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Shared dynamically small points for polynomials on average

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Abstract. Given two rational maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d over $\mathbb{C}$, DeMarco, Krieger, and Ye [Common preperiodic points for quadratic polynomials. *J. Mod. Dyn.* **18** (2022), 363–413] have conjectured that there should be a uniform bound $B = B(d) > 0$ such that either they have at most B common preperiodic points or they have the same set of preperiodic points. We study their conjecture from a statistical perspective and prove that the average number of shared preperiodic points is zero for monic polynomials of degree $d \geq 6$ with rational coefficients. We also investigate the quantity $\liminf_{x \in \overline{\mathbb{Q}}}(\widehat{h}_f(x) + \widehat{h}_g(x))$ for a generic pair of polynomials and prove both lower and upper bounds for it.

Key words: preperiodic points, energy pairing, arithmetic dynamics, unlikely intersection
2020 Mathematics Subject Classification: 37P05, 37P30 (Primary); 37P35, 37P50 (Secondary)

1. *Introduction*

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function of degree $d \geq 2$ defined over a field K. The set of preperiodic points of f, denoted by $\mathrm{Prep}(f)$, is the set of points $x \in \mathbb{P}^1(\overline{K})$ such that $f^m(x) = f^n(x)$ for two distinct integers $m > n \geq 0$, where $\overline{K}$ denotes an algebraic closure of K. Our starting point is the following finiteness theorem for the common preperiodic points of two rational maps of degree $d \geq 2$.

THEOREM 1.1. [1, Corollary 1.3, 20, Theorem 1.3] *For any pair of rational functions $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{C}$ of degree $d \geq 2$, we have*

$$|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| < \infty \quad \text{or} \quad \mathrm{Prep}(f) = \mathrm{Prep}(g).$$

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We remark that the case of f, g being defined over $\overline{\mathbb{Q}}$ had been proven previously by Mimar [14]. We now consider two possible directions in which we can study extensions of this result. The first is to ask if the number of common preperiodic points can be uniformly bounded in families of pairs (f, g).

Conjecture 1.2. [6, Conjecture 1.4] Let $d \geq 2$. There exists a constant $B = B(d)$ so that for every pair of rational functions $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\mathbb{C}$ of degree d, we have

$$|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \leq B \quad \text{or} \quad \mathrm{Prep}(f) = \mathrm{Prep}(g).$$

In this direction, DeMarco, Krieger, and Ye [6, Theorem 1.1] obtain an effective uniform bound on the family of quadratic polynomials of the form $f(z) = z^2 + c_1$, $g(z) = z^2 + c_2$ with $c_1, c_2 \in \mathbb{C}$. Mavraki and Schmidt [13, Theorem 1.3] prove a uniform bound along any one-parameter family of pairs (f, g) defined over $\overline{\mathbb{Q}}$. More recently, DeMarco and Mavraki [7] prove the existence of a Zariski open $U \subseteq \mathrm{Rat}_d \times \mathrm{Rat}_d$, where Rat_d is the space of all degree d rational maps, for which a uniform bound holds.

Following Le Boudec and Mavraki [12] and also Olechnowicz [16], we will take a statistical point of view and, instead, consider the average count of common preperiodic points over pairs of monic polynomials over $\mathbb{Q}$. We consider only monic polynomials because the exceptional locus $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ is easily described as $\{f = g\}$. We let

$$\mathcal{P} = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid a_i \in \mathbb{Q}\},$$
$$\mathcal{P}_c = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathcal{P} \mid a_{d-1} = 0\},$$

be the set of monic polynomials and *monic, centered polynomials*, respectively, with rational coefficients. For $X \geq 1$, define

$$\mathcal{P}(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid a_i \in \mathbb{Q},\ H(a_i) \leq X\},$$
$$\mathcal{P}_c(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathcal{P}(X) \mid a_{d-1} = 0\},$$

the set of monic polynomials and monic, centered polynomials, respectively, with rational coefficients of Weil height $\leq X$ (see §2). Note that any pair of monic polynomials (f, g) over $\mathbb{Q}$ can be simultaneously conjugated by a translation to lie in $\mathcal{P}_c(X) \times \mathcal{P}(X)$ for some X.

In §4, we prove the following theorem.

THEOREM 1.3. *Let $d \geq 6$ and write $\mathcal{S}(X) = \mathcal{P}_c(X) \times \mathcal{P}(X) \setminus \{(f, f) \mid f \in \mathcal{P}_c(X)\}$. Then,*

$$\lim_{X \to \infty} \frac{1}{|\mathcal{S}(X)|} \sum_{(f,g) \in \mathcal{S}(X)} |\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| = 0,$$

where $\mathrm{Prep}(f)$, $\mathrm{Prep}(g)$ *denotes all preperiodic points of f and g in* $\overline{\mathbb{Q}}$.

Remark 1.4. The condition $d \geq 6$ is purely technical and we certainly expect the average to be zero for all degrees $d \geq 2$. With more work, we believe it is possible to improve our lower bound on d. Also, many of our lemmas adapt straightforwardly to number fields K

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instead of $\mathbb{Q}$ (see §3), but it is not clear if we can obtain a version of Theorem 1.3 where we vary over polynomials over a fixed number field K instead of $\mathbb{Q}$.

Our arguments also allow us to conclude the average remains zero even when restricted to coordinate slices of $\mathcal{P}$ of large enough dimension. Let m be a positive integer and let V be a linear subspace of $\mathcal{P}$ of dimension m that can be written as an intersection of some coordinate hyperplanes $\{a_i = 0\}$. Let $x = (x_{d-1}, \ldots, x_0) \in \mathbb{Q}^{d-1}$ and let A be the affine subspace given by $x + V$. We shall assume that the projection of V onto the last coordinate a_0 is surjective. Choose m coordinates $a_{i_1}, \ldots, a_{i_m}$ that give rise to an isomorphism $V \simeq \mathbb{Q}^m$ such that i_j are decreasing and $i_m = 0$. We define

$$\mathcal{P}_A(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid (a_{d-1}, \ldots, a_0) \in A \text{ and } H(a_{i_j}) \le X)\},$$

$$\mathcal{P}_{c,A}(X) = \{z^d + a_{d-2}z^{d-2} + \cdots + a_0 \mid (0, a_{d-2}, \ldots, a_0) \in A \text{ and } H(a_{i_j}) \le X\},$$

$$\mathcal{S}_A(X) = \mathcal{P}_{c,A}(X) \times \mathcal{P}_A(X) \setminus \{(f, f) \mid f \in \mathcal{P}_{c,A}(X)\}.$$

THEOREM 1.5. *Assume that* $\dim_{\mathbb{Q}} V \ge (d/2) + 3$. *Then,*

$$\lim_{X\to\infty} \frac{1}{|\mathcal{S}_A(X)|} \sum_{(f,g)\in\mathcal{S}_A(X)} |\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| = 0.$$

In another direction, when f, g are defined over $\overline{\mathbb{Q}}$, we can reinterpret the common preperiodic points of f, g as the set of common zeroes of the Call–Silverman canonical heights of f, g [4]:

$$\mathrm{Prep}(f) \cap \mathrm{Prep}(g) = \{x \in \overline{\mathbb{Q}} \mid \widehat{h}_f(x) + \widehat{h}_g(x) = 0\}.$$

In this setting, Baker and DeMarco as well as Mimar prove the following stronger form of Theorem 1.1.

THEOREM 1.6. [14, 15, Theorem 1.8] *For any pair of rational maps* $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ *of degree* $d \ge 2$ *defined over* $\overline{\mathbb{Q}}$, *we have*

$$\liminf_{x\in\overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) > 0 \quad \text{or} \quad \widehat{h}_f = \widehat{h}_g.$$

Here, $\liminf_{x\in\overline{\mathbb{Q}}} F(x)$ refers to the infimum of all accumulation points in $\mathbb{R}$ of the set $\{F(x) \mid x \in \overline{\mathbb{Q}}\}$; equivalently,

$$\liminf_{x\in\overline{\mathbb{Q}}} F(x) = \sup_{\substack{S\subseteq\overline{\mathbb{Q}} \\ |S|<\infty}} \inf_{x\in\overline{\mathbb{Q}}\setminus S} F(x).$$

In particular, if $\widehat{h}_f \ne \widehat{h}_g$, then there exists some $\varepsilon = \varepsilon(f, g) > 0$ such that $\widehat{h}_f(x) + \widehat{h}_g(x) \le \varepsilon$ has only finitely many solutions $x \in \overline{\mathbb{Q}}$; that is, there are only finitely many points $x \in \overline{\mathbb{Q}}$ that are both small for both $\widehat{h}_f$ and $\widehat{h}_g$. A natural question to ask is how large can ε be for a generic pair (f, g) as the height of f and g grows. For any fixed $\delta > 0$, applying Zhang's inequality [21, Theorem 1.10] (see §6) gives us

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$$(\tfrac{1}{2} - \delta)(h(f) + h(g)) \leq d \liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \leq (1 + \delta)(h(f) + h(g))$$

for a generic pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ and X large enough, where $h(f)$ is the height of a polynomial f (see §2). Our next theorem improves this inequality.

THEOREM 1.7. *Fix a $\delta > 0$. We let $\mathcal{S} = \mathcal{P}_c \times \mathcal{P}$ and $\mathcal{S}(X) = \mathcal{P}_c(X) \times \mathcal{P}(X)$. Then, for all sufficiently large degrees d, there exists $a = a(d) > 0$, $A = A(d) > 0$ such that the following statement holds: there exists a subset $\mathcal{T} \subseteq \mathcal{S}$ such that for all $X \geq 1$,*

$$\frac{|\mathcal{T} \cap \mathcal{S}(X)|}{|\mathcal{S}(X)|} \geq 1 - \frac{A}{X^a}$$

and

$$(\ln 2 - \delta)(h(f) + h(g)) \leq d \liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \leq (\mathcal{C} + \delta)(h(f) + h(g))$$

for all $(f, g) \in \mathcal{T}$, where $\ln 2 = 0.69314\ldots$ and $\mathcal{C} = \log((9 + \sqrt{17})/8) + ((\sqrt{17} - 1)/8) = 0.88532\ldots$.

It will be interesting to determine the exact asymptotic, if it exists at all.

1.1. *Strategy of proof.* Our main tool for the proof of Theorems 1.3 and 1.5 is the energy pairing of Favre and Rivera-Letelier in [8], see also [17]. Given a polynomial $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over a number field K of degree $d \geq 2$, for each place $v \in M_K$, we can attach to it a corresponding equilibrium measure $\mu_{f,v}$. For Archimedean v, this is a probability measure on $\mathbb{C}$, while for non-Archimedean v, it is a probability measure on the Berkovich affine line $\mathbb{A}^1_{\mathrm{Berk},v}$. The collection of measures $\{\mu_{f,v}\}_{v \in M_K}$ gives us an adelic measure, where M_K are the places of K. For further details on $\mathbb{A}^1_{\mathrm{Berk},v}$, one may consult [2] or [3].

Given two polynomials f, g, Favre and Rivera-Letelier first define a local energy pairing between $\mu_{f,v}$ and $\mu_{g,v}$ for a place v, and then define a global energy pairing by summing the local pairings over $v \in M_K$. This gives us a quantitative way to measure a distance between f and g. Using the energy pairing and quantitative equidistribution, DeMarco, Krieger, and Ye were able to establish a uniform bound on the number of common preperiodic points for quadratic unicritical polynomials.

We follow their approach in using the adelic energy pairing, proving bounds that hold for all monic polynomials in general, but are weaker as we do not need a uniform bound. The hardest situation to handle is when both polynomials are close to each other. For example, when $f(z)$ has integer coefficients and $g(z) = f(z) + 1$, the Julia sets at every non-Archimedean place are the corresponding Gauss point $\zeta(0, 1)$ and so their local energy pairing is zero. However, if f and g have large coefficients, then they are also relatively close for the Archimedean places. To obtain a usable bound for Theorems 1.3 and 1.5, we use the fact that the moments of the equilibrium measure $\mu_{f,v}$ are determined by the coefficients of f when f is monic (see §3.4).

Now, for Theorem 1.7, we introduce a notion of genericity for pairs of polynomials in $\mathcal{P}_c(X) \times \mathcal{P}(X)$, which may be of independent interest. For an integer n, we let $\mathrm{rad}(n)$ be the product of all distinct primes that divides n.

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Definition 1.8. Given X and $\varepsilon > 0$, we will say that a pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ is *ε-ordinary* if

$$\operatorname{rad}(\operatorname{denom}(c)) \geq X^{1-2\varepsilon},$$
$$\gcd(\operatorname{denom}(c), \operatorname{denom}(c')) \leq X^{2\varepsilon}$$

for any distinct coefficients c, c' of f or g.

In other words, a pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ is ε-ordinary if the denominators of the coefficients for both f and g are large, and are mostly coprime and squarefree. Similarly, we can make an analogous definition for a single polynomial $f \in \mathcal{P}(X)$ to be ε-ordinary. As a consequence of the definitions, for most places v of bad reduction of an ε-ordinary pair, there is only one coefficient of f, g that satisfies $|c|_v > 1$.

The dynamics of an ε-ordinary polynomial f at such a place v is simple enough to be analyzed thoroughly, but still more complicated than that of a unicritical polynomial. For example, when v is non-Archimedean, elements x of the Julia set of f have three possible absolute values $|x|_v$ (see Proposition 4.3), whereas that for the Julia set of a polynomial of the form $z^d + c$ has one absolute value when $|c|_v > 1$. In §6, we study the properties of the dynamical Green's function $g_{f,v}$ and equilibrium measure $\mu_{f,v}$.

For such pairs, we control some measures of arithmetic complexity and the structure of the Julia sets in §5. The lower bound of Theorem 1.7 then becomes a consequence of the structure and the product formula. The upper bound is more involved as it requires us to construct Galois orbits whose local heights are not too large. We do so by constructing an appropriate adelic set and then applying the adelic Fekete–Szegő theorem due to Rumely [**19**].

2. *Definitions and notation*

Here, we collect some notation used throughout this paper.

The cardinality of a set S is written as $|S|$ or $\#S$.

We write $f = O(g)$ or $f \lesssim g$ for $f \leq Cg$, where $C > 0$ is an absolute constant; $f = O_{\alpha,\beta,\ldots}(g)$ or $f \lesssim_{\alpha,\beta,\ldots} g$ for $f \leq C_{\alpha,\beta,\ldots} g$, where $C_{\alpha,\beta,\ldots} > 0$ is a constant depending on parameters α, β, . . .; and $f = o(g)$ for $f/g \to 0$. In such asymptotic notation, the limit to be taken will either be explicitly stated or clear from context.

The set of preperiodic points of f is denoted by $\operatorname{Prep}(f)$.

The Weil height H on $\mathbb{Q}$ is defined by $H(a/b) = \max(|a|, |b|)$ for $a, b \in \mathbb{Z}$ with $b \neq 0$ and $\gcd(a, b) = 1$.

The radical of a positive integer n, denoted by $\operatorname{rad}(n)$, is defined as the product of all distinct primes that divide n.

For a number field K, the set of all places for K is denoted by M_K.

For each finite place $v \mid p < \infty$, the non-Archimedean norm $|\cdot|_v$ on $\mathbb{C}_v$ is normalized with $|p|_v = 1/p$. For a polynomial $f \in \mathbb{Q}[z]$ given by $f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0$, the height of f is defined as

$$h(f) = h_{\mathbb{P}^d}([1 : a_{d-1} : \cdots : a_0]) = \sum_{v \in M_{\mathbb{Q}}} \log \max(1, |a_{d-1}|_v, \ldots, |a_0|_v).$$

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We now recall some notions from non-Archimedean dynamics; we refer the reader to [3] for the theory of dynamics on Berkovich space, and to [2] for non-Archimedean potential theory.

For each finite place $v \mid p < \infty$, write $\mathbb{A}^1_{\mathrm{Berk},v}$ for the Berkovich affine line. Now, $\mathbb{C}_v$ embeds in $\mathbb{A}^1_{\mathrm{Berk},v}$ as the type I points, which form a dense set under the weak/Gelfand topology. Write $\delta(z, w)_\infty$ for the Hsia kernel on $\mathbb{A}^1_{\mathrm{Berk},v}$, the unique continuous extension of the metric $|z - w|_v$ on type I points.

Write $\mu_{f,v}$ for the invariant measure of a polynomial $f \in \mathbb{C}_v[z]$, and $g_{f,v}$ for its potential function. When f is monic, this potential is also equal to the escape rate function for f, that is,

$$g_{f,v}(\zeta) = \lim_{n\to\infty} d^{-n} \log^+[f^n]_\zeta,$$

where $\log^+ = \max\{0, \log\}$. The local energy pairing between two signed measures μ_v, μ'_v on $\mathbb{A}^1_{\mathrm{Berk},v}$ is defined as

$$(\mu_v, \mu'_v)_v = -\int_{\mathbb{A}^1_{\mathrm{Berk},v}\times\mathbb{A}^1_{\mathrm{Berk},v}\setminus\mathrm{Diag}} \log \delta(z, w)_\infty \, d\mu_v(z) \otimes d\mu'_v(w).$$

The energy of μ_v is defined as

$$I(\mu_v) = \int_{\mathbb{A}^1_{\mathrm{Berk},v}\times\mathbb{A}^1_{\mathrm{Berk},v}} \log \delta(z, w)_\infty \, d\mu_v(z) \otimes d\mu_v(w),$$

which is also equal to $-(\mu_v, \mu_v)_v$ when μ_v has continuous potential. By abuse of notation, for any compact set $E_v \subseteq \mathbb{A}^1_{\mathrm{Berk},v}$, we define the logarithmic capacity of E_v as

$$I(E_v) = \sup_{\mu_v} I(\mu_v),$$

where the supremum is taken over all probability measures μ_v supported on E_v. When this is greater than $-\infty$, we have $I(E_v) = I(\mu_{E_v})$, where μ_{E_v} is the equilibrium measure of E_v.

An adelic measure is a collection of probability measures $\{\rho_v\}_{v\in M_K}$ on $\mathbb{A}^1_{\mathrm{Berk},v}$ such that ρ_v has continuous potentials and $\rho_v = \delta_{\zeta(0,1)}$ for all but finitely many v. Following Favre and Rivera-Letelier [8], we can define a global pairing between two adelic measures ρ, ρ' by

$$\langle \rho, \rho' \rangle = \frac{1}{2} \sum_{v\in M_{\mathbb{Q}}} (\rho_v - \rho'_v, \rho_v - \rho'_v)_v.$$

For a finite $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set $F \subseteq \overline{\mathbb{Q}}$, write $[F]$ for the uniform probability distribution on F, that is,

$$[F] := \frac{1}{|F|} \sum_{w\in F} \delta_w.$$

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Furthermore, for $\varepsilon > 0$, we can define the regularized measure

$$[F]_\varepsilon := \frac{1}{|F|} \sum_{w \in F} \delta_{w,\varepsilon},$$

where $\delta_{w,\varepsilon}$ is the uniform probability measure on $\partial B(w, \varepsilon)$ for v Archimedean, or the delta mass on $\zeta(w, \varepsilon)$ for v non-Archimedean.

For any polynomial $f \in \mathbb{Q}[z]$, we will generalize the definition of the canonical height $\widehat{h}_f$ to finite $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant sets $F \subseteq \overline{\mathbb{Q}}$ by

$$\widehat{h}_f(F) := \frac{1}{|F|} \sum_{w \in F} \widehat{h}_f(w).$$

Note that since $\widehat{h}_f$ is constant on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits, we have $\widehat{h}_f(x) = \widehat{h}_f(F)$ when F is the set of Galois conjugates of $x \in \overline{\mathbb{Q}}$.

For a fixed degree d, we have the following sets:

$$\mathcal{P} = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid a_i \in \mathbb{Q}\},$$
$$\mathcal{P}_c = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathcal{P} \mid a_{d-1} = 0\},$$

of monic polynomials with rational coefficients and monic, centered polynomials with rational coefficients, respectively. For $X \geq 1$, we also define

$$\mathcal{P}(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid a_i \in \mathbb{Q},\ H(a_i) \leq X\},$$
$$\mathcal{P}_c(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathcal{P}(X) \mid a_{d-1} = 0\},$$

as subsets of $\mathcal{P}$ and $\mathcal{P}_c$ consisting of polynomials with coefficients all having Weil height $\leq X$.

3. *Upper bound on energy pairing*

Let f, g be two monic polynomials of degree $d \geq 2$ defined over $\mathbb{Q}$ and let $\mu_f = \{\mu_{f,v}\}_{v \in M_\mathbb{Q}}$, $\mu_g = \{\mu_{g,v}\}_{v \in M_\mathbb{Q}}$ be the adelic measures associated to them. Here, $\mu_{f,v}$, $\mu_{g,v}$ denotes the equilibrium measures of f and g at the place v, respectively, and they are both probability measures supported on $\mathbb{A}^1_{\mathrm{Berk},v}$.

Recall from the proof outline for Theorem 1.3 that we want to bound the adelic energy pairing

$$\langle \mu_f, \mu_g \rangle = \frac{1}{2} \sum_{v \in M_K} \int_{\mathbb{A}^1_{\mathrm{Berk},v} \times \mathbb{A}^1_{\mathrm{Berk},v} \setminus \mathrm{Diag}} \log \delta(z, w)_\infty \, d\mu_{f,v}(z) \otimes d\mu'_{g,v}(w)$$

on both sides. In this section, we give an upper bound in terms of $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)|$ and $h(f)$, $h(g)$, following the approach of DeMarco, Krieger, and Ye [6]. First, the energy pairing $\langle \cdot, \cdot \rangle^{1/2}$ is a metric on adelic measures (Lemma 3.19). Now, by the approach of Favre and Rivera-Letelier to quantitative equidistribution [8], if $F \subseteq \mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Galois-invariant and large, then for some suitable $\epsilon > 0$, the adelic measure $[F]_\varepsilon$ must be close to both μ_f and μ_g in this metric; hence, μ_f and μ_g must be close to each other, that is, $\langle \mu_f, \mu_g \rangle$ must be small.

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As input to this argument, we prove technical estimates on the Hölder continuity of the equilibrium potential $G_{f,v}$. We then finish the argument using quantitative equidistribution.

3.1. *Archimedean bounds.* We will first handle the case where v is an Archimedean place. As all computations here are local, we will omit v from our notation.

We give a uniform Hölder bound on the equilibrium potential G_f (Proposition 3.5), by an argument of Carleson and Gamelin [5, Theorem VIII.3.2]. We begin by bounding G_f near the Julia set.

PROPOSITION 3.1. *Let* $\Omega = \{z \in \mathbb{C} : \operatorname{dist}(z, K_f) \leq 1\}$. *Suppose* f *has Lipschitz constant* A *on* Ω, *that is,*

$$\text{for all } z, z' \in \Omega \; [|f(z) - f(z')| \leq A|z - z'|].$$

Also suppose that $G_f \leq M$ *on* Ω *for some* $M > 0$. *Then, for all* $z \in \Omega$, *we have*

$$G_f(z) \leq dM \operatorname{dist}(z, K_f)^{\alpha},$$

where $\alpha = \log d / \log A$.

Proof. If $z \in K_f$, then both sides of the claimed inequality are 0.

Suppose $z \in \Omega \setminus K_f$. Then, z is in the attracting basin of ∞, so there exists a minimal $n \geq 1$ for which $f^n(z) \notin \Omega$. Since K_f is invariant under f, the Lipschitz condition on f gives

$$\operatorname{dist}(f(z), K_f) \leq A \operatorname{dist}(z, K_f).$$

Now, $f^n(z) \notin \Omega$ implies $\operatorname{dist}(z, K_f) \geq A^{-n}$, and $f^{n-1}(z) \in \Omega$ implies

$$M \geq G_f(f^{n-1}(z)) = d^{n-1} G_f(z).$$

Hence,

$$G_f(z) \leq dMd^{-n} = dM(A^{-n})^{\alpha} \leq dM \operatorname{dist}(z, K_f)^{\alpha},$$

giving us our upper bound on $G_f(z)$. □

Now, we give explicit values for the various parameters in Proposition 3.1. First, we bound the size of the filled Julia set K_f in terms of the coefficients of f.

LEMMA 3.2. *Let* $R_f := 3 \max(1, |a_{d-1}|, |a_{d-2}|^{1/2}, \ldots, |a_0|^{1/d})$. *Then,* $K_f \subseteq B(0, R_f)$.

Proof. For $|z| \geq R_f$, we have for $i \geq 1$ that $|a_{d-i} z^{d-i}| \leq 3^{-i} |z|^d$. Hence,

$$|f(z)| \geq |z^d| \left(1 - \frac{1}{3} - \frac{1}{3^2} - \cdots - \frac{1}{3^d}\right) \geq \frac{1}{2}|z^d| \geq \frac{3}{2}|z|.$$

Hence, $|f^n(z)| \geq (\frac{3}{2})^n |z|$, so z lies in the attracting basin of ∞, that is, $z \notin K_f$. □

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PROPOSITION 3.3. *In Proposition 3.1, we may take*

$$M = \log(2R_f + 1), \quad A = \frac{3d}{2}(R_f + 1)^{d-1},$$

with R_f as in Lemma 3.2.

Proof. Note that $K_f \subseteq B(0, R_f)$ implies $\Omega \subseteq B(0, R_f + 1)$, so $|z - w| \leq 2R_f + 1$ for all $z \in \Omega$ and $w \in K_f$. Now, $\operatorname{supp} \mu_f = K_f$, so

$$G_f(z) = \int \log|z - w| d\mu_f(w) \leq \log(2R_f + 1),$$

which proves the first claim.

Next, the Lipschitz constant of f on Ω is at most $\sup_\Omega |f'|$. Now, for $z \in \Omega \subseteq B(0, R_f + 1)$, we have

$$\begin{aligned} |f'(z)| &\leq d|z^{d-1}| + (d-1)|a_{d-1}z^{d-2}| + \cdots + |a_1| \\ &\leq d\left(1 + \frac{1}{3} + \frac{1}{3^2} \cdots\right)(R_f + 1)^{d-1} \\ &= \frac{3d}{2}(R_f + 1)^{d-1}, \end{aligned}$$

which proves the second claim. □

Since the equilibrium potential is asymptotic to $\log|\cdot|$ far from the origin, the Hölder bound for G_f near the Julia set (Proposition 3.1) yields a global Hölder bound. In fact, we can take the same constants as in Proposition 3.3.

PROPOSITION 3.4. *Let M, A be as in Proposition 3.3 and let $\alpha = \log d / \log A$. For all $z \in \mathbb{C}$, we have*

$$G_f(z) \leq dM \operatorname{dist}(z, K_f)^\alpha.$$

Proof. Let $x = \operatorname{dist}(z, K_f)$. If $x \leq 1$, we are done by Propositions 3.1 and 3.3.

Suppose that $x \geq 1$. We have $K_f \subseteq B(0, R_f)$, so the diameter of K_f is at most $2R_f$. Hence, for every $w \in K_f$, we have $|z - w| \leq x + 2R_f$, so

$$G_f(z) = \int \log|z - w| d\mu_f(w) \leq \log(x + 2R_f).$$

Thus, it suffices to show that $\log(x + 2R_f) \leq dMx^\alpha$.

Case 1: $1 \leq x \leq A$. Now, $R_f \geq 3$, so

$$\begin{aligned} e^{dM} &= (2R_f + 1)^d \geq 7(2R_f + 1)^{d-1} \geq 6\left(\tfrac{7}{4}\right)^{d-1}(R_f + 1)^{d-1} + 2R_f \\ &\geq \frac{3d}{2}(R_f + 1)^{d-1} + 2R_f = A + 2R_f. \end{aligned}$$

Hence,

$$\log(x + 2R_f) \leq \log(A + 2R_f) \leq dM \leq dMx^\alpha. \tag{3.1}$$

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Case 2: $x \geq A$. Write $x = A^c$ with $c \geq 1$; now, $x \geq A \geq 2R_f \geq 2$, so

$$\log(x + 2R_f) \leq c \log A + \log 2 \leq (c+1) \log A \leq d^c \log A \leq dMd^c = dMx^\alpha,$$

where the last inequality follows from $\log A \leq dM$, by equation (3.1). □

We can now show uniform Hölder continuity for G_f.

PROPOSITION 3.5. *Let M, A be as in Proposition 3.3 and let $\alpha = \log d / \log A$. For all $z_1, z_2 \in \mathbb{C}$, we have*

$$|G_f(z_1) - G_f(z_2)| \leq 3dM|z_1 - z_2|^\alpha.$$

Proof. [5, Theorem VIII.3.2] Write $r_i = \operatorname{dist}(z_i, K_f)$; we may assume that $r_1 \geq r_2$. Note that $A = 3d/2(R_f + 1)^{d-1} \geq d$, so that $\alpha \leq 1$.

Case 1: $|z_1 - z_2| \geq (1/3r_1)$. Now, $0 \leq G_f(z_i) \leq dMr_i^\alpha$ by Proposition 3.4, so

$$|G_f(z_1) - G_f(z_2)| \leq dMr_1^\alpha \leq 3dM|z_1 - z_2|^\alpha.$$

Case 2: $|z_1 - z_2| = cr_1$, with $c \leq \frac{1}{3}$. Now, G_f is harmonic and non-negative on $B(z_1, r_1)$, so by Harnack's inequality, we have

$$\frac{1-c}{1+c} G_f(z_1) \leq G_f(z_2) \leq \frac{1+c}{1-c} G_f(z_1).$$

Therefore,

$$|G_f(z_1) - G_f(z_2)| \leq \frac{2c}{1-c} G_f(z_1) \leq 3cdMr_1^\alpha \leq 3dM|z_1 - z_2|^\alpha,$$

as desired. □

Remark 3.6. The arguments in Propositions 3.1 and 3.5 are essentially contained in [5, Theorem VIII.3.2]; however, we note that their argument is incomplete, since they do not control the equilibrium potential for large z as in our proof of Proposition 3.4.

The Hölder bound on G_f allows us to control the local pairing of μ_f with a regularized measure.

COROLLARY 3.7. *Let M, A be as in Proposition 3.3 and let $\alpha = \log d / \log A$. For any $\varepsilon > 0$ and any non-empty finite set $F \subseteq \mathbb{C}$, we have*

$$-(\mu_f, [F]_\varepsilon) \leq -(\mu_f, [F]) + 3dM\varepsilon^\alpha.$$

Proof. Since $|w - \xi| = \varepsilon$ for all $\xi \in \partial B(w, \varepsilon)$, we have

$$\begin{aligned} -(\mu_f, [F]_\varepsilon) &= \frac{1}{|F|} \sum_{w \in F} \int_{\mathbb{C}} G_f(\xi) d\delta_{w,\varepsilon}(\xi) \\ &\leq \frac{1}{|F|} \sum_{w \in F} (G_f(w) + 3dM\varepsilon^\alpha) = -(\mu_f, [F]) + 3dM\varepsilon^\alpha, \end{aligned}$$

as desired. □

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3.2. *Non-Archimedean bounds.* Now, we consider the non-Archimedean case $v \mid p < \infty$ with $v \in M_K$. By choosing an appropriate embedding $K \hookrightarrow \mathbb{C}_p$, we can view $|\cdot|_v = |\cdot|_p$.

We first show that the Lipschitz constant of f on a closed subset of $\mathbb{A}^1_{\mathrm{Berk},v}$ is the same as the Lipschitz constant on its type I points.

LEMMA 3.8. *Let $\Omega \subseteq \mathbb{A}^1_{\mathrm{Berk},v}$ be a closed subset. Suppose there is a constant $A > 0$ such that for all $z, z' \in \Omega \cap \mathbb{C}_v$, we have*

$$|f(z) - f(z')|_v \le A|z - z'|_v.$$

Then, for all $\zeta, \zeta' \in \Omega$, we have

$$\delta(f(\zeta), f(\zeta'))_\infty \le A\delta(\zeta, \zeta')_\infty.$$

Proof. Case 1: $f(\zeta) \ne f(\zeta')$. Take sequences $(z_n), (z_n') \subseteq \mathbb{C}_v$ such that $(z_n) \to \zeta$ and $(z_n') \to \zeta'$. Since $\delta(\cdot,\cdot)_\infty$ is continuous off the diagonal, we have

$$\begin{aligned}\delta(f(\zeta), f(\zeta'))_\infty &= \lim_{n\to\infty} |f(z_n) - f(z_n')|_v \\ &\le A \lim_{n\to\infty} |z_n - z_n'|_v \\ &= A\delta(\zeta, \zeta')_\infty.\end{aligned}$$

Case 2: $f(\zeta) = f(\zeta')$. Write $\zeta = \zeta(a, r)$ and $f(\zeta) = \zeta(b, R)$; then [3, Proposition 7.6] gives $f(B(a, r)) = B(b, R)$, so

$$\delta(f(\zeta), f(\zeta'))_\infty = R \le Ar \le A\delta(\zeta, \zeta')_\infty. \qquad \square$$

From here on, the structure of the argument for the non-Archimedean case is analogous to the Archimedean case in the previous section.

PROPOSITION 3.9. *Let $\Omega_v = \{z \in \mathbb{C}_v : \mathrm{dist}_v(z, K_{f,v}) \le 1\}$. Suppose f has Lipschitz constant $A_v > 1$ on Ω_v, that is,*

$$\text{for all } z, z' \in \Omega_v \quad [|f(z) - f(z')|_v \le A_v|z - z'|_v].$$

Also, suppose that $G_{f,v} \le M_v$ on Ω_v for some $M_v > 0$. Then, for all $z \in \Omega_v$, we have

$$G_{f,v}(z) \le dM_v \,\mathrm{dist}_v(z, K_{f,v})^{\alpha_v},$$

where $\alpha_v = \log d / \log A_v$.

Proof. In view of Lemma 3.8, we may replace Ω_v with its closure $\overline{\Omega_v}$ in $\mathbb{A}^1_{\mathrm{Berk},v}$. The proof is now completely analogous to the proof of Proposition 3.1. $\square$

PROPOSITION 3.10. *Let $R_{f,v} := \max(1, |a_{d-1}|_v, |a_{d-2}|_v^{1/2}, \ldots, |a_0|_v^{1/d})$. Then, $K_{f,v} \subseteq \overline{B}(0, R_{f,v})$.*

Proof. For $|z|_v > R_{f,v}$ and $i \ge 1$, we have $|a_{d-i}z^{d-i}|_v < |z|_v^d$. Hence, $|f(z)|_v = |z|_v^d$. By iteration, we see that $|f^n(z)|_v = |z|_v^{d^n}$, so z lies in the attracting basin of ∞, that is, $z \notin K_{f,v}$. $\square$

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PROPOSITION 3.11. *Suppose that f has bad reduction at v. In Proposition 3.9, we may take*

$$M_v = \log R_{f,v}, \quad A_v = R_{f,v}^{d-1},$$

with $R_{f,v}$ as in Proposition 3.10.

Proof. First, f having bad reduction at v implies $R_{f,v} > 1$. Now, $K_{f,v} \subseteq \overline{B}(0, R_{f,v})$ implies that $\Omega_v \subseteq \overline{B}(0, R_{f,v})$ as well, so $|z - w|_v \leq R_{f,v}$ for all $z \in \Omega_v$ and $w \in K_{f,v}$. Now, $\operatorname{supp} \mu_{f,v} = K_{f,v}$, so

$$G_{f,v}(z) = \int \log |z - w| d\mu_{f,v}(w) \leq \log R_{f,v},$$

which proves the first claim.

Next, note that for all $z \in \Omega_v \subseteq \overline{B}(0, R_{f,v})$, we have

$$|f(z)| \leq \max_i |a_{d-i} z^{d-i}|_v \leq R_{f,v}^d.$$

In other words, we have $f(\overline{B}(0, R_{f,v})) \subseteq \overline{B}(0, R_{f,v}^d)$. It now follows from [3, Proposition 3.20] that the Lipschitz constant for f on $B(0, R_{f,v})$ is bounded above by $R_{f,v}^{d-1}$. Thus, the same also holds on Ω_v, which proves the second claim. □

Remark 3.12. When f has explicit good reduction at v, we have $R_v = 1$, so $A_v = 1$; thus, $\alpha_v = \log d / \log A_v$ is not well defined and Proposition 3.9 cannot be applied. However, in this case, we have $J_{f,v} = \{\zeta(0, 1)\}$, so $\Omega_v = K_{f,v} \cap \mathbb{C}_v = \overline{B}(0, 1)$, and thus $G_{f,v} \equiv 0$ on Ω_v.

From now on in this subsection, we will assume that f has bad reduction at v.

PROPOSITION 3.13. *Let v, M_v, A_v be as in Proposition 3.11 and let $\alpha_v = \log d / \log A_v$. For all $z \in \mathbb{C}_v$, we have*

$$G_{f,v}(z) \leq dM \operatorname{dist}(z, K_{f,v})^{\alpha_v}.$$

Proof. Let $x = \operatorname{dist}(z, K_{f,v})$. If $x \leq 1$, we are done by Propositions 3.1 and 3.11.

Suppose that $x \geq 1$. We have $K_{f,v} \subseteq B(0, R_{f,v})$, so the diameter of $K_{f,v}$ is at most $R_{f,v}$. Hence, for every $w \in K_{f,v}$, we have $\delta(z, w)_\infty \leq \max(x, R_{f,v})$, so

$$G_{f,v}(z) = \int \log \delta(z, w)_\infty \, d\mu_{f,v}(w) \leq \log \max(x, R_{f,v}).$$

Thus, it suffices to show that $\log \max(x, R_{f,v}) \leq dM_v x^{\alpha_v}$.

Now, $\log R_{f,v} = M_v \leq dM_v x^{\alpha_v}$; also, writing $x = A_v^c$ for some $c \geq 0$, we have

$$\log xc(d-1) \log R_{f,v} \leq d^{c+1} \log R_{f,v} = dM_v d^c = dM_v x^{\alpha_v}.$$ □

PROPOSITION 3.14. *Let v, M_v, A_v be as in Proposition 3.11 and let $\alpha_v = \log d / \log A_v$. For all $z_1, z_2 \in \mathbb{A}^1_{\mathrm{Berk},v}$, we have*

$$|G_{f,v}(z_1) - G_{f,v}(z_2)| \leq dM_v |z_1 - z_2|_v^{\alpha_v}.$$

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Proof. Write $r_i = \operatorname{dist}_v(z_i, K_{f,v})$; we may assume that $r_1 \geq r_2$.

Case 1: $|z_1 - z_2|_v \geq r_1$. Now, $0 \leq G_{f,v}(z_i) \leq dM_v r_i^{\alpha_v}$ by Proposition 3.9, so

$$|G_{f,v}(z_1) - G_{f,v}(z_2)| \leq dM_v r_1^{\alpha_v} \leq dM_v |z_1 - z_2|^{\alpha_v}.$$

Case 2: $|z_1 - z_2|_v < r_1$. Then, for all $w \in K_{f,v} = \operatorname{supp} \mu_{f,v}$, we have

$$\delta(z_1, w)_\infty \geq r_1 > |z_1 - z_2|,$$

so that $\delta(z_1, w)_\infty = \delta(z_2, w)_\infty$. Hence,

$$G_{f,v}(z_1) = \int \log \delta(z_1, w)_\infty \, d\mu_{f,v}(w) = \int \log \delta(z_2, w)_\infty \, d\mu_{f,v}(w) = G_{f,v}(z_2). \qquad \square$$

COROLLARY 3.15. *Let M_v, A_v be as in Proposition 3.11 and let $\alpha_v = \log d / \log A_v$. For any $\varepsilon > 0$ and any non-empty finite set $F \subseteq \mathbb{C}_v$, we have*

$$-(\mu_{f,v}, [F]_\varepsilon)_v \leq -(\mu_{f,v}, [F])_v + dM_v \varepsilon^{\alpha_v}.$$

Proof. Since $\delta(\zeta(w, \varepsilon), w)_\infty = \varepsilon$, we have

$$\begin{aligned} -(\mu_{f,v}, [F]_\varepsilon)_v &= \frac{1}{|F|} \sum_{w \in F} G_{f,v}(\zeta(w, \varepsilon)) \\ &\leq \frac{1}{|F|} \sum_{w \in F} (G_{f,v}(w) + dM_v \varepsilon^{\alpha_v}) = -(\mu_{f,v}, [F])_v + dM_v \varepsilon^{\alpha_v}. \end{aligned} \qquad \square$$

3.3. *Quantitative equidistribution.* To summarize the discussion in §§3.1 and 3.2, we have shown the following theorem.

THEOREM 3.16. *Let $\varepsilon > 0$. For any place $v \in M_K$ such that either $v \mid \infty$ or f has bad reduction at v, and any non-empty finite set $F \subseteq \mathbb{C}_v$, we have*

$$-(\mu_f, [F]_\varepsilon)_v \leq -(\mu_f, [F])_v + 3dM_v \varepsilon^{\alpha_v},$$

with constants $\alpha_v = \log d / \log A_v$,

$$M_v = \begin{cases} \log(2R_{f,\infty} + 1), & v \mid \infty, \\ \log R_{f,v}, & v \nmid \infty, \end{cases} \qquad A_v = \begin{cases} \dfrac{3d}{2}(R_{f,\infty} + 1)^{d-1}, & v \mid \infty, \\ R_{f,v}^{d-1}, & v \nmid \infty, \end{cases}$$

$$R_{f,v} = \begin{cases} 3 \max(1, |a_{d-1}|_\infty, |a_{d-2}|_\infty^{1/2}, \ldots, |a_0|_\infty^{1/d}), & v \mid \infty, \\ \max(1, |a_{d-1}|_v, |a_{d-2}|_v^{1/2}, \ldots, |a_0|_v^{1/d}), & v \nmid \infty. \end{cases}$$

Proof. This is immediate from Corollaries 3.7 and 3.15. $\square$

To convert these Hölder bounds into an upper bound for the energy pairing $\langle \mu_f, \mu_g \rangle$, we show the following quantitative equidistribution result: for any large Galois-invariant subset $F \subseteq \operatorname{Prep}(f)$, there is some small adelic radius ε such that the regularized adelic measure $[F]_\varepsilon$ is close to μ_f.

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LEMMA 3.17. [8, Lemma 4.11, 9, Lemma 12] *For any place $v \in M_K$, any non-empty finite set $F \subseteq \mathbb{C}_v$, and any $\varepsilon > 0$, we have*

$$([F]_\varepsilon, [F]_\varepsilon)_v \leq ([F], [F])_v + \frac{\log(1/\varepsilon)}{|F|}.$$

PROPOSITION 3.18. *With notation as in Theorem 3.16, let $F \subseteq K$ be a finite $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set with $|F| = N \geq 2$. Define the adelic radius $\varepsilon = \{\varepsilon_v\}_{v \in M_K}$ as*

$$\varepsilon_v = \begin{cases} (dN)^{-1/\alpha_v} & \text{if } v \mid \infty \text{ or } f \text{ has bad reduction at } v, \\ 1 & \text{if } f \text{ has explicit good reduction at } v. \end{cases}$$

Then,

$$\langle \mu_f, [F]_\varepsilon \rangle \leq \widehat{h}_f(F) + O\left(d \frac{\log N}{N}(h(f) + 1)\right).$$

Proof. We extend an argument of [6, Lemma 9.2]. Let $\mathcal{B} \subseteq M_K$ be the set of places v such that either $v \mid \infty$ or f has bad reduction at v.

First, by using $(\mu_f, \mu_f)_v = 0$, we expand the left side of the claim as

$$\begin{aligned} \langle \mu_f, [F]_\varepsilon \rangle &= \frac{1}{2[K:\mathbb{Q}]} \sum_{v \in M_K} n_v ([F]_\varepsilon - \mu_f, [F]_\varepsilon - \mu_f)_v \\ &= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \left(\frac{1}{2}([F]_\varepsilon, [F]_\varepsilon)_v - (\mu_f, [F]_\varepsilon)_v \right). \end{aligned} \tag{3.2}$$

We bound the first term in the sum of equation (3.2) by Lemma 3.17 and the product formula $\sum_v n_v([F], [F])_v = 0$, to obtain

$$\begin{aligned} \frac{1}{2[K:\mathbb{Q}]} \sum_{v \in M_K} n_v ([F]_\varepsilon, [F]_\varepsilon)_v &\leq \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \frac{\log(1/\varepsilon_v)}{2N} \\ &= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{B}} n_v \frac{1 + \log N / \log d}{2N} \log A_v. \end{aligned} \tag{3.3}$$

Next, we bound the second term in the sum of equation (3.2). If $v \in \mathcal{B}$, then we can use Theorem 3.16. Suppose $v \in M_k \setminus \mathcal{B}$, that is, f has explicit good reduction at v. Then, $\varepsilon_v = 1$ and $\mu_{f,v} = \delta_{\zeta(0,1)}$, so $G_{f,v} = \log^+ |\cdot|_v$. Hence,

$$\begin{aligned} -(\mu_f, [F]_\varepsilon)_v &= \frac{1}{|F|} \sum_{w \in F} \log^+ |\zeta(w, 1)|_v = \frac{1}{|F|} \sum_{w \in F} \log^+ \max(1, |w|_v) \\ &= \frac{1}{|F|} \sum_{w \in F} \log^+ |w|_v = -(\mu_f, [F])_v. \end{aligned}$$

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The above two cases yield

$$\begin{aligned}
&\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K} n_v(-(\mu_f,[F]_\varepsilon)_v)\\
&\quad\le \frac{1}{[K:\mathbb{Q}]}\Bigg(\sum_{v\in M_K} n_v(-(\mu_f,[F])_v)+\sum_{v\in\mathcal{B}} n_v(3dM_v\varepsilon_v^{\alpha_v})\Bigg)\\
&\quad= \widehat{h}_f(F)+\frac{1}{[K:\mathbb{Q}]}\sum_{v\in\mathcal{B}} n_v(3dM_v\varepsilon_v^{\alpha_v}) = \widehat{h}_f(F)+\frac{1}{[K:\mathbb{Q}]}\sum_{v\in\mathcal{B}} n_v\frac{3M_v}{N}. \qquad (3.4)
\end{aligned}$$

Now, equations (3.2), (3.3), and (3.4) imply

$$\begin{aligned}
\langle\mu_f,[F]_\varepsilon\rangle-\widehat{h}_f(F) &\lesssim \frac{\log N}{N}\frac{1}{[K:\mathbb{Q}]}\sum_{v\in\mathcal{B}} n_v(M_v+\log A_v)\\
&\le \frac{\log N}{N}\sum_{v\in M_{\mathbb{Q}}}(M_v+\log A_v). \qquad (3.5)
\end{aligned}$$

We now bound the sum in equation (3.5). Note that

$$\sum_{v\in M_{\mathbb{Q}}}\log R_{f,v}=\sum_{v\nmid\infty}\log R_{f,v}+\log R_{f,\infty}\le h(f)+O(1).$$

For the first term M_v in equation (3.5), we have

$$\begin{aligned}
\sum_{v\in M_{\mathbb{Q}}} M_v &\le \sum_{v\nmid\infty}\log R_{f,v}+\log(2R_{f,\infty}+1)\\
&\le \sum_{v\nmid\infty}\log R_{f,v}+\log R_{f,\infty}+O(1)\le h(f)+O(1), \qquad (3.6)
\end{aligned}$$

where the first inequality follows from Theorem 3.16. Similarly, for the second term $\log A_v$ in equation (3.5), we deduce

$$\begin{aligned}
\sum_{v\in M_{\mathbb{Q}}}\log A_v &\le \sum_{v\nmid\infty}(d-1)\log R_{f,v}+(d-1)\log(R_{f,\infty}+1)\\
&\quad+\log\left(\frac{3}{2}d\right)\le (d-1)h(f)+O(d). \qquad (3.7)
\end{aligned}$$

Substituting equations (3.6) and (3.7) back into equation (3.5), we deduce

$$\langle\mu_f,[F]_\varepsilon\rangle-\widehat{h}_f(F)\lesssim d\frac{\log N}{N}(h(f)+1).$$ □

We can now show that μ_f and μ_g are close to each other, thereby proving our upper bound for the adelic energy pairing.

LEMMA 3.19. [9, Theorem 1] *For any adelic measures ρ,σ,τ, we have*

$$\langle\rho,\tau\rangle^{1/2}\le\langle\rho,\sigma\rangle^{1/2}+\langle\sigma,\tau\rangle^{1/2}.$$

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THEOREM 3.20. *Suppose* $N = |\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \geq 2$. *Then,*

$$\langle \mu_f, \mu_g \rangle \lesssim d \frac{\log N}{N} (\max(h(f), h(g)) + 1).$$

Proof. Take $F = \mathrm{Prep}(f) \cap \mathrm{Prep}(g)$, so that $\widehat{h}_f(F) = \widehat{h}_g(F) = 0$. Then, by Lemma 3.19 and Proposition 3.18, we have

$$\begin{aligned} \langle \mu_f, \mu_g \rangle &\leq (\langle \mu_f, [F]_\varepsilon \rangle^{1/2} + \langle \mu_g, [F]_\varepsilon \rangle^{1/2})^2 \lesssim d \frac{\log N}{N} (\sqrt{h(f) + 1} + \sqrt{h(g) + 1})^2 \\ &\lesssim d \frac{\log N}{N} (\max(h(f), h(g)) + 1). \end{aligned}$$

□

Remark 3.21. The proof of Theorem 3.20 in fact carries over to any pair of monic $f, g \in K[z]$, with the same implied constant independent of the field K.

We will also use the quantitative equidistribution bound of Proposition 3.18 to give an $O(\log X)$ bound in some cases where we know the measures differ significantly in a non-Archimedean place.

LEMMA 3.22. *Let $v \in M_K$ be non-Archimedean. Let $\varphi : \mathbb{A}^1_{\mathrm{Berk},v} \to \mathbb{R}$ be continuous with compact support, and let μ be a measure on $\mathbb{A}^1_{\mathrm{Berk},v}$ with total mass 0 and continuous potential. Then,*

$$\left| \int_{\mathbb{A}^1_{\mathrm{Berk},v}} \varphi d\mu \right| \leq (\triangle\varphi, \triangle\varphi)_v^{1/2} (\mu, \mu)_v^{1/2}.$$

Proof. If the right side is ∞, then there is nothing to prove. Otherwise, let $\mu = \triangle\psi$. By [2, Corollary 5.38], the expression $[F, G] := \int_{\mathbb{A}^1_{\mathrm{Berk},v}} F \triangle G$ defines a positive semidefinite symmetric bilinear form on $\{F \in \mathrm{BDV}(\mathbb{A}^1_{\mathrm{Berk},v}) : [F, F] < \infty\}$. Now,

$$[\varphi, \psi] = \int_{\mathbb{A}^1_{\mathrm{Berk},v}} \varphi d\mu, \quad [\varphi, \varphi] = (\triangle\varphi, \triangle\varphi)_v, \quad [\psi, \psi] = (\mu, \mu)_v,$$

and we are done by the Cauchy–Schwarz inequality. □

PROPOSITION 3.23. *Let $v \in M_{\mathbb{Q}}$ be non-Archimedean, with p its associated prime. If*

$$\mu_{g,v}(\mathbb{A}^1_{\mathrm{Berk},v} \setminus D_{\mathrm{an}}(0, p^{1/d})) \geq 1/d,$$

then there are at most $C_d(h(g)^2 + 1) \log^2 p$ points $x \in \mathrm{Prep}(g)$ such that $|\sigma(x)|_v \leq 1$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proof. Consider the path segment T that connects $\zeta(0, 1)$ and $\zeta(0, p^{1/d})$ in $\mathbb{A}^1_{\mathrm{Berk},v}$, namely $T = \{\zeta(0, p^{t/d}) : t \in [0, 1]\}$. Let φ be the continuous function on $\mathbb{A}^1_{\mathrm{Berk},v}$, which is locally constant on $\mathbb{A}^1_{\mathrm{Berk},v} \setminus T$ and takes values $\varphi(\zeta(0, p^{t/d})) = t$ on T. We check that

$$\triangle\varphi = \frac{d}{\log p} (\delta_{\zeta(0,1)} - \delta_{\zeta(0,p^{1/d})}).$$

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Now, if F is a non-empty $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set of preperiodic points for g contained in $\overline{B}(0,1)$, take the adelic radius ε as in Proposition 3.18. Since $0 < \varepsilon_v \le 1$, we have $\zeta(w, \varepsilon_v) \in \overline{D}_{\mathrm{an}}(0,1)$ for all $w \in F$. Hence,

$$\int_{\mathbb{A}^1_{\mathrm{Berk},v}} \varphi d[F]_{\varepsilon_v} = 0, \qquad \int_{\mathbb{A}^1_{\mathrm{Berk},v}} \varphi d\mu_{g,v} \ge \frac{1}{d}.$$

Now, Proposition 3.18 gives

$$\begin{aligned} d\frac{\log N}{N}(h(g)+1) \gtrsim \langle \mu_g, [F]_\varepsilon \rangle &= \frac{1}{2} \sum_{w \in M_{\mathbb{Q}}} (\mu_g - [F]_\varepsilon, \mu_g - [F]_\varepsilon)_w \\ &\ge \frac{1}{2}(\mu_g - [F]_\varepsilon, \mu_g - [F]_\varepsilon)_v, \end{aligned}$$

since the terms in the sum are non-negative. Then, by Lemma 3.22,

$$\begin{aligned} \frac{1}{d} \le \left| \int \varphi d[F]_{\varepsilon_v} - \int \varphi d\mu_{g,v} \right| &\le (\triangle\varphi, \triangle\varphi)_v^{1/2} ([F]_{\varepsilon_v} - \mu_{g,v}, [F]_{\varepsilon_v} - \mu_{g,v})_v^{1/2} \\ &\lesssim \left(\frac{d}{\log p}\right)^{1/2} \left(d\frac{\log N}{N}(h(g)+1)\right)^{1/2}, \end{aligned}$$

which yields $(N/\log N) \lesssim d^4(h(g)+1)\log p$. Therefore, $N \lesssim_d (h(g)^2+1)\log^2 p$. □

3.4. *Lower bound on the Archimedean energy pairing.* We now prove the lower bound on $\langle \mu_f, \mu_g \rangle$. In particular, we will give a lower bound on the local energy pairing $(\mu_f - \mu_g, \mu_f - \mu_g)_\infty$ at the Archimedean place. As usual, we will suppress the subscript ∞ in this section.

The simplest instance of our argument goes as follows: the centroid of μ_f is a constant multiple of a_{d-1}, and similarly for μ_g. Thus, if $a_{d-1} \ne b_{d-1}$, then μ_f and μ_g cannot be too close to each other, so we should get a lower bound on their energy pairing. In general, we will need to look at higher moments of μ_f, μ_g, depending on the first coefficient in which f, g differ.

LEMMA 3.24. *If, for some $k \le d$, we have $a_{d-j} = b_{d-j}$ for all $1 \le j < k$, then*

$$\int z^k d\mu_f(z) - \int z^k d\mu_g(z) = -\frac{k}{d}(a_{d-k} - b_{d-k}).$$

Proof. Case 1: $k < d$. For any $c \in \mathbb{C}$, let $\rho_1, \ldots, \rho_d$ be the roots of $f - c$. By Newton's identities for elementary symmetric polynomials, the power sum $\rho_1^k + \cdots + \rho_d^k$ is a polynomial in $a_{d-1}, \ldots, a_{d-k}$, linear in a_{d-k} with coefficient $-k$; in other words, there is some polynomial P_k such that

$$\int z^k d(f^*\delta_c)(z) = -ka_{d-k} + P_k(a_{d-1}, \ldots, a_{d-k+1}).$$

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Now, for any non-exceptional point $z_0 \in \mathbb{C}$, we consider the iterated pullback $\mu_f^{(n)} = d^{-n}(f^n)^*\delta_{z_0}$. On one hand, we have $\mu_f^{(n)} = (1/d)f^*\mu_f^{(n-1)}$, so

$$\begin{aligned}\int z^k d\mu_f^{(n)}(z) &= \frac{1}{d}\int z^k d(f^*\mu_f^{(n-1)})(z) = \frac{1}{d}\int \left(\int z^k d(f^*\delta_c)(z)\right) d\mu_f^{(n-1)}(c)\\ &= -\frac{k}{d}a_{d-k} + \frac{1}{d}P_k(a_{d-1}, \ldots, a_{d-k+1}).\end{aligned}$$

On the other hand, the sequence $\mu_f^{(n)}$ has uniformly bounded support and converges weakly to μ_f by equidistribution. Hence, we also have

$$\int z^k d\mu_f(z) = -\frac{k}{d}a_{d-k} + \frac{1}{d}P_k(a_{d-1}, \ldots, a_{d-k+1}).$$

By replacing f with g, we get an analogous equation, and by taking the difference, we obtain the claim.

Case 2: $k = d$. The constant coefficient of $f - c$ is $a_0 - c$, so we now have

$$\int z^d d(f^*\delta_c)(z) = -d(a_0 - c) + P_d(a_{d-1}, \ldots, a_1)$$

for some polynomial P_d. Arguing as before, we obtain

$$\int z^d d\mu_f(z) = -a_0 + \int c\, d\mu_f(c) + \frac{1}{d}P_d(a_{d-1}, \ldots, a_1).$$

By replacing f with g and taking the difference, we get

$$\int z^d d\mu_f(z) - \int z^d d\mu_g(z) = -(a_0 - b_0) + \left(\int z d\mu_f(z) - \int z d\mu_g(z)\right) = -(a_0 - b_0)$$

by the $k = 1$ case of this lemma. □

PROPOSITION 3.25. *Let $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ with $f \neq g$. If $a_{d-j} = b_{d-j}$ for all $1 \leq j < k$ and $a_{d-k} \neq b_{d-k}$, then*

$$\langle \mu_f, \mu_g \rangle \gtrsim_d \begin{cases} X^{-4} & \text{if } k = 1,\\ X^{-k-4} & \text{if } 2 \leq k \leq d.\end{cases}$$

Proof. Recall from Lemma 3.2 that $J_f = \operatorname{supp} \mu_f \subseteq B(0, R_f)$, where

$$R_f = 3\max(1, |a_{d-1}|, |a_{d-2}|^{1/2}, \ldots, |a_0|^{1/d}).$$

Case 1: $k \geq 2$. Now, $a_{d-1} = b_{d-1} = 0$, so $R_f, R_g \leq 3\sqrt{X} := R$.

Fix a smooth function $\psi_0(z)$ which is zero outside of $B(0, 2)$ and equal to z^k in $B(0, 1)$. Then, the function $\psi(z) = R^k\psi_0(z/R)$ satisfies $\psi(z) = z^k$ in $B(0, R)$, and

$$\|\nabla\psi\|_{L^2} = R^k\|\nabla\psi_0\|_{L^2}.$$

Next, note that a_{d-k}, b_{d-k} are distinct rationals with numerator and denominator $\leq X$, so $|a_{d-k} - b_{d-k}| \geq X^{-2}$. Hence, by Lemmas 3.22 and 3.24, we have

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$$\frac{k}{d}X^{-2} \le \frac{k}{d}|a_{d-k} - b_{d-k}| = \left|\int \psi d\mu_f - \int \psi d\mu_g\right|$$
$$\le \|\nabla\psi\|_{L^2}(\mu_f - \mu_g, \mu_f - \mu_g)_\infty^{1/2} \le (3\sqrt{X})^k \|\nabla\psi_0\|_{L^2}\langle\mu_f, \mu_g\rangle^{1/2}.$$

Note that Lemma 3.24 applies to the Archimedean setting, although it is only stated for non-Archimedean places. Rearranging, we get

$$\langle\mu_f, \mu_g\rangle \ge \left(\frac{kX^{-k/2-2}}{3^k d\|\nabla\psi_0\|_{L^2}}\right)^2 \ge \frac{X^{-k-4}}{(3^d d\|\nabla\psi_0\|_{L^2})^2} \gtrsim_d X^{-k-4}.$$

Case 2: $k = 1$. In this case, we can only bound R_f, R_g from above by $R := 3X$; however, since $a_{d-1} = 0 \ne b_{d-1}$, we have the stronger bound $|a_{d-1} - b_{d-1}| \ge X^{-1}$. Now, the same argument with these modifications yields

$$\langle\mu_f, \mu_g\rangle \ge \left(\frac{X^{-1-1}}{3d\|\nabla\psi_0\|_{L^2}}\right)^2 \gtrsim_d X^{-4}. \qquad \square$$

COROLLARY 3.26. *Let* $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ *with* $f \ne g$. *If* $a_{d-j} = b_{d-j}$ *for all* $1 \le j < k$ *and* $a_{d-k} \ne b_{d-k}$, *then* $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \lesssim_{\varepsilon,d} X^{k+4+\varepsilon}$.

Proof. Write $N = |\mathrm{Prep}(f) \cap \mathrm{Prep}(g)|$. By Proposition 3.25 and Theorem 3.20,

$$X^{-k-4} \lesssim_d \langle\mu_f, \mu_g\rangle \lesssim d\frac{\log N}{N}(\max(h(f), h(g)) + 1).$$

Hence, $(N/\log N) \lesssim_d X^{k+4}(\log X + 1) \lesssim_\varepsilon X^{k+4+\varepsilon/2}$, and thus $N \lesssim_{\varepsilon,d} X^{k+4+\varepsilon}$. $\square$

4. *Average number of common preperiodic points*

The main purpose of this section is to establish Theorem 1.3. Recall for $X \ge 1$, we have the sets

$$\mathcal{P}(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid a_i \in \mathbb{Q},\ H(a_i) \le X\},$$
$$\mathcal{P}_c(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathcal{P}(X) \mid a_{d-1} = 0\},$$
$$\mathcal{S}(X) = \mathcal{P}_c(X) \times \mathcal{P}(X) \setminus \{(f, f) \mid f \in \mathcal{P}_c(X)\}.$$

The strategy is to show that $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)|$ is small for most pairs $(f, g) \in \mathcal{S}(X)$, and not very large for the remaining pairs. In more detail, we have the following.

- For most pairs (f, g), the coefficients of f, g have a certain divisibility structure (Proposition 4.1). Often, this implies that the filled Julia sets of f, g are disjoint at some place (Corollary 4.4), so f and g cannot have common preperiodic points. Otherwise, we can still use quantitative equidistribution to show that f and g have few common preperiodic points (Corollary 4.6).
- For the remaining pairs (f, g), we adapt the approach of DeMarco, Krieger, and Ye [6] to prove

$$X^{-k-4} \lesssim_d \langle\mu_f, \mu_g\rangle \lesssim_d \frac{\log N}{N}\log X,$$

where $N = |\mathrm{Prep}(f) \cap \mathrm{Prep}(g)|$, and f, g agree in their first $k-1$ coefficients. The upper bound comes from Hölder bounds on $G_{f,v}$ when f has bad reduction at v

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(Theorem 3.16), while the lower bound comes from the moments of $\mu_{f,\infty}$ at the Archimedean place (Proposition 3.25). Together, these bounds yield $N \lesssim_{\varepsilon,d} X^{k+4+\varepsilon}$ (Corollary 3.26).

4.1. *Elementary properties for most pairs.* We first formalize the notions of 'most' and 'divisibility structure' in the proof outline: we will require that at some place, f has all coefficients with small norm, while g has some coefficient with large norm.

PROPOSITION 4.1. *Fix $f \in \mathcal{P}(X)$. Let $J \subseteq \{0, 1, \dots, d-1\}$ and choose $b_i' \in \mathbb{Q}$ for each $i \notin J$ with $h(b_i') \le \log X$. Then, for any $\varepsilon > 0$, among all polynomials $g \in \mathcal{P}(X)$ with prescribed coefficients $b_i = b_i'$ for all $i \notin J$, at most $O_{\varepsilon,d}(X^{-|J|+\varepsilon})$ proportion satisfy the following property: there does not exist a non-Archimedean place $v \in M_{\mathbb{Q}}$ such that*

$$\text{for all } i\ [|a_i|_v \le 1] \quad \text{for all } i \notin J\ [|b_i|_v \le 1] \quad \text{there exists } j \in J\ [|b_j|_v > 1].$$

(Recall that a_i, b_i are the coefficients of f, g, respectively.)

Proof. Standard results from analytic number theory (e.g. [11, §18.5]) yield

$$\#\{x \in \mathbb{Q} : h(x) \le \log X\} = \frac{2X^2}{\zeta(2)} + O(X \log X). \tag{4.1}$$

Hence, there are $O_d(X^{2|J|})$ many $g \in \mathcal{P}(X)$ with prescribed coefficients $b_i = b_i'$ for $i \notin J$, so we need to show that there are $O_{\varepsilon,d}(X^{|J|+\varepsilon})$ polynomials g with the given property.

Let $S = \{q_1, q_2, \dots, q_m\}$ be the set of all prime factors of the denominators of a_i $(0 \le i \le d-1)$ and b_i $(i \notin J)$. Then, the given property is equivalent to requiring that for each $j \in J$, every prime factor of $\mathrm{denom}(b_j)$ is in S. It now suffices to show that there are at most $O_{\varepsilon,d}(X^{\varepsilon/|J|})$ integers in $[1, X]$ with all prime divisors in S: then, for each $j \in J$, there are $O(X)$ possible numerators and $O_{\varepsilon,d}(X^{\varepsilon/|J|})$ possible denominators for b_j, so the total number of choices for $(b_j)_{j\in J}$ (and hence for g) is $O_{\varepsilon,d}(X^{1+\varepsilon/|J|})^{|J|} = O_{\varepsilon,d}(X^{|J|+\varepsilon})$.

Let $2 = p_1 \le p_2 \le \cdots$ denote the primes in increasing order; then it suffices to show that there are $O_{\varepsilon,d}(X^{\varepsilon/|J|})$ integers in $[1, X]$ with all prime divisors in $\{p_1, \dots, p_m\}$, that is, which are p_m-smooth. However, $\prod_i p_i \le \prod_i q_i \le X^{2d-|J|}$, so [11, §22.2] gives $p_m \lesssim d \log X$. Then, by [10, equation (1.19)], the number of p_m-smooth integers in $[1, X]$ is

$$\lesssim \exp\left(C_d \frac{\log X}{\log\log X}\right) \lesssim_{\varepsilon,d} X^{\varepsilon/|J|},$$

and we are done. □

By Proposition 4.1, for most pairs $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$, there is some place v where f has explicit good reduction, while g has some large coefficient. We now prove some elementary properties of the filled Julia sets and equilibrium measures of f and g in the above scenario, which will be used to carry out the first step in the proof of Theorem 1.3.

PROPOSITION 4.2. *Let $v \in M_{\mathbb{Q}}$ be non-Archimedean. If $|a_i|_v \le 1$ for all i, then $|\zeta_v| \le 1$ for all $\zeta \in K_{f,v}$.*

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Proof. By the density of type I points in $K_{f,v}$, it suffices to prove the statement for $\zeta = z \in \mathbb{C}_v$. Now, if $|z_v| > 1$, then $|f(z)|_v = |z_v|^d > 1$; hence, by induction, we have $|f^n(z)|_v = |z_v|^{d^n} \to \infty$ and, thus, $z \notin K_{f,v}$. □

PROPOSITION 4.3. *Let $v \in M_{\mathbb{Q}}$ be non-Archimedean. If $|b_i|_v \le 1$ for all $i \ne 0$ but $|b_0|_v > 1$, then $|\zeta|_v = |b_0|_v^{1/d}$ for all $\zeta \in K_{g,v}$.*

Proof. By the density of type I points in $K_{g,v}$, it suffices to prove the statement for $\zeta = z \in \mathbb{C}_v$. If $|z_v| \ne |b_0|_v^{1/d}$, then

$$|g(z)|_v = \max(|z^d|_v, |b_0|_v) > |b_0|_v^{1/d};$$

hence, by induction, we have $|g^n(z)|_v = |g(z)|_v^{d^{n-1}} \to \infty$ and, thus, $z \notin K_{g,v}$. □

Hence, most pairs (f, g) have no common preperiodic points.

COROLLARY 4.4. *Fix $f \in \mathcal{P}_c(X)$. For $1 - O_{\varepsilon,d}(X^{-1+\varepsilon})$ proportion of $g \in \mathcal{P}(X)$, there exists a non-Archimedean place $v \in M_{\mathbb{Q}}$ such that*

$$\text{for all } i\ [|a_i|_v \le 1] \quad \text{for all } j \ne 0\ [|b_j|_v \le 1], \quad |b_0|_v > 1.$$

Furthermore, in this case, we have $\mathrm{Prep}(f) \cap \mathrm{Prep}(g) = \emptyset$.

Proof. The first claim follows by applying Proposition 4.1 with $J = \{0\}$. Next, by Propositions 4.2 and 4.3, we have

$$\begin{aligned} z \in \mathrm{Prep}(f) \subseteq K_{f,v} &\implies |z|_v \le 1, \\ z \in \mathrm{Prep}(g) \subseteq K_{g,v} &\implies |z|_v = |b_0|_v^{1/d} > 1, \end{aligned}$$

so $\mathrm{Prep}(f) \cap \mathrm{Prep}(g) = \emptyset$. □

When the large coefficient of g is not the constant coefficient, we see below that the distribution of mass for $\mu_{g,v}$ differs from $\mu_{f,v} = \delta_{\zeta(0,1)}$ by a definite amount. Then, Proposition 3.23 gives an upper bound on the number of common preperiodic points.

PROPOSITION 4.5. *Let $v \in M_{\mathbb{Q}}$ be non-Archimedean, with p its associated prime. If there exists $1 \le j \le d-1$ such that $|b_j|_v > 1$, then*

$$\mu_{g,v}(\mathbb{A}^1_{\mathrm{Berk},v} \setminus D_{\mathrm{an}}(0, p^{1/d})) \ge 1/d.$$

Proof. First, we claim that for any $c \in \mathbb{C}_v$, the polynomial $g - c$ has at least one root w with $|w|_v \ge |b_j|_v^{1/(d-j)} \ge p^{1/d}$. This follows from considering the Newton polygon of $g - c$, which must have a side with slope at least $1/(d-j) \log |b_j|_v$.

Now, for any non-exceptional point $z \in \mathbb{C}_v$, we consider the iterated pullback $d^{-n}(g^n)^{\delta_z}$. On one hand, by the above argument, this measure always assigns at least $1/d$ mass outside of $D_{\mathrm{an}}(0, p^{1/d})$. On the other hand, by equidistribution, this sequence of measures converges weakly to the equilibrium measure $\mu_{g,v}$; hence, $\mu_{g,v}$ must also assign at least $1/d$ mass outside of $D_{\mathrm{an}}(0, p^{1/d})$. □

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COROLLARY 4.6. *Let $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$. If there exists a non-Archimedean place $v \in M_{\mathbb{Q}}$ such that*

$$\text{for all } i\ [|a_i|_v \le 1] \quad \text{there exists } j > 0\ [|b_j|_v > 1],$$

then, $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \lesssim_d \log^4 X$.

Proof. Let p be the prime associated to v. Then, $p \mid \mathrm{denom}(b_j)$, so $p \le X$.

Let $F = \mathrm{Prep}(f) \cap \mathrm{Prep}(g)$. Now, F is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, and $F \subseteq \overline{B}(0, 1)$ by Proposition 4.3. Hence, by Propositions 4.5 and 3.23, we have

$$|F| \lesssim_d (h(g)^2 + 1) \log^2 p \lesssim \log^4 X. \qquad \square$$

4.2. *Proof of Theorems Theorem 1.3 and 1.5.* We now restate and prove Theorem 1.3.

THEOREM 1.3. *Let $d \ge 6$ and write $\mathcal{S}(X) = \mathcal{P}_c(X) \times \mathcal{P}(X) \setminus \{(f, f) \mid f \in \mathcal{P}_c(X)\}$. Then,*

$$\lim_{X\to\infty} \frac{1}{|\mathcal{S}(X)|} \sum_{(f,g)\in\mathcal{S}(X)} |\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| = 0,$$

where $\mathrm{Prep}(f)$, $\mathrm{Prep}(g)$ *denotes all preperiodic points of f and g in $\overline{\mathbb{Q}}$.*

Proof. Fix $f \in \mathcal{P}_c(X)$.

Case 1: For some non-Archimedean place $v \in M_{\mathbb{Q}}$, we have

$$\text{for all } i\ [|a_i|_v \le 1] \quad \text{for all } j \ne 0\ [|b_j|_v \le 1], \quad |b_0|_v > 1.$$

By Corollary 4.4, f and g have no common preperiodic points.

Case 2: Case 1 does not hold, and for some non-Archimedean place $v \in M_{\mathbb{Q}}$, we have

$$\text{for all } i\ [|a_i|_v \le 1] \quad \text{there exists } j\ [|b_j|_v > 1].$$

By Corollary 4.4, this case occurs for at most $O_{\varepsilon,d}(X^{-1+\varepsilon})$ proportion of $g \in \mathcal{P}(X)$, and by Corollary 4.6, $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \lesssim_d \log^4 X$.

Case 3: Neither Case 1 nor Case 2 holds. We will further split this case by the value of k, $1 \le k \le d$, for which

$$\text{for all } i < k\ [a_{d-i} = b_{d-i}], \quad a_{d-k} \ne b_{d-k}.$$

By Proposition 4.1 with $J = \{d-k, d-k-1, \dots, 0\}$, the subcase for each value of k occurs for at most $O_{\varepsilon,d}(X^{-2(k-1)} X^{-(d-k+1)+\varepsilon}) = O_{\varepsilon,d}(X^{-(d+k-1)+\varepsilon})$ proportion of $g \in \mathcal{P}(X)$, and Corollary 3.26 gives $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \lesssim_d X^{k+4+\varepsilon}$.

Putting these three cases together, the average value of $|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)|$ over all $g \in \mathcal{P}(X) \setminus \{f\}$ is at most

$$O_{\varepsilon,d}\left(X^{-1+\varepsilon} \log^4 X + \sum_{k=1}^{d-1} X^{-(d+k-1)+\varepsilon} X^{k+4+\varepsilon}\right) = O_{\varepsilon,d}(X^{-1+2\varepsilon} + X^{5-d+\varepsilon}).$$

For $d \ge 6$ and $\varepsilon > 0$ small, this goes to 0 as $X \to \infty$, and we are done. $\square$

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We now prove Theorem 1.5 using the same argument. Let V be a linear subspace of $(a_{d-2}, \dots, a_0) \simeq \mathbb{Q}^{d-1}$ of dimension m that is given by an intersection of coordinate hyperplanes $\{a_i = 0\}$. Assume that the projection map from V to the last coordinate a_0 is surjective. Let $x = (x_{d-1}, \dots, x_0)$ be a point in $\mathbb{Q}^{d-1}$ and let A be the affine subspace $x + V$. Choose m coordinates $a_{i_1}, \dots, a_{i_m}$ such that i_j are decreasing, $i_m = 0$, and they induce an isomorphism $V \simeq \mathbb{Q}^m$. We define

$$\mathcal{P}_{c,A}(X) = \{z^d + a_{d-2}z^{d-2} + \cdots + a_0 \mid (0, a_{d-2}, \dots, a_0) \in A \text{ and } H(a_{i_j}) \le X\},$$

$$\mathcal{P}_A(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid (a_{d-1}, \dots, a_0) \in A \text{ and } H(a_{i_j}) \le X\},$$

$$\mathcal{S}_V(X) = \mathcal{P}_{c,A}(X) \times \mathcal{P}_A(X) \setminus \{(f, f) \mid f \in \mathcal{P}_{c,A}(X)\}.$$

THEOREM 1.5. *Assume that* $\dim_{\mathbb{Q}} V \ge (d/2) + 3$. *Then,*

$$\lim_{X\to\infty} \frac{1}{|\mathcal{S}_A(X)|} \sum_{(f,g)\in\mathcal{S}_A(X)} |\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| = 0.$$

Proof. The same argument of Theorem 1.3 applies. Fix $f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \in \mathcal{P}_{c,A}(X)$ and write $g(z) = z^d + b_{d-1}z^{d-1} + \cdots + b_0 \in \mathcal{P}_A(X)$. Then, we keep Case 1 as the same as Theorem 1.3. We modify Case 2 to asking for a non-Archimedean place v such that $|b_{i_k}|_v > 1$ for some k and $|a_i|_v \le 1$ for all i. For Case 3, we split by the value of $1 \le k \le m$ for which

$$\text{for all } j < k[a_{i_j} = b_{i_j}], \quad a_{i_k} \ne b_{i_k}.$$

As V is given by an intersection of coordinate hyperplanes, it follows that we can vary b_0 freely over rational numbers of height $\le X$ while keeping the rest of the coefficients fixed. Thus, Proposition 4.1 again implies Case 2 holds for $O(X^{-1+\epsilon})$ proportion of $g(z)$ values. The same bound of $O(\log^4 X)$ holds for this case as the height of each coefficient is again $O(\log X)$. Similarly, as we can vary each b_{i_k} freely while keeping the rest of the coefficients fixed, Proposition 4.1 implies that Case 3 occurs for at most

$$O_{\epsilon,d}(X^{-2(k-1)}X^{-(m-k+1)+\epsilon}) = O_{\epsilon,d}(X^{-(m+k-1)+\epsilon})$$

proportion for all possible choices for g. Now, by Corollary 3.26, since the first index j for which $a_{d-j} \ne b_{d-j}$ is at most $d - m + k$, we get

$$|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \lesssim_d X^{d-m+k+4+\epsilon}.$$

Putting it together, we get an upper bound of

$$O_{\epsilon,d}\left(X^{-1+\epsilon}\log^2(X) + \sum_{k=1}^{m} X^{-(m+k-1)+\epsilon}X^{d-m+k+4+\epsilon}\right) = O_{\epsilon,d}(X^{-1+\epsilon} + X^{5+d-2m+\epsilon}).$$

Thus, if $m \ge (d/2) + 3$, this quantity goes to zero as $X \to \infty$, as desired. □

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5. *Generic polynomials and dynamics*

In this section, we will define a notion of generic polynomials and study its properties, mainly its dynamics. For a pair of polynomials (f, g) with rational coefficients, we first consider three notions of arithmetic complexity, namely $h(f) + h(g)$, $\langle \mu_f, \mu_g \rangle$, and $\log X$, and show that these are comparable when the pair (f, g) is generic. We then make a detailed study of the equilibrium measures and potentials for generic polynomials at most places. Recall again that for a fixed degree d and $X \geq 1$, we have the sets

$$\mathcal{P}(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \mid a_i \in \mathbb{Q},\ H(a_i) \leq X\},$$
$$\mathcal{P}_c(X) = \{z^d + a_{d-1}z^{d-1} + \cdots + a_0 \in \mathcal{P}(X) \mid a_{d-1} = 0\}.$$

5.1. *Complexity of generic pairs.*

The arithmetic complexity of a pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ can be measured by the adelic energy pairing $\langle \mu_f, \mu_g \rangle$, by the heights of the polynomials $h(f) + h(g)$, or simply by the height bound $\log X$. In this section, we show that these three quantities are comparable in the generic case. We first recall the notion of ε-ordinary. For an integer n, we let $\mathrm{rad}(n)$ be the product of all distinct prime factors of n.

Definition 5.1. Given X and $\varepsilon > 0$, we will say that a pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ is *ε-ordinary* if

$$\mathrm{rad}(\mathrm{denom}(c)) \geq X^{1-2\varepsilon} \quad \text{and} \quad \gcd(\mathrm{denom}(c), \mathrm{denom}(c')) \leq X^{2\varepsilon}$$

for any distinct coefficients c, c' of f or g.

THEOREM 5.2. *For any $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$, we have*

$$\langle \mu_f, \mu_g \rangle - 2 \leq \frac{h(f) + h(g)}{d} \leq 2 \log X.$$

Moreover, if (f, g) is ε-ordinary, then

$$\langle \mu_f, \mu_g \rangle + O(d\varepsilon \log X) \geq \frac{h(f) + h(g)}{d} \geq \left(2 - O\left(d\varepsilon + \frac{1}{d}\right)\right) \log X.$$

We prove these bounds in Propositions 5.5, 5.12, 5.20, and 5.23. For convenience, we make the following definition.

Definition 5.3. For $v \in M_K$, write $\mathcal{M}_{f,v} := \max(1, |a_{d-1}|_v, \ldots, |a_0|_v)$.

Remark 5.4. Note that $\log \mathcal{M}_{f,v}$ is the local contribution to the height of f at v, that is, $h(f) = (1/[K : \mathbb{Q}]) \sum_{v \in M_K} n_v \log \mathcal{M}_{f,v}$. Also, for any place v' of a finite extension K'/K with $v' \mid v$, we have $\mathcal{M}_{f,v'} = \mathcal{M}_{f,v}$.

We begin with an easy bound for $h(f)$ in terms of $\log X$.

PROPOSITION 5.5. *For any $f \in \mathcal{P}(X)$, we have $h(f) \leq d \log X$.*

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Proof. For any $v \in M_{\mathbb{Q}}$,

$$\mathcal{M}_{f,v} = \max(1, |a_{d-1}|_v, \dots, |a_0|_v) \le \max(1, |a_{d-1}|_v) \cdots \max(1, |a_0|_v).$$

Hence,

$$\begin{aligned} h(f) &= \sum_{v \in M_{\mathbb{Q}}} \log \mathcal{M}_{f,v} \le \sum_{v \in M_{\mathbb{Q}}} (\log^+ |a_{d-1}|_v + \cdots + \log^+ |a_0|_v) \\ &= h(a_{d-1}) + \cdots + h(a_0) \le d \log X. \end{aligned}$$

□

Now, we will relate $h(f) + h(g)$ to $\langle \mu_f, \mu_g \rangle$. We first note that $\log \mathcal{M}_{f,v}$ can be read off from the equilibrium potential $G_{f,v}$ at the Gauss point.

LEMMA 5.6. *Let $v \in M_K$ be non-Archimedean. Then,*

$$G_{f,v}(\zeta(0,1)) = \frac{1}{d} \log \mathcal{M}_{f,v}.$$

Proof. It is routine to check that $f(\zeta(0,1)) = \zeta(0, \mathcal{M}_{f,v})$ and $f(\zeta(0,R)) = \zeta(0, R^d)$ for all $R \ge \mathcal{M}_{f,v}$. Hence, for all $n \ge 1$, we have

$$f^n(\zeta(0,1)) = \zeta(0, \mathcal{M}_{f,v}^{d^{n-1}}),$$

and the result follows. □

However, we can bound $G_{f,v}$ in terms of its value at the Gauss point, which in turn yields a bound on the local pairing $-(\mu_f, \mu_g)_v$.

LEMMA 5.7. *Let $v \in M_K$ be non-Archimedean. For any $\xi \in \mathbb{A}^1_{\mathrm{Berk},v}$, we have*

$$G_{f,v}(\xi) \le \log^+ |\xi|_v + G_{f,v}(\zeta(0,1)).$$

Proof. The left side of the claim is subharmonic on $\mathbb{A}^1_{\mathrm{Berk},v}$, while the right side is harmonic on $\mathbb{A}^1_{\mathrm{Berk},v} \setminus \{\zeta(0,1)\}$. Hence, it suffices to check the inequality for $\xi = \zeta(0,1)$, where it is clear, and for $|\xi|_v \to \infty$. However, $G_{f,v}(\xi) = \log^+ |\xi|_v$ for all ξ with $|\xi|_v$ sufficiently large, so we are done by observing that

$$G_{f,v}(\zeta(0,1)) \ge \inf G_{f,v} = I(\mu_{f,v}) = 0.$$

□

COROLLARY 5.8. *Let $v \in M_K$ be non-Archimedean. Then,*

$$-(\mu_f, \mu_g)_v \le \frac{1}{d}(\log \mathcal{M}_{f,v} + \log \mathcal{M}_{g,v}).$$

Proof. We have by Lemmas 5.6 and 5.7,

$$\begin{aligned} -(\mu_f, \mu_g)_v &= \int G_{f,v}(\xi) d\mu_{g,v}(\xi) \le \int \log^+ |\xi|_v d\mu_{g,v}(\xi) + G_{f,v}(\zeta(0,1)) \\ &= G_{g,v}(\zeta(0,1)) + G_{f,v}(\zeta(0,1)) = \frac{1}{d}(\log \mathcal{M}_{f,v} + \log \mathcal{M}_{g,v}). \end{aligned}$$

□

The situation at Archimedean places $v \mid \infty$ is analogous. As usual, we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and write $|\cdot|_v = |\cdot|_\infty$.

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LEMMA 5.9. *We have*

$$\max_{|w|_\infty=1} G_{f,\infty}(w) \le \frac{1}{d}\log \mathcal{M}_{f,\infty} + 1.$$

Proof. Write $|\cdot| = |\cdot|_\infty$ and $R = \max(2, \mathcal{M}_{f,\infty})$. For $|z| \ge R$, we have

$$|f(z)| \le |z|^d + R(|z|^{d-1} + \cdots + |z| + 1) \le (|z| + R)^d - R,$$

by comparing terms of the binomial expansion. By induction, we have $|f^n(z)| \le (|z| + R)^{d^n} - R$, so that by the maximum principle, we have

$$G_{f,\infty}(z) \le \log(\max(|z|, R) + R)$$

for all $z \in \mathbb{C}$.

Now, for $|w| = 1$, we have $|f(w)| \le dR + 1$. Hence,

$$d \cdot G_{f,\infty}(w) = G_{f,\infty}(f(w)) \le \log(\max(|f(w)|, R) + R) \le \log((d+1)R + 1).$$

If $1 \le \mathcal{M}_{f,\infty} \le 2$, then $R = 2$, so

$$d \cdot G_{f,\infty}(w) \le \log(2d + 3) \le d \le \log \mathcal{M}_{f,\infty} + d.$$

Otherwise, we have $\mathcal{M}_{f,\infty} > 2$ and $R = \mathcal{M}_{f,\infty}$, so

$$d \cdot G_{f,\infty}(w) \le \log \mathcal{M}_{f,\infty} + \log(d+2) \le \log \mathcal{M}_{f,\infty} + d. \qquad \square$$

LEMMA 5.10. *For any $z \in \mathbb{C}$, we have*

$$G_{f,\infty}(z) \le \log^+ |z|_\infty + \max_{|w|_\infty=1} G_{f,\infty}(w).$$

Proof. Write $|\cdot| = |\cdot|_\infty$. The left side of the claim is subharmonic on $\mathbb{C}$, while the right side is harmonic on $\mathbb{C} \setminus \partial B(0,1)$. Hence, it suffices to check the inequality on $|z| = 1$, where it is clear, and for $|z| \to \infty$. However, $G_{f,\infty}(z) - \log^+ |z| \to 0$ as $|z| \to \infty$, so we are done by observing that

$$\max_{|w|=1} G_{f,\infty}(w) \ge \inf G_{f,\infty} = I(\mu_{f,\infty}) = 0. \qquad \square$$

COROLLARY 5.11. *We have*

$$-(\mu_f, \mu_g)_\infty \le \frac{1}{d}(\log \mathcal{M}_{f,\infty} + \log \mathcal{M}_{g,\infty}) + 2.$$

Proof. We have by Lemmas 5.9 and 5.10,

$$\begin{aligned}
-(\mu_f, \mu_g)_\infty &= \int G_{f,\infty}(z) d\mu_{g,\infty}(z) \le \max_{|w|_\infty=1} G_{f,\infty}(w) + \int \log^+ |z|_\infty d\mu_{g,\infty}(z) \\
&= \max_{|w|_\infty=1} G_{f,\infty}(w) - (\mu_{S^1}, \mu_{g,\infty})_\infty \\
&= \max_{|w|_\infty=1} G_{f,\infty}(w) + \int G_{g,\infty}(e^{i\theta}) \frac{d\theta}{2\pi} \le \max_{|w|_\infty=1} G_{f,\infty}(w) + \max_{|w|_\infty=1} G_{g,\infty}(w) \\
&\le \frac{1}{d}(\log \mathcal{M}_{f,\infty} + \log \mathcal{M}_{g,\infty}) + 2. \qquad \square
\end{aligned}$$

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Combining the above results, we obtain the following bound on the adelic energy pairing $\langle \mu_f, \mu_g \rangle$.

PROPOSITION 5.12. *We have*

$$\langle \mu_f, \mu_g \rangle \leq \frac{h(f) + h(g)}{d} + 2.$$

Proof. Since $(\mu_f, \mu_f)_v = (\mu_g, \mu_g)_v = 0$, we have

$$\langle \mu_f, \mu_g \rangle = \frac{1}{2} \sum_{v \in M_{\mathbb{Q}}} (\mu_f - \mu_g, \mu_f - \mu_g)_v = \sum_{v \in M_{\mathbb{Q}}} -(\mu_f, \mu_g)_v.$$

Now, each term in the sum is bounded by Corollaries 5.8 and 5.11; summing over all places, we obtain the desired bound. □

Our next objective is to show that most pairs $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ are ε-ordinary, which requires some preparatory estimates.

PROPOSITION 5.13. [**12**, Lemma 2.9] *Fix $\varepsilon > 0$. Then, for all large enough X,*

$$\sum_{n \leq X} \frac{1}{\operatorname{rad}(n)} \leq X^{\varepsilon}.$$

COROLLARY 5.14. *There are at most $O_\varepsilon(X^{2-\varepsilon})$ rationals x such that $h(x) \leq \log X$ and* $\operatorname{rad}(\operatorname{denom}(x)) \leq X^{1-2\varepsilon}$.

Proof. By Proposition 5.13, the number of possible values of $\operatorname{denom}(x)$ is

$$\#\{n \leq X \,:\, \operatorname{rad}(n) \leq X^{1-2\varepsilon}\} \lesssim_\varepsilon X^{\varepsilon} X^{1-2\varepsilon} = X^{1-\varepsilon},$$

and for each such denominator, there are at most $2X + 1$ possibilities for x. □

LEMMA 5.15. *Let x_1, x_2 be chosen uniformly at random independently from the finite set $\{x \in \mathbb{Q} \,:\, h(x) \leq \log X\}$, and write $x_i = \alpha_i/\beta_i$ for some $|\alpha_i| \leq X$, $1 \leq \beta_i \leq X$,* $\gcd(\alpha_i, \beta_i) = 1$. *Then,*

$$\Pr(\gcd(\beta_1, \beta_2) \geq X^{2\varepsilon}) \lesssim_\varepsilon X^{-\varepsilon}.$$

Proof. Fix x_1. If $\gcd(\beta_1, \beta_2) \geq X^{2\varepsilon}$, then $\beta_2 = km$ for some k, m satisfying

$$m \mid \beta_1, \quad m \geq X^{2\varepsilon}, \quad km \leq X.$$

Since $\beta_1 \leq X$, the number of divisors of β_1 is at most $O_\varepsilon(X^{\varepsilon})$, by the divisor bound. Hence, there are at most $O_\varepsilon(X^{\varepsilon})$ choices for m, and for each such m, there are at most $X^{1-2\varepsilon}$ choices for k. Also, for each β_2, there are at most X choices for α_2. Thus, there are $\lesssim_\varepsilon X^{\varepsilon} \cdot X^{1-2\varepsilon} \cdot X = X^{2-\varepsilon}$ choices for x_2, out of $\sim X^2$ possibilities from $\{x \in \mathbb{Q} \,:\, h(x) \leq \log X\}$ by equation (4.1). Hence,

$$\Pr(\gcd(\beta_1, \beta_2) \geq X^{2\varepsilon} \mid \beta_1 \text{ is fixed}) \lesssim_\varepsilon \frac{X^{2-\varepsilon}}{X^2} = X^{-\varepsilon},$$

which implies the desired statement. □

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PROPOSITION 5.16. *The proportion of pairs* $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ *which are* ε*-ordinary is* $1 - O_\varepsilon(d^2 X^{-\varepsilon})$.

Proof. By Corollary 5.14, there is at most an $O_\varepsilon(dX^{-\varepsilon})$ proportion of polynomials $f \in \mathcal{P}_c(X)$, where $\operatorname{rad}(\operatorname{denom}(a_i)) \le X^{1-2\varepsilon}$ for some i; the analogous bound holds for g. Also, by Lemma 5.15, there is at most an $O_\varepsilon(d^2 X^{-\varepsilon})$ proportion of pairs $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ where $\gcd(\operatorname{denom}(a_i), \operatorname{denom}(b_j)) > X^{2\varepsilon}$ for some i, j.

Hence, at most $O_\varepsilon(d^2 X^{-\varepsilon})$ proportion of pairs (f, g) satisfies one of the above conditions; this is equivalent to not being ε-ordinary, and we are done. □

We now show that for ε-ordinary pairs, the reverse inequalities to Propositions 5.5 and 5.12 hold up to a small multiplicative error.

Definition 5.17. Given a pair (f, g), and $0 \le j \le d-1$, we say that a non-Archimedean place $v \in M_K$ is associated to a_j if

$$\text{for all } i \ne j \ [|a_i|_v \le 1] \quad \text{for all } i \ [|b_i|_v \le 1], \quad |a_j|_v > 1.$$

We define places associated to b_j similarly.

We call a non-Archimedean place $v \in M_K$ a good place for the pair (f, g) if v is associated to some coefficient a_j or b_j, or a bad place for (f, g) otherwise. (We will always treat Archimedean places separately.)

Remark 5.18. Each $v \in M_K$ is associated to at most one coefficient of f or g. If v is associated to a_j, then $\mathcal{M}_{f,v} = |a_j|_v > 1$ and $\mathcal{M}_{g,v} = 1$, and similarly for b_j.

We first show the reverse bound to Proposition 5.5 for $h(f) + h(g)$ and $\log X$.

LEMMA 5.19. *For any* ε*-ordinary pair* $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$*, we have*

$$\frac{1}{[K:\mathbb{Q}]} \sum_{\substack{v \in M_K \\ v \text{ assoc. } a_j}} n_v \log |a_j|_v \ge (1 - O(d\varepsilon)) \log X,$$

and similarly for b_j.

Proof. We may assume that $K = \mathbb{Q}$. Let A_i, B_i denote the denominators of a_i, b_i, respectively. Now, $v \in M_{\mathbb{Q}}$ is associated to a_j if and only if v divides A_j, but no other A_i or B_i; hence,

$$\begin{aligned}\sum_{v \text{ assoc. } a_j} \log |a_j|_v &\ge \log \operatorname{rad}(A_j) - \sum_{i \ne j} \log \gcd(A_i, A_j) - \sum_i \log \gcd(B_i, A_j) \\ &\ge (1 - 2\varepsilon) \log X - (2d-1) 2\varepsilon \log X \ge (1 - 4d\varepsilon) \log X.\end{aligned}$$

The argument for b_j is analogous. □

PROPOSITION 5.20. *For any* ε*-ordinary pair* $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$*, we have*

$$\frac{h(f) + h(g)}{d} \ge \left(2 - O\left(d\varepsilon + \frac{1}{d}\right)\right) \log X.$$

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Proof. Using Lemma 5.19, we have

$$h(f)+h(g)=\sum_{v\in M_{\mathbb{Q}}}(\log \mathcal{M}_{f,v}+\log \mathcal{M}_{g,v})\geq \sum_{j}\sum_{v \text{ assoc. } a_j}\log|a_j|_v+\sum_{j}\sum_{v \text{ assoc. } b_j}\log|b_j|_v$$
$$\geq (2d-1)(1-4d\varepsilon)\log X\geq 2d\left(1-O\left(d\varepsilon+\frac{1}{d}\right)\right)\log X,$$

and we are done after dividing by d. □

We now proceed to show the reverse bound to Proposition 5.12 for $h(f)+h(g)$ and $\langle \mu_f, \mu_g\rangle$. We begin by observing an important equality case for Corollary 5.8.

PROPOSITION 5.21. *Let $v\in M_K$ be non-Archimedean. If v is good for (f,g), then*

$$-(\mu_f,\mu_g)_v=\frac{1}{d}(\log \mathcal{M}_{f,v}+\log \mathcal{M}_{g,v}).$$

Proof. Assume that g has explicit good reduction at v; in particular, $\mathcal{M}_{g,v}=1$ and $\mu_{g,v}=\delta_{\zeta(0,1)}$. Now, by Lemma 5.6,

$$-(\mu_f,\mu_g)_v=\int G_{f,v}d\mu_{g,v}=G_{f,v}(\zeta(0,1))$$
$$=\frac{1}{d}\log \mathcal{M}_{f,v}=\frac{1}{d}(\log \mathcal{M}_{f,v}+\log \mathcal{M}_{g,v}).$$

The argument when f has explicit good reduction is analogous. □

Next, for an ε-ordinary pair (f,g), we show that the local contribution to $h(f)+h(g)$ from bad places is small.

PROPOSITION 5.22. *For any ε-ordinary pair $(f,g)\in \mathcal{P}_c(X)\times\mathcal{P}(X)$, we have*

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\substack{v\in M_K\\ v \text{ bad}}} n_v(\log \mathcal{M}_{f,v}+\log \mathcal{M}_{g,v})\lesssim d^2\varepsilon\log X.$$

Proof. We may assume that $K=\mathbb{Q}$. Write A_i, B_i for the denominator of a_i, b_i, respectively. Let p_v denote the prime associated to a place v. Now,

$$\sum_{v \text{ bad}}\log \mathcal{M}_{f,v}\leq \sum_{v \text{ bad}}\log p_v+\sum_{v \text{ bad}}\max_i(0,\log|a_i|_v-\log p_v). \tag{5.1}$$

Note that v is a bad place if and only if p_v divides more than one of the A_i or B_i. Hence, the first sum on the right in equation (5.1) is

$$\sum_{v \text{ bad}}\log p_v\leq \sum_{i,j}\log\gcd(A_i,B_j)+\sum_{i<j}\log\gcd(A_i,A_j)$$
$$+\sum_{i<j}\log\gcd(B_i,B_j)\leq 4d^2\varepsilon\log X.$$

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Also, the second term on the right in equation (5.1) is

$$\begin{aligned}\sum_{v \text{ bad}} \max_i(0, \log |a_i|_v - \log p_v) &\le \sum_i \sum_{v \nmid \infty} \max(0, \log |a_i|_v - \log p_v) \\ &= \sum_i (\log A_i - \log \operatorname{rad}(A_i)) \\ &\le \sum_i (\log X - \log X^{1-2\varepsilon}) \le 2d\varepsilon \log X.\end{aligned}$$

Substituting back into equation (5.1), we get $\sum_{v \text{ bad}} \log \mathcal{M}_{f,v} \lesssim d^2\varepsilon \log X$. The same argument works for g. □

PROPOSITION 5.23. *For any ε-ordinary pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$, we have*

$$\langle \mu_f, \mu_g \rangle \ge \frac{h(f) + h(g)}{d} - O\left(\left(d\varepsilon + \frac{1}{d}\right) \log X\right).$$

Proof. We may assume that $K = \mathbb{Q}$. Note that $(\mu_f, \mu_f)_v = (\mu_g, \mu_g)_v = 0$ for any $v \in M_{\mathbb{Q}}$, so $0 \le (\mu_f - \mu_g, \mu_f - \mu_g)_v = -2(\mu_f, \mu_g)_v$. Hence, by Proposition 5.21,

$$\begin{aligned}\langle \mu_f, \mu_g \rangle &\ge \sum_{v \text{ good}} -(\mu_f, \mu_g)_v = \frac{1}{d} \sum_{v \text{ good}} (\log \mathcal{M}_{f,v} + \log \mathcal{M}_{g,v}) \\ &= \frac{h(f) + h(g)}{d} - \frac{1}{d} \sum_{v \text{ bad}} (\log \mathcal{M}_{f,v} + \log \mathcal{M}_{g,v}).\end{aligned}$$

Using Proposition 5.22, we get the above expression is

$$\begin{aligned}&= \frac{h(f) + h(g)}{d} - \frac{1}{d}(\log \mathcal{M}_{f,\infty} + \log \mathcal{M}_{g,\infty}) \\ &\ge \frac{h(f) + h(g)}{d} - O(d\varepsilon \log X) - O\left(\frac{1}{d} \log X\right).\end{aligned}$$

□

5.2. *Dynamics on generic polynomials.* For ε-ordinary pairs (f, g), since the denominators of the coefficients are almost pairwise coprime, most primes which divide one such denominator would not divide any other denominator. Hence, for most places v where f, g do not both have explicit good reduction, it will be the case that one of them has explicit good reduction, while the other has exactly one large coefficient.

We know in the former that the equilibrium measure and potential are $\delta_{\zeta(0,1)}$ and $\log^+ |\cdot|_v$, respectively; in this section, we will study the equilibrium measure and potential for the latter case.

First, we show that the equilibrium measure lies in three distinct strata (cf. Proposition 4.3, which corresponds to the case $j = 0$).

PROPOSITION 5.24. *Let $v \in M_{\mathbb{Q}}$ be non-Archimedean with $|a_0|_v = 1$. If there exists $1 \le j \le d - 1$ such that*

$$\text{for all } i \ne j \ [|a_i|_v \le 1], \quad |a_j|_v > 1,$$

then $|\zeta|_v \in \{|a_j|_v^{1/(d-j)}, |a_j|_v^{1/j(1/(d-j)-1)}, |a_j|_v^{-1/j}\}$ for all $\zeta \in K_{f,v}$.

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Proof. By the density of type I points in $K_{f,v}$, it suffices to prove the statement for $\zeta = z \in \mathbb{C}_v$. We will prove the contrapositive statement, so suppose that $|z|_v$ is not one of the listed values.

Case 1a: $|z|_v \geq 1$ and $j < d-1$. Then, $|z^d|_v \neq |a_j z^j|_v$ by assumption, so

$$|f(z)|_v = \max(|z^d|_v, |a_j z^j|_v) > |a_j|_v^{1/(d-j)}.$$

By induction, we have $|f^n(z)|_v = |f(z)|_v^{d^{n-1}}$; in particular, $f^n(z) \to \infty$, so $z \notin K_{f,v}$.

Case 1b: $|z|_v \geq 1$ and $j = d-1$. Note that $|z|_v \neq |a_j|_v^{1/j(1/(d-j)-1)} = 1$, so $|z|_v > 1$, and now the argument of Case 1a works verbatim.

Case 2: $|z|_v < 1$. Then, $|a_j z^j|_v \neq 1$ and $|a_0|_v = 1$ by assumption, so

$$|f(z)|_v = \max(|a_j z^j|_v, 1) \geq 1.$$

Furthermore, $|a_j z^j|_v \neq |a_j|_v^{1/(d-j)}$ by assumption, so $f(z) \notin K_{f,v}$ by Cases 1a and 1b. Thus, $z \notin K_{f,v}$ as well. □

The next result shows that if some subset of $\mathbb{A}^1_{\mathrm{Berk},v}$ decomposes into finitely many strata, then the equilibrium measure restricted to each stratum is a multiple of the equilibrium measure on that stratum.

LEMMA 5.25. *Let $r_1 > r_2 > \cdots > r_k > 0$ and let $K_i \subseteq \overline{D}_{\mathrm{an}}(0, r_i) \setminus D_{\mathrm{an}}(0, r_i)$ be compact sets in $\mathbb{A}^1_{\mathrm{Berk},v}$ with equilibrium measures μ_i. Then, the equilibrium measure μ on $K = \bigcup_{i=1}^k K_i$ is a convex combination of the μ_i.*

Proof. Write $\mu = \sum_{i=1}^k \alpha_i \nu_i$ for some $\alpha_i \in [0,1]$ and ν_i probability measures supported on $\overline{D}_{\mathrm{an}}(0, r_i) \setminus D_{\mathrm{an}}(0, r_i)$. Now, for all $i < j$ and all $\zeta_i \in K_i$, $\zeta_j \in K_j$, we have $\delta(\zeta_i, \zeta_j)_\infty = \log r_i$, so

$$I(\mu) = \sum_{i=1}^k \alpha_i^2 I(\nu_i) + \sum_{1 \leq i < j \leq k} 2\alpha_i \alpha_j \log r_i. \tag{5.2}$$

Since μ is the equilibrium measure on K, the above expression is minimal over all choices of ν_i supported on K_i. However, the minimizer of $I(\nu_i)$ for ν_i supported on K_i is precisely μ_i; hence, $\nu_i = \mu_i$ when $\alpha_i \neq 0$, and so $\mu = \sum_{i=1}^k \alpha_i \mu_i$. □

Remark 5.26. The above result is known in more general contexts, cf. [**18**, Proposition 4.1.27, **19**, Proposition A.12].

THEOREM 5.27. *Let f, v, j be as in Proposition 5.24, and write*

$$r_1 = |a_j|_v^{1/(d-j)}, \quad r_2 = |a_j|_v^{1/j(1/(d-j)-1)}, \quad r_3 = |a_j|_v^{-1/j}.$$

Let $\mu_{f,v} = \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3$ for some $\alpha_i \in [0,1]$ and μ_i probability measures supported on the strata $S_i := \overline{D}_{\mathrm{an}}(0, r_i) \setminus D_{\mathrm{an}}(0, r_i)$. Then,

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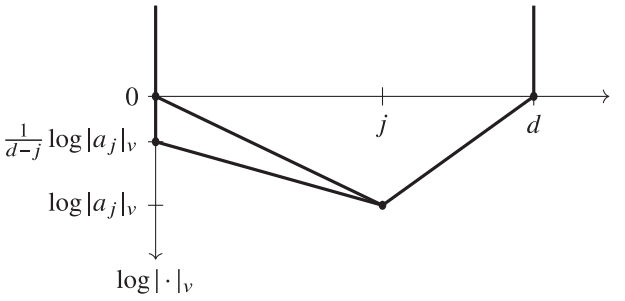


FIGURE 1. Newton polygon for $f-c$ for $c \in K_{f,v} \cap \mathbb{C}_v$, when $|c|_v = r_1$ (below), or $|c|_v = r_2$ or r_3 (above).

$$\alpha_1 = \frac{d-j}{d}, \qquad I(\mu_1) = -\frac{j}{(d-j)^2}\log|a_j|_v,$$
$$\alpha_2 = \frac{j(d-j)}{d^2}, \qquad I(\mu_2) = -\left(\frac{1}{j} + \frac{1}{(d-j)^2}\right)\log|a_j|_v,$$
$$\alpha_3 = \frac{j^2}{d^2}, \qquad I(\mu_3) = -\left(\frac{1}{j} + \frac{1}{j^2}\right)\log|a_j|_v.$$

Proof. For $c \in K_{f,v} \cap \mathbb{C}_v$, we consider the Newton polygon for the polynomial $f-c$. This is the same as the Newton polygon for f except when $|c|_v = r_1 > 1$, where the constant coefficient then has absolute value r_1. The two possibilities are shown in Figure 1.

We conclude that:

- if $|c|_v = r_1$, then $f-c$ has $d-j$ roots with absolute value r_1, and j roots with absolute value r_2;
- if $|c|_v = r_2$ or r_3, then $f-c$ has $d-j$ roots with absolute value r_1, and j roots with absolute value r_3.

Hence, for any measure ρ with finite support in the type I points within the three strata $S_1 \cup S_2 \cup S_3$, we have

$$\begin{pmatrix}(f^*\rho)(S_1)\\(f^*\rho)(S_2)\\(f^*\rho)(S_3)\end{pmatrix} = \begin{pmatrix}d-j & d-j & d-j\\ j & 0 & 0\\ 0 & j & j\end{pmatrix}\begin{pmatrix}\rho(S_1)\\\rho(S_2)\\\rho(S_3)\end{pmatrix}.$$

Now, let $\rho = d^{-n}(f^n)^*\delta_z$ for any non-exceptional $z \in \mathbb{C}_v$ with (say) $|z|_v = r_1$. Taking $n \to \infty$ and using equidistribution, we see that the above identity holds for $\rho = \mu_{f,v}$ as well. However, $f^*\mu_{f,v} = d \cdot \mu_{f,v}$, so we have

$$d\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}d-j & d-j & d-j\\ j & 0 & 0\\ 0 & j & j\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}.$$

Combined with $\alpha_1+\alpha_2+\alpha_3 = 1$, we find that, indeed,

$$\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \frac{1}{d^2}\begin{pmatrix}d(d-j)\\ j(d-j)\\ j^2\end{pmatrix}.$$

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We now compute the energies $I(\mu_i)$. By equation (5.2),

$$0 = I(\mu_{f,v}) = I(\alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3) = \alpha_1^2 I(\mu_1) + \alpha_2^2 I(\mu_2) + \alpha_3^2 I(\mu_3) + 2\alpha_1\alpha_2 \log(r_1) + 2\alpha_1\alpha_3 \log(r_1) + 2\alpha_2\alpha_3 \log(r_2).$$

However, taking the sum of α_i times the ith expression above, we recover the expression for $I(\mu_{f,v})$, which has value 0; hence, each of the above expressions is 0. Now, we get the claimed values of $I(\mu_i)$ by isolating them within each expression. □

COROLLARY 5.28. *In the setting of Theorem 5.27, we have*

$$G_{f,v}(\xi) \le \begin{cases} \log|\xi|_v & \text{if } |\xi|_v \in [r_1, \infty), \\ \dfrac{j}{d}\log|\xi|_v + \dfrac{1}{d}\log|a_j|_v & \text{if } |\xi|_v \in [r_2, r_1], \\ \dfrac{j^2}{d^2}\log|\xi|_v + \dfrac{j+1}{d^2}\log|a_j|_v & \text{if } |\xi|_v \in [r_3, r_2], \\ \dfrac{1}{d^2}\log|a_j|_v & \text{if } |\xi|_v \in [0, r_3], \end{cases}$$

with equality when $|\xi|_v \notin \{r_1, r_2, r_3\}$.

Proof. We have $G_{f,v} = \alpha_1 G_{\mu_1} + \alpha_2 G_{\mu_2} + \alpha_3 G_{\mu_3}$, where G_{μ_i} is the equilibrium potential for μ_i. Since $|w|_v = r_i$ for all $w \in \operatorname{supp}\mu_i$, we have

$$G_{\mu_i}(\xi) = \int \delta(\xi, w)_\infty d\mu_i(w) \le \log\max(|\xi|_v, r_i),$$

with equality when $|\xi|_v \ne r_i$. The claim follows by routine computation. □

We conclude this section by computing the capacities and potentials on some sets we will use in §6.

LEMMA 5.29. *Let* $s_1 > s_2 > 0$ *and let* $K_i \subseteq \overline{D}_{\mathrm{an}}(0, s_i) \setminus D_{\mathrm{an}}(0, s_i)$ *be compact sets with equilibrium measures* ν_i. *Then, the equilibrium measure* ν *on* $K = K_1 \cup K_2$ *is given by* $\nu = \alpha_1\nu_1 + \alpha_2\nu_2$ *for some* $\alpha_1, \alpha_2 \in [0, 1]$.

Moreover, we have

$$I(\nu) = \frac{\log^2 s_1 - I(\nu_1)I(\nu_2)}{2\log s_1 - I(\nu_1) - I(\nu_2)}.$$

Proof. By Lemma 5.25, ν has the given form. Now,

$$I(\nu) = a\alpha_1^2 + b\alpha_2^2 + 2c\alpha_1\alpha_2, \tag{5.3}$$

where $a = I(\nu_1)$, $b = I(\nu_2)$, and $c = \log s_1$. It is easy to verify (e.g. by Lagrange multipliers) that this quadratic form is minimized on $\alpha_1 + \alpha_2 = 1$ at

$$(\alpha_1, \alpha_2) = \left(\frac{c-b}{2c-a-b}, \frac{c-a}{2c-a-b}\right).$$

Since $K_1, K_2 \subseteq \overline{D}_{\mathrm{an}}(0, s_1)$, we have $a, b \le V(\overline{D}_{\mathrm{an}}(0, s_1)) = c$, so $\alpha_1, \alpha_2 \in [0, 1]$. Finally, substituting the values of α_1, α_2 into equation (5.3) yields $I(\nu) = (c^2 - ab)/(2c - a - b)$. □

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PROPOSITION 5.30. *In the setting of Theorem 5.27, define* $\mathcal{R} = |a_j|_v^{j/(d^2-j^2)}$,

$$S_{f,v}^{\cup} = \{\zeta(0,\mathcal{R})\} \cup \operatorname{supp}\mu_1,\ S_{f,v}^{\cap} = \operatorname{supp}\mu_2 \cup \operatorname{supp}\mu_3.$$

Then,

$$\sup_{S_{f,v}^{\cup}}(G_{f,v} + \log^+|\cdot|_v) = \frac{1}{d-j}\log|a_j|_v, \qquad V(S_{f,v}^{\cup}) = \frac{1}{2(d-j)}\log|a_j|_v,$$

$$\sup_{\overline{D}_{\mathrm{an}}(0,1)}(G_{f,v} + \log^+|\cdot|_v) = \frac{1}{d}\log|a_j|_v, \qquad V(\overline{D}_{\mathrm{an}}(0,1)) = 0,$$

$$\sup_{S_{f,v}^{\cap}}(G_{f,v} + \log^+|\cdot|_v) = 0, \qquad V(S_{f,v}^{\cap}) = -\frac{1}{j}\log|a_j|_v.$$

Proof. Write $\nu_{f,v}^{\cup}$, $\nu_{f,v}^{\cap}$ for the equilibrium measures on $S_{f,v}^{\cup}$, $S_{f,v}^{\cap}$, respectively. Then,

$$\nu_{f,v}^{\cup} = \alpha\mu_1 + \beta\delta_{\zeta(0,\mathcal{R})}, \quad \nu_{f,v}^{\cap} = \alpha'\mu_2 + \beta'\mu_3.$$

By Corollary 5.28, the values of $G_{f,v}$ and $\log^+|\cdot|_v$ on each of the pieces are given as follows. These yield the identities in the left column of the claim.

	$G_{f,v}$	$\log^+\lvert\cdot\rvert_v$
$\zeta(0,\mathcal{R})$	$\frac{d}{d^2-j^2}\log\lvert a_j\rvert_v$	$\frac{j}{d^2-j^2}\log\lvert a_j\rvert_v$
$\operatorname{supp}\mu_1$	0	$\frac{1}{d-j}\log\lvert a_j\rvert_v$
$\overline{D}_{\mathrm{an}}(0,1)$	$\leq \frac{1}{d}\log\lvert a_j\rvert_v$	0
$\operatorname{supp}\mu_2$	0	0
$\operatorname{supp}\mu_3$	0	0

Next, we have $V(\overline{D}_{\mathrm{an}}(0,1)) = \log 1 = 0$. Also, by Lemma 5.29,

$$I(\nu_{f,v}^{\cup}) = \frac{\log^2 r_1 - I(\mu_1)\log\mathcal{R}}{2\log r_1 - I(\mu_1) - \log\mathcal{R}}, \quad I(\nu_{f,v}^{\cap}) = \frac{\log^2 r_2 - I(\mu_2)I(\mu_3)}{2\log r_2 - I(\mu_2) - I(\mu_3)}.$$

By Theorem 5.27, these expressions simplify to the right column of the claim. □

6. *Shared small points on average*

Let f, g be polynomials of degree $d \geq 2$ over $\mathbb{Q}$. We will now prove Theorem 1.7 on the essential minimum of $\widehat{h}_f + \widehat{h}_g$ for generic pairs (f, g). We first start off with some results that apply to all pairs (f, g), before specializing to the generic case.

Let $\overline{L}_f$ be the canonical adelic line bundle associated to our polynomial $f \in \mathbb{Q}[z]$ of degree $d \geq 2$. For background on adelic line bundles, readers may refer to [17]. Then, the height associated to $\overline{L}_f$ is the canonical height h_f, and so applying Zhang's inequality [21, Theorem 1.10] to the line bundle $\overline{L}_{f\otimes g} = \overline{L}_f \otimes \overline{L}_g$ gives us

$$\liminf_{\mathbb{P}^1(\overline{\mathbb{Q}})}(\widehat{h}_f + \widehat{h}_g) + \inf_{\mathbb{P}^1(\overline{\mathbb{Q}})}(\widehat{h}_f + \widehat{h}_g) \leq \langle\mu_f, \mu_g\rangle \leq 2\liminf_{\mathbb{P}^1(\overline{\mathbb{Q}})}(\widehat{h}_f + \widehat{h}_g).$$

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Since f and g are both polynomials, ∞ is a common preperiodic point and thus we have $\inf(\widehat{h}_f + \widehat{h}_g) = 0$. We then get

$$\tfrac{1}{2}\langle \mu_f, \mu_g \rangle \leq \liminf_{\mathbb{P}^1(\overline{\mathbb{Q}})} (\widehat{h}_f + \widehat{h}_g) \leq \langle \mu_f, \mu_g \rangle. \tag{6.1}$$

When $\langle \mu_f, \mu_g \rangle = 0$, this implies that $\liminf(\widehat{h}_f + \widehat{h}_g) = 0$, but in general only gives a possible range. We now give a different proof of equation (6.1) using equidistribution and the energy pairing. We first prove the upper bound, which is an easy consequence of the Arakelov–Zhang pairing in [17].

PROPOSITION 6.1. *We have the upper bound* $\liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \leq \langle \mu_f, \mu_g \rangle$.

Proof. Let (x_n) be a sequence of pairwise distinct points in $\overline{\mathbb{Q}}$ with $\widehat{h}_f(x_n) \to 0$ (for instance, from $\mathrm{Prep}(f)$, which is infinite). By [17, Theorem 1], we have $\widehat{h}_g(x_n) \to \langle \mu_f, \mu_g \rangle$, so $\langle \mu_f, \mu_g \rangle$ is a limit point for the values of $\widehat{h}_f + \widehat{h}_g$ on $\overline{\mathbb{Q}}$. □

We next do the lower bound.

PROPOSITION 6.2. *We have the lower bound* $\liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \geq \tfrac{1}{2}\langle \mu_f, \mu_g \rangle$.

Proof. Given $x \in \overline{\mathbb{Q}}$, write $F \subseteq \overline{\mathbb{Q}}$ for its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit and let $N = |F|$. Let K be a number field containing F and let $\varepsilon = \{\varepsilon_v\}_{v \in M_K}$ be the adelic radius given in the proof of Theorem 3.20.

Recall the parallelogram law for bilinear forms $(\cdot, \cdot)$:

$$(a + b, a + b) + (a - b, a - b) = 2(a, a) + 2(b, b).$$

Applying this for $a = (\mu_f + \mu_g)/2 - [F]_\varepsilon$, $b = (\mu_f - \mu_g)/2$, $(\cdot, \cdot) = (\cdot, \cdot)_v$, we have

$$\begin{aligned}
&(\mu_f - [F]_\varepsilon, \mu_f - [F]_\varepsilon)_v + (\mu_g - [F]_\varepsilon, \mu_g - [F]_\varepsilon)_v \\
&\quad = 2\left(\frac{\mu_f + \mu_g}{2} - [F]_\varepsilon, \frac{\mu_f + \mu_g}{2} - [F]_\varepsilon \right)_v + \frac{1}{2}(\mu_f - \mu_g, \mu_f - \mu_g)_v \\
&\quad \geq \frac{1}{2}(\mu_f - \mu_g, \mu_f - \mu_g)_v.
\end{aligned} \tag{6.2}$$

Summing over $v \in M_K$ and using Proposition 3.18, we obtain

$$\begin{aligned}
\frac{1}{2}\langle \mu_f, \mu_g \rangle &\leq \langle \mu_f, [F]_\varepsilon \rangle + \langle \mu_g, [F]_\varepsilon \rangle \leq \widehat{h}_f(x) + \widehat{h}_g(x) + O\left(d \frac{\log N}{N} (h(f) + h(g) + 1) \right) \\
&\leq \widehat{h}_f(x) + \widehat{h}_g(x) + C_{d,f,g} \frac{\log N}{N}.
\end{aligned}$$

Now, for any $\delta > 0$, choose M sufficiently large such that $(\log m/m) \leq (\delta / C_{d,f,g})$ for all $m \geq M$. Then, we have the implication

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$$\widehat{h}_f(x) + \widehat{h}_g(x) \le \tfrac{1}{2}\langle \mu_f, \mu_g \rangle - \delta \implies N = [\mathbb{Q}(x) : \mathbb{Q}] < M.$$

By Northcott's theorem, there are only finitely many $x \in \overline{\mathbb{Q}}$ satisfying both the hypothesis and our degree bound. Hence,

$$\liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \ge \tfrac{1}{2}\langle \mu_f, \mu_g \rangle - \delta,$$

and we are done since $\delta > 0$ is arbitrary. □

The proof of Proposition 6.2 also explains why it is not an equality in general. For equation (6.2) to be an equality, we must have $[F] \approx \frac{1}{2}(\mu_f + \mu_g)$ at every place v, but then $\sum_{v \in M_K} N_v([F], F)_v \approx \sum_{v \in M_K} N_v \langle \mu_f, \mu_g \rangle_v$, which is positive whenever $\mu_f \neq \mu_g$, contradicting the product formula.

We now move onto the case of a generic pair. We start by controlling the Archimedean place. Recall from Lemma 3.2 that $K_{f,\infty} \subseteq B(0, R_{f,\infty})$, where $R_{f,\infty} = 3\max(1, |a_{d-1}|_\infty, |a_{d-2}|_\infty^{1/2}, \ldots, |a_0|_\infty^{1/d})$.

LEMMA 6.3. *For any ε-ordinary pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$, we have $R_{f,\infty} \le 3X^{\varepsilon}$ and $R_{g,\infty} \le 3X^{2\varepsilon}$.*

Proof. Since $f \in \mathcal{P}(X)$, the numerator of a_j has absolute value at most X. Also, since (f, g) is ε-ordinary, we have $\mathrm{denom}(a_j) \ge X^{1-2\varepsilon}$. Hence, $|a_j|_\infty \le X^{2\varepsilon}$, and now $a_{d-1} = 0$ implies $R_{f,\infty} \le 3X^{\varepsilon}$. The argument for g is analogous. □

The key step in the lower bound for $\liminf_{\overline{\mathbb{Q}}}(\widehat{h}_f + \widehat{h}_g)$ is the following estimate on the sum of local heights, with an extra term $-\frac{1}{2}\log|x|_v$ which vanishes by the product formula when we sum over all places.

LEMMA 6.4. *For any ε-ordinary pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$ and any $v \in M_K$ associated to a_j, we have*

$$\inf_{x \in \mathbb{A}^1_{\mathrm{Berk},v}} \left(G_{f,v}(x) + G_{g,v}(x) - \frac{1}{2}\log|x|_v \right) = \begin{cases} \dfrac{1}{2(d-j)}\log|a_j|_v, & j \le \dfrac{d-1}{2}, \\ \dfrac{d-j-1}{2j(d-j)}\log|a_j|_v, & j \ge \dfrac{d-1}{2}. \end{cases}$$

The same statement holds with a_j replaced by b_j.

Proof. Let $\psi_v(\zeta) = G_{f,v}(\zeta) + G_{g,v}(\zeta) - \frac{1}{2}\log|\zeta|_v$. Since $\psi_v(\zeta) \to \infty$ as $|\zeta|_v \to \infty$, the minimum value of ψ_v occurs on the support of the positive part of $\Delta\psi_v$, which is $J_{f,v} \cup \{\zeta(0,1)\}$. Now, v is associated to a_j, so g has explicit good reduction at v and, thus, $G_{g,v} = \log^+|\cdot|_v$. Hence, we can compute the value of ψ_v at each of the three strata of $J_{f,v}$ (Theorem 5.27) and at $\zeta(0,1)$ (Corollary 5.28), as follows:

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$\lvert\zeta\rvert_v$	$G_{f,v}(\zeta)$	$G_{g,v}(\zeta)$	$\log\lvert\zeta\rvert_v$	$\psi_v(\zeta)$
r_1	0	$\dfrac{1}{d-j}\log\lvert a_j\rvert_v$	$\dfrac{1}{d-j}\log\lvert a_j\rvert_v$	$\dfrac{1}{2(d-j)}\log\lvert a_j\rvert_v$
1	$\dfrac{1}{d}\log\lvert a_j\rvert_v$	0	0	$\dfrac{1}{d}\log\lvert a_j\rvert_v$
r_2	0	0	$\dfrac{1}{j}\left(\dfrac{1}{d-j}-1\right)\log\lvert a_j\rvert_v$	$\dfrac{1}{2j}\left(1-\dfrac{1}{d-j}\right)\log\lvert a_j\rvert_v$
r_3	0	0	$-\dfrac{1}{j}\log\lvert a_j\rvert_v$	$\dfrac{1}{2j}\log\lvert a_j\rvert_v$

Since $(2/d) \geq \min(1/j, 1/(d-j))$ and $1/j > 1/j(1-1/(d-j))$, we have

$$\inf_{\mathbb{A}^1_{\mathrm{Berk},v}} \psi_v = \min\left(\frac{1}{2j}\left(1-\frac{1}{d-j}\right), \frac{1}{2(d-j)}\right)\log|a_j|_v,$$

which is equivalent to the claim. The argument for b_j is analogous. □

THEOREM 6.5. *For any ε-ordinary pair $(f, g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$, we have*

$$\inf_{x\in\overline{\mathbb{Q}}\setminus\{0\}} (\widehat{h}_f(x)+\widehat{h}_g(x)) \geq 2\left(\log 2 - O\left(d^2\varepsilon + \frac{\log d}{d}\right)\right)\log X - O(1).$$

Proof. Let K be a number field with $x \in K \setminus \{0\}$. For any $v \in M_K$, write $\psi_v(x) = G_{f,v}(x) + G_{g,v}(x) - \frac{1}{2}\log|x|_v$. Then, by the product formula $\sum_{v\in M_K} n_v \log|x|_v = 0$, we have

$$\widehat{h}_f(x)+\widehat{h}_g(x) = \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K} n_v\psi_v(x).$$

We split the sum over good, bad, and Archimedean places. For good places, by Lemma 6.4, we have

$$\begin{aligned}
&\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K \text{ good}} n_v\psi_v(x)\\
&\quad\geq \frac{1}{[K:\mathbb{Q}]}\Bigg(\sum_{\substack{v \text{ assoc. } a_j\\ 0<j\leq((d-1)/2)}} n_v\frac{1}{2(d-j)}\log|a_j|_v + \sum_{\substack{v \text{ assoc. } b_j\\ 0<j\leq((d-1)/2)}} n_v\frac{1}{2(d-j)}\log|b_j|_v\\
&\qquad+\sum_{\substack{v \text{ assoc. } a_j\\ ((d-1)/2)<j<d}} n_v\frac{d-j-1}{2j(d-j)}\log|a_j|_v + \sum_{\substack{v \text{ assoc. } b_j\\ ((d-1)/2)<j<d}} n_v\frac{d-j-1}{2j(d-j)}\log|b_j|_v\Bigg).
\end{aligned}$$

By Lemma 5.19, this is

$$\geq 2(1-O(d\varepsilon))\left(\sum_{0<j\leq((d-1)/2)}\frac{1}{2(d-j)} + \sum_{((d-1)/2)<j<d}\frac{d-j-1}{2j(d-j)}\right)\log X. \quad (6.3)$$

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We recognize the first sum in equation (6.3) as a Riemann sum:

$$\sum_{0<j\le((d-1)/2)} \frac{1}{2(d-j)} \ge \int_0^{1/2} \frac{1}{2(1-t)}dt - O\left(\frac{1}{d}\right) = \frac{\log 2}{2} - O\left(\frac{1}{d}\right). \tag{6.4}$$

The summand in the second sum of equation (6.3) splits into $1/2j - 1/2j(d-j)$. As before,

$$\sum_{((d-1)/2)<j<d} \frac{1}{2j} \ge \int_{1/2}^{1} \frac{1}{2t}dt - O\left(\frac{1}{d}\right) = \frac{\log 2}{2} - O\left(\frac{1}{d}\right), \tag{6.5}$$

while

$$\sum_{((d-1)/2)<j<d} \frac{1}{2j(d-j)} \le \sum_{((d-1)/2)<j<d} \frac{1}{j(d-j)} \le O\left(\frac{\log d}{d}\right). \tag{6.6}$$

Substituting equations (6.4), (6.5), and (6.6) back into equation (6.3), we have

$$\begin{aligned}\frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K \text{ good}} n_v\psi_v(x) &\ge 2(1-O(d\varepsilon))\left(\log 2 - O\left(\frac{\log d}{d}\right)\right)\log X \\ &\ge 2\left(\log 2 - O\left(d\varepsilon + \frac{\log d}{d}\right)\right)\log X. \end{aligned}\tag{6.7}$$

For bad places, we have $K_{f,v} \subseteq \overline{D}_{\text{an}}(0, R_{f,v}) \subseteq \overline{D}_{\text{an}}(0, \mathcal{M}_{f,v})$ by Proposition 3.10, and similarly for g. Hence, $G_{f,v}(x) = G_{g,v}(x) = \log^+ |x|_v$ for all x such that $|x|_v > \max(\mathcal{M}_{f,v}, \mathcal{M}_{g,v})$. Therefore,

$$\psi_v(x) \ge \begin{cases} \frac{3}{2}\log|x|_v & \text{if } |x|_v > \max(\mathcal{M}_{f,v}, \mathcal{M}_{g,v}), \\ -\frac{1}{2}\log\max(\mathcal{M}_{f,v}, \mathcal{M}_{g,v}) & \text{if } |x|_v \le \max(\mathcal{M}_{f,v}, \mathcal{M}_{g,v}). \end{cases}$$

Hence, using Proposition 5.22,

$$\begin{aligned}\frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K \text{ bad}} n_v\psi_v(x) &\ge -\frac{1}{2[K:\mathbb{Q}]} \sum_{v\in M_K \text{ bad}} n_v(\log \mathcal{M}_{f,v} + \log \mathcal{M}_{g,v}) \\ &\ge -O(d^2\varepsilon)\log X. \end{aligned}\tag{6.8}$$

For Archimedean places $v \mid \infty$, we split into two cases. If $|x|_v \le R_{f,\infty} + 1$, then

$$\psi_v(x) \ge -\tfrac{1}{2}\log|x|_v \ge -\tfrac{1}{2}\log(R_{f,\infty}+1).$$

If $|x|_v \ge R_{f,\infty} + 1$, then for all $w \in K_{f,v}$, we have $|w/x|_v \le R_{f,\infty}/(R_{f,\infty}+1)$, so

$$G_{f,v}(x) - \log|x|_v \ge \int \log\left|\frac{x-w}{x}\right|_v d\mu_{f,v}(w) \ge -\log(R_{f,\infty}+1)$$

and, thus,

$$\psi_v(x) \ge \tfrac{1}{2}(G_{f,v}(x) - \log|x|_v) \ge -\tfrac{1}{2}\log(R_{f,\infty}+1).$$

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In either case, by Lemma 6.3, we have

$$\frac{1}{[K:\mathbb{Q}]}\sum_{v|\infty} n_v \psi_v(x) \geq -\frac{1}{2}\log(R_{f,\infty}+1) = -O(\varepsilon \log X + 1). \tag{6.9}$$

Adding equations (6.7), (6.8), and (6.9) yields the desired bound. □

We now prove our upper bound of $\liminf \widehat{h}_f(x) + \widehat{h}_g(x)$, which improves Proposition 6.1 by a constant. As in the proof of Proposition 6.1, we need to construct infinitely many $x \in \overline{\mathbb{Q}}$ such that $\widehat{h}_f(x) + \widehat{h}_g(x)$ is small. Our main tool will be Theorem 6.7, which is a special case of the adelic Fekete–Szegő theorem [**19**, Theorem 0.4].

Definition 6.6. A compact Berkovich adelic set $\mathbb{E} = (E_v)_{v\in M_\mathbb{Q}}$ is a collection of compact affinoids $E_v \subseteq \mathbb{A}^1_{\mathrm{Berk},v}$ for each place $v \in M_\mathbb{Q}$, with $E_v = \overline{D}_{\mathrm{an}}(0,1)$ for all but finitely many v. The Robin constant $V(\mathbb{E})$ of $\mathbb{E}$ is $V(\mathbb{E}) := \sum_{v\in M_\mathbb{Q}} V(E_v)$.

THEOREM 6.7. *Let $\mathbb{E} = (E_v)_{v\in M_\mathbb{Q}}$ be a compact Berkovich adelic set. If $V(\mathbb{E}) \geq 0$, then there exists infinitely many $x \in \overline{\mathbb{Q}}$ such that the* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-orbit of x lies inside E_v at every place v.*

Our strategy is to construct, given any ε-ordinary (f,g), a compact Berkovich adelic set $\mathbb{E}$ with $V(\mathbb{E}) > 0$ and $\sum_v n_v \sup_{x\in E_v}(G_{f,v}(x) + G_{g,v}(x))$ small.

THEOREM 6.8. *For any ε-ordinary pair $(f,g) \in \mathcal{P}_c(X) \times \mathcal{P}(X)$, we have*

$$\liminf_{x\in\overline{\mathbb{Q}}}(\widehat{h}_f(x) + \widehat{h}_g(x)) \leq 2\left(\mathcal{C} + O\left(d\varepsilon + \frac{1}{d}\right)\right)\log X + O(1),$$

where $\mathcal{C} = \log((9+\sqrt{17})/8) + (\sqrt{17}-1)/8 = 0.88532\ldots$.

Proof. Let $\alpha := (\sqrt{17}-1)/8$ and let $c > 0$ be a small parameter to be chosen later. We consider the adelic set $\mathbb{E} = (E_v)_{v\in M_\mathbb{Q}}$ given by

$$E_v := \begin{cases} S^{\cup}_{f,v}, & v \text{ assoc. } a_j \text{ with } 0 < \dfrac{j}{d} < \alpha + c, \\ S^{\cup}_{g,v}, & v \text{ assoc. } b_j \text{ with } 0 < \dfrac{j}{d} < \alpha + c, \\ S^{\cap}_{f,v}, & v \text{ assoc. } a_j \text{ with } 2\alpha < \dfrac{j}{d} < 1, \\ S^{\cap}_{g,v}, & v \text{ assoc. } b_j \text{ with } 2\alpha < \dfrac{j}{d} < 1, \\ \overline{D}_{\mathrm{an}}(0,1) & \text{otherwise,} \end{cases}$$

with $S^{\cup}_{f,v}$, $S^{\cap}_{f,v}$ as defined in Proposition 5.30. Using Proposition 5.30 and Lemma 5.19, we bound $V(\mathbb{E})$ from below:

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$$V(\mathbb{E}) = \sum_{v\in M_{\mathbb{Q}}} V(E_v) = \sum_{\substack{v \text{ assoc. } a_j \\ 0<(j/d)<\alpha+c}} \frac{1}{2(d-j)} \log|a_j|_v + \sum_{\substack{v \text{ assoc. } b_j \\ 0<(j/d)<\alpha+c}} \frac{1}{2(d-j)} \log|b_j|_v$$
$$- \sum_{\substack{v \text{ assoc. } a_j \\ 2\alpha<(j/d)<1}} \frac{1}{j} \log|a_j|_v - \sum_{\substack{v \text{ assoc. } b_j \\ 2\alpha<(j/d)<1}} \frac{1}{j} \log|b_j|_v$$
$$\geq (2 - O(d\varepsilon)) \log X \sum_{0<(j/d)<\alpha+c} \frac{1}{2(d-j)} - 2\log X \sum_{2\alpha<(j/d)<1} \frac{1}{j}.$$

We recognize the sums above as Riemann sums:

$$\sum_{0<(j/d)<\alpha+c} \frac{1}{2(d-j)} \geq \int_0^{\alpha+c} \frac{1}{2(1-t)} dt - O\left(\frac{1}{d}\right) = \frac{\log(1-\alpha-c)}{2} - O\left(\frac{1}{d}\right),$$
$$\sum_{2\alpha<(j/d)<1} \frac{1}{j} \geq \int_{2\alpha}^1 \frac{1}{t} dt - O\left(\frac{1}{d}\right) = \log(2\alpha) - O\left(\frac{1}{d}\right).$$

Hence,

$$V(\mathbb{E}) \geq \left(- (1 - O(d\varepsilon)) \log(1-\alpha-c) - 2\log(2\alpha) - O\left(\frac{1}{d}\right)\right) \log X$$
$$\geq \left(\Theta(c) - O\left(d\varepsilon + \frac{1}{d}\right)\right) \log X,$$

where we have used the identity $-\frac{1}{2}\log(1-\alpha) = \log(2\alpha)$. Hence, we may choose $c = O(d\varepsilon + 1/d)$ so that $V(\mathbb{E}) \geq 0$.

We want to apply the adelic Fekete–Szegő theorem to $\mathbb{E}$, but the issue is that $S^{\cup}_{f,v}$, $S^{\cap}_{f,v}$ are not Berkovich affinoids. To fix this, we construct a new adelic set $\widetilde{\mathbb{E}} = (\widetilde{E}_v)_{v\in M_{\mathbb{Q}}}$ by replacing the sets $S^{\cup}_{f,v}$, $S^{\cap}_{f,v}$ in the definition of $\mathbb{E}$ by compact affinoids $\widetilde{S}^{\cup}_{f,v} \supseteq S^{\cup}_{f,v}$ and $\widetilde{S}^{\cap}_{f,v} \supseteq S^{\cap}_{f,v}$ such that

$$\sup_{\widetilde{S}^{\cup}_{f,v}} (G_{f,v} + \log^+ |\cdot|_v) \leq \frac{1}{d-j} \log|a_j|_v + \eta, \quad \sup_{\widetilde{S}^{\cap}_{f,v}} (G_{f,v} + \log^+ |\cdot|_v) \leq \eta, \quad (6.10)$$

where $\eta > 0$ is a small parameter to be chosen later. (This is possible by Proposition 5.30 and the fact that $G_{f,v} + \log^+ |\cdot|_v$ is continuous on $\mathbb{A}^1_{\mathrm{Berk},v}$.) Similarly, we also replace $S^{\cup}_{g,v}$, $S^{\cap}_{g,v}$ with closed neighborhoods $\widetilde{S}^{\cup}_{g,v}$, $\widetilde{S}^{\cap}_{g,v}$.

Now, $\widetilde{\mathbb{E}}$ is a compact Berkovich adelic set and $V(\widetilde{\mathbb{E}}) \geq V(\mathbb{E}) \geq 0$, so by Theorem 6.7, we can find infinitely many distinct $x_n \in \overline{\mathbb{Q}}$ whose $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit F_n lies entirely in $\widetilde{\mathbb{E}}$.

Let K_n be a number field containing F_n. The local contribution to $\widehat{h}_f(x_n)$ at $v \in M_{\mathbb{Q}}$ is

$$\frac{1}{[K_n:\mathbb{Q}]} \sum_{\substack{v'\in M_{K_n} \\ v'|v}} n_{v'} G_{f,v'}(x_n) = \frac{1}{|F_n|} \sum_{x\in F_n} G_{f,v}(x) \leq \sup_{\widetilde{E}_v} G_{f,v},$$

and similarly for g. Hence, we are left to bound $\sup_{\widetilde{E}_v}(G_{f,v} + G_{g,v})$ from above at good, bad, and Archimedean places $v \in M_{\mathbb{Q}}$.

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For good places $v \in M_{\mathbb{Q}}$, if v is associated to a_j, then g has good reduction at v, so $G_{f,v} + G_{g,v} = G_{f,v} + \log^+ |\cdot|_v$, and we are in the setting of Proposition 5.30 and equation (6.10). If v is associated to b_j, we use the analogous statements with f, g swapped. Working through the possible cases in terms of j, we have

$$\sup_{\widetilde{E}_v}(G_{f,v} + G_{g,v}) \leq \begin{cases} \dfrac{1}{d-j}\log|a_j|_v + \eta, & v \text{ assoc. } a_j \text{ with } 0 < \dfrac{j}{d} < \alpha + c, \\ \dfrac{1}{d-j}\log|b_j|_v + \eta, & v \text{ assoc. } b_j \text{ with } 0 < \dfrac{j}{d} < \alpha + c, \\ \dfrac{1}{d}\log|a_j|_v, & v \text{ assoc. } a_j \text{ with } \alpha + c < \dfrac{j}{d} < 2\alpha, \\ \dfrac{1}{d}\log|b_j|_v, & v \text{ assoc. } b_j \text{ with } \alpha + c < \dfrac{j}{d} < 2\alpha, \\ \eta, & v \text{ assoc. } a_j \text{ or } b_j \text{ with } 2\alpha < \dfrac{j}{d} < 1; \end{cases}$$

or bad places $v \in M_{\mathbb{Q}}$, we have by Lemmas 5.7 and 5.6,

$$\sup_{\overline{D}_{\mathrm{an}}(0,1)}(G_{f,v} + G_{g,v}) \leq G_{f,v}(\zeta(0,1)) + G_{g,v}(\zeta(0,1)) = \frac{1}{d}(\log \mathcal{M}_{f,v} + \log \mathcal{M}_{g,v}).$$

Finally, for the Archimedean place, we have $\widetilde{E}_\infty = \partial B(0,1)$, so for any $z \in \widetilde{E}_\infty$ and $w \in K_{f,\infty}$, we have $|z - w| \leq R_{f,\infty} + 1$, with $R_{f,\infty}$ as in Lemma 3.2. Thus,

$$G_{f,\infty}(z) = \int \log|z - w|_\infty d\mu_{f,\infty}(w) \leq \log(R_{f,\infty} + 1),$$

and similarly for g. Hence, by Lemma 6.3, we have

$$\sup_{\partial B(0,1)}(G_{f,\infty} + G_{g,\infty}) \leq \log(R_{f,\infty} + 1) + \log(R_{g,\infty} + 1) \leq O(\varepsilon \log X + 1).$$

Now, we choose $\eta \leq \#\{\text{good places } v \in M_{\mathbb{Q}}\}^{-1}$, so

$$\begin{aligned} \widehat{h}_f(x_n) + \widehat{h}_g(x_n) \leq & \sum_{\substack{v \text{ assoc. } a_j \\ 0<(j/d)<\alpha+c}} \frac{1}{d-j}\log|a_j|_v + \sum_{\substack{v \text{ assoc. } b_j \\ 0<(j/d)<\alpha+c}} \frac{1}{d-j}\log|b_j|_v \\ & + \sum_{\substack{v \text{ assoc. } a_j \\ \alpha+c<(j/d)<2\alpha}} \frac{1}{d}\log|a_j|_v + \sum_{\substack{v \text{ assoc. } b_j \\ \alpha+c<(j/d)<2\alpha}} \frac{1}{d}\log|b_j|_v \\ & + \frac{1}{d}\sum_{v \text{ bad}}(\log \mathcal{M}_{f,v} + \log \mathcal{M}_{g,v}) + O(\varepsilon \log X + 1). \end{aligned}$$

By Proposition 5.22, this is

$$\leq \left(2\sum_{0<(j/d)<\alpha+c}\frac{1}{d-j} + 2\sum_{\alpha+c<(j/d)<2\alpha}\frac{1}{d} + O(d\varepsilon)\right)\log X + O(1).$$

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The first sum is a Riemann sum as before, and the second sum is easy to bound:

$$\widehat{h}_f(x_n) + \widehat{h}_g(x_n) \le \left(-2\log(1-\alpha-c) + 2(\alpha - c) + O\left(d\varepsilon + \frac{1}{d}\right)\right)\log X + O(1)$$
$$= 2\left(\mathcal{C} + O\left(c + d\varepsilon + \frac{1}{d}\right)\right)\log X + O(1),$$

where $\mathcal{C} = -\log(1-\alpha) + \alpha$, and we are done since $c = O(d\varepsilon + 1/d)$. □

Putting together Theorems 6.5 and 6.8, we obtain Theorem 1.7. Recall that we define $\mathcal{P}$ and $\mathcal{P}_c$ to be the space of monic polynomials and monic centered polynomials with rational coefficients, respectively.

THEOREM 1.7. *Fix a $\delta > 0$. We let $\mathcal{S} = \mathcal{P}_c \times \mathcal{P}$ and $\mathcal{S}(X) = \mathcal{P}_c(X) \times \mathcal{P}(X)$. Then, for all sufficiently large degrees d, there exists $a = a(d) > 0$, $A = A(d) > 0$ such that the following statement holds: there exists a subset $\mathcal{T} \subseteq \mathcal{S}$ such that for all $X \ge 1$,*

$$\frac{|T \cap \mathcal{S}(X)|}{|\mathcal{S}(X)|} \ge 1 - \frac{A}{X^a}$$

and

$$(\ln 2 - \delta)(h(f) + h(g)) \le d \liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \le (\mathcal{C} + \delta)(h(f) + h(g)),$$

for all $(f, g) \in \mathcal{T}$, where $\ln 2 = 0.69314\ldots$ and $\mathcal{C} = \log((9+\sqrt{17})/8) + (\sqrt{17} - 1/8) = 0.88532\ldots$.

Proof. Let $\delta > 0$ be given. Then, for all large enough d depending on δ, we can choose ϵ small enough depending on d such that the terms $O(d\epsilon + 1/d)$, $O(d^2\epsilon + \log d/d)$, and $O(d\epsilon + 1/d)$ occurring in Proposition 5.20, Theorems 6.5, and 6.8, respectively, are bounded above by $\delta/3$.

Let X be sufficiently large such that for the terms $O(1)$ occurring in Theorems 6.5 and 6.8, we have $\delta/3 \log X \ge O(1)$. Then, by Theorems 6.5 and 6.8, we know that for any ε-ordinary pair $(f, g) \in \mathcal{S}(X)$, we have

$$2(\ln 2 - \tfrac{2}{3}\delta)\log X \le \liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \le 2(\mathcal{C} + \tfrac{2}{3}\delta)\log X.$$

Now, Proposition 5.20 implies that for an ε-ordinary pair of $\mathcal{S}(X)$, we have

$$h(f) + h(g) \ge d(2 - \tfrac{1}{3}\delta)\log X.$$

We also have the trivial upper bound

$$h(f) + h(g) \le 2d \log X$$

since each coefficient of f and g have height bounded by $\log X$, and we have at most $2d$ different coefficients. We now assume δ is small enough so that

$$(2 - \tfrac{1}{3}\delta)(\mathcal{C} + \delta) \ge 2(\mathcal{C} + \tfrac{2}{3}\delta).$$

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This is possible since $\mathcal{C} < 1$. Then, substituting in our bounds of $h(f) + h(g)$ against $\log X$, we obtain

$$(\ln 2 - \delta)(h(f) + h(g)) \leq d \liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \leq (\mathcal{C} + \delta)(h(f) + h(g))$$

for all ε-ordinary pairs $(f, g) \in \mathcal{S}(X)$, if X is sufficiently large. By Proposition 5.16, the proportion of ε-ordinary pairs in $\mathcal{S}(X)$ is at least $1 - O_\epsilon(d^2 X^{-\epsilon})$. Clearly, we can take $\epsilon = (c/d^2)$ for some constant $c > 0$ that depends on δ and so the proportion of ε-ordinary pairs will be $1 - O_d(d^2 X^{-c/d^2})$. Finally, we set $\mathcal{T}$ to be all the pairs (f, g) which are ε-ordinary for some X which is sufficiently large for our previous bounds to hold. Then,

$$\frac{|\mathcal{T} \cap \mathcal{S}(X)|}{|\mathcal{S}(X)|} \geq 1 - \frac{A}{X^a}$$

for some constants $a = a(d) > 0$, $A = A(d) > 0$ that depend on the degree d, and

$$(\ln 2 - \delta)(h(f) + h(g)) \leq d \liminf_{x \in \overline{\mathbb{Q}}} (\widehat{h}_f(x) + \widehat{h}_g(x)) \leq (\mathcal{C} + \delta)(h(f) + h(g))$$

for all $(f, g) \in \mathcal{T}$, as desired. □

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References

[1] M. Baker and L. Demarco. Preperiodic points and unlikely intersections. *Duke Math. J.* **159**(1) (2011), 1–29.
[2] M. Baker and R. Rumely. *Potential Theory and Dynamics on the Berkovich Projective Line (Mathematical Surveys and Monographs, 159)*. American Mathematical Society, Providence, RI, 2010.
[3] R. Benedetto. *Dynamics in One Non-Archimedean Variable (Graduate Studies in Mathematics, 198)*. American Mathematical Society, Providence, RI, 2019.
[4] G. S. Call and J. H. Silverman. Canonical heights on varieties with morphisms. *Compos. Math.* **89**(2) (1993), 163–205.
[5] L. Carleson and T. W. Gamelin. *Complex Dynamics*. Springer, New York, 1993.
[6] L. DeMarco, H. Krieger and H. Ye. Common preperiodic points for quadratic polynomials. *J. Mod. Dyn.* **18** (2022), 363–413.
[7] L. DeMarco and N. M. Mavraki. Dynamics on $\mathbb{P}^1$: preperiodic points and pairwise stability. *Preprint*, 2023, arXiv:2212.13215 [math.DS].
[8] C. Favre and J. Rivera-Letelier. Equidistribution quantitative des points de petite hauteur Sur la droite projective. *Math. Ann.* **335**(2) (2006), 311–361.
[9] P. Fili. A metric of mutual energy and unlikely intersections for dynamical systems. *Preprint*, 2017, arXiv:1708.08403.
[10] A. Granville. Smooth numbers: computational number theory and beyond. *Algorithmic Number Theory: Lattices, Number Fields, Curves and Cryptography (Mathematical Sciences Research Institute Publications, 44)*. Ed. J. P. Buhler and P. Stevenhagen. Cambridge University Press, Cambridge, 2008, pp. 267–323.
[11] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*, 6th edn. Oxford University Press, 2008.
[12] P. Le Boudec and N. M. Mavraki. Arithmetic dynamics of random polynomials. *Preprint*, 2021, arXiv:2112.12005.
[13] N. M. Mavraki and H. Schmidt. On the dynamical Bogomolov conjecture for families of split rational maps. *Preprint*, 2022, arXiv:2201.10455.

https://doi.org/10.1017/etds.2025.10213 Published online by Cambridge University Press

[14] A. Mimar. On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve. *PhD Thesis*, Columbia University, 1997.
[15] A. Mimar. On the preperiodic points of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which lie on a curve. *Trans. Amer. Math. Soc.* **365**(1) (2013), 161–193.
[16] M. Olechnowicz. Distribution of preperiodic points in one-parameter families of rational maps. *Preprint*, 2023, arXiv:2306.00933 [math.NT].
[17] C. Petsche, L. Szpiro and T. J. Tucker. A dynamical pairing between two rational maps. *Trans. Amer. Math. Soc.* **364** (2012), 1687–1710.
[18] R. Rumely. *Capacity Theory on Algebraic Curves (Lecture Notes in Mathematics, 1378)*. Springer, Berlin, 1989.
[19] R. Rumely. *Capacity Theory with Local Rationality: The Strong Fekete–Szegő Theorem on Curves (Mathematical Surveys and Monographs, 193)*. American Mathematical Society, Providence, RI, 2013.
[20] X. Yuan and S.-W. Zhang. The arithmetic Hodge index theorem for adelic line bundles II. *Preprint*, 2021, arXiv:1304.3539 [math.NT].
[21] S.-W. Zhang. Small points and adelic metrics. *J. Algebraic Geom.* **4**(2) (1995), 281–300.

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