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Bernstein's theorem for compact, connected Lie groups

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1. *Introduction*

In what follows, G will denote a compact, connected Lie group.

Definition. A complex-valued function $f \in L^2(G)$ lies in the space $A(G)$ if it can be written in the form

$$f = \sum_1^\infty g_j * h_j, \tag{1}$$

where

$$\sum_1^\infty \|g_j\|_2 \, \|h_j\|_2 < \infty. \tag{2}$$

The $A(G)$ norm of f is the infimum of all sums (2) with respect to all decompositions (1).

The space $A(G)$ is the same as the space of functions having absolutely convergent Fourier series ([7], 34·4). It is also the case that every element of $A(G)$ can be written as a single convolution $g * h$, where $g, h \in L^2(G)$. [4]. Suppose that an Ad-invariant norm is chosen on the Lie algebra $\mathfrak{g}$ of G.

Definition. Let k be a positive integer and V an element of $\mathfrak{g}$. If $f \in L^2(G)$ and $s > 0$, write

$$\Delta_{sV}^k f(x) = \sum_0^k (-1)^{k+j} \binom{k}{j} f(x \exp jsV)$$

and, if $t > 0$,

$$\omega_{\mathfrak{z}}(t, f \ L^2) = \sup_{\substack{0 < s \leqslant t \\ \|V\| = 1}} \|\Lambda_{sV}^k f\|_2.$$

If $\alpha > 0$, k is an integer, $k > \alpha$, and $f \in L^2(G)$, we say that f belongs to the Besov space $\Lambda_{1,2}^\alpha$ if

$$\int_0^1 \frac{\omega_k(t, f, L^2}{t^{\alpha+1}} \, dt < \infty.$$

In that case, the norm $N_{k,\alpha}(f)$ of f is defined to be

$$N_{k,\alpha}(f) = \|f\|_2 + \int_0^1 \frac{\omega_k(t, f, L^2)}{t^{\alpha+1}} \, dt.$$

Remarks. (*a*) The kth-order difference $\Delta_{sV}^k f$ is the same as $(\Delta_{sV}^1)^k f$.

(*b*) The Besov space $\Lambda_{1,2}^\alpha$ is unchanged if k is changed to another integer $k' > \alpha$. The norms $N_{k,\alpha}$ and $N_{k',\alpha}$ are equivalent.

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(c) If $\beta > \alpha$ then $\Lambda^{\beta}_{1,2} \subseteq \Lambda^{\alpha}_{1,2}$.

These, and other, basic facts can be found in [9] for the particular case $G = SU(2)$. The proof in the general case follows similar lines. A good general reference to Besov spaces is [1].

We are now in a position to state our result.

THEOREM. *Let d denote the dimension of G. Then $\Lambda^{d/2}_{1,2}$ is continuously embedded in $A(G)$.*

The proof will be broken up into a sequence of lemmas.

Remarks. (a) The classical Bernstein theorem for $G = \mathbf{T}$ ([3], 10·6·2) states that if

$$\omega_{\infty} f(t) = \sup_{0<s\leqslant t} \|f(x+s)-f(x)\|_{\infty} = O(t^{\frac{1}{2}+\epsilon})$$

from some $\epsilon > 0$, then $f \in A(\mathbf{T})$. It is implied by our theorem.

(b) Bernstein's theorem for the Euclidean space $\mathbf{R}^n$ is due to Herz ([6], theorem 1); for the Heisenberg group, it is due to Inglis [6]; for $G = SU(2)$ to Pini [7].

2. *Construction of an approximate identity on G*

We fix a maximal torus T in G, with Lie algebra $\mathfrak{t}$, of dimension r. Denote the dimension of G by d. For each $\rho > 0$, let $B(\rho)$ denote the open ball of radius ρ in $\mathfrak{t}$. If ρ is sufficiently small, then the exponential mapping is one-to-one from $B(\rho)$ on to an open subset of T.

In this section we shall construct a central approximate identity $\{\tilde{\Phi}_n\}$ on G whose form is well adapted to the problem in hand. Specifically, it will be such that we shall be able to estimate $\|f * \tilde{\Phi}_n - f * \tilde{\Phi}_{n-1}\|_2$ in terms of the kth-order modulus of continuity of f.

Recall that a continuous function $\tilde{\Psi}$ is central on G if $\tilde{\Psi}(gh) = \tilde{\Psi}(hg)$ for all $g, h \in G$. Equivalently, $\tilde{\Psi} * f = f * \tilde{\Psi}$ for all $f \in L^1(G)$. If the compact Lie group is connected, then $G = \bigcup_{h\in G} hTh^{-1}$ ([2], 21·7·7, 21·7·4). From this it follows that, if we begin with a continuous function $\tilde{\Psi}'$ on T, it can be extended to a continuous central function $\tilde{\Psi}$ on G (by making it constant on conjugacy classes of T) if and only if it satisfies the consistency condition $\tilde{\Psi}(w(t)) = \tilde{\Psi}(t)$ for all $t \in T$ and all inner automorphisms w of G that leave T stable. The group W of such transformations w is finite and can be identified with $N(T)/Z(T)$, where $N(T)$ and $Z(T)$ are respectively the normalizer and the centralizer of T; it is known as the *Weyl group*. See [2], 21·13·5. Each element of the Weyl group can be thought of as acting on the Lie algebra $\mathfrak{g}$, in which case it stabilizes $\mathfrak{t}$.

The estimates we are going to make involve some use of root systems, via the Weyl integration formula. Explicit use of these ideas appears only in the proof of Lemma 1; this means that any reader unfamiliar with root systems can simply accept Lemma 1 without proof and proceed to the remainder of the paper.

Fix a measure $dm_{G/T}$ on the space of left cosets of T in G that is left invariant under the action of G.

LEMMA 1. *There exists a continuous function $|\Delta^2|$ on $\mathfrak{t}$ having the following properties.*

(i) *$|\Delta^2|$ is Weyl-group-invariant.*

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(ii) *There exists $\rho > 0$ such that, if $h \in L^1(G)$ is supported in a sufficiently small neighbourhood of the identity, then*

$$\int_G h(g)\,dm(g) = \int_{G/T} dm_{G/T}(z) \int_{B(\rho)} h(z \exp y z^{-1})\,|\Delta^2(y)|\,dy \tag{3}$$

for a suitably normalized choice of Lebesgue measure on t.

(iii) *For each positive integer k, there exist constants A and B such that*

$$A(2^{-n})^{k(d-r)+r} \leqslant \int_{B(2^{-n})} \prod_{s=1}^{k} |\Delta(sy)|^2\,dy = B(2^{-n})^{k(d-r)+r} \tag{4}$$

for all sufficiently large n.

(iv) *For each positive integer k there is a constant C such that*

$$\int_{B(2^{-n})} \prod_{\substack{s=1 \\ s \neq j}}^{k} |\Delta(sy)|^4\,|\Delta(jy)|^2\,dy \leqslant C(2^{-n})^{(2k-1)(d-r)+r}$$

for all sufficiently large n.

Proof. Make a choice P^+ of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let $l = |P^+|$, $r = \operatorname{rank}(G) = \dim(T)$, $d = \dim(G)$. Then $d = 2l + r$.

(i) For each positive root α, let e^α be the function on T such that

$$e^\alpha(\exp_T y) = \exp(\alpha(y)), \tag{5}$$

$\exp_T$ denoting the exponential mapping from $\mathfrak{t}$ to T. The formula (5) makes sense since α is the differential of a character of T.

Define

$$|\tilde{\Delta}^2| = \prod_{\alpha \in P^+} |e^\alpha - 1|^2,$$

$$\begin{aligned} |\Delta^2|(y) &= |\tilde{\Delta}^2|(\exp_T y) \\ &= \prod_{\alpha \in P^+} |\exp(\alpha(y)) - 1|^2. \end{aligned}$$

The invariance of $|\tilde{\Delta}^2|$ and of $|\Delta^2|$ with respect to the Weyl group action follows from ([2], 21·11·6 and 21·11·8) for instance.

(ii) The Weyl integration formula ([2], 21·15·4) permits us to write

$$\int_G h(g)\,dm_G = \int_{G/T} dm_{G/T}(z) \int_T h(ztz^{-1})\,|\tilde{\Delta}(t)|^2\,dt \tag{6}$$

for an appropriate choice of normalized Haar measure on T, and arbitrary $h \in L^1(G)$. Now if ρ is sufficiently small,

$$\int_T \phi(t)\,dt = \int_{B(\rho)} \phi(\exp y)\,dy$$

if ϕ is integrable on T and supported in $\exp(B(\rho))$. Cf. [5], theorem 1·14. So (3) follows from (6).

(iii) Observe that

$$|\exp(\alpha(y)) - 1|^2 = 4 \sin^2 i\alpha(y)$$

and that there exist positive constants K_1 and K_2 such that

$$K_1 |\alpha(y)|^2 \leqslant \sin^2(i\alpha(y)) \leqslant K_2 |\alpha(y)|^2$$

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if $0 \leqslant |\alpha(y)| = \pi/2$. Hence, if n is large enough,

$$\int_{B(2^{-n})} \prod_{s=1}^{k} |\Delta(sy)|^2 \, dy$$

can be bounded from above and below by appropriate positive multiples of

$$\int_{B(2^{-n})} \prod_{\alpha \in P^+} \prod_{s=1}^{k} |\alpha(sy)|^2 \, dy = \int_0^{2^{-n}} \tau^{2kl+r-1} \, d\tau \int_{|\omega|=1} \prod_{\alpha \in P^+} \prod_{s=1}^{k} |\alpha(s\omega)|^2 \, d\omega$$
$$= C(2^{-n})^{2kl+r} = C(2^{-n})^{k(d-r)+r},$$

where C is the positive constant

$$\frac{(k!)^2}{(2kl+r)} \int_{|\omega|=1} \prod_{\alpha \in P^+} |\alpha(\omega)|^{2k} \, d\omega.$$

(The integrals over the unit sphere are with respect to surface measure.)

(iv) As in part (iii), this amounts to estimating

$$\int_0^{2^{-n}} \tau^{4l(k-1)+2l+r-1} \, d\tau \int_{|\omega|=1} \prod_{\alpha \in P^+} \left\{ \prod_{\substack{s=1 \\ s \neq j}}^{k} |d(s\omega)|^4 \, |\alpha(j\omega)|^2 \right\} d\omega,$$

which is a constant multiple of

$$(2^{-n})^{4l(k-1)+2l+r} = (2^{-n})^{(d-r)(2k-1)+r}.$$

Construction of $(\tilde{\Phi}_n)$. For each integer j such that $1 \leqslant j \leqslant k$, k being a fixed integer with $k > d/2$, let

$$g_j(y) = \prod_{\substack{s=1 \\ s \neq j}}^{k} |\Delta(sy/j)|^2. \tag{7}$$

Observe that

$$g_j(jy) \, |\Delta(jy)|^2 = \prod_{s=1}^{k} |\Delta(sy)|^2.$$

For each sufficiently large positive integer n, let Φ_n be the function on t defined by the formula

$$\Phi_n(y) = \frac{1}{c_n} \sum_{j=1}^{k} (-1)^{k+j} \binom{k}{j} j^{-r} \chi_{B(j2^{-n})}(y) \, g_j(y), \tag{8}$$

$\chi_{B(j2^{-n})}$ denoting the characteristic function of the set in question. The constant c_n is to be chosen in accordance with Lemma 2.

Define the central function $\tilde{\Phi}_n$ on G, for all sufficiently large n, as follows:

$$\tilde{\Phi}_n(x) = \begin{cases} \Phi_n(y) & \text{if} \quad x = z(\exp y) z^{-1}, \; z \in G, \; y \in B(k2^{-n}), \\ 0 & \text{if} \quad x \notin \bigcup_{z \in G} z \exp(B(k2^{-n})) z^{-1}. \end{cases} \tag{9}$$

The formula (9) makes sense since Φ_n is Weyl-group-invariant.

In the statement of the next lemma, saying that $\{\tilde{\Phi}_n\}$ is an approximate identity means that

(*a*) $\tilde{\Phi}_n$ is integrable for each n, and

$$\sup_n \|\tilde{\Phi}_n\|_1 < +\infty;$$

(*b*) $\int_G \tilde{\Phi}_n \, dm_G = 1$ for all n;

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(c) for each fixed neighbourhood U of e in G,

$$\lim_{n\to+\infty}\int_{G\setminus U}|\tilde{\Phi}_n(g)|\,dm(g)=0.$$

LEMMA 2. *Let c_n be chosen so that*

$$\int_G \tilde{\Phi}_n\,dx=1.$$

Then (i) *there exist positive constants A_1 and A_2 such that*

$$A_1(2^{-n})^{k(d-r)+r}\leqslant|c_n|\leqslant A_2(2^{-n})^{k(d-r)+r}$$

for all sufficiently large n;

(ii) *the sequence $\{\tilde{\Phi}_n\}$ is an approximate identity on G.*

Proof. (i) It follows from Lemma 1 (ii) that

$$\begin{aligned}c_n&=\sum_{j=1}^{k}(-1)^{k+j}\binom{k}{j}\int_{G/T}dm_{G/T}\int_{B(2^{-n})}\prod_{s=1}^{k}|\Delta(sy)|^2\,dy\\&=(-1)^{k+1}\int_{G/T}dm_{G/T}\int_{B(2^{-n})}\prod_{s=1}^{k}|\Delta(sy)|^2\,dy.\end{aligned}$$

Part (i) now follows immediately from Lemma 1 (iii).

(ii) A similar proof to that just given shows that

$$\sup_n\|\tilde{\Phi}_n\|_1<+\infty.$$

Also, given any neighbourhood U of e in G, it is simple to see that the $\tilde{\Phi}_n$ are supported in V from a certain stage on. So $\{\tilde{\Phi}_n\}$ satisfies properties (a)–(c) of an approximate identity.

The next lemma goes to show how the particular form chosen for our approximate identity leads to useful estimates for $\|f*\tilde{\Phi}_n-f*\tilde{\Phi}_{n-1}\|_2$ in terms of the kth-order modulus of continuity.

LEMMA 3. (i) *For all sufficiently large n,*

$$(f*\tilde{\Phi}_n+(-1)^k f)(x)=c_n^{-1}\int_{G/T}dm_{G/T}(z)\int_{B(2^{-n})}\Delta^k_{Ad(z)y}f(x)\prod_{s=1}^{k}|\Delta(sy)|^2\,dy.$$

(ii) *There is a constant C such that, for all sufficiently large n,*

$$\|f*\tilde{\Phi}_n-f*\tilde{\Phi}_{n-1}\|_2\leqslant C\omega_k(2^{-n},f,L^2).$$

Proof. (i) Since G is compact, and $\tilde{\Phi}_n(u^{-1})=\tilde{\Phi}_n(u)$, we have, by Lemma 1 (ii),

$$\begin{aligned}&f*\tilde{\Phi}_n(x)\\&\quad=\int_G f(xu)\,\tilde{\Phi}_n(u)\,du\\&\quad=c_n^{-1}\sum_{j=1}^{k}(-1)^{k+j}\binom{k}{j}\int_{G/T}dm_{G/T}(z)\int_{B(2^{-n})}j^{-r}f(x\exp(Ad(z)\,y))\,g_j(y)\,|\Delta(y)|^2\,dy.\end{aligned}$$

The change of variables $y\to jy$ followed by an integration in polar coordinates gives (i).

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(ii) It follows from (i) that

$$\|f*\tilde{\Phi}_n+(-1)^k f\|_2 \leqslant Dc_n^{-1}\int_{B(2^{-n})}\omega_k(2^{-n},f,L^2)\prod_{s=1}^{k}|\Delta(sy)|^2\,dy$$
$$\leqslant D'\omega_k(2^{-n},f,L^2) \tag{10}$$

by Lemma 2 (i) and Lemma 1 (iii), for suitable constants D, D'. Using (10), we see that

$$\|f*\tilde{\Phi}_n-f*\tilde{\Phi}_{n-1}\|_2 \leqslant D'\{\omega_k(2^{-n},f,L^2)+\omega_k(2^{-n+1},f,L^2)\}$$
$$\leqslant 3D'\omega_k(2^{-n},f,L^2).$$

LEMMA 4. *The series*

$$\sum_{n_0+1}^{\infty} f*P_n*Q_n,$$

where $P_n=\tilde{\Phi}_n-\tilde{\Phi}_{n-1}$ *and* $Q_n=\tilde{\Phi}_n+\tilde{\Phi}_{n-1}$, *and* n_0 *is sufficiently large for all the terms to be defined, converges in* L^2 *to* $f-f*\tilde{\Phi}_{n_0}*\tilde{\Phi}_{n_0}$.

Proof. If $N > n_0+1$,

$$\sum_{n_0+1}^{N} f*(\tilde{\Phi}_n-\tilde{\Phi}_{n-1})*(\tilde{\Phi}_n+\tilde{\Phi}_{n-1})=\sum_{n_0+1}^{N}\{f*\tilde{\Phi}_n*\tilde{\Phi}_n-f*\tilde{\Phi}_{n-1}*\tilde{\Phi}_{n-1}\}$$

since the $\tilde{\Phi}_n$ are central. The telescopic series on the right clearly converges to

$$\lim_{N\to\infty}\{f*\tilde{\Phi}_N*\tilde{\Phi}_N-f*\tilde{\Phi}_{n_0}*\tilde{\Phi}_{n_0}\}=f-f*\tilde{\Phi}_{n_0}*\tilde{\Phi}_{n_0},$$

which limit exists in L^2 by simple approximate-identity arguments.

LEMMA 5. *There is a constant* B *such that*

$$\sum_{n_0+1}^{\infty}\|f*P_n\|_2\,\|Q_n\|_2 \leqslant BN_{k,\,d/2}(f)$$

for all $f\in\Lambda_{1,2}^{d/2}$.

Proof. Let $(\chi_{B(j2^{-n})}g_j)^{\sim}$ be defined in the same way as $\tilde{\Phi}_n$ is defined in relation to Φ_n. Then

$$\|\tilde{\Phi}_n\|_2 \leqslant c_n^{-1}\sum_{j=1}^{k!}\binom{k}{j}j^{-r}\,\|(\chi_{B(j2^{-n})}g_j)^{\sim}\|_2. \tag{11}$$

But

$$\|(\chi_{B(j2^{-n})}g_j)^{\sim}\|_2^2=\int_{G/T}dm_{G/T}\int_{B(j2^{-n})}\prod_{\substack{s=1\\ s\neq j}}^{k}\left|\Delta\left(\frac{sy}{j}\right)\right|^4|\Delta(y)|^2\,dy$$
$$=j^r\int_{G/T}dm_{G/T}\int_{B(2^{-n})}\prod_{\substack{s=1\\ s\neq j}}^{k}|\Delta(su)|^4\,|\Delta(ju)|^2\,du, \tag{12}$$
$$\leqslant C_3(2^{-n})^{(2k-1)(d-r)+r}$$

for a suitable choice of constant C_3, by Lemma 1 (iv). It follows from (11) and (12) that

$$\|\tilde{\Phi}_n\|_2 \leqslant C_4c_n^{-1}(2^{-n})^{\{(2k-1)(d-r)+r\}/2}$$
$$\leqslant C_5(2^{-n})^{-d/2},$$

by Lemma 2 (i). So

$$\|Q_n\|_2 \leqslant C_6(2^{-n})^{-d/2}. \tag{13}$$

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It follows from Lemma 3 (ii) and (13) that, for a suitable sequence of constants,

$$\sum_{n_0+1}^{\infty} \|f * P_n\|_2 \, \|Q_n\|_2 \leqslant C_7 \sum_{n_0+1}^{\infty} (2^{-n})^{-d/2} \omega_k(2^{-n}, f, L^2)$$
$$\leqslant C_8 \int_0^1 \frac{\omega_k(t, f, L^2)}{t^{1+d/2}} \, dt.$$

Conclusion of proof.

$$\|f\|_A \leqslant \|f * \tilde{\Phi}_{n_0+1} * \tilde{\Phi}_{n_0+1}\|_A + \sum_{n_0+1}^{\infty} \|f * P_n\|_2 \, \|Q_n\|_2$$
$$\leqslant C \left\{ \|f\|_2 + \int_0^1 \frac{\omega_k(t, f, L^2)}{t^{1+d/2}} \, dt \right\},$$

where C depends on the L^2-norm of $\tilde{\Phi}_{n_0+1}$.

3. *The index $d/2$ is sharp*

The Bernstein theorem shows that, if $f \in \Lambda^{d/2}_{1,2}$, then f is equal amost everywhere to a function in $A(G)$. We now prove, by adapting an example of Herz ([6], p. 290), that one cannot decrease the index $d/2$. An analogous construction for $G = SU(2)$ appears in [9].

THEOREM. *If $0 < \alpha < d/2$, there exists a function $f \in \Lambda^{\alpha}_{1,2}(G)$ such that f is not equal almost everywhere to a function in $A(G)$.*

Proof. Choose a function f in $C^{\infty}(\mathfrak{g})$ such that:

(i) $f(0) = 1$;

(ii) $\operatorname{supp} f \subseteq \{y \in \mathfrak{g} : \|y\| < h\}$ where h is so small that exp is a diffeomorphism on $B(h)$;

(iii) $f(y) \geqslant 0$ for all $y \in \mathfrak{g}$.

Define the function F on G by specifying that

$$F(x) = \begin{cases} f(y) & \text{when} \quad x = \exp y, \; y \in B(h), \\ 0 & \text{otherwise.} \end{cases}$$

Form the family of dilates $\{f^m\}_{m=1}^{\infty}$ and the corresponding functions $\{F^m\}$ on G, where

$$f^m(y) = f(my).$$

Observe that

$$\operatorname{supp} f^m \subseteq B\left(\frac{h}{m}\right).$$

We now proceed to estimate $N_{k,\alpha}(f^m)$; in doing so, we shall use the following basic properties of the difference operators.

(a) For each $V \in \mathfrak{g}$ and C^{∞} function F on G, let VF denote the derivative of F in the direction V:

$$VF(x) = \frac{d}{dt} F(x \exp tV)|_{t=0}.$$

Then

$$\Delta^k_{sV} F(x) = \int_0^s dt_1 \ldots \int_0^s dt_k \, V^k F(x \exp(t_1 + \ldots + t_k) \, V). \tag{14}$$

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(*b*) There exist C^∞ functions $c_1, \ldots, c_d$ on $\mathfrak{g}$ such that

$$VF(x) = \sum_{j=1}^{d} c_j(y) \frac{\partial f}{\partial y_j}(y), \quad y \in B(h). \tag{15}$$

It follows by induction from (15) that there exist C^∞ functions $\{c_\beta\}_{1 \leqslant |\beta| \leqslant k}$ on $\mathfrak{g}$ such that

$$V^k F(x) = \sum_{1 \leqslant |\beta| \leqslant k} c_\beta(y) D^\beta f(y), \quad y \in B(h), \tag{16}$$

where

$$D^\beta f = \frac{\partial^\beta f}{\partial y_1^{\beta_1} \ldots \partial y_d^{\beta_d}}.$$

The formulas (14)–(16) are in particular valid for each f^m.

(*c*) As a consequence of (*a*) and (*b*), there is a constant C_1 such that

$$\omega_k(t, F^m, L^2) \leqslant C_1 t^k \sup_{1 \leqslant |\beta| \leqslant k} \|D^\beta f^m\|_2. \tag{17}$$

(*d*) In any event,

$$\omega_k(t, F, L^2) \leqslant 2^k \|F\|_2. \tag{18}$$

Now, by definition of the Haar measure on G, there is a C^∞ function J on $\mathfrak{g}$ such that

$$\begin{aligned} \|D^\beta f^m\|_2 &= \left(\int_{\mathfrak{g}} |D^\beta f^m(y)|^2 J(y)\, dy \right)^{\frac{1}{2}} \\ &= m^{|\beta|} \left(\int_{\mathfrak{g}} |D^\beta f(my)|^2 J(y)\, dy \right)^{\frac{1}{2}} \\ &= m^{|\beta| - d/2} \left(\int_{\mathfrak{g}} |D^\beta f(y)|^2 J\left(\frac{y}{m}\right) dy \right)^{\frac{1}{2}} \\ &\leqslant C_2 m^{|\beta| - d/2} \end{aligned} \tag{19}$$

for some constant C_2 and all m. The equality (19) applies also when $|\beta| = 0$. It therefore follows from (17)–(19) that

$$\begin{aligned} N_{k,\alpha}(F^m) &= \|F^m\|_2 + \int_0^1 \frac{\omega_k(t, F^m, L^2)}{t^{\alpha+1}}\, dt \\ &\leqslant C_2 m^{-d/2} + \int_0^{1/m} + \int_{1/m}^1 \frac{\omega_k(t, F^m, L^2)}{t^{\alpha+1}}\, dt \\ &\leqslant C_3 \{ m^{-d/2} + m^{k-d/2} m^{\alpha-k} + m^{-d/2}(1 - m^\alpha) \} \\ &= O(1) \quad \text{as} \quad m \to \infty. \end{aligned}$$

Using a well-known property of null sequences, we can choose a sequence $\{a_m\}$ of real numbers such that $a_m \geqslant 0$,

$$\sum_1^\infty a_m = +\infty$$

and

$$\sum_1^\infty a_m N_{k,\alpha}(F^m) < +\infty.$$

The partial sums of the series $\sum_1^\infty a_m F^m$ converge in $\Lambda^\alpha_{1,2}(G)$ to a function H, say. They also converge monotonically, uniformly on every compact set, and to $+\infty$ at e. The sum function is evidently continuous away from e and unbounded. Therefore H cannot be equal almost everywhere to a function in $A(G)$.

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