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A nonlinear partial differential system arising in thermoelectricity

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In this paper we prove the existence of a solution for a system of nonlinear parabolic partial differential equations arising from thermoelectric modelling of metallurgical electrodes undergoing a phase change. The model consists of an electromagnetic problem for eddy current computation coupled with a Stefan problem for temperature. The proof uses a regularized problem obtained by truncating the source term in temperature equation. Passing to the limit requires fine a priori estimates leading to compactness.

1 Introduction

In this paper we consider the following system of nonlinear parabolic differential equations posed in a bounded domain of $\mathbb{R}^3$:

$$\partial_t(\mu(x)\mathbf{h}) + \nabla \times \left(\frac{1}{\sigma(x,\theta)} \nabla \times \mathbf{h}\right) = \mathbf{0} \quad \text{in} \quad \Omega \times]0, T[, \tag{1.1}$$

$$\partial_t e - \nabla \cdot (k(x)\nabla\theta) = \frac{1}{\sigma(x,\theta)} |\nabla \times \mathbf{h}|^2 \quad \text{in} \quad \Omega \times]0, T[, \tag{1.2}$$

together with

$$e(x,t) \in g(x, \theta(x,t)) \text{ a.e.} \quad \text{in} \quad \Omega \times]0, T[. \tag{1.3}$$

Equation (1.1) is the eddy current model in a time domain formulation. It is obtained from Maxwell's equations by neglecting the term involving the displacement current in Ampère's law. Equation (1.2), with e given by (1.3), is the heat transfer equation with a heat source due to the Joule effect. We notice that enthalpy can be a multi-valued function of temperature because a phase change is allowed.

This system appears when modelling heat transfer in devices where thermal dependence of electric conductivity and the Joule effect are significant for their behaviour. Sometimes these devices are called "thermistors" (see [17]).

For 2D models, the term $\nabla \times (\frac{1}{\sigma} \nabla \times \mathbf{h})$ is replaced by $\nabla \cdot (\frac{1}{\sigma} \nabla h)$. Such models are considered by Clain & Touzani [18, 19] for alternating currents, and also in Parietti & Rappaz [29] for unsteady cases. Although Rodrigues [31] also considers a 2D case, an alternating current is not assumed. Bermúdez & Muñoz [8] approach the alternating current model from a different perspective by considering axisymmetry and using cylindrical

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coordinates. Phase changes in 2D models are taken into account less frequently. Although Yuan [36] is among these that do, he considers direct current.

In general, no differences between 2D and 3D models need to be established for direct currents, since the current conservation law is used instead of (1.1). With regard to steady solutions, Cimatti & Prodi [17] show the existence with Dirichlet boundary conditions and σ being a given function. Howison *et al.* [25] revisit the problem, with rather general hypotheses. Antontsev & Chipot [4] study the existence, uniqueness and regularity of solutions for transient thermal models with direct current. In the article by Shi *et al.* [32], a 3D model for a direct current "thermistor" with phase change is considered. Xu & Shillor [34] include a convective term in this model.

The articles most relevant to our work are those studying a transient thermal model coupled with eddy current in 3D. Yin [35] analyzed this model together with $\nabla \cdot \mathbf{h} = 0$. In the same year, Bossavit & Rodrigues [14] extended Yin's results to the case where thermal and electric conductivities may depend on position and temperature; a priori estimates were obtained using Boccardo–Gallouët techniques [13]. Although g is the identity in both articles, we have taken Bossavit & Rodrigues [14] as the starting point for our work.

Since tackling the Stefan problem (1.2)–(1.3) is the main difficulty, we are particularly concerned with the solution of nonlinear parabolic partial differential equations. Grange & Mignot [24] dealt with $\mathrm{d}(Bu)/\mathrm{d}t + Au = f$, with A and B being subgradients of continuous convex functionals. DiBenedetto & Showalter [23], Bermúdez & Saguez [9] and Bermúdez *et al.* [7] have studied different generalizations of the problem. Blanchard & Francfort [10, 11] analyzed the particular case where $Bu = b(u)$, b being a locally Lipschitz-continuous monotone increasing function with some time regularity of the forcing term f assumed. Their proof uses Lemma 1.5 of Alt & Luckhaus [3], which is related to our Lemma A.3. In Blanchard & Francfort [12] they also consider some multi-valued cases for B.

Another difficulty stems from the fact that, since the natural setting for solution $\mathbf{h}$ of problem (1.1) is the space $L^2(0, T; \mathbf{H}(\mathbf{curl} : \Omega))$, the right-hand side of the nonlinear parabolic equation (1.2) is only in $L^1(\Omega \times]0, T[)$. Thus we must truncate the source term and obtain suitable estimates for the truncated problem. In the previously mentioned paper [12] and in papers by Decarreau & Rakotoson [21] and Abergel *et al.* [1], the source term is general enough to include functions in $L^1(\Omega \times]0, T[)$. However, their operator B is single-valued and does not explicitly depend on x.

In short, while a few "thermistor" articles deal with other types of electricity than direct currents, they assume simple expressions for the enthalpy. In the present article, however, we are concerned with low frequency currents (which include both direct and alternating currents) with a (possibly multi-valued) enthalpy depending on both point x and temperature. Our objective is to model electrodes for production of "silicon metal" in reduction furnaces. In particular we are interested in the numerical calculation of the ELSA electrode developed by FerroAtlántica (see Figure 1). This consists of a graphite core surrounded by a layer of paste particles that becomes firm during baking. This phase change is the key to understanding the behaviour of the electrodes, their efficiency and possible breakage. Since the ELSA electrode is compound and the thermoelectric parameters vary widely from the graphite core to the paste, their spatial dependence should be considered. See Bermúdez *et al.* [6] for further details.

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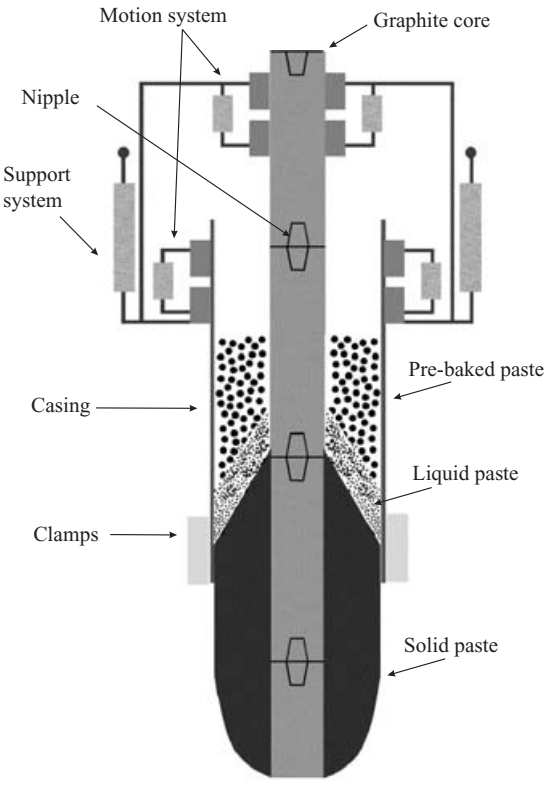


FIGURE 1. Sketch of ELSA electrode.

The model that we shall study is system (1.1)–(1.3) consisting of an eddy current equation coupled with a transient heat equation with a phase change. Boundary conditions for the electromagnetic equation correspond to the free exit of electricity on the part of the surface touching the electric arc and to a given tangential component of **h** on the rest of the boundary. For the sake of simplicity, homogeneous Dirichlet conditions for temperature are assumed. We shall consider two different cases for the function g: either g is a Lipschitz-continuous function with a positive and bounded derivative or it is a continuous function except at the sole discontinuity point. We also assume additional conditions on regularity and monotonicity.

This paper is organized as follows: § 2 and 3 introduce the model and establish assumptions about parameters and boundary conditions. Before introducing the existence problem, in § 4 we first discuss two weak formulations because proving the theorem requires an auxiliary problem with truncated heat source. In § 5, the existence of a solution to this new problem is achieved by using Schauder's fixed point theorem together with some estimates. Adapting Boccardo–Gallouët's techniques [13] in § 6 we obtain another estimate for temperature which is independent of the truncation parameter. Then, in § 7, we show pointwise convergence of the temperature, both in single-valued and multi-valued cases.

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With this information, it is possible to pass to the limit in the truncation parameter, which is done in § 8. The Appendix contains some results used throughout.

1.2 Statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^3$ with a locally Lipschitz-continuous boundary denoted by Γ. Let Γ_D be a measurable subset of Γ and $\Gamma_C = \Gamma \setminus \Gamma_D$. For $T > 0$, we denote $Q_T = \Omega \times]0, T[$. We assume that $\overline{\Omega}$ is decomposed as $\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i$ where Ω_i, $i = 1, \dots, N$, are mutually disjoint domains with locally Lipschitz-continuous boundaries. Physically, Ω_i is the part of Ω occupied by the i-th material.

We shall deal with the following coupled electromagnetic-thermal problem: Find $\mathbf{h} : Q_T \mapsto \mathbb{R}^3$, $\theta : Q_T \mapsto \mathbb{R}$ and $e : Q_T \mapsto \mathbb{R}$, satisfying equations (1.1)–(1.3) together with the electromagnetic boundary conditions

$$\mathbf{h} \times \mathbf{n} = \mathbf{f} \qquad \text{on } \Gamma_D \times]0, T[, \tag{1.4}$$

$$(\nabla \times \mathbf{h}) \times \mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_C \times]0, T[, \tag{1.5}$$

the thermal boundary condition

$$\theta = 0 \quad \text{on} \quad \Gamma \times]0, T[, \tag{1.6}$$

and the initial conditions

$$\mathbf{h}(0) = \mathbf{h}_0 \quad \text{in} \quad \Omega, \tag{1.7}$$

$$e(0) = e_0 \quad \text{in} \quad \Omega. \tag{1.8}$$

Equation (1.1) comes from the eddy current model of Maxwell's equations, where μ, σ and $\mathbf{h}$ stand for magnetic permeability, electric conductivity and magnetic field respectively. Equation (1.2) is the heat equation, where k, θ and e denote, respectively, thermal conductivity, temperature and enthalpy. The term on the right-hand side of equation (1.2) accounts for Joule heating. The operator $g(x, \cdot)$ can be multi-valued in (1.3). We have denoted by $\mathbf{n}$ the unit normal vector to the surface Γ, pointing outside Ω. The Dirichlet boundary data $\mathbf{f}$ must satisfy $\mathbf{f} \cdot \mathbf{n} = 0$ on Γ_D.

2 Functional spaces and data assumptions

In this section we introduce some functional spaces and assumptions on physical parameters and boundary conditions.

2.1 Functional spaces

For a bounded domain $\Omega \subset \mathbb{R}^3$ with a locally Lipschitz-continuous boundary, we consider the classical Lebesgue spaces $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, and the Sobolev spaces $W^{1,p}(\Omega)$. As usual, we denote by $W_0^{1,p}(\Omega)$ the closure of $\mathscr{D}(\Omega)$ in $W^{1,p}(\Omega)$, $H^1(\Omega) = W^{1,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. Moreover, $\mathbf{L}^p(\Omega) = [L^p(\Omega)]^3$,

$$\mathbf{V} = \mathbf{H}(\mathbf{curl} : \Omega) = \left\{ \mathbf{f} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{f} \in \mathbf{L}^2(\Omega) \right\}$$

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and

$$\mathbf{V}_0 = \left\{ \mathbf{f} \in \mathbf{V} : \ \mathbf{f} \times \mathbf{n} = \mathbf{0} \text{ in } [H_{00}^{-1/2}(\Gamma_D)]^3 \right\}$$

where $H_{00}^{-1/2}(\Gamma_D)$ is the dual space of $H_{00}^{1/2}(\Gamma_D)$. The latter is defined by

$$H_{00}^{1/2}(\Gamma_D) = \left\{ v \in H^{1/2}(\Gamma_D) : \tilde{v} \in H^{1/2}(\partial\Omega) \right\},$$

where $\tilde{v}$ is the extension of v to $\partial\Omega$ vanishing on $\partial\Omega \backslash \Gamma_D$. We have $\mathbf{V}_0 \subset \mathbf{L}^2(\Omega)$ with dense inclusion and identify $\mathbf{L}^2(\Omega)$ with its dual space, so $\mathbf{V}_0 \subset \mathbf{V} \subset \mathbf{L}^2(\Omega) \subset \mathbf{V}_0'$.

Let $T > 0$ and X be a Banach space. We also use the classical Banach spaces $L^p(0, T; X)$, $1 \leqslant p \leqslant \infty$. In particular, we have $L^p(0, T; L^p(\Omega)) \cong L^p(Q_T)$.

We denote by $\mathscr{C}^0([0, T]; X)$ the space of continuous functions from $[0, T]$ into X, and $W^{1,p}(0, T; X) = \{v \in L^p(0, T; X);\ \frac{dv}{dt} \in L^p(0, T; X)\}$, where the derivative is understood in the sense of the distributions on $]0, T[$ in X. Finally, we denote $H^1(0, T; X) = W^{1,2}(0, T; X)$.

2.2 Assumptions on coefficients

- $\mu : \Omega \mapsto \mathbb{R}$ and $k : \Omega \mapsto \mathbb{R}$ are measurable and there are constants μ_*, μ^*, k_*, k^* such that

$$0 < \mu_* \leqslant \mu(x) \leqslant \mu^* \text{ a.e. } x \in \Omega. \tag{2.1}$$

$$0 < k_* \leqslant k(x) \leqslant k^* \text{ a.e. } x \in \Omega. \tag{2.2}$$

- $\sigma : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function and there are two constants σ_*, σ^* such that

$$0 < \sigma_* \leqslant \sigma(x, s) \leqslant \sigma^* \quad \forall s \in \mathbb{R}, \quad \text{a.e. } \ x \in \Omega. \tag{2.3}$$

2.3 Assumptions on boundary and initial conditions

We make the following assumption concerning Dirichlet boundary data $\mathbf{f}$:
There exists a field $\mathbf{h}_\mathbf{f} \in L^2(0, T; \mathbf{V}) \cap \mathscr{C}^0([0, T]; \mathbf{L}^2(\Omega))$, with $\partial_t(\mu \mathbf{h}_\mathbf{f}) \in L^2(0, T; \mathbf{V}_0')$ such that

$$\mathbf{h}_\mathbf{f} \times \mathbf{n} = \mathbf{f} \text{ in } L^2\big(0, T; [H_{00}^{-1/2}(\Gamma_D)]^3\big). \tag{2.4}$$

Concerning the initial conditions we assume

$$\mathbf{h}_0 \in \mathbf{L}^2(\Omega), \tag{2.5}$$

$$e_0 \in L^1(\Omega). \tag{2.6}$$

2.4 Enthalpy dependence with respect to temperature

Here we distinguish two cases:

1. *The single-valued case.* We assume that $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying the

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following conditions:

- $x \mapsto g(x,s)$ is constant on each Ω_i, $i = 1, \dots, N$, for all $s \in \mathbb{R}$, (2.7)
- $g(x,\cdot)$ is Lipschitz-continuous in $\mathbb{R}$ a.e. $x \in \Omega$ and there are two constants, g_*, g^*, such that $0 < g_* \leqslant \partial_s g(x,s) \leqslant g^*$ a.e. $s \in \mathbb{R}$, a.e. $x \in \Omega$. (2.8)
- $g(x,0) = 0$ a.e. $x \in \Omega$. (2.9)

2. *The multi-valued case.* Maximal monotone operators in $\mathbb{R}$ are induced by nondecreasing functions. Hence we denote with the same letter a nondecreasing function and the maximal monotone operator induced by it. We assume that:

- For all $i = 1, \dots, N$, there is a maximal monotone operator g_i with $D(g_i) = \mathbb{R}$ such that $g(x,.) = g_i \ \forall x \in \Omega_i$. (2.10)
- For all $i = 1, \dots, N$, function $g_i : \mathbb{R} \mapsto \mathbb{R}$ is continuous everywhere except, at most, at one point $\hat{\theta}_i \neq 0$. (2.11)

Let $g_i^- = g_{i|(-\infty,\hat{\theta}_i]}$ and $g_i^+ = g_{i|[\hat{\theta}_i,+\infty)}$ (where $g_i^{\pm}(\hat{\theta}_i) = g_i(\hat{\theta}_i \pm 0)$).

- Function g_i^- (resp. g_i^+) is Lipschitz-continuous on $(-\infty, \hat{\theta}_i]$ (resp. $[\hat{\theta}_i, +\infty)$), for all $i = 1, \dots, N$. (2.12)
- We have $R(g(x,\cdot)) = \mathbb{R}$ and the inverse operator $j(x,\cdot) = (g(x,\cdot))^{-1}$ is single-valued a.e. $x \in \Omega$ (Hence, $j(x,\cdot)$ is a continuous function defined in the whole $\mathbb{R}$). We denote $j_i = g_i^{-1}$ so that $j(x,\cdot) = j_i$ for $x \in \Omega_i$. (2.13)
- $j(x,\cdot)$ is Lipschitz-continuous (we denote its Lipschitz constant by K_1). (2.14)
- $g(x,0) = 0$ a.e. $x \in \Omega$. (2.15)

Remark 2.1 *Clearly, assumptions* (2.7)–(2.9) *imply* (2.10)–(2.15)*, so the framework for the multi-valued case is an extension of that for the single-valued case. Nevertheless, some proofs for the single-valued case are different and easier to obtain.*

Remark 2.2 *Assumptions* (2.10)–(2.15) *guarantee that* $g(x,\cdot)$ *is a strongly monotone operator, uniformly in* x*. In fact we have, for any* $s_1, s_2 \in \mathbb{R}$*,*

$$(\xi_1 - \xi_2)(s_1 - s_2) \geqslant K_1^{-1}|s_1 - s_2|^2 \ \forall \xi_1 \in g(x,s_1) \ \forall \xi_2 \in g(x,s_2) \quad a.e.\ x \in \Omega. \tag{2.16}$$

In both cases we define the (possibly multi-valued) operator G in $L^2(\Omega)$ by

$$e \in G(\theta) \ \text{ if and only if } \ e(x) \in g(x,\theta(x)) \text{ a.e. in } \Omega. \tag{2.17}$$

Assumptions (2.10)–(2.12) ensure that G is a maximal monotone operator in $L^2(\Omega)$, its domain $D(G) = L^2(\Omega)$ and there are two constants $K_2 > 0$ and $K_3 > 0$ such that

$$\|e\|_{L^2(\Omega)} \leqslant K_2 \|\theta\|_{L^2(\Omega)} + K_3 \quad \forall \theta \in L^2(\Omega) \quad \forall e \in G(\theta), \tag{2.18}$$

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so G is bounded. In the single-valued case, G is also single-valued and continuous from $L^2(\Omega)$ into itself. Let $J = G^{-1}$. Taking into account (2.13), we can write

$$\theta \in J(e) \text{ if and only if } \theta(x) = j(x, e(x)) \quad \text{a.e. in} \quad \Omega, \tag{2.19}$$

so J is single-valued. Moreover, thanks to (2.14) and since $j(x,.) = j_i \ \forall x \in \Omega_i$, we have $R(G) = D(J) = L^2(\Omega)$. Let

$$\varphi(x,s) := \int_0^s g(x,r)\,\mathrm{d}r. \tag{2.20}$$

In view of (2.10) we have $\varphi(x,s) = \varphi_i(s) \quad \forall x \in \Omega_i$, where $\varphi_i(s) = \int_0^s g_i(r)\,\mathrm{d}r, \quad i = 1,\dots,N$. Then $\varphi_i \in \mathscr{C}^0(\mathbb{R})$, it is convex and $\partial\varphi_i = g_i$ for $i = 1,\dots,N$. Owing to assumptions (2.10)–(2.12) there are two constants $K_4 > 0$ and $K_5 > 0$ such that

$$|\varphi_i(s)| \leqslant K_4 s^2 + K_5 \quad \forall s \in \mathbb{R} \quad \forall i = 1,\dots,N. \tag{2.21}$$

Let $\Phi : L^2(\Omega) \mapsto \mathbb{R}$ be the functional defined by

$$\Phi(\theta) := \int_\Omega \varphi(x, \theta(x))\,\mathrm{d}x \ = \ \sum_{i=1}^N \int_{\Omega_i} \varphi_i(\theta(x))\,\mathrm{d}x. \tag{2.22}$$

Continuity of functions φ_i and growth property (2.21) yield $\Phi \in \mathscr{C}^0(L^2(\Omega);\mathbb{R})$. Besides, Φ is convex and $\partial\Phi = G$.

3 Weak formulation of the coupled problem. An existence result

Using a Green's formula and standard manipulations, the following weak form of problem (1.1)–(1.8) can be derived:

(WP1) *Find* $\mathbf{h} \in L^2(0,T;\mathbf{V})$, $\theta \in L^a(0,T;W_0^{1,a}(\Omega))$ *and* $e \in L^a(Q_T)$, *with* $1 < a < \infty$, *such that:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu\mathbf{h}\cdot\mathbf{v} + \int_\Omega \frac{1}{\sigma(\theta)}(\nabla\times\mathbf{h})\cdot(\nabla\times\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V_0}, \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e\,w + \int_\Omega k\nabla\theta\cdot\nabla w = \int_\Omega \frac{1}{\sigma(\theta)}|\nabla\times\mathbf{h}|^2\,w \quad \forall w \in W_0^{1,a'}(\Omega)\cap L^\infty(\Omega), \tag{3.2}$$

$$\mathbf{h}\times\mathbf{n} = \mathbf{f} \ \textit{in} \ L^2\big(0,T;[H_{00}^{-1/2}(\Gamma_D)]^3\big), \tag{3.3}$$

$$e(x,t) \in g(x,\theta(x,t)) \ \textit{a.e. in} \ Q_T \tag{3.4}$$

$$\mathbf{h}(0) = \mathbf{h}_0 \quad \textit{in} \quad \mathbf{L}^2(\Omega), \tag{3.5}$$

$$e(0) = e_0 \quad \textit{in} \quad W^{-1,s}(\Omega), \quad \textit{with} \ s \in \left(1, \frac{3}{2}\right), \ s \leqslant a. \tag{3.6}$$

Equations (3.1) and (3.2) are understood in the sense of distributions on $]0,T[$.

In (3.2) and throughout the whole paper, we denote by a' the conjugate exponent of a $\left(\frac{1}{a} + \frac{1}{a'} = 1\right)$. We must justify that equations (3.5) and (3.6) are meaningful. The next Lemma is a straightforward consequence of assumption (2.3).

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Lemma 3.1 *(i) For each $u \in L^1(\Omega)$, the operator $A(u) : \mathbf{V} \to \mathbf{V}_0'$ defined by*

$$\langle A(u)\mathbf{v}, \mathbf{z}\rangle_{\mathbf{V}_0', \mathbf{V}_0} = \int_\Omega \frac{1}{\sigma(u)}(\nabla \times \mathbf{v}) \cdot (\nabla \times \mathbf{z}) \quad \forall \mathbf{z} \in \mathbf{V}_0,$$

is linear and continuous from $\mathbf{V}$ into $\mathbf{V}_0'$. Moreover, there are three constants $C, \alpha, \beta > 0$ independent of u such that $\|A(u)\|_{\mathscr{L}(\mathbf{V}, \mathbf{V}_0')} \leqslant C$ and

$$\langle A(u)\mathbf{v}, \mathbf{v}\rangle_{\mathbf{V}_0', \mathbf{V}_0} + \alpha\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} \geqslant \beta\|\mathbf{v}\|^2_{\mathbf{V}_0} \ \forall\ \mathbf{v} \in \mathbf{V}_0. \tag{3.7}$$

(ii) The operator $\eta \in W_0^{1,a}(\Omega) \to -\nabla \cdot (k\nabla\eta) \in W^{-1,a}(\Omega)$ is linear and continuous.

Remark 3.1 *Let $\mathbf{h} \in L^2(0, T; \mathbf{V})$ and $\theta \in L^1(Q_T)$ satisfy equations* (3.1) *and* (3.3). *Since $\mu \in L^\infty(\Omega)$, we have $\mu\mathbf{h} \in \mathbf{L}^2(Q_T)$. Then from* (3.1)

$$\partial_t(\mu\mathbf{h}) = -A(\theta(t))\mathbf{h} \in L^2(0, T; \mathbf{V}_0').$$

Thanks to assumption (2.4), $\mathbf{h} \in \mathscr{C}^0([0, T]; \mathbf{L}^2(\Omega))$. *Thus the initial condition* (3.5) *is meaningful. Moreover, if $\theta \in L^a(0, T; W_0^{1,a}(\Omega))$ and $e \in L^a(Q_T)$ satisfy* (3.2), *we have*

$$\partial_t e = -\nabla \cdot (k\nabla\theta) + \frac{1}{\sigma(\theta)}|\nabla \times \mathbf{h}|^2 \in L^a(0, T; W^{-1,a}(\Omega)) + L^1(0, T; L^1(\Omega))$$

Since $L^1(\Omega) \subset W^{-1,s}(\Omega)$ if $s' > 3$, we get $e \in \mathscr{C}^0([0, T]; W^{-1,s}(\Omega))$ for any $s \in (1, \frac{3}{2})$, $s \leqslant a$. Hence equation (3.6) *is meaningful.*

Another weak formulation of problem (1.1)–(1.8) is the following:

(WP2) *Find $\mathbf{h} \in L^2(0, T; \mathbf{V})$, $\theta \in L^a(0, T; W_0^{1,a}(\Omega))$ and $e \in L^a(Q_T)$, with $1 < a < \infty$, such that equations* (3.3) *and* (3.4) *hold and*

$$-\int_{Q_T} \mu\mathbf{h} \cdot \partial_t\boldsymbol{\zeta} + \int_{Q_T} \frac{1}{\sigma(\theta)}(\nabla \times \mathbf{h}) \cdot (\nabla \times \boldsymbol{\zeta}) = \int_\Omega \mu\mathbf{h}_0 \cdot \boldsymbol{\zeta}(0) \qquad \forall \boldsymbol{\zeta} \in \mathbf{Z}_{\mathbf{V}}, \tag{3.8}$$

$$-\int_{Q_T} e\,\partial_t\phi + \int_{Q_T} k\nabla\theta \cdot \nabla\phi = \int_{Q_T} \frac{1}{\sigma(\theta)}|\nabla \times \mathbf{h}|^2\phi + \int_\Omega e_0\,\phi(0) \qquad \forall \phi \in \Phi_{a'}, \tag{3.9}$$

where we have used the following spaces of test functions:

$$\mathbf{Z}_{\mathbf{V}} = \left\{\boldsymbol{\zeta} \in L^2(0, T; \mathbf{V}_0) :\ \partial_t\boldsymbol{\zeta} \in \mathbf{L}^2(Q_T),\ \boldsymbol{\zeta}(T) = 0\right\}, \tag{3.10}$$

$$\Phi_{a'} = \left\{\phi \in L^{a'}(0, T; W_0^{1,a'}(\Omega)) \cap \mathscr{C}^0\big([0, T]; L^\infty(\Omega)\big) :\ \partial_t\phi \in L^{a'}(Q_T),\ \phi(T) = 0\right\}. \tag{3.11}$$

Remark 3.2 *Using standard arguments and the fact that the space of finite sums $\sum_i \psi_i\mathbf{v}_i$, with $\mathbf{v}_i \in \mathbf{V}_0$ and $\psi_i \in \mathscr{C}^\infty([0, T])$, $\psi_i(T) = 0$, is dense in $\mathbf{Z}_{\mathbf{V}}$ (see Lions [27]), one can readily see that, for $\mathbf{h} \in L^2(0, T; \mathbf{V})$ satisfying* (3.3) *and $\theta \in L^1(Q_T)$, equations* (3.1) *and* (3.5) *are equivalent to* (3.8). *In the same way, for $\mathbf{h} \in L^2(0, T; \mathbf{V})$, $\theta \in L^a(0, T; W_0^{1,a}(\Omega))$ and $e \in L^a(Q_T)$, equation* (3.9) *implies* (3.2) *and* (3.6). *Hence if $(\mathbf{h}, \theta, e)$ is a solution of **(WP2)**, then it is also a solution of **(WP1)**.*

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The main result of this paper is the following.

Theorem 3.1 *Under the assumptions* (2.1)–(2.6) *and* (2.10)–(2.15), *problem* **(WP2)** *has at least one solution with* $a < \frac{5}{4}$.

Remark 3.3 *Our results also cover the case of a thermal conductivity depending solely on temperature (only one material is allowed). Assuming* $k : \mathbb{R} \mapsto \mathbb{R}$ *is continuous, positive, upper bounded and bounded away from zero, Kirchoff's transformation,*

$$u = \beta(\theta) := \int_0^\theta k(s)\, ds, \tag{3.12}$$

allows us to rewrite (1.2)-(1.3) *as*

$$\partial_t e - \Delta u = \frac{1}{\sigma}|\nabla \times \mathbf{h}|^2 \quad in \quad Q_T, \tag{3.13}$$

$$e(x,t) \in g(\beta^{-1}(u(x,t))) \ a.e. \ in \ Q_T, \tag{3.14}$$

Now, Theorem 3.1 can be applied.

The proof of theorem 3.1 follows the structure of the main theorem in Bossavit & Rodrigues [14] and is divided into several steps. In step one, we consider a weak problem obtained from **(WP1)** by truncation of the Joule heating term and some approximation of the thermal initial condition. We prove the existence of a solution for this truncated problem by means of Schauder's Fixed Point Theorem. To do this, we obtain initial estimates for this solution, but estimates for θ depend on the truncation parameter. In the second step, by using the techniques in Boccardo & Gallouët [13] we obtain additional estimates for θ independent of the truncation parameter. In the next step, we prove a compactness property of temperature. This step is easy to carry out in the single-valued case, but in the multi-valued case it needs more elaborated estimates. In the fourth step, we consider a new weak truncated problem which can be seen as a truncation of problem **(WP2)**. The solution of the weak truncated problem considered in the first step is a solution of this new one. Finally, by passing to the limit with respect to the truncation parameter, we obtain a solution of problem **(WP2)**.

4 The truncated problem

In this section we introduce a truncated problem obtained by approximating in $L^1(Q_T)$ the heat source term by functions in $L^\infty(Q_T)$. For $0 < M < \infty$ we set

$$T_M(s) = \max(-M, \min(s, M)). \tag{4.1}$$

Lemma 4.1 *Let* $e_0 \in L^1(\Omega)$. *There is a sequence* $e_0^M \in L^\infty(\Omega)$, *with* $e_0^M \in G(\theta_0^M)$ *and* $\theta_0^M \in H_0^1(\Omega)$, *such that* $\lim_{M\to+\infty} e_0^M = e_0$ *strongly in* $L^1(\Omega)$.

Proof We approximate e_0 in $L^1(\Omega)$ by a sequence of functions $e_0^M \in \prod_{i=1}^N \mathscr{D}(\Omega_i)$. Let $\theta_0^M = J(e_0^M)$, that is, $\theta_0^M(x) = j_i(e_0^M(x))$ a.e. $x \in \Omega_i$. From (2.13)–(2.15) we have $\theta_0^M \in \prod_{i=1}^N H_0^1(\Omega_i) \subset H_0^1(\Omega)$. □

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For each $M > 0$, we consider the following *weak truncated problem*:

(WT1)$_M$ *Find* $\mathbf{h}_M \in L^2(0,T;\mathbf{V}) \cap \mathscr{C}^0([0,T];\mathbf{L}^2(\Omega))$, $\theta_M \in L^2(0,T;H_0^1(\Omega))$ *and* $e_M \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ *such that:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu \mathbf{h}_M \cdot \mathbf{v} + \int_\Omega \frac{1}{\sigma(\theta_M)}(\nabla \times \mathbf{h}_M)\cdot(\nabla \times \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V_0}, \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e_M\, w + \int_\Omega k\nabla\theta_M \cdot \nabla w = \int_\Omega \frac{1}{\sigma(\theta_M)} T_M(|\nabla \times \mathbf{h}_M|^2) w \qquad \forall w \in H_0^1(\Omega), \tag{4.3}$$

$$\mathbf{h}_M \times \mathbf{n} = \mathbf{f} \quad in \quad L^2\big(0,T;[H_{00}^{-1/2}(\Gamma_D)]^3\big), \tag{4.4}$$

$$e_M(t) \in G(\theta_M(t)) \ a.e.\ in\ [0,T] \tag{4.5}$$

$$\mathbf{h}_M(0) = \mathbf{h}_0 \quad in \quad \mathbf{L}^2(\Omega), \tag{4.6}$$

$$e_M(0) = e_0^M \quad in \quad H^{-1}(\Omega). \tag{4.7}$$

Equations (4.2) and (4.3) are understood in the sense of distributions in $]0,T[$.

4.1 The existence of a solution to the truncated problem

To prove the existence of a solution to problem **(WT1)$_M$**, we shall use a fixed point argument. To do this, we first verify the existence and uniqueness of a solution to the uncoupled electromagnetic and thermal problems. Later on, we shall verify the hypothesis of Schauder's Theorem.

4.1.1 *The uncoupled electromagnetic problem*

Lemma 4.2 *For any* $\gamma \in L^1(Q_T)$ *there is a unique*

$$\mathbf{g}_\gamma \in L^2(0,T;\mathbf{V}) \cap \mathscr{C}^0([0,T];\mathbf{L}^2(\Omega)) \tag{4.8}$$

satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu \mathbf{g}_\gamma \cdot \mathbf{v} + \int_\Omega \frac{1}{\sigma(\gamma)}(\nabla \times \mathbf{g}_\gamma)\cdot(\nabla \times \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V_0} \quad in \quad \mathscr{D}'(]0,T[), \tag{4.9}$$

$$\mathbf{g}_\gamma \times \mathbf{n} = \mathbf{f} \quad in \quad L^2\big(0,T;[H_{00}^{-1/2}(\Gamma_D)]^3\big), \tag{4.10}$$

$$\mathbf{g}_\gamma(0) = \mathbf{h}_0 \quad in \quad \mathbf{L}^2(\Omega). \tag{4.11}$$

Proof It suffices to write $\tilde{\mathbf{g}}_\gamma = \mathbf{g}_\gamma - \mathbf{h}_\mathbf{f}$, to take into account (2.4) and to apply the existence and uniqueness Theorems stated in Chapter XVIII of Dautray & Lions [20], taking $\big(\mathbf{L}^2(\Omega),(\cdot,\cdot)_\mu\big)$ as pivot space, where $(\cdot,\cdot)_\mu$ denotes the (equivalent to the usual) scalar product given by $(\mathbf{v},\mathbf{w}) = \int_\Omega \mu \mathbf{v}\cdot\mathbf{w}$. □

From this lemma we can define the mapping

$$S_1 : \gamma \in L^1(Q_T) \to S_1\gamma = \mathbf{g}_\gamma \in L^2(0,T;\mathbf{V}).$$

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Lemma 4.3 *There exists a constant $C > 0$ independent of γ such that*

$$\|S_1(\gamma)\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|S_1(\gamma)\|_{L^2(0,T;\mathbf{V})} \leqslant C \qquad \forall \gamma \in L^1(Q_T). \tag{4.12}$$

Proof It suffices to multiply (4.9) by $\tilde{\mathbf{g}}_\gamma(t)$ and recall assumptions (2.1)–(2.4), and equation (4.11). □

Lemma 4.4 *The mapping S_1 is continuous from $L^1(Q_T)$ into $L^2(0,T;\mathbf{V})$ when both spaces are endowed with their respective strong topologies.*

Proof The proof follows the same lines as that of Lemma 2 of Bossavit & Rodrigues [14]. □

4.1.2 *The uncoupled thermal problem*

Lemma 4.5 *Let $\gamma \in L^1(Q_T)$ and $\mathbf{g}_\gamma = S_1(\gamma)$. For $M > 0$, let e_0^M be defined as in Lemma 4.1. Then there exists a unique couple (τ_γ, e_γ), fulfilling $\tau_\gamma \in L^2(0,T;H_0^1(\Omega))$, $e_\gamma \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(Q_T)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e_\gamma\, w + \int_\Omega k\nabla\tau_\gamma \cdot \nabla w = \int_\Omega \frac{1}{\sigma(\gamma)} T_M(|\nabla \times \mathbf{g}_\gamma|^2)\, w \quad \forall w \in H_0^1(\Omega), \tag{4.13}$$

$$e_\gamma(t) \in G(\tau_\gamma(t)) \quad \textit{a.e. in } [0,T], \tag{4.14}$$

$$e_\gamma(0) = e_0^M \quad \textit{in} \quad H^{-1}(\Omega). \tag{4.15}$$

Proof This problem fits into the framework of Theorem 2 of DiBenedetto & Showalter [23]. Referring to the notations in that paper, we take

- the spaces $V = H_0^1(\Omega)$, $W = L^2(\Omega)$ and the injection $i : V \hookrightarrow W$.
- the convex function $\varphi = \Phi : L^2(\Omega) \mapsto \mathbb{R}$ (defined in Subsection 2.4). Remember that $\partial\Phi = G$, so operator $\mathscr{A}$ in [23] is given by $\mathscr{A} = i^* \circ \partial\Phi \circ i = i^* \circ G \circ i$.
- the operator $\mathscr{B} : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ given by $\langle \mathscr{B}u, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega k\nabla u \nabla v$.

Recall that $e_0^M \in L^2(\Omega)$ and $e_0^M \in G(\theta_0^M)$ with $\theta_0^M \in H_0^1(\Omega)$. This is equivalent to saying that $e_0^M \in R(i^* \circ G \circ i) = R(\mathscr{A})$. The rest of the hypotheses of the aforementioned Theorem can be easily checked. Thus, we have the existence of a solution $(\tau_\gamma, e_\gamma) \in L^2(0,T;H_0^1(\Omega)) \times H^1(0,T;H^{-1}(\Omega))$ of (4.13)–(4.15). We also have uniqueness because $\mathscr{B}$ is a linear continuous self-adjoint operator (see Theorem 5 of DiBenedetto & Showaalter [23]).

We note that, $L^2(0,T;L^2(\Omega))$ regularity for e_γ is not a direct consequence of Theorem 2 of DiBenedetto & Showalter [23] as it is stated, but it stems from a careful examination of its proof. □

Given Lemma 4.5, we can define mappings $S_2 : \gamma \in L^1(Q_T) \mapsto \tau_\gamma \in L^2(0,T;H_0^1(\Omega))$ and $S_3 : \gamma \in L^1(Q_T) \mapsto e_\gamma \in H^1(0,T;H^{-1}(\Omega)) \cap L^2(Q_T)$.

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Lemma 4.6 *The following quantities are bounded for all* $\gamma \in L^1(Q_T)$*:*

$$\|\tau_\gamma\|_{L^2(0,T;H_0^1(\Omega))}, \qquad \|e_\gamma\|_{L^2(Q_T)}, \qquad \|\partial_t e_\gamma\|_{L^2(0,T;H^{-1}(\Omega))}, \tag{4.16}$$

with bounds depending on the truncation parameter M*.*

Proof We take $\tau_\gamma \in L^2(0,T;H_0^1(\Omega))$ as the test function in (4.13). We obtain

$$\begin{aligned}&\langle \partial_t e_\gamma(t), \tau_\gamma(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + k_* \int_\Omega |\nabla \tau_\gamma(t)|^2 \\ &\quad \leqslant C_\varepsilon \| \frac{1}{\sigma(\gamma)} T_M(|\nabla \times \mathbf{g}_\gamma(t)|^2)\|^2_{L^2(\Omega)} + \varepsilon \|\tau_\gamma(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{4.17}$$

for a.e. $t \in [0,T]$. We take ε small enough so that, from Poincare's inequality,

$$k_* \|\nabla \tau_\gamma(t)\|^2_{L^2(\Omega)} - \varepsilon \|\tau_\gamma(t)\|^2_{L^2(\Omega)} \geqslant C_1 \|\tau_\gamma(t)\|^2_{H_0^1(\Omega)}\,, \tag{4.18}$$

C_1 being a positive constant. By using this lower bound, integrating in $[0,t]$ and applying Lemma A.4 with $\psi(s) = s$ and j introduced in (2.13), we get

$$Q(e_\gamma(t)) + C_1 \|\tau_\gamma\|^2_{L^2(0,t;H_0^1(\Omega))} \leqslant Q(e_0^M) + C_\varepsilon \left\| \frac{1}{\sigma(\gamma)} T_M(|\nabla \times \mathbf{g}_\gamma|^2) \right\|^2_{L^2(0,t;L^2(\Omega))},$$

a.e. $t \in [0,T]$, where Q is defined through equations (A 6),(A 7) and (A 14).

Now we notice that functional Q takes nonnegative values because $j(x,0) = 0$ and, moreover, condition (2.14) ensures that Q is everywhere finite in $L^2(\Omega)$. In particular, $Q(e_0^M) < +\infty$ because $e_0^M \in L^2(\Omega)$. Thus we obtain

$$C_1 \|\tau_\gamma\|^2_{L^2(0,t;H_0^1(\Omega))} \leqslant Q(e_0^M) + \frac{C_\varepsilon\, M^2\, T\, \mathrm{meas}(\Omega)}{\sigma_*^2} \quad \text{a.e. } t \in [0,T],$$

and, since the left hand side is continuous at $t = T$, we obtain $\|\tau_\gamma\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C_M$. The second bound of (4.16) holds because G is a bounded (possibly multi-valued) operator from $L^2(Q_T)$ into itself. Finally, $\partial_t e_\gamma$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ because of equation (4.13). □

Lemma 4.7 *The set* $\{\tau_\gamma : \gamma \in L^2(Q_T)\}$ *is relatively compact in* $L^2(Q_T)$*.*

Proof Let γ_m be a sequence in $L^2(Q_T)$. With the help of Aubin's Theorem (see Lions [28], Chapter I, Theorem 5.1) and Lemma 4.6, we deduce that there is a subsequence still denoted γ_m such that $e_{\gamma_m} \to e$ strongly in $L^2(0,T;H^{-1}(\Omega))$ and $\tau_{\gamma_m} \rightharpoonup \tau$ weakly in $L^2(0,T;H_0^1(\Omega))$.

We recall that i denotes the injection $i : H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. Since Φ is continuous in $L^2(\Omega)$, we have $i^* \circ G \circ i = i^* \circ \partial\Phi \circ i = \partial(\Phi \circ i)$. Thus $i^* \circ G \circ i$ is a maximal monotone operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ and hence its realization is a maximal monotone operator from $L^2(0,T;H_0^1(\Omega))$ to $L^2(0,T;H^{-1}(\Omega))$. From Proposition 2.5 in Chapter II of Brézis [16],

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we conclude that $e \in G(\tau)$. Besides

$$\int_0^T \int_\Omega (e_{\gamma_m} - e)(\tau_{\gamma_m} - \tau) = \int_0^T \langle e_{\gamma_m} - e, \tau_{\gamma_m} - \tau \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \to 0, \tag{4.19}$$

Now, by using equation (2.16) we obtain the strong convergence of τ_{γ_m} in $L^2(Q_T)$. □

Lemma 4.8 *Let i be the injection $L^2(0,T;H_0^1(\Omega)) \hookrightarrow L^2(Q_T)$. The mapping $i \circ S_2$ is continuous and compact from $L^2(Q_T)$ into itself for the strong topology.*

Proof We start by showing continuity. Let $\gamma_m \to \gamma$ strongly in $L^2(Q_T)$. Recall that $S_2(\gamma_m) = \tau_{\gamma_m}$, where $(\tau_{\gamma_m}, e_{\gamma_m}) \in L^2(0,T;H_0^1(\Omega)) \times \left(L^2(Q_T) \cap H^1(0,T;H^{-1}(\Omega))\right)$ is the unique solution of the analogous equations to (4.13)–(4.15) with γ replaced by γ_m. From Lemmas 4.6 and 4.7 and Aubin's Theorem we deduce that there is a subsequence of γ_m, still denoted by γ_m, and two functions $\tau \in L^2(0,T;H_0^1(\Omega))$, $e \in L^2(0,T;H^{-1}(\Omega)) \cap L^2(Q_T)$, such that: $\gamma_m \to \gamma$ a.e. in Q_T, e_{γ_m} converges to e strongly in $L^2(0,T;H^{-1}(\Omega))$, weakly in $L^2(Q_T)$ and weakly in $H^1(0,T;H^{-1}(\Omega))$, and τ_{γ_m} converges to τ strongly in $L^2(Q_T)$ and weakly in $L^2(0,T;H_0^1(\Omega))$.

We must show that $\tau = S_2(\gamma)$. Firstly, as in the proof of Lemma 4.7, we have $e \in G(\tau)$. Secondly, the above limits ensure passing to the limit on the left-hand side of (4.13). Continuity of S_1 gives $\mathbf{g}_m \to \mathbf{g}$ strongly in $L^2(0,T;\mathbf{V})$. This fact, together with Lebesgue's Theorem and a.e. convergence of γ_m, yield to the convergence of the Joule term. Finally, since $e_{\gamma_m} \rightharpoonup e$ weakly in $H^1(0,T;H^{-1}(\Omega))$ and this space is embedded in $\mathscr{C}^0([0,T];H^{-1}(\Omega))$, $e_0^M = e_{\gamma_m}(0) \rightharpoonup e(0)$ weakly in $H^{-1}(\Omega)$, so $e(0) = e_0^M$. Thus, we have seen that couple $\tau_\gamma \in L^2(0,T;H_0^1(\Omega))$ and $e_\gamma \in L^2(Q_T) \cap H^1(0,T;H^{-1}(\Omega))$ satisfy equations (4.13)–(4.15) written for γ. Therefore $\tau = S_2(\gamma)$ and the whole sequence τ_{γ_m} converges strongly to τ in $L^2(Q_T)$. Compactness is immediate from Lemma 4.7. □

4.1.3 *Application of Schauder's fixed point theorem*

Theorem 4.1 *Under assumptions* (2.1)–(2.6) *and* (2.10)–(2.15)*, problem* ***(WT1)***$_M$ *has at least one solution.*

Proof The mapping $\mathscr{T} = i \circ S_2 : L^2(Q_T) \mapsto L^2(Q_T)$ is continuous and compact and, from Lemma 4.6, there is a constant $C_M > 0$ such that $\mathscr{T}$ maps $L^2(Q_T)$ into the closed ball $B_M = \{\phi \in L^2(Q_T) : \ \|\phi\|_{L^2(Q_T)} \leqslant C_M\}$. From Schauder's fixed point theorem, $\mathscr{T}$ has a fixed point $\theta_M \in B_M$. Then the triple $(S_1(\theta_M), S_2(\theta_M), S_3(\theta_M))$ is a solution of problem **(WT1)**$_M$. □

5 Boccardo-Gallouët like estimates

In this section we obtain estimates independent of the truncation parameter M.

In the following we shall denote by $(\mathbf{h}_M, \theta_M, e_M)$ a solution of **(WT1)**$_M$ and by C a positive constant independent of M but whose values may be different at each occurrence.

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From Lemma 4.3 we obtain the following bound for the magnetic field:

$$\|\mathbf{h}_M\|_{L^2(0,T;\mathbf{V})} \leqslant C. \tag{5.1}$$

Although the estimate in Lemma 4.6 provides a bound for temperature θ_M, this bound depends on the truncation parameter. Hence one cannot pass to the limit as $M \to \infty$. In order to obtain estimates for temperature θ_M (and also for enthalpy e_M), independent of M, we shall apply the ideas in section IV of Boccardo & Gallouët [13] to the thermal part of problem **(WT1)**$_M$.

Lemma 5.1 *For any $1 < a < \frac{5}{4}$ there is a constant C_a independent of M such that*

$$\|\theta_M\|_{L^a(0,T;W_0^{1,a}(\Omega))} \leqslant C_a, \tag{5.2}$$

We also have the estimates

$$\|e_M\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C, \tag{5.3}$$

$$\|\theta_M\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C. \tag{5.4}$$

Proof This is similar to that of Theorem 4 of Boccardo & Gallouët [13], where enthalpy dependence with respect to temperature is linear. Let the truncation function be $\psi = T_1$ (see eqn. (4.1)). Taking $\psi(\theta_M(t))$ as the test function in (4.3) we obtain, a.e. $t \in [0, T]$,

$$\langle \partial_t e_M(t),\, \psi(\theta_M(t))\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega k\nabla\theta_M(t)\,\nabla\psi(\theta_M(t)) = \int_\Omega f_M(t)\,\psi(\theta_M(t)), \tag{5.5}$$

where $f_M = \frac{1}{\sigma(\theta_M)} T_M(|\nabla \times \mathbf{h}_M|^2)$. By integrating in $[0,t]$ and applying Lemma A.4 in the Appendix, with $\psi = T_1$ and j introduced in (2.13), we get

$$Q(e_M(t)) \leqslant Q(e_M^0) + \|f_M\|_{L^1(Q_T)} \quad \text{a.e. } t \in [0,T]. \tag{5.6}$$

Moreover, using (2.13)–(2.15), we deduce that $q(x,r) = \int_0^r \psi(j(x,s))\,\mathrm{d}s$ satisfies

$$|r| - K \leqslant q(x,r) \leqslant |r| \;\; \forall r \in \mathbb{R} \quad \text{a.e. } x \in \Omega,$$

with some constant $K \geqslant 0$. Hence,

$$\int_\Omega |e_M(t)| \leqslant K\,\mathrm{meas}(\Omega) + \int_\Omega |e_0^M| + \|f_M\|_{L^1(Q_T)} \leqslant C \quad \text{a.e. } t \in [0,T],$$

the last inequality being due to (5.1) and strong convergence of e_0^M in $L^1(\Omega)$. Thus we have (5.3), and because of the properties of j also (5.4).
For $m \in \mathbb{N}$, let us denote $B_m = \{(x,t) \in Q_T \;:\; m \leqslant |\theta_M(x,t| \leqslant m+1\}$ and consider ψ defined by

$$\psi(s) = \begin{cases} 0 & \text{if } |s| \leqslant m, \\ (|s| - m)\,\mathrm{sign}\,(s) & \text{if } m \leqslant |s| \leqslant m+1, \\ \mathrm{sign}\,(s) & \text{if } |s| \geqslant m+1. \end{cases} \tag{5.7}$$

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By taking $\psi(\theta_M(t))$ as the test function in (4.3), integrating in $[0,t]$ and applying Lemma A.4 with ψ given by (5.7) and the same j as before, we obtain

$$Q(e_M(t)) + \int_0^t \int_\Omega k\psi'(\theta_M)|\nabla\theta_M|^2 = Q(e_M^0) + \int_0^t \int_\Omega f_M\,\psi(\theta_M) \text{ a.e. } t \in [0,T]. \tag{5.8}$$

The properties of j and ψ imply that Q is nonnegative and $Q(e_0^M) \leqslant \int_\Omega |e_0^M|$. Moreover, we have

$$\lim_{t\to T} \int_0^t \int_\Omega k\psi'(\theta_M)|\nabla\theta_M|^2 = \int_0^T \int_\Omega k\psi'(\theta_M)|\nabla\theta_M|^2 = \int_{B_m} k|\nabla\theta_M|^2$$

Considering (5.1), strong convergence of e_0^M in $L^1(\Omega)$ and assumption (2.2), we obtain

$$\int_{B_m} |\nabla\theta_M|^2 \leqslant C, \tag{5.9}$$

where C is a constant independent of both M and m. Inequalities (5.4) and (5.9) are those labeled (63) and (64) with $p = 2$ in the proof of Theorem 4 in Boccardo & Gallouët [13]. Then we obtain (5.2) by specializing equation (68) of that paper for $p = 2$ and $N = 3$. □

6 Pointwise convergence of the temperature

By using the estimates in the previous section, we will show pointwise convergence of the temperature as $M \to \infty$. For this purpose, we distinguish between single-valued and multi-valued cases.

6.1 The single-valued case

Lemma 6.1 *Under assumptions* (2.1)–(2.9)*, we have*

- *Sequence* e_M *is bounded in* $\prod_{i=1}^p L^a(0,T;W^{1,a}(\Omega_i))$ *for* $1 < a < \frac{5}{4}$. (6.1)
- *There is a subsequence of* θ_M *(still denoted by* θ_M*) and a function* $\theta \in L^a(0,T;W_0^{1,a}(\Omega))$ *for* $1 < a < \frac{5}{4}$*, such that*

$$\theta_M \to \theta \text{ a.e. in } Q_T, \tag{6.2}$$

$$\theta_M \rightharpoonup \theta \text{ weakly in } L^a(0,T;W_0^{1,a}(\Omega)) \text{ for } 1 < a < \frac{5}{4}. \tag{6.3}$$

- *Function* e *defined by* $e(x,t) = g(x,\theta(x,t))$ *belongs to* $L^a(Q_T)$ *for* $1 < a < \frac{5}{4}$ *and there is a subsequence of* e_M *(still denoted by* e_M*) such that*

$$e_M \to e \text{ strongly in } L^a(Q_T) \text{ for } 1 < a < \tfrac{5}{4}. \tag{6.4}$$

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Proof Property (6.1) can be easily derived from estimate (5.2) and the properties of g (see equations (2.7)–(2.9)). Equation (4.3) yields

$$\partial_t e_M = -\nabla \cdot (k \nabla \theta_M) + f_M.$$

Moreover, f_M is bounded in $L^1(Q_T)$ due to (5.1) and θ_M is bounded in $L^a(0, T; W_0^{1,a}(\Omega))$ for any $a < \frac{5}{4}$ by (5.2). Besides, $L^1(\Omega) \subset W^{-1,a}(\Omega)$. Therefore

$$\partial_t e_M \text{ is bounded in } L^1(0, T; W^{-1,a}(\Omega)). \tag{6.5}$$

Let $B_0 = \prod_{i=1}^p W^{1,a}(\Omega_i)$, $B = L^s(\Omega)$ and $B_1 = W^{-1,a}(\Omega)$, with $1 < s < \frac{3a}{3-a}$. It can be checked that B_0 is compactly embedded in B and $B \subset B_1$. A compactness result by Simon (see Simon [33], Corollary 4) together with (6.1) and (6.5) imply that there is a subsequence, still denoted by e_M, strongly convergent to a limit e in $L^a(0, T; L^s(\Omega))$. Put $\theta(x, t) = j(x, e(x, t))$. Recalling (2.14) and (5.2), we easily obtain (6.2) and (6.3). □

6.2 The multi-valued case

The technique used to prove pointwise convergence for temperature in the single-valued case consisted of obtaining space regularity estimates for temperature and time regularity estimates for enthalpy. Then, we derived space regularity estimates for enthalpy by using the fact that enthalpy is a regular function of temperature. This led successively to $L^1(Q_T)$-compactness of enthalpy, pointwise convergence of enthalpy and pointwise convergence of temperature. However, in the multi-valued case we cannot transfer space regularity from temperature to enthalpy because enthalpy is not even a single-valued function of temperature (see § 7.1).

The key idea is to construct a suitable function of enthalpy which can be written as a single-valued function solely of temperature. Here, assumption (2.11) plays a crucial role. For this new function we will be able to derive time regularity estimates from enthalpy and spatial ones from temperature.

Remark 6.1 *Since θ is a single-valued function of e, the thermal problem can be rewritten in terms of enthalpy as $\partial_t e - \nabla \cdot (k(x) \nabla j(x, e)) = f$ in Q_T, together with boundary condition $j(x, e) = 0$ in $\Gamma \times]0, T[$ and initial condition $e(x, 0) = e_0(x)$ in Ω, where $f = \frac{1}{\sigma(\theta)} |\nabla \times \mathbf{h}|^2$. In the truncated thermal problem, f must be replaced by $f_M = \frac{1}{\sigma(\theta_M)} T_M(|\nabla \times \mathbf{h}|^2)$. The results about the compactness of Green's operator stated in Díaz & Vrabie [22] cannot be applied in our context because, among other reasons, we only know that f_M is bounded in $L^1(Q_T)$.*

We refer to the Appendix for the definition and justification of the notation $\bullet$. For $a, b \in \mathbb{R}$, $a < b$, we set

$$T_{a,b}(s) = \max(a, \min(s, b)), \qquad T_{a,+\infty}(s) = \max(a, s), \qquad T_{-\infty,a}(s) = \min(a, s).$$

We recall that $\hat{\theta}_i \neq 0$ for any $i = 1, \ldots, N$. In what follows we consider $i \in \{1, \ldots, N\}$ fixed. We shall expose the case $\hat{\theta}_i > 0$ in detail.

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Lemma 6.2 *For any $R > 0$, there is a constant $C = C_R > 0$ independent of M such that*

$$\| T_{-R,\hat{\theta}_i}(\theta_M)\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C_R. \tag{6.6}$$

Moreover, for $1 < a < \frac{5}{4}$, there is a constant $C_{R,a} > 0$ independent of M such that

$$\| T_{-R,\hat{\theta}_i}(\theta_M) \bullet \partial_t e_M \|_{L^1(0,T;W^{-1,a}(\Omega))} \leqslant C_{R,a}. \tag{6.7}$$

Proof We take $\psi = T_{-R,\hat{\theta}_i}$ as the truncation function. Then ψ is Lipschitz-continuous and nondecreasing, and $\psi(0) = 0$ because $\hat{\theta}_i > 0$. Taking $\psi(\theta_M(t))$ as the test function in (4.3) and following the same lines used to obtain (5.9), we get

$$\int_{D_R} |\nabla \theta_M|^2 \leqslant C \max(R, \hat{\theta}_i) = C_R,$$

where $D_R = \{(x,t) \in Q_T : -R \leqslant \theta_M(x,t) \leqslant \hat{\theta}_i\}$. Since C is a constant independent of M, so is C_R. Because of $T_{-R,\hat{\theta}_i}$ is Lipschitz-continuous, using the chain rule we have

$$\int_{Q_T} |\nabla(T_{-R,\hat{\theta}_i}(\theta_M))|^2 \;=\; \int_{D_R} |\nabla \theta_M|^2 \leqslant C_R, \tag{6.8}$$

which yields (6.6).

Let $a < \frac{5}{4}$ be fixed and $a' > 4$ its conjugate. Let w be an arbitrary function in $W_0^{1,a'}(\Omega)$. Since $T_{-R,\hat{\theta}_i}(\theta_M(t)) \in H_0^1(\Omega)$ a.e. $t \in [0,T]$, Lemma A.1 ensures that $wT_{-R,\hat{\theta}_i}(\theta_M(t)) \in H_0^1(\Omega)$, hence it can be taken as the test function in (4.3). We get

$$\begin{aligned} &\left\langle \partial_t e_M(t), wT_{-R,\hat{\theta}_i}(\theta_M(t)) \right\rangle_{H^{-1}(\Omega)', H_0^1(\Omega)} + \int_\Omega k T'_{-R,\hat{\theta}_i}(\theta_M(t)) |\nabla \theta_M(t)|^2 w \\ &\quad + \int_\Omega k T_{-R,\hat{\theta}_i}(\theta_M(t)) \nabla \theta_M(t) \nabla w \;=\; \int_\Omega f_M(t) T_{-R,\hat{\theta}_i}(\theta_M(t)) w \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{6.9}$$

Recalling that $W^{1,a'}(\Omega) \subset \mathscr{C}^0(\overline{\Omega})$, we have

$$\left| \int_\Omega f_M(t) T_{-R,\hat{\theta}_i}(\theta_M(t)) w \right| \leqslant C_R \| f_M(t) \|_{L^1(\Omega)} \| w \|_{W^{1,a'}(\Omega)} \quad \text{a.e. } t \in [0,T] \tag{6.10}$$

$$\left| \int_\Omega k T_{-R,\hat{\theta}_i}(\theta_M(t)) \nabla \theta_M(t) \nabla w \right| \leqslant C_R \| \theta_M(t) \|_{W^{1,a}(\Omega)} \| w \|_{W^{1,a'}(\Omega)} \quad \text{a.e. } t \in [0,T], \tag{6.11}$$

and taking into account that

$$h(x,t) := k(x) T'_{-R,\hat{\theta}_i}(\theta_M(x,t)) |\nabla \theta_M(x,t)|^2 \geqslant 0 \quad \text{a.e. in } \; Q_T, \tag{6.12}$$

we also have

$$\left| \int_\Omega k T'_{-R,\hat{\theta}_i}(\theta_M(t)) |\nabla \theta_M(t)|^2 w \right| \leqslant C \| w \|_{W^{1,a'}(\Omega)} \int_\Omega h(x,t). \tag{6.13}$$

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Now, (6.12) and (6.8) yield

$$\int_0^T \int_\Omega h(x,t) = \int_{Q_T} k T'_{-R,\hat{\theta}_i}(\theta_M(t))|\nabla\theta_M(t)|^2 \leqslant k^* \int_{D_R} |\nabla\theta_M|^2 \leqslant C_R, \tag{6.14}$$

so $\int_\Omega h(x,t) < +\infty$ a.e. $t \in [0,T]$. Note that $T_{-R,\hat{\theta}_i}(\theta_M(t)) \in H_0^1(\Omega)$ and $\partial_t e_M(t) \in H^{-1}(\Omega)$ a.e. $t \in [0,T]$, then, by Remark A.1 in Appendix, $T_{-R,\hat{\theta}_i}(\theta_M(t)) \bullet \partial_t e_M(t) \in W^{-1,a}(\Omega)$. Moreover, since functions $t \in [0,T] \mapsto T_{-R,\hat{\theta}_i}(\theta_M(t)) \in H^1(\Omega)$ and $t \in [0,T] \mapsto \partial_t e_M(t) \in H^{-1}(\Omega)$ are measurable and the bilinear form b introduced in (A 3) is continuous, function $t \in [0,T] \mapsto T_{-R,\hat{\theta}_i}(\theta_M(t)) \bullet \partial_t e_M(t) \in W^{-1,a}(\Omega)$ is measurable. We recall that

$$\left\langle T_{-R,\hat{\theta}_i}(\theta_M(t)) \bullet \partial_t e_M(t), w \right\rangle_{W^{-1,a}(\Omega), W_0^{1,a'}(\Omega)} = \left\langle \partial_t e_M(t), w T_{-R,\hat{\theta}_i}(\theta_M(t)) \right\rangle_{H^{-1}(\Omega)', H_0^1(\Omega).} \tag{6.15}$$

This equality, together with (6.9)–(6.13), yields

$$\| T_{-R,\hat{\theta}_i}(\theta_M(t)) \bullet \partial_t e_M(t) \|_{W^{-1,a}(\Omega)} \leqslant C_R \left(\|f_M(t)\|_{L^1(\Omega)} + \|\theta_M(t)\|_{W^{1,a}(\Omega)} + \int_\Omega h(x,t) \right),$$

a.e. $t \in [0,T]$. Finally, by integrating on $[0,T]$, recalling that $\|f_M\|_{L^1(Q_T)}$ is bounded and using (5.2) and (6.14) we obtain (6.7). □

Remark 6.2 *A naïve approach to estimate* $\| T_{-R,\hat{\theta}_i}(\theta_M) \bullet \partial_t e_M \|_{L^1(0,T;W^{-1,a}(\Omega))}$ *would consist in using* (A 3) *and handling* $T_{-R,\hat{\theta}_i}(\theta_M)$ *and* $\partial_t e_M$ *separately. Although* (6.5) *holds, as in the single-valued case, this approach does not work because estimate* (6.7) *cannot be obtained from* (6.6) *and* (6.5). *Estimate* (6.7) *is nontrivial and the technique developed in our proof seems to be essential to obtain it.*

6.2.1 *Study of some auxiliary functions*

Let us denote

$$q(x,s) = \int_0^s T_{-R,\hat{\theta}_i}(j(x,r))\, dr. \tag{6.16}$$

From Lemma A.5, estimate (6.7) can be rewritten as

$$\|\partial_t q(x, e_M)\|_{L^1(0,T;W^{-1,a}(\Omega))} \leqslant C_{R,a}. \tag{6.17}$$

Let us focus our attention on Ω_i. Recall that $j(x,r) = j_i(r)$ for all $r \in \mathbb{R}$ and $x \in \Omega_i$, so $q(x,s) = q_i(s)$, where $q_i(s) = \int_0^s T_{-R,\hat{\theta}_i}(j_i(r))dr$. For a.e. $(x,t) \in \Omega_i \times [0,T]$, we have the relation $e \in g_i(\theta)$, which is equivalent to $\theta = j_i(e)$. Thus $T_{-R,\hat{\theta}_i}(\theta) = T_{-R,\hat{\theta}_i}(j_i(e))$, which is equivalent to $e \in \left(T_{-R,\hat{\theta}_i} \circ j_i\right)^{-1} \left(T_{-R,\hat{\theta}_i}(\theta)\right)$. (Note that $\left(T_{-R,\hat{\theta}_i} \circ j_i\right)^{-1}$ is always multi-valued.) Since $T_{-R,\hat{\theta}_i}(\theta) = q_i'(e) = \partial q_i(e)$, the relationship

$$q_i(e) = e T_{-R,\hat{\theta}_i}(\theta) - q_i^* \left(T_{-R,\hat{\theta}_i}(\theta)\right) \tag{6.18}$$

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holds, where q_i^* is the conjugate of the convex function q_i. The left-hand side depends only on e and because e is not a single-valued function of θ, the right-hand side cannot be expressed as a single-valued function of θ. However, our assumption (2.11) asserts that the only point where the dependence of e on θ is multi-valued is $\theta = \hat{\theta}_i$. To avoid this multi-valued feature we observe that the previous equation is equivalent to

$$q_i(e) - \hat{\theta}_i e \;=\; e\left(T_{-R,\hat{\theta}_i}(\theta) - \hat{\theta}_i\right) - q_i^*\left(T_{-R,\hat{\theta}_i}(\theta)\right) \tag{6.19}$$

Now the left-hand side depends only on e and the right-hand side can be written as a function depending only on θ. Let us define the function p_i by

$$p_i = p_i(\theta) = \begin{cases} g_i(\theta)\left(T_{-R,\hat{\theta}_i}(\theta) - \hat{\theta}_i\right) - q_i^*\left(T_{-R,\hat{\theta}_i}(\theta)\right) & \text{if } \theta < \hat{\theta}_i, \\ -q_i^*(\hat{\theta}_i) & \text{if } \theta \geqslant \hat{\theta}_i. \end{cases} \tag{6.20}$$

Since $e \in g_i(\theta)$, equation (6.19) can be rewritten as

$$q_i(e) - \hat{\theta}_i e \;= p_i(\theta). \tag{6.21}$$

Actually, the previous ideas apply to functions θ_M and e_M. To investigate the properties of function p_i, we must first study function q_i^*.

Proposition 6.1 *The effective domain of q_i^* is $dom(q_i^*) = [-R, \hat{\theta}_i]$. Moreover $q_i^* \in \mathscr{C}^1([-R, \hat{\theta}_i])$ and*

$$q_i^*(r) = q_i^*(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{r} g_i(s)\, ds \;\forall r \in [-R, \hat{\theta}_i]. \tag{6.22}$$

This proposition can be proved by using the equalities

$$\partial q_i^* = (\partial q_i)^{-1} = \left(T_{-R,\hat{\theta}_i} \circ j_i\right)^{-1}. \tag{6.23}$$

Indeed, one only needs to compute the explicit expression for $T_{-R,\hat{\theta}_i} \circ j_i$ and its inverse, and then to reconstruct q_i^* from ∂q_i^* by integration.

Now we study function $p_i : \mathbb{R} \mapsto \mathbb{R}$ defined by (6.20). It is easy to check that p_i is continuous in $\mathbb{R}$. From the representation (6.22) of q_i^* and assumption (2.12), we find

$$p_i(\theta) = \begin{cases} g_i(\theta)(-R - \hat{\theta}_i) - q^*(\hat{\theta}_i) - \int_{\hat{\theta}_i}^{-R} g_i(s)ds & \text{if } \theta \leqslant -R, \\ -q_i^*(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta} (s - \hat{\theta}_i) g_i'(s)ds & \text{if } -R \leqslant \theta \leqslant \hat{\theta}_i, \\ -q_i^*(\hat{\theta}_i) & \text{if } \theta \geqslant \hat{\theta}_i. \end{cases} \tag{6.24}$$

Proposition 6.2 *Function p_i is Lipschitz-continuous in $\mathbb{R}$ and its restriction to $(-\infty, \hat{\theta}_i]$,*

$$p_{i|(-\infty,\hat{\theta}_i]} : (-\infty, \hat{\theta}_i] \mapsto [-q_i^*(\hat{\theta}_i), +\infty),$$

is strictly decreasing and bijective. Moreover there is a constant $C = C(R, \hat{\theta}_i, \|g_i'\|_{L^\infty(-\infty,\hat{\theta}_i)})$

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such that

$$|p_i'(\theta)| \leqslant C \quad a.e.\ \theta \in \mathbb{R} \tag{6.25}$$

This proposition can be proved by using (6.24) and assumptions (2.12)–(2.13).

Remark 6.3 *As a consequence of the previous Proposition, the mapping*

$$\left(p_{i|(-\infty,\hat{\theta}_i]}\right)^{-1} : [-q_i^*(\hat{\theta}_i), +\infty) \mapsto (-\infty, \hat{\theta}_i]$$

is continuous and strictly decreasing. Furthermore

$$\left(p_{i|(-\infty,\hat{\theta}_i]}\right)^{-1}(p_i(\theta)) \;=\; T_{-\infty,\hat{\theta}_i}(\theta) \quad \forall \theta \in \mathbb{R}. \tag{6.26}$$

6.2.2 *Convergence of the temperature and enthalpy*

Let us go back to pointwise convergence of the temperature. First, we have the following.

Lemma 6.3 *There is a subsequence of θ_M, still denoted θ_M, such that $T_{-\infty,\hat{\theta}_i}(\theta_M)$ converges a.e. in $\Omega_i \times]0, T[$.*

Proof As in the single-valued case, $\partial_t e_M$ is bounded in $L^1(0, T; W^{-1,a}(\Omega))$ which, together with (6.17), the fact that $q(x, s) = q_i(s)$ for $x \in \Omega_i$ and Remark A.3, give

$$\partial_t\left(q_i(e_M) - \hat{\theta}_i e_M\right) \text{ is bounded in } L^1(0, T; W^{-1,a}(\Omega_i)). \tag{6.27}$$

Then, owing to (6.21), we have the equality

$$q_i(e_M) - \hat{\theta}_i e_M = p_i(\theta_M) \;\text{ a.e. in }\quad \Omega_i \times]0, T[. \tag{6.28}$$

The Lipschitz continuity of p_i together with estimates (6.25) and (5.2) imply that

$$p_i(\theta_M) \text{ is bounded in } L^a(0, T; W^{1,a}(\Omega_i)). \tag{6.29}$$

Using a compactness result (see Simon [33], Corollary 4), we deduce that the set $\{p_i(\theta_M)\}_{M\in\mathbb{N}}$ is relatively compact in $L^a(0, T; L^a(\Omega_i)) = L^a(\Omega_i \times]0, T[)$. Hence, there is a subsequence, still denoted by $p_i(\theta_M)$, and some $\eta_i^- \in L^a(\Omega_i \times]0, T[)$ such that $p_i(\theta_M)$ converges to η_i^- strongly in $L^a(\Omega_i \times]0, T[)$ and a.e. in $\Omega_i \times]0, T[$. Finally, since $\left(p_{i|(-\infty,\hat{\theta}_i]}\right)^{-1} \circ p_i = T_{-\infty,\hat{\theta}_i}$ and $\left(p_{i|(-\infty,\hat{\theta}_i]}\right)^{-1}$ is continuous, we have the pointwise convergence of $T_{-\infty,\hat{\theta}_i}(\theta_M) \to \left(p_{i|(-\infty,\hat{\theta}_i]}\right)^{-1}(\eta_i^-)$ a.e. in $\Omega_i \times]0, T[$. □

Remark 6.4 *We must also truncate the temperature below the discontinuity point $\hat{\theta}_i$. For this purpose we cannot use $T_{\hat{\theta}_i,R}$ because $T_{\hat{\theta}_i,R}(0) = \hat{\theta}_i > 0$ and then $T_{\hat{\theta}_i,R}(\theta_M(t))$ is not an admissible test function in* (4.3). *But we can take $\psi = T_{\hat{\theta}_i,R} - \hat{\theta}_i$ as ψ is Lipschitz-continuous, nondecreasing and $\psi(0) = 0$. Just as in Lemma 6.2, we can establish for any $R > \hat{\theta}_i$ and $1 < a < \frac{5}{4}$ the following estimates:*

$$\| T_{\hat{\theta}_i,R}(\theta_M) - \hat{\theta}_i \|_{L^2(0,T;H_0^1(\Omega))} \leqslant C_R, \tag{6.30}$$

$$\| \left(T_{\hat{\theta}_i,R}(\theta_M) - \hat{\theta}_i\right) \bullet \partial_t e_M \|_{L^1(0,T;W^{-1,a}(\Omega))} \leqslant C_{R,a}. \tag{6.31}$$

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Defining q by

$$q(x,s) = \int_0^s \left(T_{\hat{\theta}_i,R}(j(x,r)) - \hat{\theta}_i\right) dr, \tag{6.32}$$

we deduce from Lemma A.5 that estimate (6.31) *can be rewritten in the form* (6.17). *Now we have* $q(x,s) = q_i(s) \quad \forall r \in \mathbb{R} \quad \forall x \in \Omega_i$, *where* $q_i(s) = \int_0^s \left(T_{\hat{\theta}_i,R}(j_i(r)) - \hat{\theta}_i\right) dr$.

Moreover, from equalities $\theta = j_i(e)$ *and* $T_{\hat{\theta}_i,R}(\theta) - \hat{\theta}_i = q_i'(e)$ *we derive the equation*

$$q_i(e) \;=\; e\left(T_{\hat{\theta}_i,R}(\theta) - \hat{\theta}_i\right) - q_i^*\left(T_{\hat{\theta}_i,R}(\theta) - \hat{\theta}_i\right). \tag{6.33}$$

Defining now the function p_i *by*

$$p_i = p_i(\theta) = \begin{cases} -q_i^*(0) & \text{if } \theta \leqslant \hat{\theta}_i, \\ g_i(\theta)\left(T_{\hat{\theta}_i,R}(\theta) - \hat{\theta}_i\right) - q_i^*\left(T_{\hat{\theta}_i,R}(\theta) - \hat{\theta}_i\right) & \text{if } \theta > \hat{\theta}_i, \end{cases} \tag{6.34}$$

we have the relationship

$$q_i(e) = p_i(\theta). \tag{6.35}$$

Following the above methodology, we can prove that $p_i : \mathbb{R} \mapsto \mathbb{R}$ *is Lipschitz-continuous,* $p_{i|[\hat{\theta}_i,+\infty)} : [\hat{\theta}_i, +\infty) \mapsto [-q_i^*(0), +\infty)$ *is strictly increasing and bijective and* $\left(p_{i|[\hat{\theta}_i,+\infty)}\right)^{-1}(p(\theta)) = T_{\hat{\theta}_i,+\infty}(\theta) \;\forall \theta \in \mathbb{R}$. *Finally, we can obtain the a.e. convergence in* $\Omega_i \times]0, T[$ *of a subsequence of* $T_{\hat{\theta}_i,+\infty}(\theta_M)$.

Remark 6.5 *Until now we have focused on the case of* $\hat{\theta}_i > 0$. *The case of* $\hat{\theta}_i < 0$ *can be treated using the same process by first taking* $\psi = T_{-R,\hat{\theta}_i} - \hat{\theta}_i$ *and then* $\psi = T_{\hat{\theta}_i,R}$ *in the definition of function test* $\psi(\theta_M(t))$ *to be inserted in* (4.3).

Lemma 6.4 *Under assumptions* (2.1)–(2.6) *and* (2.10)–(2.15), *we have*

- *There is a subsequence of* θ_M *(still denoted by* θ_M*) and a function* $\theta \in L^a(0,T; W_0^{1,a}(\Omega))$, *for* $1 < a < \frac{5}{4}$, *such that*

$$\theta_M \to \theta \text{ a.e. in } Q_T, \tag{6.36}$$

$$\theta_M \to \theta \text{ strongly in } L^a(Q_T) \text{ and weakly in } L^a(0,T; W_0^{1,a}(\Omega)) \text{ for } 1 < a < \frac{5}{4}, \tag{6.37}$$

- *There is a subsequence of* e_M *(still denoted by* e_M*) and a function* $e \in L^a(Q_T)$ *for* $1 < a < \frac{5}{4}$, *such that*

$$e_M \rightharpoonup e \text{ weakly in } L^a(Q_T) \text{ for } 1 < a < \tfrac{5}{4} \tag{6.38}$$

Proof As a consequence of Lemma 6.3 and Remarks 6.4 and 6.5, we know that there is a subsequence of θ_M (still denoted by θ_M) such that, for any $i = 1, \ldots, N$, $T_{-\infty,\hat{\theta}_i}(\theta_M)$ and

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$T_{\hat{\theta}_i,+\infty}(\theta_M)$ converge a.e. in $\Omega_i\times]0,T[$. Using the identity

$$T_{-\infty,\hat{\theta}_i}(s)+T_{\hat{\theta}_i,+\infty}(s)-\hat{\theta}_i=s$$

we obtain that θ_M converges a.e. in Q_T. Let θ be the a.e. pointwise limit of θ_M. Using (5.2) and Vitali's theorem we infer that $\theta\in L^a(0,T;W_0^{1,a}(\Omega))$ for any $a<\frac{5}{4}$. Thus, (6.37) holds.

Concerning enthalpy, as a consequence of (2.10)–(2.12) and estimate (5.2) we have that e_M is bounded in $L^a(Q_T)$, hence there is $e\in L^a(Q_T)$ such that (6.38) holds. □

7 Passing to the limit in the truncation parameter

7.1 An equivalent formulation of the weak truncated problem

We consider the following formulation of the truncated weak problem:

(WT2)$_M$ *Find* $\mathbf{h}_M\in L^2(0,T;\mathbf{V})\cap\mathscr{C}^0([0,T];\mathbf{L}^2(\Omega))$, $\theta_M\in L^2(0,T;H_0^1(\Omega))$ *and* $e_M\in H^1(0,T;H^{-1}(\Omega))\cap L^2(0,T;L^2(\Omega))$ *such that:*

$$-\int_{Q_T}\mu\mathbf{h}_M\cdot\partial_t\boldsymbol{\zeta}+\int_{Q_T}\frac{1}{\sigma(\theta_M)}(\nabla\times\mathbf{h}_M)\cdot(\nabla\times\boldsymbol{\zeta})=\int_\Omega\mu\mathbf{h}_0\cdot\boldsymbol{\zeta}(0)\quad\forall\boldsymbol{\zeta}\in\mathbf{Z}_{\mathbf{V}},\tag{7.1}$$

$$-\int_{Q_T}e_M\,\partial_t\phi+\int_{Q_T}k\nabla\theta_M\cdot\nabla\phi=\int_{Q_T}\frac{1}{\sigma(\theta_M)}T_M(|\nabla\times\mathbf{h}_M|^2)\phi+\int_\Omega e_0^M\,\phi(0)\quad\forall\phi\in\Phi_2,\tag{7.2}$$

$$\mathbf{h}_M\times\mathbf{n}=\mathbf{f}\quad in\quad L^2\big(0,T;[H_{00}^{-1/2}(\Gamma_D)]^3\big),\tag{7.3}$$

$$e_M(t)\in G(\theta_M(t))\ a.e.\ in[0,T],\tag{7.4}$$

where notation for the spaces of test functions is that established in equations (3.10) *and* (3.11).

Lemma 7.1 *Any solution* $(\mathbf{h}_M,\theta_M,e_M)$ *of* ***(WT1)$_M$*** *is a solution of* ***(WT2)$_M$*** *and conversely.*

Proof The proof is similar to Remark 3.2. In addition, the temperature and enthalpy of **(WT1)$_M$** are also a solution of **(WT2)$_M$** because the finite sums $\sum_i\psi_i v_i$, with $v_i\in H_0^1(\Omega)$ and $\psi_i\in\mathscr{C}^\infty([0,T])$, $\psi_i(T)=0$, are dense (see Lions [27]) in

$$Z_{H_0^1(\Omega)}=\left\{\phi\in L^2(0,T;H_0^1(\Omega)):\partial_t\phi\in L^2(Q_T),\ \phi(T)=0\right\},$$

which contains Φ_2. □

7.2 Passing to the limit

From (5.1) we have that $\{\mathbf{h}_M\}$ is bounded in $L^2(0,T;\mathbf{V})$ so there is a subsequence weakly convergent in $L^2(0,T;\mathbf{V})$, still denoted $\mathbf{h}_M$. Using also the pointwise convergence of θ_M, we pass to the limit in the electromagnetic equation (7.1) as $M\to\infty$. Then $\mathbf{h}$ satisfies (3.8). It is clear that $\mathbf{h}$ also satisfies (3.3).

Now we deal with the thermal part. Since $a<\frac{5}{4}$, then $a'>5$ and, in particular, the space of test functions $\Phi_{a'}$ is included in Φ_2. Let $\phi\in\Phi_{a'}$. Firstly, we pass to the limit

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as $M \to \infty$ in equation (7.2). Since $\partial_t \phi \in L^{a'}(Q_T)$, (6.38) implies the convergence of the first term of the left-hand side of (7.2). For the second term, convergence is obtained by using the weak convergence of $\nabla\theta_M$ to $\nabla\theta$ in $L^a(Q_T)$. As in the proof of Lemma 4.4, we can deduce that $\mathbf{h}_M \to \mathbf{h}$ strongly in $\mathbf{L}^2(0,T;\mathbf{V})$ because $\theta_M \to \theta$ strongly in $L^1(Q_T)$ from (6.37). Hence $|\nabla \times \mathbf{h}_M|^2 \to |\nabla \times \mathbf{h}|^2$ strongly in $L^1(Q_T)$. Now this implies $T_M\big(|\nabla \times \mathbf{h}_M|^2\big) \to |\nabla \times \mathbf{h}|^2$ strongly in $L^1(Q_T)$. Thanks to the pointwise convergence of $\sigma(\theta_M)$ and to Lebesgue's Theorem, we have

$$\frac{1}{\sigma(\theta_M)} T_M\big(|\nabla \times \mathbf{h}_M|^2\big) \to \frac{1}{\sigma(\theta)}\big(|\nabla \times \mathbf{h}|^2\big) \quad \text{strongly in } L^1(Q_T)$$

and, since $\phi \in \mathscr{C}^0\big([0,T];L^\infty(\Omega)\big)$, we can pass to the limit in the first term on the right-hand side of (7.2). The second term is handled by recalling $\phi(0) \in L^\infty(\Omega)$ and the strong convergence of e_0^M in $L^1(Q_T)$. Thus e and θ satisfy (3.9). Whereas (3.4) has been obtained in the single-valued case, in the multi-valued case it follows from relationship $e_M(x,t) \in g(x,\theta_M(x,t))$ a.e. in Q_T, weak convergence (6.38), a.e. convergence (6.36) and the fact that g_i is a maximal monotone operator. Indeed, one can apply Lemma 3 of Brézis [15] in each measure space $\Omega_i \times]0,T[$, $i = 1,\dots,N$. Thus, we have shown that the triple $(\mathbf{h},\theta,e)$ is a solution of **(WP2)**, which consequently proves Theorem 3.1.

8 Conclusion

We have studied a model consisting of the eddy current equation coupled with transient heat equation with a phase change. Mixed boundary conditions for the electromagnetic equation and Dirichlet conditions for temperature are taken.

We have considered an enthalpy depending on both position and temperature. Regarding temperature, we have dealt with two different cases: either it is a Lipschitz-continuous function with a positive and bounded derivative or it is a continuous function except at a single discontinuity point.

The existence of a solution is obtained as the limit of a truncated problem. Temperature estimates, independent of the truncation parameter, are derived using adapted Boccardo–Gallouët techniques. Moreover, in the multi-valued case we construct a function of the enthalpy which can be written as a single-valued function solely of temperature, to achieve a.e. convergence of temperatures.

Several open problems arise from this work: the uniqueness and regularity of the solution, the existence of a solution when thermal conductivity depends on both temperature and position or more general forms of the enthalpy operator.

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Appendix A

Lemma A.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ with locally Lipschitz-continuous boundary. Let $p \in (n,+\infty)$, $p \geqslant 2$. If $z \in H^1(\Omega)$ and $w \in W_0^{1,p}(\Omega)$ then $zw \in H_0^1(\Omega)$,*

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$\frac{\partial}{\partial x_i}(zw) = w\frac{\partial z}{\partial x_i} + z\frac{\partial w}{\partial x_i}$ *for* $i = 1, \ldots, n$, *and there is a constant* C *depending only on* Ω *and* p *such that*

$$\|zw\|_{H^1(\Omega)} \leqslant C\|z\|_{H^1(\Omega)}\|w\|_{W^{1,p}(\Omega)}. \tag{A 1}$$

The previous lemma is a consequence of a Sobolev embedding. (See Adams [2], Theorems 5.4 and 3.18).

Remark A.1 *Lemma A.1 still hold if we assume* $z \in H_0^1(\Omega)$ *and* $w \in W^{1,p}(\Omega)$.

Thanks to Lemma A.1, the following definition makes sense.

Definition A.1 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with locally Lipschitz-continuous boundary and* $p \in (n, +\infty)$, $p \geqslant 2$. *For* $\ell \in H^{-1}(\Omega)$ *and* $z \in H^1(\Omega)$, *its product* $z \bullet \ell \in W^{-1,p'}(\Omega)$ *is defined by*

$$\langle z \bullet \ell, w\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} := \langle \ell, zw\rangle_{H^{-1}(\Omega)',H_0^1(\Omega)} \quad \forall w \in W_0^{1,p}(\Omega). \tag{A 2}$$

We notice that the bilinear form

$$b : (z, \ell) \in H^1(\Omega) \times H^{-1}(\Omega) \mapsto z \bullet \ell \in W^{-1,p'}(\Omega) \tag{A 3}$$

is continuous. Besides, when $\ell \in L^2(\Omega)$ the linear form defined by (A 2) coincides with that induced over $W_0^{1,p}(\Omega)$ by the pointwise product $z\ell \in L^1(\Omega)$.

Definition A.2 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with locally Lipschitz-continuous boundary and* $\omega \subset \Omega$ *be a domain also having a locally Lipschitz-continuous boundary. For* $p \in [1, \infty)$ *and* $\ell \in W^{-1,p'}(\Omega)$, *we define its restriction* $\ell_{|\omega} \in W^{-1,p'}(\omega)$ *by*

$$\langle \ell_{|\omega}, w\rangle_{W^{-1,p'}(\omega),W_0^{1,p}(\omega)} := \langle \ell, \tilde{w}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \forall\ w \in W_0^{1,p}(\omega) \tag{A 4}$$

where $\tilde{w}$ *is the extension of* w *to* Ω *by zero outside* ω.

Remark A.2 *If* $\ell \in H^{-1}(\Omega)$, $z \in H^1(\Omega)$, *and* $p \in (n, \infty)$, $p \geqslant 2$, *a computation using* (A 4) *and* (A 2) *shows that* $(z \bullet \ell)_{|\omega} = z_{|\omega} \bullet \ell_{|\omega}$ *as elements of* $W^{-1,p'}(\omega)$.

Remark A.3 *Since the mapping* $\ell \in W^{-1,p'}(\Omega) \mapsto \ell_{|\omega} \in W^{-1,p'}(\omega)$ *is linear and continuous, it naturally induces a restriction mapping from* $L^r(0, T; W^{-1,p'}(\Omega))$ *into* $L^r(0, T; W^{-1,p'}(\omega))$, *for any* $r \in [1, \infty]$. *Moreover, if* $u \in L^1(0, T; L^2(\Omega))$ *with* $\partial_t u \in L^1(0, T; W^{-1,p'}(\Omega))$, *then* $\partial_t(u_{|\omega\times]0,T[}) = (\partial_t u)_{|\omega\times]0,T[} \in L^1(0, T; W^{-1,p'}(\omega))$

Throughout the present paper, the restriction $\ell_{|\omega}$ is simply denoted by ℓ.

The next lemma can be obtained from Lemma 2.2 in DiBenedetto & Showalter [23] by considering the sub-differential as a multi-valued operator acting from a space into its dual space.

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Lemma A.2 *Let X be a real Hilbert space. Let $\varphi : X \mapsto (-\infty, +\infty]$ be a lower semi-continuous convex proper functional. If $u \in H^1(0,T;X)$, $v \in L^2(0,T;X')$ and $v(t) \in \partial u(t)$ a.e. $t \in [0,T]$, then function $t \mapsto \varphi(u(t)))$ is absolutely continuous in $[0,T]$ and the following "chain rule" holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u(t)) = \left\langle w, u'(t)\right\rangle_{X',X} \quad \text{for all } w \in \partial\varphi(u(t)) \quad \text{a.e. } t \in [0,T]. \tag{A 5}$$

Lemma A.3 *Let Ω be a bounded domain of $\mathbb{R}^n$ with locally Lipschitz-continuous boundary. Let $j : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a Carathéodory function such that $j(x,\cdot)$ is nondecreasing a.e. $x \in \Omega$. Let $\psi : \mathbb{R} \mapsto \mathbb{R}$ be a Lipschitz-continuous nondecreasing function such that $\psi(0) = 0$. For any $s \in \mathbb{R}$ and a.e. $x \in \Omega$ we define functions β, q, and q^* by*

$$\beta(x,s) = \psi(j(x,s)), \tag{A 6}$$

$$q(x,s) = \int_0^s \beta(x,r)\,\mathrm{d}r, \tag{A 7}$$

$$q^*(x,s) = \sup_{r\in\mathbb{R}}\{rs - q(x,r)\}.$$

Assume $q^*(\cdot,0) \in L^1(\Omega)$. Let $p \in (n,+\infty)$, $p \geqslant 2$, and $w \in W^{1,p}(\Omega)$ be such that $w(x) > 0 \ \forall\ x \in \Omega$. Let $Q_w : L^2(\Omega) \mapsto (-\infty,\infty]$ be the convex lower semi-continuous proper functional defined by

$$Q_w(e) = \begin{cases} \int_\Omega w(x)q(x,e(x))\,\mathrm{d}x & \text{if } w(\cdot)q(\cdot,e(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise .} \end{cases} \tag{A 8}$$

Let i denote the injection $i : H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $i^* : L^2(\Omega) \to H^{-1}(\Omega)$ its transposed operator. Then there is a convex lower semi-continuous proper functional $\Upsilon_w : H^{-1}(\Omega) \to (-\infty,+\infty]$ such that, if $\theta \in H_0^1(\Omega)$, $e \in L^2(\Omega)$ and $\theta(x) = j(x,e(x))$ a.e. $x \in \Omega$, then the following relations hold:

$$w\psi(\theta) \in \partial\Upsilon_w(i^*(e)), \tag{A 9}$$

$$\Upsilon_w(i^*(e)) = Q_w(e). \tag{A 10}$$

Proof Since j is a Carathéodory function and ψ is continuous, β and q are Caratheodory functions and $q(x,\cdot) \in \mathscr{C}^1(\mathbb{R})$ a.e. $x \in \Omega$. Moreover, $\beta(x,\cdot)$ is nondecreasing. Hence it is a single-valued maximal monotone operator in $\mathbb{R}$ and $q(x,\cdot)$ is convex and finite valued a.e. $x \in \Omega$. Thus $q : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a normal convex integrand.

Since $w \in \mathscr{C}^0(\overline{\Omega})$ (because $p > n$) and $w(x) > 0 \ \forall\ x \in \Omega$, function $q_w : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by $q_w(x,s) = w(x)q(x,s)$ is also a normal convex integrand and its conjugate

$$q_w^*(x,s) = \sup_{r\in\mathbb{R}}\{rs - q_w(x,r)\}$$

is related to q^* by

$$q_w^*(x,s) = w(x)q^*\left(x, \frac{s}{w(x)}\right) \quad \forall s \in \mathbb{R} \quad \text{a.e. } x \in \Omega$$

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Using Proposition 2.7 of Chapter 2 in Barbu & Precupanu [5] and Proposition 2 and Theorem 2 in Rockafellar [30], we infer that:

- Functional Q_w is convex lower semi-continuous and proper.
- $q_w^* : \Omega \times \mathbb{R} \mapsto (-\infty, +\infty]$ is a normal convex integrand and $Q_w^* : L^2(\Omega) \mapsto (-\infty, +\infty]$ (the conjugate functional of Q_w) is given by

$$Q_w^*(z) = \begin{cases} \int_\Omega q_w^*(x, z(x))\,\mathrm{d}x & \text{if } q_w^*(\cdot, z(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

- Q_w^* is convex lower semi-continuous and proper.
- For every $z \in L^2(\Omega)$

$$\partial Q_w^*(z) = \{v \in L^2(\Omega); v(x) \in \partial q_w^*(x, z(x)) \ \text{ a.e. } x \in \Omega\}. \tag{A 11}$$

We set $\beta_w(x,s) = w(x)\psi(j(x,s))$. Since $w(x) > 0$, $\beta_w(x,\cdot)$ is a single-valued maximal monotone operator. Then we define $\alpha_w(x,\cdot) = (\beta_w(x,\cdot))^{-1}$. We have $\beta_w(x,\cdot) = \partial q_w(x,\cdot)$ so $\alpha_w(x,\cdot) = \partial q_w^*(x,\cdot)$.

Let us introduce $\Lambda_w = Q_w^* \circ i$. Then Λ_w is convex and lower semi-continuous in $H_0^1(\Omega)$ and it is proper because $q_w^*(\cdot,0) \in L^1(\Omega)$. Let $\Upsilon_w = \Lambda_w^* : H^{-1}(\Omega) \to (-\infty, +\infty]$ be the conjugate functional of Λ_w. To prove (A 9), we shall stablish

$$i^*(e) \in \partial \Lambda_w(w\psi(\theta)). \tag{A 12}$$

Since $\theta \in H_0^1(\Omega)$, ψ is Lipschitz-continuous, $\psi(0) = 0$ and $w \in W^{1,p}(\Omega)$ with $p \geqslant 2$ and $p \in (n, \infty)$, using Remark A.1 we have $w\psi(\theta) \in H_0^1(\Omega)$. Besides, $e \in L^2(\Omega)$ and $w(x)\psi(\theta(x)) = w(x)\psi(j(x, e(x)) = \beta_w(x, e(x))$, so $e(x) \in \alpha_w(x, w(x)\psi(\theta(x))) = \partial q_w^*(x, w(x)\psi(\theta(x)))$.Thus recalling (A 11) we deduce

$$e \in \partial Q_w^*(i(w\psi(\theta))) \tag{A 13}$$

which together with the inclusion $\partial \Lambda_w(w\psi(\theta)) \supset i^*\left(\partial Q_w^*(i(w\psi(\theta)))\right)$ implies (A 12). Now, from relationships (A 12) and (A 13), we deduce

$$\begin{aligned} \Upsilon_w(i^*(e)) + Q_w^*(i(w\psi(\theta))) &= \Lambda_w^*(i^*(e)) + \Lambda_w(w\psi(\theta)) \\ &= \langle i^*(e), w\psi(\theta)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = (e, i(w\psi(\theta)))_{L^2(\Omega)} = Q_w(e) + Q_w^*(i(w\psi(\theta))). \end{aligned}$$

so assertion (A 10) holds. □

Lemma A.4 *Let Ω, j, ψ, β, q and q^* be defined as in the previous Lemma. Assume $q^*(\cdot,0) \in L^1(\Omega)$. Let $Q : L^2(\Omega) \mapsto (-\infty, +\infty]$ be the convex lower semi-continuous proper functional defined by (A 8) with $w = 1$, namely,*

$$Q_(e) = \begin{cases} \int_\Omega q(x, e(x))\,\mathrm{d}x & \textit{if } q(\cdot, e(\cdot)) \in L^1(\Omega), \\ +\infty & \textit{otherwise.} \end{cases} \tag{A 14}$$

Then, if $\theta \in L^2(0, T, H_0^1(\Omega))$, $e \in L^1(0, T; L^2(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$, $\theta(x,t) = j(x, e(x,t))$

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a.e. $(x,t) \in \Omega \times (0,T)$, $e_0 = e(0) \in L^2(\Omega)$ *and* $\theta_0(x) = j(x, e_0(x))$ *satisfies* $\theta_0 \in H_0^1(\Omega)$, *we have*

$$\int_0^t \langle \partial_t e, \psi(\theta) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = Q(e(t)) - Q(e_0) \quad \text{a.e. } t \in [0,T]. \tag{A 15}$$

Proof We first apply Lemma A.3 with weight function $w = 1$. We denote the corresponding functional Υ_w just by Υ. Since ψ is Lipschitz-continuous, $\psi(0) = 0$ and $\theta \in L^2(0,T,H_0^1(\Omega))$, the function $\psi(\theta) \in L^2(0,T,H_0^1(\Omega)))$. Since $e \in H^1(0,T;H^{-1}(\Omega))$ and $\psi(\theta(t)) \in \partial \Upsilon(i^*(e(t)))$ a.e. $t \in [0,T]$, we can apply Lemma A.2. Thus we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \Upsilon(i^*(e(t))) = \langle \partial_t e(s), \psi(\theta(s)) \rangle_{H^{-1}(\Omega)', H_0^1(\Omega)} \quad \text{a.e. } t \in [0,T] \tag{A 16}$$

Integrating this equality from 0 to t we obtain

$$\int_0^t \langle \partial_t e(s), \psi(\theta(s)) \rangle_{H^{-1}(\Omega)', H_0^1(\Omega)} \, ds = \Upsilon(i^*(e(t))) - \Upsilon(i^*(e_0)) \tag{A 17}$$

Now, using the assumptions on e, θ, e_0 and (A 10), we deduce that $\Upsilon(i^*(e_0)) = Q(e_0)$ and $\Upsilon(i^*(e(t))) = Q(e(t))$ a.e. $t \in [0,T]$ from which (A 15) follows. □

Remark A.4 *In Lemmas A.3 and A.4, if* $j(x,0) = 0$ *a.e.* $x \in \Omega$, *then* $q(x,s) \geqslant 0$ $\forall s \in \mathbb{R}$ *a.e.* $x \in \Omega$ *and* $q^*(\cdot,0) = 0$. *If* ψ *is bounded or* j *satisfies* $|j(x,s)| \leqslant C(1+|s|)$ $\forall s \in \mathbb{R}$ *a.e.* $x \in \Omega$, *for some constant* $C > 0$, *then* $Q \in \mathscr{C}^1(L^2(\Omega);\mathbb{R})$ *(see Kavian [26], Chapter I, Lemma 17.1).*

Remark A.5 *Lemma A.4 is closely related to the Alt–Luckhaus' Lemma [3]). However, the latter applies when* $e = g(\theta)$, *with* $g : \mathbb{R} \mapsto \mathbb{R}$ *being a continuous function. Even if* j *were independent of* x, *the operator* $g = j^{-1}$ *could be multi-valued and hence, in general, Alt–Luckhaus' Lemma cannot be applied under the assumptions in Lemma A.4.*

Lemma A.5 *Let* Ω, j, ψ, β, q *and* q^* *be defined as in Lemma A.3. Assume further that* ψ *is bounded and* $q^*(\cdot,0) \in L^1(\Omega)$. *If* $\theta \in L^2(0,T,H_0^1(\Omega))$, $e \in L^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, $\theta(x,t) = j(x,e(x,t))$ *a.e.* $(x,t) \in \Omega \times (0,T)$, *then, for any* $p \in (n, +\infty)$, $p \geqslant 2$, *we have*

$$\partial_t q(x,e) = \psi(\theta) \bullet \partial_t e \quad \text{in} \quad \mathscr{D}'(]0,T[; W^{-1,p'}(\Omega)). \tag{A 18}$$

Proof Since q is a Carathéodory function satisfying $|q(x,s)| \leqslant \|\psi\|_{L^\infty(\mathbb{R})}|s|$ for all $s \in \mathbb{R}$, a.e. $x \in \Omega$ and $e \in L^2(0,T;L^2(\Omega))$, function $u = q(x,e)$ belongs to $L^2(0,T;L^2(\Omega))$ and hence $\partial_t u \in \mathscr{D}'(]0,T[;L^2(\Omega)) \subset \mathscr{D}'(]0,T[;W^{-1,p'}(\Omega))$. As in Lemma A.4, $\psi(\theta) \in L^2(0,T;H_0^1(\Omega))$. This fact, together with $\partial_t e \in L^2(0,T;H^{-1}(\Omega))$ and (A 3) imply $\psi(\theta) \bullet \partial_t e \in L^1(0,T;W^{-1,p'}(\Omega))$. Moreover, for any $w \in W_0^{1,p}(\Omega)$ and any $\phi \in \mathscr{D}(]0,T[)$,

$$\left\langle \int_0^T u(t)\phi'(t)\,\mathrm{d}t, w \right\rangle_{W^{-1,p'}(\Omega)), W_0^{1,p}(\Omega)} = \int_0^T (u(t),w)_{L^2(\Omega)} \phi'(t)\,\mathrm{d}t \tag{A 19}$$

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and, if $w > 0$ on Ω, we also have

$$(u(t), w)_{L^2(\Omega)} \;=\; \int_\Omega w(x) q(x, e(x,t))\,\mathrm{d}x \;=\; Q_w(e(t)), \tag{A 20}$$

where Q_w is given by (A 8). Applying Lemma A.3 and taking into account the assumptions on θ and e, we deduce that there is a convex lower semi-continuous proper functional $\Upsilon_w : H^{-1}(\Omega) \to (-\infty, +\infty]$ and a zero measure set $\mathscr{N}_w \subset [0,T]$ such that

$$w\psi(\theta(t)) \in \partial \Upsilon_w(i^*(e(t))) \;\; \forall t \in [0,T]\backslash\mathscr{N}_w, \tag{A 21}$$

$$\Upsilon_w(i^*(e(t))) = Q_w(e(t)) \;\; \forall t \in [0,T]\backslash\mathscr{N}_w. \tag{A 22}$$

Since $e \in H^1(0,T;H^{-1}(\Omega))$ and $w\psi(\theta) \in L^2(0,T,H_0^1(\Omega))$, it stems from Lemma A.2 that function $t \in [0,T] \mapsto \Upsilon_w(i^*(e(t))) \in \mathbb{R}$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_w(i^*(e(t))) = \langle \partial_t e, w\psi(\theta(t))\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \tag{A 23}$$

a.e. $t \in [0,T]$. Hence (A 23) also holds in the sense of $\mathscr{D}'(]0,T[)$. Using successively (A 20), (A 22), (A 23) and (A 2), we get

$$-\int_0^T (u(t), w)_{L^2(\Omega)} \phi'(t)\,\mathrm{d}t \;=\; \int_0^T \langle \psi(\theta(t)) \bullet \partial_t e, w\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\, \phi(t)\,\mathrm{d}t, \tag{A 24}$$

Since these integrals depend continuously on $w \in W_0^{1,p}(\Omega)$, the set $\{w \in W_0^{1,p}(\Omega); w > 0 \;\text{ in }\; \Omega\}$ is dense in $K = \{w \in W_0^{1,p}(\Omega); w \geqslant 0 \;\text{ in }\; \Omega\}$ and any $w \in W_0^{1,p}(\Omega)$ admits the decomposition $w = w^+ - w^-$, where $w^+ = \max(w,0)$, $w^- = -\min(w,0)$, and $w^\pm \in K$, equation (A 24) holds for any $w \in W_0^{1,p}(\Omega)$. This, together with (A 19), implies that

$$-\int_0^T u(t)\phi'(t)\,\mathrm{d}t \;=\; \int_0^T \phi(t)\,\psi(\theta(t)) \bullet \partial_t e\,\mathrm{d}t \;\;\text{ in }\;\; W^{-1,p'}(\Omega) \quad \forall \phi \in \mathscr{D}(]0,T[) \tag{A 25}$$

Thus we have (A 18) and $u \in W^{1,1}(0,T;W^{-1,p'}(\Omega))$. □

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