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Noetherian Banach Jordan pairs†

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0. *Introduction*

An associative or alternative algebra A is Noetherian if it satisfies the ascending chain condition on left ideals. Sinclair and Tullo [**21**] showed that a complex Noetherian Banach associative algebra is finite dimensional. This result was extended by Benslimane and Boudi [**5**] to the alternative case.

For a Jordan algebra J or a Jordan pair V, the suitable Noetherian condition is the ascending chain condition on inner ideals. In a recent work Benslimane and Boudi [**6**] proved that a complex Noetherian Banach Jordan algebra is finite dimensional.

Here we show the following results:

(i) the Jacobson radical of a Noetherian Banach Jordan pair is finite dimensional;
(ii) nondegenerate Noetherian Banach Jordan pairs have finite capacity;
(iii) complex Noetherian Banach Jordan pairs are finite dimensional.

After a preliminary section whose main interest is to fix notation and recall some basic results on Jordan algebras and pairs, we study in Section 2 the small radical of a Jordan pair under chain conditions.

In the course of the proof of his fundamental Nilpotence Theorem [**22**], Zelmanov proved that if a nondegenerate linear Jordan algebra J has the chain condition on annihilators contained inside a nil ideal N, then $N = 0$. This result, as well as the Nilpotence Theorem, would be extended by McCrimmon [**17**] to the quadratic case. In both McCrimmon and Zelmanov's proofs, the strategy is to assume $N \neq 0$ and then find a nonzero element z in N whose annihilator is maximal, which yields a contradiction. Such an element z is a particular case of the so-called uniform elements introduced in [**10**], which were characterized in [**12**] as those nonzero elements a in a nondegenerate Jordan algebra J such that the local algebra J_a of J at a is a Jordan domain.

We show in Proposition 2·3 that the same characterization of uniform elements remains true for nondegenerate Jordan pairs. Then we use this local characterization to prove Corollary 2·5 that, under either the ascending or descending chain condition

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on inner ideals, the small radical of a Jordan pair coincides with the properly nil radical. Moreover, by using a combinatorial description of the inner ideal generated by a finite number of elements, we show, Lemma 2·6, that if $\mathrm{rad}(V) = (S^+, S^-)$ is the small radical of a Noetherian Jordan pair V over a field Φ, then $S^\pm$ has a countable basis as a vector space over Φ.

In Section 3 we deal with Banach Jordan pairs and prove the main results (i), (ii) and (iii) of the paper. A key point in the proof of these results is the fact that the small radical of a Noetherian Banach Jordan pair is closed which, together with Lemma 2·6, proves via the Baire Category Theorem that the small radical is finite dimensional. This reduces the study of a Noetherian Banach Jordan pair V to the case that V is nondegenerate. The proof of (ii) lies in two fundamental facts: the abundance of uniform elements in Noetherian Jordan pairs and Lemma 3·5 that asserts that Noetherian Banach Jordan domains J are division Jordan algebras (the complex field in the complex case). As a consequence of (ii), we have that the Jacobson radical of a Noetherian Banach Jordan pair coincides with the small radical and therefore it is finite dimensional. Finally, (iii) follows from (ii) and Loos' classical list of simple Jordan pairs of finite capacity, previously refined according to an idea of McCrimmon [**19**] in order to identify the centroids.

Unlike the complex case, real Noetherian Banach Jordan pairs (even real division Banach Jordan algebras) are not necessarily finite dimensional.

This paper is dedicated to the memory of Khalid Bouhya.

1. *Preliminaries*

In this paper we shall deal with Jordan pairs and Jordan algebras over a ring of scalars Φ. Nevertheless, we shall be mainly interested in the *linear* case ($\frac{1}{2} \in \Phi$), and very specially in the case that Φ is the real or complex field. The reader is referred to [**15**] and [**13**] for notation, conventions and basic results. In particular, the numbering of identities JPn refers to [**15**]. We shall record in this section some of that notation and those results.

Given a Jordan pair $V = (V^+, V^-)$, we write $Q: V^\pm \to \mathrm{Hom}_\Phi(V^\mp, V^\pm)$ to denote the quadratic maps of V. The multiplication $Q_x y$ is quadratic in x and linear in y and has linearizations

$$\{x\,y\,z\} = Q(x,z)y = D(x,y)z = Q_{x+z}y - Q_x y - Q_z y.$$

Note that $\{x\,y\,x\} = 2Q_x y$, so we only need to consider the triple product in the linear case.

A typical example of a Jordan pair is given by taking $V^+ = \mathrm{Hom}_\Delta(X, Y)$, $V^- = \mathrm{Hom}_\Delta(Y, X)$, linear maps between right vector spaces X and Y over a division associative Φ-algebra Δ, with $Q_a b = aba$.

Any associative, alternative or Jordan algebra A gives rise to a Jordan pair (A, A) with quadratic multiplication xyx or $U_x y$, with U denoting the usual U-operator of the Jordan algebra.

In the opposite direction, given a Jordan pair $V = (V^+, V^-)$ and an element $y \in V^{-\sigma}$ we can define a Jordan algebra on V^σ by

$$U_a^{(y)} = Q_a Q_y \quad \text{and} \quad a^{(2,y)} = Q_a y.$$

This Jordan algebra, denoted by $(V^\sigma)^{(y)}$, is called the *y-homotope* of V^σ. We shall

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write $\{\ldots\}^{(y)}$ to denote the triple product in $(V^\sigma)^{(y)}$. If V is a linear Jordan system, we just need to define the linear product $x \cdot_y z = \frac{1}{2}\{x\, y\, z\}$ on $(V^\sigma)^{(y)}$.

Local algebras of a Jordan pair. Let $V = (V^+, V^-)$ be a Jordan pair and $y \in V^{-\sigma}$. By [**2**, 1·2·2], the set

$$\mathrm{Ker}\,(y) = \{x \in V^\sigma : Q_y x = 0,\ Q_y Q_x y = 0\}$$

is an ideal of $(V^\sigma)^{(y)}$ and the quotient $(V^\sigma)^{(y)}/\mathrm{Ker}\,(y)$ is a Jordan algebra called the *local algebra* of V at y which we denote by V_y. As pointed out in [**2**, 1·2·4(ii)], the condition $Q_y Q_x y = 0$ is superfluous if V is linear.

A Jordan pair V is *nondegenerate* if it has no nonzero *absolute zero divisors*, i.e. $Q_a = 0$ implies $a = 0$. Nondegenerate Jordan pairs V are *semiprime*, i.e. $Q_{I^\pm} I^\mp = 0$ implies $I = 0$, for I ideal of V, but the converse is not true in general, even for Jordan algebras. A Jordan pair V is *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$ for B and C ideals of V and it is called *strongly prime* if it is prime and nondegenerate.

(**1·1**) By JP3, if V is nondegenerate then so are all its local algebras. Moreover [**4**, 3·2], V is strongly prime if and only if V is nondegenerate and V_y is strongly prime for every $y \in V^-$.

Recall that a submodule $K \subset V^\sigma$ is an inner ideal if $Q_K V^{-\sigma} \subset K$. A Jordan pair V is called *Noetherian* if it satisfies the ascending chain condition (acc) on all inner ideals, equivalently, every inner ideal is finitely generated as an inner ideal. Since the inner ideals of V_y are precisely those inner ideals of V^σ contained in $Q_y V^+$ [**2**, 2·4], we have:

(**1·2**) If V is a Noetherian Jordan pair, then all its local algebras are also Noetherian.

Let V be a Jordan pair and let $x_1, \ldots, x_r$ be elements of V^σ. Set $\langle x_1, \ldots, x_r\rangle = \sum_{n\in\mathbb{N}^*} \langle x_1, \ldots, x_r\rangle_n$, where the subspaces $\langle x_1, \ldots, x_r\rangle_n$ are defined by recurrence as follows: $\langle x_1, \ldots, x_r\rangle_1 = \sum_{i=1}^r K x_i$ and for $n \geqslant 1$, $\langle x_1, \ldots, x_r\rangle_n$ is defined as:

(i) if $n = 2m$ is even, then $\langle x_1, \ldots, x_r\rangle_{2m}$ is the linear span of the set

$$\{Q_y V^{-\sigma},\ Q(x, x')V^{-\sigma} :\ y \in \langle x_1, \ldots, x_r\rangle_m,\ x \in \langle x_1, \ldots, x_r\rangle_i \text{ and } x' \in \langle x_1, \ldots, x_r\rangle_j \text{ with } i + j = 2m\}.$$

(ii) if n is odd, then $\langle x_1, \ldots, x_r\rangle_n$ is the linear span of the set

$$\{Q(x, x')V^{-\sigma} : x \in \langle x_1, \ldots, x_r\rangle_i \quad \text{and} \quad x' \in \langle x_1, \ldots, x_r\rangle_j \text{ with } i + j = n\}.$$

Clearly, $\langle x_1, \ldots, x_r\rangle$ is an inner ideal of V^σ. In fact, we have

Lemma 1·3. *Let V be a Jordan pair and $x_1, \ldots, x_r \in V^\sigma$. Then $\langle x_1, \ldots, x_r\rangle$ is the inner ideal generated by $x_1, \ldots, x_r$.*

Annihilators. Following [**15**, p. 104], the *annihilator* of a subset X of V^σ is the inner ideal $\mathrm{ann}\,(X) \subset V^{-\sigma}$ ($\mathrm{ann}\,(X)$ when V need not be specified) of all $a \in V^{-\sigma}$ satisfying:

$$Q_a X = 0, \quad Q_X a = 0, \quad Q_a Q_X = D(a, X) = 0 \quad \text{and} \quad Q_X Q_a = D(X, a) = 0.$$

An additional condition: $a \circ X = 0$, where $a \circ x = (a + x)^2 - a^2 - x^2$, is required for the annihilator of a subset X of a Jordan algebra J. By [**18**], the annihilator of an

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ideal I of a nondegenerate Jordan algebra J has an easy expression:

$$\text{and} \qquad \left.\begin{aligned} \operatorname{ann}_J(I) &= \{a \in J : U_a I = 0\} \\ I \bigcap \operatorname{ann}_J(I) &= 0. \end{aligned}\right\} \tag{1·4}$$

It is not difficult to check (see [**17**]) that the correspondence $X \to \operatorname{ann}(X)$ is a Galois one:

$$X \subset \operatorname{ann}(\operatorname{ann}(X)) \quad \text{and} \quad \operatorname{ann}(\operatorname{ann}(\operatorname{ann}(X))) = \operatorname{ann}(X).$$

Thus either chain condition on annihilators implies the other and in this case we say that V has chain condition on annihilators (cca). In particular, if V has acc or dcc on all inner ideals, then it automatically has cca.

Radicals. Following [**15**, p. 33], we denote by $\operatorname{rad}(V)$ the *small radical* of a Jordan pair V, i.e. $\operatorname{rad}(V)$ is the smallest ideal I of V such that V/I is nondegenerate.

Let B and C be ideals of V and let $T(B,C)^\sigma$ be the set of all finite sums of elements $z \in V^\sigma$ such that $Q_z B^{-\sigma} \subset C^\sigma$. By [**15**, 4.6], $T(B,C) = (T(B,C)^+, T(B,C)^-)$ is an ideal of V containing C. In particular, $T(V,0)$ is an ideal of V contained in $\operatorname{rad}(V)$ which consists of all finite sum of absolute zero divisors. Set $T_0(V) = 0$ and $T_i(V) = T(V, T_{i-1}(V))$ for $i \geqslant 1$. Then the $T_i(V)$ form an increasing sequence of ideals of V. Hence, it follows from [**15**, 4·7]:

(**1·5**) If V is Noetherian, then there exists $n \geqslant 0$ such that $\operatorname{rad}(V) = T_n(V)$.

We write $\operatorname{Rad}(S)$ to denote the *Jacobson radical* of a Jordan system S (algebra or pair). We say that S is *semiprimitive* if $\operatorname{Rad}(S) = 0$.

Following [**15**, 3·8], given $(x,y) \in V$, we denote by $x^{(n,y)}$ the nth power of x in the Jordan algebra $(V^+)^{(y)}$ and by $y^{(n,x)}$ the n-th power of y in $(V^-)^{(x)}$. We have the formula:

$$x^{(n+1,y)} = Q_x y^{(n,x)}. \tag{1·6}$$

A pair $(x,y) \in V$ is called *nilpotent* if x is nilpotent in $(V^+)^{(y)}$, equivalently, if y is nilpotent in $(V^-)^{(x)}$.

A Jordan pair V is called *properly nil* (nil in the nomenclature of [**15**]) if all $(x,y) \in V$ are nilpotent. If I is an ideal of V then V is properly nil if and only if I and V/I are properly nil. Hence the sum of all properly nil ideal of V is a properly nil ideal, called the *properly nil radical* and denoted by $\operatorname{PNil}(V)$. As noted in [**15**, 4·14], $\operatorname{PNil}(V)^\sigma$ consists of *properly nilpotent elements*, that is, those elements $x \in V^\sigma$ which are nilpotent in every homotope $(V^\sigma)^{(y)}$, $y \in V^{-\sigma}$. By [**15**, 4·15],

$$\operatorname{rad}(V) \subset \operatorname{PNil}(V) \subset \operatorname{Rad}(V). \tag{1·7}$$

Since for a Jordan algebra J, the odd powers of $z \in J$ are just all the powers of z in the z-homotope, $z^{(n,z)} = z^{2n+1}$, proper nilness implies ordinary nilness.

Socle and capacity. For a nondegenerate Jordan pair V, $(\operatorname{Soc}(V^+), \operatorname{Soc}(V^-))$, where $\operatorname{Soc}(V^\pm)$ denotes the sum of all minimal inner ideals of $V^\pm$, is an ideal of V called the *socle* of V. By [**16**, theorem 1], for $x \in V^+$, the following conditions are equivalent: (i) x is in the socle; (ii) x is von Neumann regular and if $x = e^+$ for some idempotent e then the Peirce space $V_2(e)$ has *finite capacity* (the lengths of its chains of principal

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inner ideals are bounded) and (iii) the inner ideal $\langle x\rangle$ generated by x has dcc on principal inner ideals.

The following local characterization of the socle can be found in [**20**, 0·7(b)]:

(**1·8**) Let V be a nondegenerate Jordan pair and $x \in V^\sigma$. Then $x \in \mathrm{Soc}\,(V^\sigma)$ if and only if V_x has finite capacity. Moreover, $u \in V^\sigma$ generates a minimal inner ideal if and only if V_u is a division Jordan algebra.

Centroid. Following [**15**, p. 10], the centroid $\Gamma(V)$ of a Jordan pair V is defined as the set of all $\varphi = (\varphi^+, \varphi^-) \in \mathrm{End}\,(V^+) \times \mathrm{End}\,(V^-)$ such that:

$$\begin{aligned} \varphi^\sigma Q_x &= Q_x \varphi^{-\sigma}, \\ \varphi^\sigma D(x,y) &= D(x,y)\varphi^\sigma, \\ Q_{\varphi^\sigma x} &= (\varphi^\sigma)^2 Q_x. \end{aligned}$$

Note that in the linear case the last condition is superfluous.

Clearly, if $\lambda \in K$ then $(\lambda id, \lambda id)$ belongs to $\Gamma(V)$. If every element of $\Gamma(V)$ is of this form, V is said to be *central*. The reader is referred to [**19**] for general results on the centroid. In particular we have (see [**19**, 2·7 and 2·8]) that the centroid of a simple Jordan pair is a field and the centroid $\Gamma(V)$ of a prime Jordan pair V is a commutative domain acting faithfully on V.

Recall that an element $0 \neq c \in V^\sigma$ is called *reduced* if $Q_c V^{-\sigma} = \Phi c$. Note that if Φ is a field, then every reduced element of V lies in the socle.

Lemma 1·9 [**19**, 2·10]. *If a prime Jordan pair V contains a reduced element, then V is central.*

2. *Nilness and uniform elements*

Let V be a Jordan pair. A nonzero element $a \in V^\sigma$ will be called *uniform* if $\mathrm{ann}_V(a) = \mathrm{ann}_V(Q_a x)$ for all $0 \neq Q_a x \in Q_a V^{-\sigma}$.

To study uniform elements through their local algebras we need some computations which we include in the following lemma. This lemma is similar to that proved in [**12**, 8·2] for the case of a Jordan algebra.

Lemma 2·1. *Let V be a Jordan pair and $a \in V^\sigma$. For $x, y \in V^{-\sigma}$, set $\overline{x} = x + \mathrm{Ker}\,(a)$, $\overline{y} = y + \mathrm{Ker}\,(a)$. If $\overline{x} \in \mathrm{ann}_{V_a}(\overline{y})$, then $Q_x Q_a Q_{V^{-\sigma}} Q_a Q_x Q_a V^{-\sigma} \subset \mathrm{ann}_V(Q_a y)$.*

Proof. Take $r, s \in V^{-\sigma}$. Set $b = Q_x Q_a Q_r Q_a Q_x Q_a s$ and $c = Q_a y$. We must show that $b \in \mathrm{ann}_V(c)$. It is not difficult to see that $Q_b c = 0$ and $Q_c b = 0$. Now we have

(i) $Q_b Q_c V^{-\sigma} = Q_x Q_a Q_{Q_r Q_a Q_x Q_a s} Q_a Q_x Q_{Q_a y} V^{-\sigma}$, but

$$Q_a Q_x Q_{Q_a y} V^{-\sigma} = Q_a (U_x^{(a)} U_y^{(a)} V^{-\sigma}) = 0$$

since $\overline{x} \in \mathrm{ann}_{V_a}(\overline{y})$.

(ii) Now if $z \in V^{-\sigma}$, we have by linearization of JP2,

$$\begin{aligned} \{b\,c\,z\} &= \{Q_{Q_x Q_a r} Q_a s\, Q_a y\, z\} \\ &= \{Q_x Q_a r\, \{Q_a s\, Q_x Q_a r\, Q_a y\}\, z\} - \{Q_{Q_x Q_a r} Q_a y\, Q_a s\, z\}, \end{aligned}$$

but

$$\{Q_a s\, Q_x Q_a r\, Q_a y\} = Q_a \{s\, U_x^{(a)} r\, y\}^{(a)} = 0$$

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and

$$Q_{Q_xQ_ar}Q_ay = Q_xQ_aQ_rQ_a(U_x^{(a)}y) = 0$$

since $\overline{x} \in \mathrm{ann}_{V_a}(\overline{y})$. Similarly, $Q_cQ_bV^\sigma = \{c\,b\,V^\sigma\} = 0$. Therefore, $b \in \mathrm{ann}_V(c)$.

Recall that a Jordan algebra J is called a *Jordan domain* if every nonzero element $x \in J$ is *regular*, i.e. U_x is injective. Notice that J may not be unital with this definition.

Given a Jordan algebra J, we shall write J' to denote the *unital hull* of J, i.e. $J' = J$ if J is unital and $J' = \Phi \oplus J$, the *unitization* of J, otherwise. Note that in general the unital hull of a Jordan (even associative and commmutative) domain is not always a domain. Consider for example the unitization $\mathbb{Z} \oplus 2\mathbb{Z}$ of $2\mathbb{Z}$. However, this is true for Jordan domains which are algebras over a field.

LEMMA 2·2. *Let J be a Jordan algebra over a field Φ. If J is a Jordan domain, then J' is also a Jordan domain.*

Proof. Clearly, we may assume that J is not unital and it is easy to see that $J' = \Phi \oplus J$ is a nondegenerate Jordan algebra. Let us show that $\mathrm{ann}_{J'}(J) = 0$. Indeed, suppose that there exists $0 \neq \alpha + x \in \mathrm{ann}_{J'}(J)$. Then $\alpha \neq 0$ and $1 + \alpha^{-1}x \in \mathrm{ann}_{J'}(J)$. Thus, for any y in J, we have

$$(1 + \alpha^{-1}x) \circ y = 2y + \alpha^{-1}x \circ y = 0$$

and

$$U_{1+\alpha^{-1}x}y = y + U_{\alpha^{-1}x}y + \alpha^{-1}x \circ y = 0,$$

and hence we obtain (i) $U_{-\alpha^{-1}x}y = U_{\alpha^{-1}x}y = y$. Since $U_y(1 + \alpha^{-1}x) = 0$, we also have (ii) $U_y(-\alpha^{-1}x) = y$. It follows from (i) and (ii) that $-\alpha^{-1}x$ is a unit element for J, which is a contradiction. Note that in the linear case, i.e. $\frac{1}{2} \in \Phi$, just the condition $(1 + \alpha^{-1}x) \cdot y = 0$ is enough to assure that $-\alpha^{-1}x$ is a unit element for J. Since J is an ideal of J', we have shown by (1·4) that $U_{\alpha+x}J \neq 0$ for any nonzero $\alpha + x \in J'$.

Let $\alpha + x$ and $\beta + y$ be two nonzero elements in J' such that $U_{\alpha+x}(\beta + y) = 0$. By the above, there exist $0 \neq u \in U_{\alpha+x}J \subset J$ and $0 \neq v \in U_{\beta+y}J \subset J$. Since J is a domain, $U_vu \neq 0$ and therefore $U_{\beta+y}u \neq 0$. But, since $U_{\alpha+x}(\beta + y) = 0$, it follows by JP3,

$$U_uU_{U_{\beta+y}u}J = U_{U_u(\beta+y)}U_{\beta+y}J = 0,$$

which is a contradiction since J is a domain.

PROPOSITION 2·3. *Let V be a nondegenerate Jordan algebra and $0 \neq a \in V^\sigma$. The following conditions are equivalent*:

(i) *the element a is uniform*;

(ii) *the local algebra V_a of V at a is a Jordan domain.*

Proof. (i) $\Rightarrow$ (ii) According to [**12**, 8·3], to show that V_a is a domain it is enough to see that every nonzero element of V_a has zero annihilator. Now if $x, y \in V^{-\sigma}$ have images $\overline{x}, \overline{y} \in V_a$ with $\overline{y} \neq 0$ and $\overline{x} \in \mathrm{ann}_{V_a}(\overline{y})$, then, by Lemma 2·1, $Q_xQ_aQ_V^{-\sigma}Q_aQ_xQ_aV^{-\sigma} \subset \mathrm{ann}_V(Q_ay)$. Since a is uniform and $Q_ay \neq 0$, $Q_xQ_aQ_V^{-\sigma}Q_aQ_xQ_aV^{-\sigma} \subset \mathrm{ann}_V(a)$. Thus $0 = Q_aQ_xQ_aQ_V^{-\sigma}Q_aQ_xQ_aV^{-\sigma} =$

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$Q_{Q_a x} Q_V^{-\sigma} Q_{Q_a x} V^{-\sigma} = Q(Q_{Q_a x} V^{-\sigma}) V^{-\sigma}$. Therefore $Q_a x = 0$ by nondegeneracy of V and $\overline{x} = 0$.

(ii) $\Rightarrow$ (i) If a is not uniform, then there exists $0 \neq Q_a x$ such that $\operatorname{ann}_V(a)$ is strictly contained in $\operatorname{ann}_V(Q_a x)$. Take a $y \in \operatorname{ann}_V(Q_a x)$ not in $\operatorname{ann}_V(a)$. Then

$$Q_a y = Q_a Q_y V^{\sigma} = 0 \quad \text{and} \quad Q_y a = Q_y Q_a V^{-\sigma} = 0. \tag{$*$}$$

Indeed, $Q_{Q_a x} Q_y V^{\sigma} = 0$ since $Q_y V^{\sigma} \subset \operatorname{ann}_V(Q_a x)$, so in the local algebra V_a of V at a we have $U_{\overline{x}} \overline{Q_y V^{\sigma}} = 0$ with $\overline{x} \neq 0$. Since V_a is a domain, $\overline{Q_y V^{\sigma}} = 0$, which implies $Q_a Q_y V^{\sigma} = 0$. The remaining equalities follow by nondegeneracy of V.

Since y does not belong to $\operatorname{ann}_V(a)$, we have by $(*)$ that there exists $t \in V^{-\sigma}$ such that $\{y\, a\, t\} \neq 0$. Take $z \in V^{\sigma}$ such that $Q_{\{y\, a\, t\}} z \neq 0$. Replacing z by t, y by a, and x by y in the Jordan operator identity JP20 of [15, p. 19], and making it act on the element z, we obtain

$$0 \neq Q_{\{y\, a\, t\}} z = Q_y Q_a Q_t z + Q_t Q_a Q_y z + \{y\, Q_a \{t\, z\, y\}\, t\} - \{Q_y a\, z\, Q_t a\}.$$

Hence, by $(*)$, $Q_a \{t\, z\, y\} \neq 0$, equivalently, $\overline{\{t\, z\, y\}} \neq 0$. On the other hand, replacing x by $Q_a x$ in JP11 of [15, p. 14], and making it act on t, we obtain that

$$Q_{Q_a x} \{t\, z\, y\} = \{\{Q_a x\, t\, z\}\, y\, Q_a x\} - \{z\, t\, Q_{Q_a x} y\} = 0,$$

since $y \in \operatorname{ann}_V(Q_a x)$. Therefore, $U_{\overline{x}} \overline{\{t\, z\, y\}} = 0$ with $\overline{x} \neq 0$ and $\overline{\{t\, z\, y\}} \neq 0$, which is a contradiction.

Lemma 2·4. *Let V be a nondegenerate Jordan pair and let $K \subset V^{\sigma}$ be an inner ideal of V. If V satisfies the ascending chain condition on* $\operatorname{ann}_V(x)$ *for $0 \neq x \in K$, and K is properly nil, then $K = 0$.*

Proof. Among all the annihilators $\operatorname{ann}_V(x)$ for $0 \neq x$ in K, choose a maximal one. Then x is a uniform element of V and hence V_x is a Jordan domain by Proposition 2·3. In particular, V_x contains no nonzero nilpotent elements. Let $\overline{y} \in V_x$ be a non-nilpotent element. Then $Q_x y^{(n,x)} \neq 0$ for all positive integers n and hence by (1·6), $x^{(n+1,y)} = Q_x y^{(n,x)} \neq 0$, which is a contradiction since x is properly nilpotent.

Corollary 2·5. *If a Jordan pair V has acc or dcc on inner ideals, then* $\operatorname{rad}(V) = \operatorname{PNil}(V)$.

Proof. As pointed out in (1·7), $\operatorname{rad}(V) \subset \operatorname{PNil}(V)$ for any Jordan pair V. Now, if V has acc or dcc on inner ideals, then $\overline{V} = V/\operatorname{rad}(V)$ is a nondegenerate Jordan pair having acc or dcc on inner ideals and therefore cca. Hence $\operatorname{rad}(V) = \operatorname{PNil}(V)$ by Lemma 2·4.

Something else can be said about the small radical of a Noetherian Jordan pair if the ring of scalars is a field.

Lemma 2·6. *Let V be a Noetherian Jordan pair over a field Φ and let* $\operatorname{rad}(V) = (S^+, S^-)$ *be its small radical. Then S^{σ} has a countable basis as vector space over Φ.*

Proof. By subsection 1·5, $\operatorname{rad}(V) = T_n(V)$ for some n. Also, $T_{i+1}(V)/T_i(V) = T_1(V/T_i(V))$ and $V/T_i(V)$ is a Noetherian Jordan pair. Hence it suffices to show that the vector space $T_1(V)^{\sigma}$ has a countable basis.

By hypothesis, the inner ideal $T_1(V)^{\sigma}$ is generated by a finite set $\{z_1, \ldots, z_r\}$ of

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elements of V^σ. But $T_1(V)^\sigma$ is the set of all finite sums of absolute zero divisors of V^σ, hence we can assume that $z_1, \ldots, z_r$ are absolute zero divisors. By Lemma 1·3, $T_1(V)^\sigma = \sum_{n\in\mathbb{N}^*} \langle z_1, \ldots, z_r\rangle_n$. We shall show by induction that $\langle z_1, \ldots, z_r\rangle_i$ is finite dimensional. Clearly, $\langle z_1, \ldots, z_r\rangle_1$ is a finite dimensional vector space over Φ. Suppose now that for every $m \leqslant n$, $\langle z_1, \ldots, z_r\rangle_m$ is finite dimensional and that $\langle z_1, \ldots, z_r\rangle_m = \sum_{i=1}^{t_m} \Phi w_i$, where $t_m \in \mathbb{N}^*$ and $w_1, \ldots, w_{t_m}$ are absolute zero divisors. Then, there exists a finite set $\{d_1, \ldots, d_s\}$ of absolute zero divisors such that the vector space $\langle z_1, \ldots, z_r\rangle_{n+1}$ is contained in the linear span of $\bigcup_{1\leqslant i,j\leqslant s} Q(d_i, d_j)V^{-\sigma}$. But $Q(d_i, d_j)V^{-\sigma}$ is an inner ideal of V which consists of absolute zero divisors (JP21). Hence, using the fact that V is Noetherian, we obtain that $Q(d_i, d_j)V^{-\sigma}$ is finite dimensional. Finally, $\langle z_1, \ldots, z_r\rangle_{n+1}$ is finite dimensional and $T_1(V)^\sigma$ has a countable basis.

3. *Noetherian Banach Jordan pairs*

In this section we shall prove: (i) the Jacobson radical of a (real or complex) Noetherian Banach Jordan pair is finite dimensional (ii) nondegenerate Noetherian Banach Jordan pairs have finite capacity and (iii) complex Noetherian Banach Jordan pairs are finite dimensional. However, unlike the complex case, real Noetherian Banach Jordan pairs are not necessarily finite dimensional.

By a normed Jordan algebra we mean a (real or complex) Jordan algebra J such that, as a vector space, is endowed with a norm $\|.\|$ making continuous the product of J. If the norm is complete, then J is said to be a Banach Jordan algebra. Of course, we can always renorm J with an equivalent norm $\|.\|'$ so that $\|x \cdot y\|' \leqslant \|x\|'\|y\|'$ for $x, y \in J$.

We note that if J is a normed (Banach) Jordan algebra, the unitization of J is also a normed (Banach) Jordan algebra with the norm $\|\alpha + x\| = |\alpha| + \|x\|$.

By a *normed Jordan pair* we shall mean a Jordan pair $V = (V^+, V^-)$ where the vector spaces V^+ and V^- are endowed with norms making continuous the triple products of V. If these norms are complete, then we shall say that V is a Banach Jordan pair.

Given a normed vector space H and a subset $X \subset H$, we denote by $\mathrm{cl}\,(X)$ and $\delta(X)$ respectively the closure and the boundary of X.

LEMMA 3·1. *Let V be a normed (Banach) Jordan pair. Then every local algebra V_y of V is a normed (Banach) Jordan algebra.*

Proof. It is clear that for $y \in V^{-\sigma}$, the homotope $(V^\sigma)^{(y)}$, with the same norm as V, is a normed (Banach) Jordan algebra. Moreover, $\mathrm{Ker}\,(a)$ is a closed ideal of $(V^\sigma)^{(y)}$. Hence, for the quotient norm, the local algebra V_y is a normed (Banach) Jordan algebra.

Let V be a nondegenerate complex normed Jordan pair containing a minimal inner ideal $K \subset V^\sigma$. Taking $0 \neq x \in K$, it follows from subsection 1·8 that V_x is a division Jordan algebra. Moreover, by Lemma 3·1, V_x is a complex normed division algebra. Hence V_x is isomorphic to the complex field by the Mazur–Gelfand theorem for Jordan algebra. Now, by [**2**, 1·7·1], V_x is isomorphic to $Q_x V^{-\sigma}$ as vector spaces. Then $Q_x V^{-\sigma} = \mathbb{C}x$ and x is a reduced element of V. This result and Lemma 1·9 imply:

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PROPOSITION 3·2. *Let V be a strongly prime complex normed Jordan pair with nonzero socle. Then V is central.*

LEMMA 3·3. *Let V be a Noetherian Banach Jordan pair. Then the small radical* $\mathrm{rad}\,(V)$ *is finite dimensional.*

Proof. Since $\mathrm{rad}\,(V)$ has a countable basis by Lemma 2·6, we only need to show by the Baire Category Theorem that $\mathrm{rad}\,(V)$ is closed. Set $\mathrm{rad}\,(V) = (S^+, S^-)$ and take $x \in V^\sigma$. Let $y \mapsto \pi(y) = y + \mathrm{Ker}\,(x)$ denote the canonical projection of $(V^{-\sigma})^{(x)}$ onto V_x. Since $\mathrm{rad}\,(V) \subset \mathrm{PNil}\,(V)$ by (1·7) (Corollary 2·5 is not really necessary here), any element $y \in S^{-\sigma}$ is properly nilpotent. In particular, it is nil in $(V^{-\sigma})^{(x)}$ and hence $\pi(S^{-\sigma}) \subset \mathrm{Nil}\,(V_x)$. But V_x is a Noetherian Jordan algebra, so $\mathrm{Nil}\,(V_x)$ is nilpotent by [**22**, theorem 1], which implies that $\mathrm{Nil}\,(V_x)$ is closed. Then $\pi(\mathrm{cl}\,(S^{-\sigma})) \subset \mathrm{cl}\,(\mathrm{Nil}\,(V_x)) = \mathrm{Nil}\,(V_x)$ by the continuity of π. Then, by (1·6), every element of $y \in \mathrm{cl}\,(S^{-\sigma})$ is nilpotent in $(V^{-\sigma})^{(x)}$. Therefore, $\mathrm{cl}\,(S^{-\sigma})$ consists uniquely of properly nilpotent elements. Hence $\mathrm{cl}\,(S^{-\sigma}) = S^{-\sigma}$ by Corollary 2·5.

Let J be a unital Jordan algebra over the complex field $\mathbb{C}$. Recall that the *spectrum* of an element $x \in J$ is defined by set:

$$\mathrm{Sp}_J(x) = \{\lambda \in \mathbb{C} \colon \lambda - x \quad \text{is not invertible in } J\}.$$

For a real Jordan algebra J, the spectrum of an element $x \in J$ is defined to be the spectrum of x in the complexification $J_\mathbb{C}$ of J: $\mathrm{Sp}_J(x) = \mathrm{Sp}_{J_\mathbb{C}}(x)$.

It follows from [**7**] that if A is a unital complex Jordan Banach algebra and $x \in A$, then there exists a closed associative subalgebra $C_A(x)$ of A containing x which is full in the following sense: $y \in C_A(x)$ is invertible in A if and only if it is invertible in $C_A(x)$. Hence, by [**8**, theorem 8, p. 70], if J is a real unital Banach Jordan algebra and $x \in J$, then $\lambda \in \mathrm{Sp}_J(x)$ if and only if $(\lambda - x) \cdot (\overline{\lambda} - x)$ is not invertible in J, where $\overline{\lambda}$ denotes the complex conjugate of λ.

LEMMA 3·4. *Let J be a real unital Banach Jordan algebra and let $y \in J$. If $\lambda \in \delta(\mathrm{Sp}_J(y))$ is such that $U_{(\overline{\lambda}-y)\cdot(\lambda-y)}J$ is closed, then $(\overline{\lambda} - y) \cdot (\lambda - y)$ is not regular in J.*

Proof. Let λ be in $\delta(\mathrm{Sp}_J(y)) = \delta(\mathrm{Sp}_{J_\mathbb{C}}(y))$. By [**8**, theorem 14, p. 13], there exists a sequence $\{x_n\}$ of elements of $C_{J_\mathbb{C}}(y) \subset J_\mathbb{C}$ such that $\|x_n\| = 1$ and $\lim_n (\lambda - y)x_n = 0$. Hence

$$\lim_n (\overline{\lambda} - y)(\lambda - y)x_n(\overline{\lambda} - y)(\lambda - y) = 0.$$

Now, it follows as in [**1**, lemma 3, p. 75] that $U_{(\overline{\lambda}-y)\cdot(\lambda-y)} \colon J_\mathbb{C} \to J_\mathbb{C}$ is not injective. Indeed, suppose on the contrary that it is injective. Since $U_{(\overline{\lambda}-y)\cdot(\lambda-y)}J_\mathbb{C}$ is closed, because so is $U_{(\overline{\lambda}-y)\cdot(\lambda-y)}J$, we have by the Inverse Mapping Theorem that there is a continuous linear operator $T \colon U_{(\overline{\lambda}-y)\cdot(\lambda-y)}J_\mathbb{C} \to J_\mathbb{C}$ such that $TU_{(\overline{\lambda}-y)\cdot(\lambda-y)}a = a$ for all $a \in J_\mathbb{C}$. Thus,

$$1 = \|x_n\| = \|TU_{(\overline{\lambda}-y)\cdot(\lambda-y)}x_n\| \leqslant |\, T \,|\, \|U_{(\overline{\lambda}-y)\cdot(\lambda-y)}x_n\|$$

which yields a contradiction since $\lim_n U_{(\overline{\lambda}-y)\cdot(\lambda-y)}x_n = 0$. Therefore, $(\overline{\lambda} - y) \cdot (\lambda - y)$ is not regular in $J_\mathbb{C}$ and hence neither in J.

LEMMA 3·5. *Every Noetherian Banach Jordan domain J is a division Jordan algebra. Moreover, if J is complex then it is the complex field.*

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Proof. If J is complex then it is finite dimensional by [**6**, theorem 6] and hence J is the complex field by the classification of simple finite dimensional Jordan algebras over an algebraically closed field [**13**, corollary 2, p. 204]. Suppose then that J is a real algebra. Following the methods of the proof of [**6**, corollary], and with the help of Lemma 3·4, we shall prove that J is algebraic and semiprimitive.

Clearly, the normed unital hull J' of J is a Noetherian Banach Jordan algebra, and by Lemma 2·2 it is a Jordan domain. Now, given $x \in J$, $\mathrm{Sp}_{J'}(x)$ is not empty and hence neither is $\delta(\mathrm{Sp}_{J'}(x))$. Thus we can take $\lambda \in \delta(\mathrm{Sp}_{J'}(x))$. Since J' is Noetherian, it follows from [**6**, proposition 3] that $U_{(\lambda-x)\cdot(\overline{\lambda}-x)}J'$ is closed. Hence $(\lambda - x)\cdot(\overline{\lambda} - x)$ is not regular in J' by Lemma 3·4. But J' is a domain, so $(\lambda - x)\cdot(\overline{\lambda} - x) = 0$, and therefore x is algebraic.

To see that J is semiprimitive, let $z \in \mathrm{Rad}\,(J')$. Then 0 is in the boundary of $\mathrm{Sp}_{J'}(z)$ and as above we obtain that $z = 0$. Thus, $\mathrm{Rad}\,(J') = 0$ and therefore $\mathrm{Rad}\,(J) = 0$ as well.

Now we have by [**9**, theorem 4] that J is unital and has finite capacity, but a Jordan domain cannot have nonzero pairwise orthogonal idempotents. Therefore, J is a division Jordan algebra.

THEOREM 3·6. *Every nondegenerate Noetherian Banach Jordan pair has finite capacity.*

Proof. Let V be a nondegenerate Noetherian Banach Jordan pair. By [**11**, 4·2], V is an essential subdirect sum of finitely many strongly prime Noetherian Banach Jordan pairs V_i. Now we show that each V_i is a simple Jordan pair of finite capacity. By the Noetherian condition, there exists a uniform element $b \in V_i^{-\sigma}$. Then, by subsection 1·2, Proposition 2·3 and Lemma 3·1, $(V_i)_b$ is a Noetherian Banach Jordan domain and hence, by Lemma 3·5, a division Jordan algebra. Then V_i has nonzero socle by subsection 1·8. Moreover, V_i is idempotent finite since it is Noetherian. Therefore, $\mathrm{Soc}\,(V_i)$ has a finite capacity. Let $e_1, \ldots, e_n$ be a maximal sequence of pairwise orthogonal division idempotents in $\mathrm{Soc}\,(V_i)$ and set $e = e_1 + \cdots + e_n$. Write $W = V_i$ and consider the Peirce decomposition of W with respect to e [**15**, p. 44]. Here the Peirce-0-space $W_0(e)$ is a nondegenerate Noetherian Banach Jordan pair and hence $W_0(e) = 0$ by the maximality of the sequence $e_1, \ldots, e_n$. Then, it follows from [**16**, theorem 2(*b*)] and the strong primeness of W, that $W = W_2(e) \oplus W_1(e) = \mathrm{Soc}\,(W)$ is a simple Jordan pair of finite capacity. Since V contains an essential ideal of the direct sum of the V_i, V is actually the direct sum of the V_i, and therefore the whole V has finite capacity.

Remark. For a complex Jordan pair V a more elementary proof of Theorem 3·6 can be given. This alternative proof uses the fact that complex Noetherian Banach Jordan algebras are finite dimensional instead of Lemma 3·5. Indeed, given $y \in V^{-\sigma}$, we have (subsections 1·1, 1·2 and Lemma 3·1) that the local algebra V_y of V at y is a complex nondegenerate Noetherian Banach Jordan algebra. Hence V_y is finite dimensional by [**6**, theorem 6]. In particular, V_y has a finite capacity. Then $y \in \mathrm{Soc}\,(V^{\sigma})$ by subsection 1·8. Therefore, V is equal to its socle and hence it follows from [**16**, theorem 2], together with the acc, that V is a direct sum of finitely many simple ideals V_i each of which is a Noetherian Banach Jordan pair having finite capacity.

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COROLLARY 3·7. *The Jacobson radical of a Noetherian Banach Jordan pair V is finite dimensional and hence nilpotent.*

Proof. By (1·7), $\operatorname{rad}(V) \subset \operatorname{PNil}(V) \subset \operatorname{Rad}(V)$, where $\operatorname{rad}(V) = \operatorname{PNil}(V)$ is finite dimensional by Corollary 2·5 and Lemma 3·3. Thus, $\operatorname{rad}(V)$ is closed and $V/\operatorname{rad}(V)$ is a nondegenerate Noetherian Banach Jordan pair. Hence $V/\operatorname{rad}(V)$ has finite capacity by Theorem 3·6. Now it follows from [**15**, 10·8] that the Jacobson radical coincides with the small radical and hence it is finite dimensional. Finally, $\operatorname{Rad}(V)$ is nilpotent by [**15**, 4·13 and 14·11].

We do not know whether the Nilpotence Theorem is true for arbitrary Noetherian Jordan pairs, i.e. whether the properly nil radical of a Noetherian Jordan pair is nilpotent.

We end this section by proving that complex Noetherian Banach Jordan pairs are finite dimensional.

THEOREM 3·8. *For a complex Banach Jordan pair V, the following conditions are equivalent*:

(i) *V is finite dimensional;*
(ii) *every inner ideal of V is closed;*
(iii) *V is Noetherian.*

Proof. (i) $\Rightarrow$ (ii) is clear and (ii) $\Rightarrow$ (iii) can be obtained by a standard application of the Baire Category Theorem (see [**1**, corollary 6, p. 77]).

(iii) $\Rightarrow$ (i). Clearly, $V/\operatorname{rad}(V)$ is a nondegenerate Noetherian Jordan pair. Moreover, by Lemma 3·3, $\operatorname{rad}(V)$ has finite dimension (in particular, closed), so $V/\operatorname{rad}(V)$ is a Banach Jordan pair which will be finite dimensional if and only if V is too. Thus, we may assume from the beginning that V is nondegenerate. Then V has finite capacity by Theorem 3·6 and hence by [**16**, Theorem 2] it is a direct sum of finitely many simple ideals V_i each of which is a Notherian Banach Jordan pair having finite capacity. Moreover, we have by Proposition 3·2 that the V_is are central Jordan pairs. Looking at Loos' classical list of the central Jordan pairs of finite capacity [**15**, 12·12], but as presented in [**19**, 5·9] in order to identify the centroids, we have one of the following possibilities for each V_i:

(i) $A(M, R, f)^J$ for a central simple Artinian associative algebra $R \cong M_n(\Delta)$ with Δ a central division associative algebra over $\mathbb{C}$;
(ii) $(A_n(\mathbb{C}), A_n(\mathbb{C}))$, the Jordan pair of alternating $n \times n$ complex matrices with $(n \geqslant 5)$;
(iii) $(H_n(\Delta, *), H_n(\Delta, *))$, the Jordan pair of a Jordan algebra $H_n(\Delta, *)$ of hermitian $n \times n$ matrices over a division associative $*$-algebra over the complex field $(n \geqslant 2)$;
(iv) (J, J), the Jordan pair of the Jordan algebra J defined by a nondegenerate quadratic form over the complex vector space;
(v) $(M_{1,2}(\mathscr{C}), M_{1,2}(\mathscr{C}))$ where $\mathscr{C}$ is the Cayley algebra over the complex field;
(vi) $(\mathscr{A}, \mathscr{A})$, the Jordan pair of the complex 27-dimensional Albert algebra $\mathscr{A}$.

Clearly, if V_i is in one of the cases (ii), (v) or (vi), then V_i is finite dimensional. Also, if V_i is as in (iii) or (iv), then V_i is the Jordan pair of a complex Noetherian Banach Jordan algebra and hence V_i is finite dimensional by [**6**, theorem 6]. Finally, if V_i is

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as in (i), we have by [**15**, 11·17] that it is isomorphic to $(M_{p,q}(\Delta), M_{p,q}(\Delta^{op}))$, where Δ is a complex division associative algebra. Moreover, by [**3**, 4·1], $\Delta \cong (V_i)_b$ for a certain rank one element b of V_i. Hence Δ is the complex field by the Mazur–Gelfand Theorem for Jordan algebras and therefore V_i is finite dimensional.

Remark. Unlike the complex case, a real Noetherian Banach Jordan pair is not necessarily finite dimensional. In fact, as shown in [**14**], a real division Banach Jordan algebra need not be finite dimensional. Consider the Jordan Banach algebra given by a continuous negative definite symmetric bilinear form on an infinite dimensional real Banach space.

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