time $A = \frac{3H^5}{16} + \frac{A}{16}$ or $A = \frac{H^5}{5}$ as we expected from integral calculus.

Now we can go on and find new formulae for increasing powers using the different methods for odd and even powers. For even powers we have to use the argument using dilations and reflection symmetry whilst for odd powers we use the rotation trick and thus reduce the degree of the power function by one.

In the two articles [1] and [2] Rainer Kaenders presents an approach where affine transformations play the role of symmetry in this article. Also there a new road to integration is opened. For another approach to integration see Kirfel [3].

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101.06 Finding some integrals using an interesting self-similar substitution

After recently completing a section of work on techniques of integration I was asked by one of my students if the following integral could be found using a substitution other than a trigonometric (or hyperbolic) substitution:

$$\int \frac{dx}{(1 - x)\sqrt{1 - x^2}}.$$

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Using the 'obvious' substitution of either $x = \sin u$ or $x = \cos u$ leads to an integral that is rational in terms of sine or cosine, depending on whether the former or latter substitution is used. A tangent half-angle substitution of $t = \tan \frac{1}{2}u$ (a so-called 't-substitution') is then needed before the integral can finally be found.

In my classes I often make the point of telling students a t-substitution should only ever be used as a substitution of last resort and is most likely the reason why the student asked the question in the first place. While a t-substitution is the standard rationalising substitution used to convert any integral of a rational function consisting of sine and cosine into a rational function in terms of the substitution variable used, the resulting rational function tends to require tedious partial fraction decompositions before the integral can be found.

Having not thought of such a question before, intrigued, I decided to give the question some thought. While it is apparent many non-trigonometric substitutions are possible, one particularly interesting substitution came to mind. It is

$$u = \frac{1 - x}{1 + x}.$$

The substitution is noteworthy for the fact that if u is considered as a function of x, the function is equal to its own inverse. The function is therefore an example of an *involution* with the substitution said to be *self-similar*.

We may rewrite the integral as

$$\begin{aligned}\int \frac{dx}{(1 - x)\sqrt{1 - x^2}} &= \int \frac{dx}{(1 - x)\sqrt{(1 - x)(1 + x)}} \\ &= \int \frac{1}{(1 - x)\sqrt{(1 - x)(1 + x)}} \cdot \frac{\sqrt{1 - x}}{\sqrt{1 - x}}\, dx \\ &= \int \sqrt{\frac{1 - x}{1 + x}} \cdot \frac{dx}{(1 - x)^2}.\end{aligned}$$

As the substitution is self-similar one has $x = \dfrac{1 - u}{1 + u}$, giving $dx = \dfrac{-2}{(1 + u)^2} du$. On noting that $1 - x = \dfrac{2u}{1 + u}$, the integral becomes

$$\begin{aligned}\int \frac{dx}{(1 - x)\sqrt{1 - x^2}} &= \int \sqrt{u} \cdot \left(\frac{1 + u}{2u}\right)^2 \cdot \frac{-2}{(1 + u)^2}\, du \\ &= -\frac{1}{2} \int u^{-3/2}\, du = \frac{1}{\sqrt{u}} + C = \sqrt{\frac{1 + x}{1 - x}} + C.\end{aligned}$$

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A famous appearance

One place where the self-similar substitution used above makes an important appearance is in the evaluation of the following integral known as *Serret's* integral [1]:

$$t = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx.$$

It is named after the French mathematician Joseph-Alfred Serret (1819-1885) who first evaluated the integral in 1844 [2]. Serret himself used a trigonometric substitution of $x = \tan\varphi$. While the integral can be found in older texts (see, for example, [3, p. 76]), more recently it rose to fame by its appearance in 2005 in the Sixty-Sixth William Lowell Putnam Mathematical Competition, an annual mathematics competition open to undergraduate students studying in Canada and the United States. Problem A5 of that year asked contestants to evaluate Serret's integral (though the name of the integral was not given) and, judging from the statistics provided, most contestants found it to be a particularly difficult problem. Of the top 196 contestants (out of 3545), 132 did not submit a solution while only 20 scored 3 or more marks out of a possible 10 [4].

Under our self-similar substitution, when applied to Serret's integral, one finds

$$\begin{aligned} I &= \int_0^1 \frac{(1+u)^2}{2(1+u^2)} \cdot \ln\left(\frac{2}{1+u}\right) \cdot \frac{2}{(1+u)^2}\,du \\ &= \int_0^1 \ln\left(\frac{2}{1+u}\right) \cdot \frac{du}{(1+u^2)} \\ &= \ln 2 \int_0^1 \frac{du}{1+u^2} - I \end{aligned}$$

which yields

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2.$$

Two other interesting integrals closely connected to Serret's integral are the following:

$$J_1 = \int_0^1 \frac{\tan^{-1}x}{1+x}\,dx \qquad \text{and} \qquad J_2 = \int_0^1 \frac{\cot^{-1}x}{1+x}\,dx.$$

If the value for Serret's integral is known in advance, values for each of the above integrals quickly follow by integrating Serret's integral by parts. If on the other hand the value for Serret's integral is not known beforehand, each can be evaluated using our self-similar substitution.

For the first of the integrals, application of our self-similar substitution reduces J_1 to

$$J_1 = \int_0^1 \tan^{-1}\left(\frac{1-u}{1+u}\right)\frac{du}{1+u}.$$

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From the angle difference identity for the tangent function, namely

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b},$$

setting $a = \tan^{-1}\alpha$ and $b = \tan^{-1}\beta$ gives

$$\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha - \beta}{1 + \alpha\beta}\right).$$

If $\alpha = 1$ and $\beta = u$ in the above identity, the inverse tangent term appearing in the integral for J_1 can be expressed as

$$\tan^{-1}\left(\frac{1 - u}{1 + u}\right) = \frac{\pi}{4} - \tan^{-1} u.$$

Thus

$$J_1 = \frac{\pi}{4}\int_0^1 \frac{du}{1 + u} - \int_0^1 \frac{\tan^{-1} u}{1 + u}\, du = \frac{\pi}{4}\ln 2 - J_1,$$

or

$$\int_0^1 \frac{\tan^{-1} x}{1 + x}\, dx = \frac{\pi}{8}\ln 2.$$

Proceeding in a similar manner for the second of the integrals, application of our self-similar substitution reduces J_2 to

$$J_2 = \int_0^1 \cot^{-1}\left(\frac{1 - u}{1 + u}\right)\frac{du}{1 + u}.$$

From the angle difference identity for the cotangent function, namely

$$\cot(a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a},$$

setting $a = \cot^{-1}\alpha$ and $b = \cot^{-1}\beta$ gives

$$\cot^{-1}\alpha - \cot^{-1}\beta = \cot^{-1}\left(\frac{\alpha\beta + 1}{\beta - \alpha}\right) = -\cot^{-1}\left(\frac{1 + \alpha\beta}{\alpha - \beta}\right),$$

since $\cot^{-1}(-x) = -\cot^{-1} x$ for $x \neq 0$ when a domain restriction of $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ for cotangent is used to define its inverse. If $\alpha = 1$ and $\beta = u$ in the above identity, the inverse cotangent term appearing in the integral for J_2 can be expressed as

$$-\cot^{-1}\left(\frac{1 + u}{1 - u}\right) = \frac{\pi}{4} - \cot^{-1} u. \tag{1}$$

On the interval of integration $0 < u < 1$, as the argument $\dfrac{1 + u}{1 - u}$ for the cotangent function is positive, using the result

$$\cot^{-1} x = \frac{\pi}{2} - \cot^{-1}\left(\frac{1}{x}\right), \text{ for } x > 0,$$

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one can write (1) as

$$\cot^{-1}\left(\frac{1-u}{1+u}\right) = \frac{3\pi}{4} - \cot^{-1}u.$$

Thus

$$J_2 = \frac{3\pi}{4}\int_0^1 \frac{du}{1+u} - \int_0^1 \frac{\cot^{-1}u}{1+u}\,du = \frac{3\pi}{4}\ln 2 - J_2,$$

or

$$\int_0^1 \frac{\cot^{-1}x}{1+x}\,dx = \frac{3\pi}{8}\ln 2.$$

Note that if the value of J_1 is known, the value for J_2 follows immediately from J_1 on application of the identity

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x,\ x > 0.$$

Two identities

From the definite integral definition for the inverse hyperbolic tangent function, namely

$$\tanh^{-1}x = \int_0^x \frac{dt}{1-t^2},\ |x| < 1,$$

on applying the self-similar substitution of $t = \dfrac{1-u}{1+u}$ one has

$$\tanh^{-1}x = -\int_1^{\frac{1-x}{1+x}} \frac{2}{(1+u)^2}\cdot\frac{(1+u)^2}{4u}\,du = -\frac{1}{2}\int_1^{\frac{1-x}{1+x}}\frac{du}{u} = -\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right),$$

or

$$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ |x| < 1.$$

This is of course the well-known logarithmic representation for the inverse hyperbolic tangent function.

For the second identity, from the definite integral definition for the inverse tangent function, namely

$$\tan^{-1}x = \int_0^x \frac{dt}{1+t^2}$$

on application of the same self-similar substitution as used above, one has

$$\tan^{-1}x = -\int_1^{\frac{1-x}{1+x}} \frac{2}{(1+u)^2}\cdot\frac{(1+u)^2}{2(1+u^2)}\,du = -\int_1^{\frac{1-x}{1+x}}\frac{du}{1+u^2} = \left[\cot^{-1}x\right]_1^{\frac{1-x}{1+x}},$$

or

$$\tan^{-1}x = \cot^{-1}\left(\frac{1-x}{1+x}\right) - \frac{\pi}{4}. \tag{2}$$

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For what values of x such an identity is valid depends on the definition one chooses to use for the inverse cotangent function. If a domain restriction of $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ for the cotangent function is used to define its inverse, then (2) will be valid for all $x < -1$ while if instead a domain restriction of $0 < x < \pi$ is used, (2) will be valid for all $x > -1$.

Further examples

Some further examples of integrals that can be found or evaluated using the self-similar substitution considered in this note are collected below.

(a) $\displaystyle\int \frac{dx}{(1 - x^2)^{3/2}}$ (b) $\displaystyle\int \frac{\tan^{-1} x}{(1 + x)^2}\, dx$

(c) $\displaystyle\int \ln\left(\frac{1 + x}{1 - x}\right)\frac{dx}{1 - x^2}$ (d) $\displaystyle\int \frac{dx}{\sqrt[3]{(1 + x)^2(1 - x)^4}}$

(e) $\displaystyle\int_0^1 \frac{(1 + 2x - x^2)\ln(1 + x)}{(1 + x^2)^2}\, dx$ (f) $\displaystyle\int_{-1/2}^2 \frac{\tan^{-1} x}{x^2 + x + 2}\, dx$

(g) $\displaystyle\int_{-1/2}^3 \frac{\ln(x + 2)}{(x + 1)\left[\ln(x + 3) + \ln(x + 2) - \ln(x + 1)\right]}\, dx$

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101.07 Cauchy's mean value theorem meets the logarithmic mean

The arithmetic and geometric means

Almost everybody knows that the average, or *arithmetic mean*, of two numbers a and b is

$$A(a, b) = \frac{a + b}{2}.$$

Fewer people know that for a and b non-negative, their *geometric mean* is

$$G(a, b) = \sqrt{ab}.$$

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