The representation of GT component maps using Mach numbers

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ABSTRACT

Component maps are produced under certain environmental conditions using air as the working fluid during static ground operation. Any changes of the component characteristics when operating under different temperature conditions and/or with different working fluid are partially taken into account, because of the existence of the gas constant and the ratio of the specific heats in the non-dimensional mass flow and rotational speed. This provides a second order correction for the component characteristics, which may be adequate for the initial modeling of engines. However, for rigorous performance calculations correction factors are applied to the non-dimensional mass flow, rotational speed and pressure ratio distributions of a map, when deviations from the reference conditions under which it was extracted, are experienced. In the current study, a different approach is considered in order to eliminate the inaccuracies caused by the varying temperature and chemical composition. It makes direct use of inlet and circumferential Mach numbers based upon stagnation temperature in conjunction with dimensionless enthalpy variation. A sensitivity analysis against gas property variations is conducted to quantify the benefits gained in precision. Generally, the well-known relationships correlating the Mach number with total and static properties are based on the assumption of perfect gas and constant gas properties. Introducing dependency on temperature and/or chemical composition for the caloric properties of the semi-perfect gas, proper mean values are defined and some theoretical corrections are provided for the well-known equations. The mass flow compatibility equation is then based on the 'corrected' expression correlating dimensionless mass flow and Mach number and takes full account of gas property variations.

NOMENCLATURE

A	area
C	specific heat
D	diameter
far	fuel to air ratio
h	specific enthalpy
M	Mach number
N	rotational speed
p	pressure
R	gas constant
T	temperature
U	circumferential velocity
V	velocity
α	angle
γ	ratio of specific heats
π	pressure ratio
ρ	density

Subscripts

ax	axial
C	compressor
h	enthalpy mean valued
p	constant pressure
t	total
T	turbine
u	circumferential
φ	Gibb's equation mean valued

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1.0 INTRODUCTION

When carrying out whole engine gas turbine performance calculations, it is common practice to use component performance maps with corrected or pseudo-non-dimensional mass flow and speed. The selection of the appropriate scaling parameter group is chosen on the basis of including variations of gas properties or absolute component size and depends on the problem. Various scaling groups are proved in Ref. 1 and they are also tabulated in Ref. 2, together with the original non-dimensional parameters. The non-dimensional mass flow and rotational speed are extracted on the basis of inlet and circumferential Mach number similarity, respectively. This is justified by the fact that, having fixed Mach numbers for the working fluid and the rotating blades results in the incidence angles on the blades and vanes being fixed and corresponds to particular values for the pressure loss coefficients, the efficiency and the work input[2], as long as the Reynolds number is above its critical value. The Mach number similarity and its correlation with stage loading and the flow-coefficient is also analysed in Ref. 3 and referred to in Refs 4 and 5. In Ref. 6 the appropriate representation of performance characteristics is examined under extreme part-load conditions and the basic methodologies based on Mach number similarity are briefly summarised including some second order effects. In a similar manner, Ref. 7 investigates a number of compressor modeling methods and compares their relative merits. More importantly, caution is paid to how the parametric representation of the characteristics is affected by the introduction of certain suggested parameters, which are then examined for their dependency on gas properties.

Generally, working with pressure ratio and scaling groups of mass flow and rotational speed provides a rather convenient form, since mass flow, rotational speed, total temperature and pressure can be experimentally measured during rig testing in ground facilities with air being the working fluid. For the same working fluid, operation in different temperature conditions, than the one of the testing facility, can lead to small differences in turbomachinery performance, which are usually referred to as second-order temperature effects. Gas properties and the delivered component performance are further differentiated, when changes in the chemical composition are also encountered at the inlet of the components.

In order to encounter the problem, it is common practice correction factors to be applied on turbomachinery characteristics, whenever the operating conditions and/or the working fluid deviate from the reference ones. One approach to overcome the inaccuracies due to varying caloric properties is for example to raise temperature in the mass flow parameter by an exponent somewhat different than exactly 0·50[2]. The theoretical background allowing the estimation of the exponent is outlined in Ref. 8. A typical example of significant gas property variation is the digestion of atmospheric gaseous humidity by the engine, which is examined extensively in Ref. 9. Specifically, relations correlating quasi-dimensional rotational speed and mass flow for the different working fluids are provided on the basis of Mach number equality. Moreover, a variety of commonly used map correction procedures are introduced and compared.

Well-known relationships, which correlate pressure with temperature ratios or the non-dimensional mass flow with axial Mach number, are shown in Refs 9 and 10 and textbooks. Even though they are derived from the First Thermodynamic Law under the assumption of constant gas properties, they are applied to gases with varying caloric properties. When utilised, it is not clear at what temperature the gas properties are evaluated; for example whether gamma is calculated at the total or the static temperature, at the inlet or the outlet of a component or at an appropriately averaged temperature.

Seeking justification, a theoretical approach is initially followed in the current paper to identify the appropriate mean values for the gas properties involved in the well-known equations. Applying the proper mean values, when proving the fundamental equations from the beginning, leads to correction factors being identified in certain relationships. Such are the ones interrelating stagnation with static properties and the non-dimensional mass flow with inlet Mach number. In an attempt to quantify the effect of gas properties on mass flow compatibility, the exchange rate of non-dimensional mass flow versus gamma is plotted as a function of inlet Mach number later on.

Most importantly, Mach numbers are proved to directly relate to the work parameter of a turbomachinery stage. The relationship is shown to be independent from gas properties, as long as the work parameter is the non-dimensional enthalpy variation, and this justifies their direct use in turbomachinery characteristics. Sensitivity analysis for differentiations of the delivered pressure ratio due to gamma variations for the same non-dimensional operating point are inspected separately for a compression and an expansion and the results are provided.

2.0 PARAMETERS AFFECTING CALORIC PROPERTY MODELING

Temperature dependency of specific heat and gamma for individual chemical species constitutes the semi-perfect gas assumption. The thermodynamic properties of a gas, can be considered as a weighted average of the thermodynamic properties of the individual species comprising the mixture; the weight factors being the mass or mol fraction of each chemical component, depending on the units being utilised. As a consequence, the caloric properties of a mixture are considered to be a function of temperature and chemical composition.

In the case of air and its combustion products with kerosene, assuming lean combustion — which is actually the case for modern gas turbine engines — the chemical composition is controlled by the mass ratio of fuel to air, which is the parameter to be quoted in the text to follow. Introducing a pressure dependency for air would do very little to improve accuracy, but would severely complicate the fluid modeling. Especially for combustion products, absolute pressure is considered to affect the chemical composition of the dissociated products, which are initially observed at temperatures above 1,800K.

In the text to follow, pressure dependency under such specialised conditions is not going to be considered. Instead, gas properties such as enthalpy, specific heat and gamma shall be treated as a function of temperature and chemical composition or equivalently fuel to air ratio for the combustion of air and kerosene, utilised in gas turbine engines.

3.0 CALCULATION OF MEAN VALUES

There are various mean values for the specific heat, which can be obtained. Integrating the enthalpy definition of Equation 1 and the Gibbs Law of Equation 2 during an isentropic variation in the temperature range from T_1 to T_2, gives:

$$h_2 - h_1 = \int_{T_1}^{T_2} C_p \mathrm{d}T \qquad \ldots (1)$$

$$0 = \int_{T_1}^{T_2} \frac{C_p \mathrm{d}T}{T} - R \ln\left(\frac{p_2}{p_1}\right) \qquad \ldots (2)$$

$$\phi_2 - \phi_1 = \int_{T_1}^{T_2} \frac{C_p \mathrm{d}T}{T} \qquad \ldots (3)$$

If adequate mean values for each of the two equations are assumed, Equations 1 and 2 can be rewritten as follows.:

$$h_2 - h_1 = \overline{C^{12}_{p,h}}(T_2 - T_1) \qquad \ldots (4)$$

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$$0 = \overline{C_{p,\phi}^{12}} \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) \quad \ldots (5)$$

and the following mean values for the specific heat are extracted:

$$\overline{C_{p,h}^{12}} = \frac{h_2 - h_1}{(T_2 - T_1)} \quad \ldots (6)$$

$$\overline{C_{p,\phi}^{12}} = \frac{\phi_2 - \phi_1}{\ln\left(\frac{T_2}{T_1}\right)} \quad \ldots (7)$$

In Equations 6 and 7, enthalpy h and the ϕ function are considered polynomial functions of temperature and fuel to air ratio, according to relationships stated in Ref. 2 using kerosene as fuel. Mean values for specific heat in J/kg/K are provided in the table below for T_1 = 300K. Higher differences occur between the mean values of specific heat with an increase of the temperature range and the fuel-to-air ratio.

Table 1
Differences between mean values of specific heat

far	$T_2(K)$	$\overline{C_{p,\phi}^{12}}$	$\overline{C_{p,h}^{12}}$	$\frac{(\overline{C_{p,h}^{12}} - \overline{C_{p,\phi}^{12}})}{\overline{C_{p,\phi}^{12}}}\%$
0·00	600	1,020·01	1022.67	0·261
	900	1,043·59	1,054·48	1·044
	1,200	1,065·30	1,086·15	1·957
	1,500	1,083·03	1,113·08	2·775
	1,800	1,097·39	1,135·37	3·461
0·03	600	1,053·80	1,057·42	0·343
	900	1,082·24	1,095·51	1·226
	1,200	1,107·69	1,132·53	2·243
	1,500	1,128·56	1,164·41	3·177
	1,800	1,145·54	1,191·51	4·013

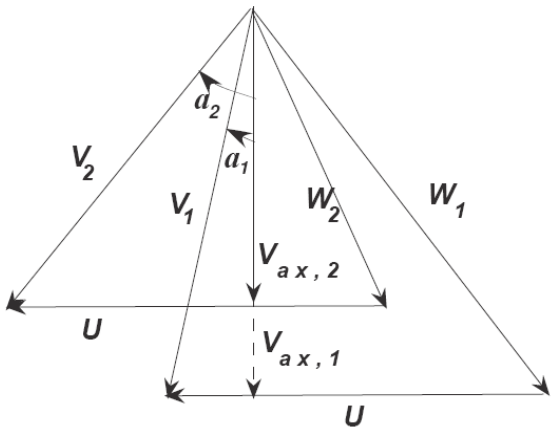


Figure 1. Typical compressor velocity triangle.

4.0 FORMULATION OF BASIC THERMODYNAMIC RELATIONSHIPS

From Equation 5 the isentropic relation between temperature and pressure ratio takes the form:

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{R}{\overline{C_{p,\phi}^{12}}}} \quad \ldots (8)$$

In the aforementioned equations states 1 and 2 can individually represent total or static conditions. These states will be explicitly defined in the equations to follow.

Introducing γ at the local temperature and $\overline{\gamma_h}$ as the mean value extracted by the definition of enthalpy lead to the following:

$$\gamma = \gamma(T, far) \quad \ldots (9)$$

$$\overline{\gamma_h} = \frac{\overline{C_{p,h}}}{\overline{C_{p,h}} - R} \quad \ldots (10)$$

The definition of total enthalpy is written as:

$$h_t = h + \frac{V^2}{2} \Leftrightarrow$$

$$\overline{C_{p,h}} T\left(\frac{T_t}{T} - 1\right) = \frac{V^2}{2} \Leftrightarrow \quad \ldots (11(a))$$

$$\frac{T_t}{T} = 1 + \frac{\overline{\gamma_h} - 1}{2}\frac{V^2}{\overline{\gamma_h} RT} \quad \ldots (11(b))$$

with "$\Leftarrow\Rightarrow$" implying that the initial and final forms of Equation 11 are equivalent, since no assumptions have been made during the deduction of Equation 11(b) from Equation 11(a).

By definition, Mach number at a point of static temperature T is expressed as:

$$\mathrm{M} = \frac{V}{\sqrt{\gamma RT}} \quad \ldots (12)$$

Combining the last two equations it is concluded that:

$$\frac{T_t}{T} = 1 + \frac{\overline{\gamma_h} - 1}{2}\frac{\gamma}{\overline{\gamma_h}}\mathrm{M}^2$$

or equivalently

$$\frac{T_t}{T} = 1 + \frac{R}{2\overline{C_{p,h}}}\gamma \mathrm{M}^2 \quad \ldots (13)$$

with $\overline{\gamma_h}$ and $\overline{C_{p,h}}$ being averaged between the stagnation and the static temperature of the specified station.

5.0 NON-DIMENSIONAL ANALYSIS AND PRINCIPLES

5.1 Adiabatic work exchange

In order to reach safe conclusions for the dimensionless quantities being interrelated during an adiabatic process and their dependency on gas properties, a unified analysis for the case of a single stage compressor and turbine is presented.

Figures 1 and 2 present typical velocity triangles of a compressor and a turbine rotor respectively, where work exchange is occurring.

Assuming that a common circumferential velocity exists for the inlet and outlet stations, the Euler equation between the respective stations is expressed as:

$$\Delta h_{t12} = U\left(V_{u2} - V_{u1}\right) \Leftrightarrow$$

$$\Delta h_{t12} = UV_{ax1}\left(\frac{V_{u2}}{V_{ax2}}\frac{V_{ax2}}{V_{ax1}} - \frac{V_{u1}}{V_{ax1}}\right) \Leftrightarrow$$

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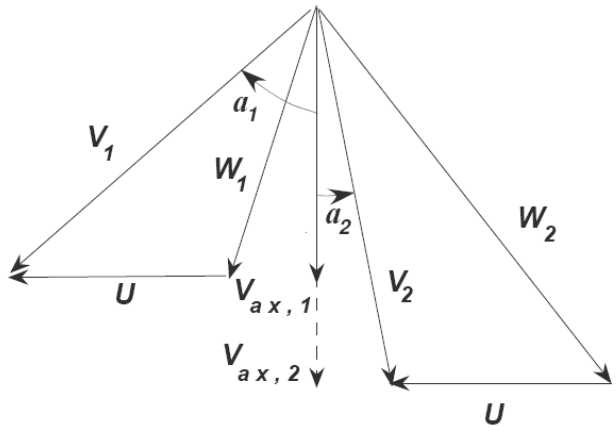


Figure 2. Typical turbine velocity triangle.

$$\Delta h_{t12} = UV_{ax1}\left(\mathrm{T\,an}\,a_2 \cdot \frac{V_{ax2}}{V_{ax1}} - \mathrm{T\,an}\,a_1\right) \quad \ldots (14)$$

In Equation 14, the ratio of axial velocities between inlet and outlet depends on the pressure ratio and the annulus geometry.

Equation 14 is divided by the velocity of sound based upon stagnation temperature, since the dimensionless enthalpy variation formed can be directly correlated to pressure ratio:

$$\frac{\Delta h_{t12}}{\gamma_{t1}RT_{t1}} = \frac{U}{\sqrt{\gamma_{t1}RT_{t1}}}\frac{V_{ax1}}{\sqrt{\gamma_{t1}RT_{t1}}}\left(\mathrm{T\,an}\,a_2 \cdot \frac{V_{ax2}}{V_{ax1}} - \mathrm{T\,an}\,a_1\right) \quad \ldots (15)$$

Axial and circumferential Mach numbers are defined upon stagnation temperature at the rotor entry, as follows:

$$\mathrm{M}_{ax,t} = \frac{V_{ax}}{\sqrt{\gamma_t RT_t}} \quad \ldots (16)$$

$$\mathrm{M}_{ut} = \frac{U}{\sqrt{\gamma_t RT_t}} \quad \ldots (17)$$

Substituting the Mach number definitions in Eq.15, a relationship is extracted for the dimensionless enthalpy variation:

$$\frac{\Delta h_{t12}}{\gamma_{t1}RT_{t1}} = \mathrm{M}_{ut1}\mathrm{M}_{ax1,t}\left(\mathrm{T\,an}\,a_2 \cdot \frac{V_{ax2}}{V_{ax1}} - \mathrm{T\,an}\,a_1\right) \quad \ldots (18)$$

From the latter, it is evident that dimensionless enthalpy variation is related to dimensionless quantities only and no dependency on gas properties exist, when the Mach numbers are calculated on the basis of stagnation temperatures.

If the Mach numbers were to be based upon static temperatures, Equation 18 is transformed in the following way:

$$\frac{\Delta h_{t12}}{\gamma_{t1}RT_{t1}} = \mathrm{M}_{u1}\mathrm{M}_{ax1}\frac{\gamma_1 T_1}{\gamma_{t1}T_{t1}}\left(\mathrm{T\,an}\,a_2 \cdot \frac{V_{ax\,2}}{V_{ax\,1}} - \mathrm{T\,an}\,a_1\right) \quad \ldots (19)$$

In that case, dependency of dimensionless enthalpy variation on gas properties is attained through the ratio of static to total temperature and Equation 13.

Dimensionless enthalpy variation is directly related to pressure ratio.

Assuming adiabatic compression:

$$\frac{\Delta h_{t12}}{\gamma_{t1}RT_{t1}} = \frac{1}{\eta_{is,C}}\frac{1}{\overline{\gamma_h} - 1}\frac{\overline{\gamma_h}}{\gamma_{t1}}\left(\pi_C^{R/\overline{C_{p\phi}}} - 1\right) \quad \ldots (20)$$

Similarly, for adiabatic expansion:

$$\frac{\Delta h_{t12}}{\gamma_{t1}RT_{t1}} = \eta_{is,T}\frac{1}{\overline{\gamma_h} - 1}\frac{\overline{\gamma_h}}{\gamma_{t1}}\left(1 - \left(1/\pi_T\right)^{R/\overline{C_{p\phi}}}\right) \quad \ldots (21)$$

Setting the right hand sides of Equations 19 and 20, 21 equal, expressions for the pressure ratio of a compression and an expansion can be derived:

$$\pi_C = \left\{1 + \frac{\gamma_{t1}}{\overline{\gamma_h}}\left(\overline{\gamma_h} - 1\right)\mathrm{M}_{ut1}\mathrm{M}_{ax1,t}\underbrace{\left(\mathrm{T\,an}\,a_2 \cdot \frac{V_{ax\,2}}{V_{ax1}} - \mathrm{T\,an}\,a_1\right)\eta_{is\,C}}_{A}\right\}^{\overline{C_{p\phi}}/R} \quad \ldots (22)$$

$$\pi_T = \left\{1 - \frac{\gamma_{t1}}{\overline{\gamma_h}}\left(\overline{\gamma_h} - 1\right)\mathrm{M}_{ut1}\mathrm{M}_{ax1,t}\underbrace{\left(\mathrm{T\,an}\,a_2 \cdot \frac{V_{ax\,2}}{V_{ax1}} - \mathrm{T\,an}\,a_1\right)\frac{1}{\eta_{is\,T}}}_{A}\right\}^{\overline{C_{p\phi}}/R} \quad \ldots (23)$$

In Equations 20 to 23 the mean values of $\overline{\gamma_h}$ and $\overline{C_{p\phi}}$ are extracted between the total temperatures of the inlet and outlet stations 1 and 2, respectively.

Term A in Equations 22 and 23 depends on the operating condition of the component, which is mainly defined by the inlet and circumferential Mach numbers. In Term A the effect of the Reynolds number is also introduced, since it influences the isentropic efficiency of the adiabatic process. Most importantly, pressure ratio depends on gas properties because of the existence of gamma and specific heat in the right hand side of Equations 22 and 23. This is the reason for the compressor and the turbine maps, which utilise pressure ratio as the work parameter, being derived on the basis of reference conditions, which include working fluid and ISA deviations for temperature, humidity index and pressure. This leads to correction factors being applied to maps, whenever the conditions and/or the chemical composition vary from the reference values under which the map was initially extracted.

5.2 Dimensionless mass flow

Having introduced the proper mean values for specific heat and gamma, in order to express the dimensionless mass flow as a function of inlet Mach number, a short proof is stated that makes use of the mass flow definition:

$$\dot{m} = \rho AV\,\mathrm{Cos}\,a = \frac{p}{RT}AV\,\mathrm{Cos}\,a \Leftrightarrow$$

$$\frac{\dot{m}}{A} = \frac{p}{p_t}\sqrt{\frac{T_t}{T}}\frac{p_t}{\sqrt{T_t}}\frac{V}{\sqrt{\gamma}\sqrt{R}\sqrt{R}}\frac{\sqrt{\gamma}}{\sqrt{T}}\,\mathrm{Cos}\,a \Leftrightarrow$$

$$\frac{\dot{m}\sqrt{T_t}}{p_t A} = \sqrt{\frac{\gamma}{R}}\left(\frac{p}{p_t}\sqrt{\frac{T_t}{T}}\right)\mathrm{M}\cdot\mathrm{Cos}\,a \Leftrightarrow$$

$$m_{nd} = \frac{\dot{m}\sqrt{T_t}}{p_t A}\sqrt{\frac{R}{\gamma}} = \mathrm{M}\left(1 + \frac{\overline{\gamma_h} - 1}{2}\frac{\gamma}{\overline{\gamma_h}}\mathrm{M}^2\right)^{\frac{1}{2} - \frac{\overline{C_{p\phi}}}{R}}\mathrm{Cos}\,a \quad \ldots (24)$$

In Equation 24, inlet Mach number based upon static temperature is also a function of gamma and Mach number upon stagnation temperature, as follows:

$$\frac{V^2}{\gamma RT} = \frac{V^2}{\gamma_t RT_t}\cdot\frac{\gamma_t}{\gamma}\cdot\frac{T_t}{T} \Leftrightarrow$$

$$\mathrm{M}^2 = \mathrm{M}_t^2\cdot\frac{\gamma_t}{\gamma}\cdot\left(1 + \frac{\overline{\gamma_h} - 1}{2}\frac{\gamma}{\overline{\gamma_h}}\mathrm{M}^2\right) \Leftrightarrow$$

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$$\mathrm{M} = \sqrt{\frac{\gamma_t}{\gamma}} \frac{\mathrm{M}_t}{\sqrt{1 - \frac{\overline{\gamma_h} - 1}{2} \frac{\gamma_t}{\overline{\gamma_h}} \mathrm{M}_t^2}} \quad \ldots (25)$$

In Equations 24 and 25, $\overline{\gamma_h}$ and $\overline{C_{p,\phi}}$ are averaged between the static and stagnation temperatures at the particular station. The left hand side of Equation 24 is the non-dimensional mass flow. Although it is considered to provide a correction for gas property variation because of the term $\sqrt{R/\gamma}$, there is still a weak influence of gas properties on the non-dimensional mass flow due to the existence of γ and R in the right hand side of the same equation.

5.3 Dimensionless rotational speed

By definition:

$$U = \frac{pi \cdot ND}{60} \quad \ldots (26)$$

Dividing both hand sides of Equation 26 by the velocity of sound based upon stagnation temperature at the inlet and making use of the definition of Equation 17, it is concluded that:

$$\frac{U}{\sqrt{\gamma_t RT_t}} = \frac{pi \cdot D}{60} \frac{N}{\sqrt{\gamma_t RT_t}} \Leftrightarrow$$

$$\mathrm{M}_{u,t} = \frac{U}{\sqrt{\gamma_t RT_t}} = \frac{pi}{60}\left(\frac{ND}{\sqrt{\gamma_t RT_t}}\right) \quad \ldots (27)$$

Unlike the case of the non-dimensional mass flow capacity, using the non-dimensional rotational speed shown in the parenthesis in the right hand side of Equation 27 provides full cover for variations of temperature and chemical composition. Still, the use of circumferential Mach number upon total conditions provides better physical understanding.

6.0 ERROR SENSITIVITY ANALYSIS

Scope of the present section is the derivation of exchange rates, which provide:

- the sensitivity of dimensionless mass flow versus gamma for a particular inlet Mach number based upon stagnation temperature.
- the sensitivity of the pressure ratio of a compression and an expansion against the variation of gamma, for a given component operating condition, namely constant Mach numbers upon stagnation temperature.

In the previous sections, the semi-perfect gas assumption leads to the introduction of proper mean values and their implementation in well-known equations. This affected among others, relationships, which correlate total with static properties, temperature ratios with pressure ratios and dimensionless mass flow with inlet Mach number. In order to facilitate the extraction of the exchange rates, the simplified form of these equations shall be used, which is derived when using the perfect gas assumption. Therefore, gamma is treated hereafter as a parametric constant.

6.1 Dimensionless mass flow versus inlet Mach number

It is common practice dimensionless mass flow to be used in place of axial Mach number and a one-to-one relationship is assumed to exist, since the variation of gas properties on the right hand side of Equation 24 is considered to cause practically insignificant variations. Scope of the following section is the derivation of the exchange rate for dimensionless mass flow given a change in the gas properties, when considering constant inlet Mach number.

Assuming axial flow at the entry of the component and keeping the same inlet Mach number based upon total temperature, dimensionless mass flow is then a function of gamma only, as follows:

$$m_{nd} = \mathrm{M}\left(1 + \frac{\gamma - 1}{2}\mathrm{M}^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \quad \ldots (28)$$

Or

$$m_{nd}(\gamma) = \mathrm{M}(\gamma)\, f(\gamma)^{g(\gamma)} \quad \ldots (29)$$

Where

$$f(\gamma) = 1 + \frac{\gamma - 1}{2}\left(\mathrm{M}(\gamma)\right)^2 \quad \ldots (30)$$

$$g(\gamma) = -\frac{\gamma + 1}{2(\gamma - 1)} \quad \ldots (31)$$

$$\mathrm{M}(\gamma) = \frac{\mathrm{M}_t}{\sqrt{1 - \frac{\gamma - 1}{2}\mathrm{M}_t^2}} \quad \ldots (32)$$

The first order derivatives of Equations 30 to 32 are expressed according to the following relationships:

$$\frac{\mathrm{dM}}{\mathrm{d}\gamma} = \frac{\mathrm{M}_t^3}{4}\left(1 - \frac{\gamma - 1}{2}\mathrm{M}_t^2\right)^{-3/2} \quad \ldots (33)$$

$$\frac{\mathrm{d}f}{\mathrm{d}\gamma} = \frac{\mathrm{M}^2}{2} + (\gamma - 1)\mathrm{M}\frac{\mathrm{dM}}{\mathrm{d}\gamma} \quad \ldots (34)$$

$$\frac{\mathrm{d}g}{\mathrm{d}\gamma} = \frac{1}{(\gamma - 1)^2} \quad \ldots (35)$$

Then

$$\frac{\mathrm{d}m_{nd}}{\mathrm{d}\gamma} = \frac{\mathrm{dM}}{\mathrm{d}\gamma} f^g + \mathrm{M}\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(f^g\right) \quad \ldots (36)$$

The chain rule applies:

$$\frac{\mathrm{d}\left(f^g\right)}{\mathrm{d}\gamma} = \frac{\mathrm{d}\left(e^{g \cdot \ln f}\right)}{\mathrm{d}\gamma} = e^{g \cdot \ln f}\frac{\mathrm{d}(g \cdot \ln f)}{\mathrm{d}\gamma} = f^g\left(\ln f \frac{\mathrm{d}g}{\mathrm{d}\gamma} + \frac{g}{f}\frac{\mathrm{d}f}{\mathrm{d}\gamma}\right) \quad \ldots (37)$$

The exchange rate is formed as:

$$\frac{\mathrm{d}m_{nd}/m_{nd}}{\mathrm{d}\gamma/\gamma} = \frac{\gamma}{m_{nd}}\left(\frac{\mathrm{dM}}{\mathrm{d}\gamma} f^g + \mathrm{M}\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(f^g\right)\right) \quad \ldots (38)$$

Making the necessary algebraic calculations and substituting the values of Equations 33 to 37 in Equation 38, typical values for the exchange rate of dimensionless mass flow versus gamma are derived and they are plotted against inlet Mach number in Fig. 3.

The inlet Mach number in Fig. 3 is based upon stagnation temperature and is used as the *x*-axis parameter. The absolute value of the exchange rate continuously increases with inlet Mach number, while the exact value of gamma plays a secondary role. Typical Mach number values during engine operation are much below 0·8, showing that a one percent increase in gamma would cause a reduction of less than 0·1% in dimensionless mass flow. The very small exchange rate verifies that a one-to-one relationship between inlet Mach number and dimensionless mass flow can be practically assumed.

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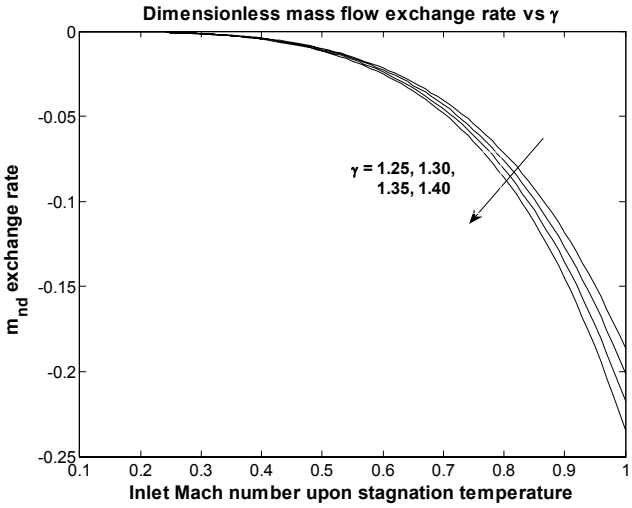


Figure 3. Dimensionless mass flow exchange rate versus gamma.

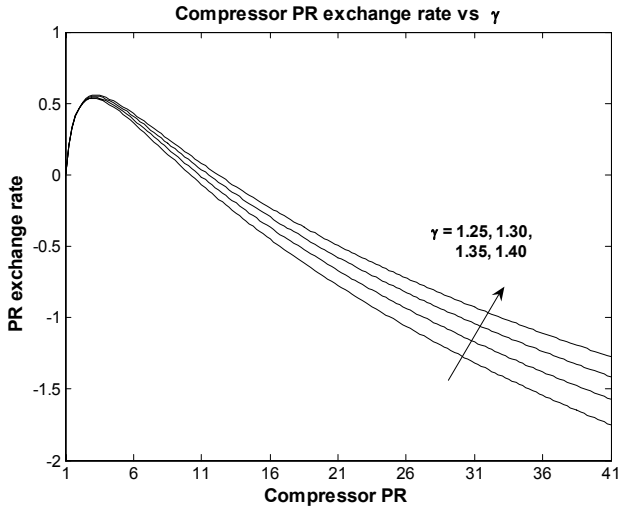


Figure 4. Compressor pressure ratio exchange rate versus gamma.

6.2 Pressure ratio dependency on gamma

For the analysis to follow, the same component geometry and operating Mach numbers are considered. Variation to the value of gamma is made only, in order to extract the sensitivity of pressure ratio against gas properties exclusively. Under this frame, Equations 22 and 23 can be rewritten as:

$$\pi_C(\gamma) = \left\{1 + k_C(\gamma - 1)\right\}^{\frac{\gamma}{\gamma-1}} \quad \ldots (39)$$

$$\pi_T(\gamma) = \left\{1 - k_T(\gamma - 1)\right\}^{-\frac{\gamma}{\gamma-1}} \quad \ldots (40)$$

where k_C and k_T are treated as a constants.

Following the chain rule of Equation 37 and making the necessary algebra, the following exchange rate of pressure ratio versus gamma is derived for the cases of compression and expansion, respectively:

$$\frac{d\pi_C/\pi_C}{d\gamma/\gamma} = \frac{\gamma}{\gamma-1}\left\{\frac{\gamma}{\gamma-1}\cdot\frac{\pi_C^{\frac{\gamma-1}{\gamma}} - 1}{\pi_C^{\frac{\gamma-1}{\gamma}}} - \frac{\ln \pi_C}{\gamma}\right\} \quad \ldots (41)$$

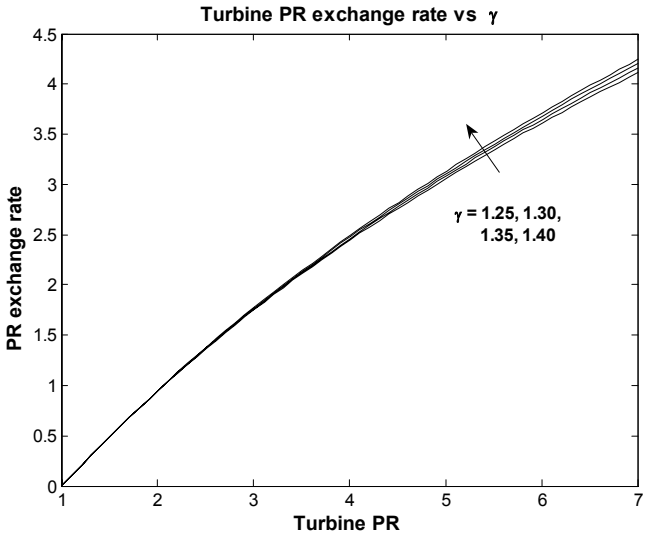


Figure 5. Turbine pressure ratio exchange rate versus gamma.

$$\frac{d\pi_T/\pi_T}{d\gamma/\gamma} = \frac{\gamma}{\gamma-1}\left\{\frac{\gamma}{\gamma-1}\cdot\frac{1 - \pi_T^{-\frac{\gamma-1}{\gamma}}}{\pi_T^{-\frac{\gamma-1}{\gamma}}} - \frac{\ln \pi_T}{\gamma}\right\} \quad \ldots (42)$$

It is of importance to notice that the exchange rate is independent of any Mach numbers or parameters dealing with the component geometry and only depends on gamma and pressure ratio.

According to Fig. 4, an increase or reduction of the compressor pressure ratio — given a change in gas properties for a particular operating point — depends on the absolute value of the pressure ratio. Given a zero value of the exchange rate for a pressure ratio of approximately 11, the exchange rate absolute values do not exceed unity for existing individual compressors; this can be interpreted as one percent change of the pressure ratio given a 1% change in gamma.

For the case of turbines, pressure ratios vary in a range from 2 for single-stage turbines up to approximately 6 for multistage LP turbines. In contrast to the compressor case, the turbine exchange rate is characterised by values above unity, which rapidly increase with pressure ratio up to values of 3·5, as graphically presented in Fig. 5.

Generally, compressors experience a narrow variation of gas properties at their inlet, due to changes in ambient conditions and/or operating point of other compressors upstream. The aforementioned results show the necessity to apply corrections to the pressure ratio distribution of a compressor map, especially when variations in water index may lead to higher variations in gas properties.

However, the sensitivity of the turbine pressure ratio to variation of gas properties, due to changes in either temperature or chemical composition, is much higher compared to the sensitivity of the compressor pressure ratio. Additionally, the differences in chemical composition and temperature between ground rig-testing and actual operation are much higher than the ones encountered by a compressor due to temperature and humidity index variation at its inlet. The attempt to avoid large correction factors in the case of turbines justifies current practice, according to which, the turbine work parameter is directly represented as dimensionless enthalpy drop instead of pressure ratio.

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7.0 COMPATIBILITY EQUATIONS

7.1 Mass flow compatibility

For reasons of simplicity, the function Q_{nd} is introduced as follows:

$$Q_{nd} = \mathrm{M}\left(1+\frac{\overline{\gamma_h}-1}{2}\frac{\gamma}{\overline{\gamma_h}}\mathrm{M}^2\right)^{\frac{1}{2}\frac{\overline{C_{p,\phi}}}{R}} \quad \ldots (43)$$

It has to be noted that Q_{nd} is a function of Mach number, static or total temperature and fuel to air ratio, with the last two parameters being used for the exact calculation of the caloric properties. In order to remind the reader, these parameters are going to be enclosed in parenthesis following the function. The mass flow compatibility relation can then be written between two engine stations i and j as follows:

$$\frac{\dot{m}_i}{\dot{m}_j}\sqrt{\frac{T_{t,i}}{T_{t,j}}}\frac{p_{t,j}}{p_{t,i}}\sqrt{\frac{R_i}{R_j}\frac{\gamma_j}{\gamma_i}} = \frac{A_i}{A_j}\frac{Q_{nd}(\mathrm{M}_i,T_i,far_i)}{Q_{nd}(\mathrm{M}_j,T_j,far_j)} \quad \ldots (44)$$

Or

$$\frac{\dot{m}_i}{\dot{m}_j} = \sqrt{\frac{T_{t,j}}{T_{t,i}}}\frac{p_{t,i}}{p_{t,j}}\sqrt{\frac{R_j}{R_i}\frac{\gamma_i}{\gamma_j}}\cdot\frac{A_i}{A_j}\frac{Q_{nd}(\mathrm{M}_i,T_i,far_i)}{Q_{nd}(\mathrm{M}_j,T_j,far_j)} \quad \ldots (45)$$

The ratio of mass flows calculated from the mass flow compatibility equation depends on the operating points of the components between stations i, j, that is on their pressure ratio and inlet Mach number, or equivalently for turbomachinery components on their pressure ratio and rotational speed. In case no mass flow storage is assumed, the left hand side of Equation 45 also has to satisfy continuity of mass flow. Such are cases of either steady state operation or transient operation during which mass flow storage is considered negligible. Such can be the transient behavior of aero-engines, where the transients are mostly experienced in terms of high rate of change of their rotational speed.

It is worthwhile to refer to the mass flow compatibility equation making use of the non-dimensional mass flow, as follows:

$$\frac{\dot{m}_i}{\dot{m}_j} = \sqrt{\frac{T_{t,j}}{T_{t,i}}}\frac{p_{t,i}}{p_{t,j}}\sqrt{\frac{R_j}{R_i}\frac{\gamma_i}{\gamma_j}}\cdot\frac{m_{nd,i}}{m_{nd,j}} \quad \ldots (46)$$

Comparison of Equations 45 and 46 shows that from the thermodynamics point of view, it is the dimensionless mass flow of significance. For example, if the exit of a component experiences a fixed value of dimensionless mass flow, it is of no importance whether this is due to the component downstream being large and operating at a low inlet Mach number, or being smaller and operating at a higher inlet Mach number. Consequently, the definition of size and axial Mach number mostly affects the aerodynamics of the component, rather than the thermodynamic performance of the whole system. However, use of Equation 45 takes full account of gas property variations through the direct use of the Q_{nd} function, which is not the case with Equation 46. Moreover, the calculation of the effective flow area is not required when non-dimensional mass flow is utilised in Equation 46, however it is necessary when using inlet Mach number as the flow parameter in Equation 45. This additional task is accomplished during the Design Point calculation and presupposes the specification of inlet Mach numbers. However, additional flexibility is provided in terms of modeling the Reynolds effects and/or component deterioration, since they both affect the boundary layer thickness and consequently the area the flow experiences.

In either case, the mass flow compatibility equation depends on ratios of mass flow, temperature and pressure, thus showing its major dependence on cycle parameters rather than engine size. Absolute values for temperature and chemical composition play a secondary role under the assumption of the semi-perfect gas.

7.2 Rotational speed compatibility

Considering that positions i and j can correspond to the inlets of two components rotating under the same rotational speed, for example a compressor and a turbine connected on the same spool, the compatibility relation for the rotational speed takes the form:

$$\frac{\mathrm{M}_{u,i}}{\mathrm{M}_{u,j}} = \frac{D_i}{D_j}\sqrt{\frac{(\gamma_t RT_t)_j}{(\gamma_t RT_t)_i}} \quad \ldots (47)$$

As in the case of mass flow compatibility, the rotational speed compatibility also depends on ratios of engine dimensions rather than absolute engine size.

8.0 DISCUSSION

The assumption of temperature and chemical composition dependency for gas properties, has lead to the introduction of proper mean values and their implementation in well-known equations. A non-dimensional analysis for the work exchange of a rotor row has revealed that the dimensionless enthalpy variation is the proper non-dimensional parameter for work. The latter is independent of any gas properties when interrelated to axial and circumferential Mach numbers based upon the stagnation temperature at the component's inlet. Although such an implementation provides unique turbomachinery characteristics, which are independent from the actual working fluid and conditions of the rig-testing procedure, differences to common practice are encountered.

Under this scope, a sensitivity analysis for dimensionless mass flow against gamma is undertaken. The exchange rate for dimensionless mass flow is rather small and a one-to-one relationship to inlet Mach number can be justified. Under these terms, dimensionless mass flow is easier to extract as value, since temperature, pressure and humidity index deviations from ISA and absolute mass flow can be directly measured during the rig-test.

As far as the work parameter is concerned, it is common practice to use pressure ratio and dimensionless enthalpy drop for compressors and turbines respectively. This can be explained on the basis of the pressure ratio exchange rate versus gamma, which for a compression takes lower values when compared to the respective expansion value. Additionally, the range of variation for the compressor inlet temperature and air composition is relatively narrow. On the contrary, for a turbine not only is the exchange rate higher, but also the relative difference in chemical composition and inlet temperature is enormous between ground rig-testing using cold air and actual operation. Therefore, the application of correction factors to compressor characteristics, which utilise pressure ratio, is common industrial practice when deviations from the reference conditions are encountered. And although this may work well for a compressor, it may appear problematic for the case of a turbine, justifying the use of dimensionless enthalpy drop instead.

Following the above, whole engine system behavior can be predicted, based on the modeling of individual components and on their interrelation. Component performance has been proven to be a function of dimensionless quantities, such as inlet and circumferential Mach numbers and dimensionless enthalpy variation. This function is unique when neglecting any secondary effects due to Reynolds number and is represented by the component's characteristics. Additionally, equations interrelating the component behavior, such as mass flow compatibilities, depend with first order precision on cycle parameters, such as ratios of pressure, temperature, dimensionless and absolute mass flow.

As a result steady state operation can be considered to depend on dimensionless, cycle parameters. Absolute temperature and chemical composition play a secondary role under the semi-perfect gas assumption only, while absolute engine mass flow or equivalently size also plays a minor role in the cases of constant power off-takes and the correction of component characteristics due to Reynolds number variation.

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9.0 CONCLUSIONS

In the current study a representation of component maps, which uses the inlet and circumferential Mach numbers, is discussed. Main purpose is the elimination of inaccuracies caused by varying gas properties, due to changes in temperature and/or chemical composition. Non-dimensional mass flow, rotational speed and work parameters are mathematically correlated with inlet and circumferential Mach numbers. Additionally, sensitivity analysis is carried out to identify the effects of varying gas properties. The following can be concluded:

1. Different mean values for specific heat can be derived from the definition of enthalpy and the Gibb's equation. Applying the appropriate mean values when proving the equations, differentiations are identified in well-known relationships, which correlate temperature with pressure ratios and static with stagnation properties.
2. The effect of varying gas properties is eliminated in terms of mass flow, when inlet Mach number is used as the flow parameter in the 'corrected' expression for the non-dimensional mass flow.
3. The non-dimensional speed directly relates to the circumferential Mach number, without the need of any correction factor being applied.
4. The practice of utilising pressure ratio instead of dimensionless enthalpy drop is justified for compressors only and it is not suitable for turbines, even when correction factors are applied. Firstly, this is due to the significantly higher sensitivity of turbine pressure ratio against gamma for a given dimensionless enthalpy drop. Secondly, because of the relatively high difference between the gas properties of cold air used during rig-testing and of the hot exhaust gases, which occur during actual operation.
5. Maps utilising dimensionless enthalpy variation and Mach numbers, which are based upon stagnation temperatures, are independent of the working fluid and the conditions under which the characteristic was extracted. This is beneficial when compared to the use of pressure ratio as the work parameter, since the application of correction factors is not required when deviations from the characteristic's reference conditions and/or working fluid are encountered.
6. The 'corrected' relationship, which correlates the dimensionless mass flow with inlet Mach number, is used to form the mass flow compatibility equation, when Mach number is the flow parameter.
7. As a consequence, the use of axial Mach number necessitates the calculation of the effective flow areas at the turbomachinery components' inlet, during the design point calculation. This is not necessary when dimensionless mass flow or an appropriate mass flow scaling parameter is utilised. However, the appearance of the effective flow area provides additional capability in terms of modeling the Reynolds number effects and/or component deterioration.

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