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RESEARCH ARTICLE

Darboux formulae for linear hyperbolic equations in the discrete case

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Abstract

In the second half of the 19th century, Darboux obtained determinant formulae that provide the general solution for a linear hyperbolic second-order PDE with the finite Laplace series. These formulae played an important role in his study of the theory of surfaces and, in particular, in the theory of conjugate nets. During the last three decades, discrete analogues of conjugate nets (Q-nets) were actively studied. Laplace series can be defined also for hyperbolic difference operators. We prove discrete analogues of Darboux formulae for discrete and semi-discrete hyperbolic operators with finite Laplace series.

1. Introduction

Linear hyperbolic second-order differential operators had been extensively studied by Darboux [1], Goursat [2] and their followers at the end of the 19th century in relation to classical differential geometry: parameterisation of a hypersurface in 3-dimensional Euclidean space such that its second fundamental form is diagonal is called a *conjugate net* in the theory of surfaces; it is known to satisfy a hyperbolic second-order partial differential equation. Conjugate nets are considered in spaces of higher dimension both in affine and in projective settings [3].

Various problems in classical differential geometry can be reformulated in terms of solutions of linear hyperbolic equations. Therefore, Darboux, Goursat and their followers were interested in developing techniques that allowed them to obtain exact solutions of such equations in explicit form. Some of them are based on the so-called *Laplace cascade method* that allows expressing solutions of one hyperbolic equation in terms of solutions of another one whenever they are related by a finite sequence of *Laplace transformations*. If at least one of the *Laplace invariants* for a linear hyperbolic operator $\mathcal{L}$ is zero, then the general solution of the equation $\mathcal{L}\psi = 0$ can be found explicitly. Combining this with the cascade method, one can obtain the general solution for hyperbolic equations that can be converted to the ones with a zero Laplace invariant by a finite number of Laplace transformations. There are two different Laplace transformations that can be applied to any hyperbolic operator with non-zero Laplace invariants. If the application of a finite number of each of them leads to an operator with a zero Laplace invariant, then the initial operator is said to have a *finite Laplace series*. There are several ways to express the general solution of a hyperbolic PDE with the finite Laplace series. Darboux [1] gives an elegant way to represent the general solution of all hyperbolic equations having finite Laplace series in terms of determinants that contain an appropriate number of arbitrary functions of one variable (*the Darboux determinant formula*). From a geometrical point of view, the Laplace transformation is a transformation of a surface with a conjugate coordinate net. Thus, the Darboux formula gives a nice representation of all conjugate nets with finite Laplace series in a convenient form. The Darboux formulae, the Laplace cascade method and related things in the theory of integrable systems are discussed from a modern perspective in [4], but unfortunately, this book is available only in Russian.

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During the last three decades, classical differential geometry has been discretised by Bobenko, Suris, Doliwa, Schief and many others. Exactly as in the continuous case, a linear hyperbolic second-order difference operator defines a *Q-net*, which is a discrete analogue of a conjugate net [3]. Laplace invariants are also defined in the discrete case [5–7]. Discrete Laplace transformations provide transformations of Q-nets [8]. Semi-discrete analogues of conjugate nets, where one of the independent variables is continuous and the other one is discrete, had also been considered by specialists in discrete differential geometry [9]. Although the discrete case is rather similar to the continuous one, discrete analogues of the Darboux formulae seem to be missing in the existing literature. This paper is aimed at filling this gap.

Classical Laplace transformations play an important role not only in the theory of surfaces but also in famous integrable systems such as the two-dimensional Toda lattice and the Sine-Gordon equation. Laplace invariants of hyperbolic operators in a Laplace series satisfy nonlinear hyperbolic equations that are equivalent to the Toda system; finite Laplace series correspond to the so-called *open Toda lattice* (also known as the *A-series Toda lattice*), and the Sine-Gordon equation corresponds to the simplest periodic closure of the Laplace series. Semi-discrete and entirely discrete versions of the two-dimensional Toda lattice are related to (semi)-discrete Laplace transformations in the same way as in the continuous case [6] and are known to be Darboux integrable [10].

In the continuous case, symmetry reductions of the open Toda lattice lead to exponential systems corresponding to the Cartan matrices of B- and C-type [11, 12]. From the point of view of the Laplace transformations, these reductions correspond to the Laplace series of the Goursat equation and the Moutard equation, respectively [1, 4]. From a geometrical point of view, B-type reduction corresponds to a special type of conjugate net known as *Königs nets*. Darboux also provides a procedure to describe the general solutions of these equations in the case when their Laplace series is finite [1]. In the discrete case, the situation with symmetry reduction is more complicated: the C-series Toda lattice is known to be a reduction on the A-series lattice [13, 14], but for B-series systems this is not clear. Moreover, it is not known whether (semi)-discrete B-series Toda systems introduced in [15, 16] within the frame of a general approach to discretisation of exponential systems are related to the discrete version of the Moutard equation that is studied in [17–20] and to discrete Königs nets [3]. We leave these questions, as well as obtaining the discrete analogue of the Darboux procedure for finding general solutions of the reduced systems in explicit form, for future research.

In Section 2, we review the basic notions such as the Laplace invariants and the Laplace transformations. In Section 3, we discuss the Darboux formulae for the general solution of a hyperbolic equation with a finite Laplace series in the continuous case. In Section 4, we recall the notions of the Laplace invariants and the Laplace transformations in the discrete case. Section 5 contains the discrete analogues of the Darboux formulae and their detailed proof. In Section 6, we provide semi-discrete versions of these formulae.

2. Laplace invariants and Laplace series

In this section, we review the main notions and facts concerning the Laplace invariants, the Laplace transformations and the Laplace series in the continuous case that we need to prove discrete analogues of the Darboux formulae. All propositions, theorems and statements of this section are known to experts in the area. Detailed proofs can be found in the original book by Darboux [1], which is in French, or in the classical book [21]. The cascade method had also been discussed in a well-known book [22] in a bit of an old-fashioned way and in papers [23–25] from a modern perspective. Laplace invariants for differential operators of a more general form had been studied in [26–29].

Consider the linear hyperbolic differential operator

$$\mathcal{L} = \partial_x \partial_y + a\partial_x + b\partial_y + c, \tag{2.1}$$

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where a, b and c are functions of the variables x and y. The functions $k = c - ab - a_x$ and $h = c - ab - b_y$ are called *the Laplace invariants* of (2.1). They are invariant under gauge transformations $\mathcal{L} \mapsto \tilde{\mathcal{L}} = \omega^{-1}\mathcal{L}\omega$, where $\omega = \omega(x, y)$ is an arbitrary function and control factorisability of such an operator: since

$$\mathcal{L} = (\partial_x + b)(\partial_y + a) + k = (\partial_y + a)(\partial_x + b) + h,$$

the operator $\mathcal{L}$ is factorisable if and only if at least one of its Laplace invariants is zero. Straightforward calculation shows that if $h = 0$, then the general solution of $\mathcal{L}\psi = 0$ has the form

$$\psi(x, y) = \left(\int \left(X(x) e^{\int b(x,y)dx - \int a(x,y)dy} \right) dx + Y(y) \right) e^{-\int b(x,y)dx} \tag{2.2}$$

and if $k = 0$, then the general solution has the form

$$\psi(x, y) = \left(\int \left(Y(y) e^{\int a(x,y)dy - \int b(x,y)dx} \right) dy + X(x) \right) e^{-\int a(x,y)dy}, \tag{2.3}$$

where X and Y are arbitrary functions of one variable.

Two linear hyperbolic differential operators

$$\mathcal{L} = \partial_x\partial_y + a\partial_x + b\partial_y + c \quad \text{and} \quad \hat{\mathcal{L}} = \partial_x\partial_y + \hat{a}\partial_x + \hat{b}\partial_y + \hat{c}$$

are related by a *Darboux–Laplace transformation* if there exist first-order operators $\mathcal{D} = \alpha\partial_x + \beta\partial_y + \gamma$ and $\hat{\mathcal{D}} = \hat{\alpha}\partial_x + \hat{\beta}\partial_y + \hat{\gamma}$ such that the following relation is satisfied:

$$\hat{\mathcal{L}} \circ \mathcal{D} = \hat{\mathcal{D}} \circ \mathcal{L}. \tag{2.4}$$

Two particular cases $\mathcal{D} = \partial_x + b$ and $\mathcal{D} = \partial_y + a$ correspond to classical *Laplace transformations*, which were widely used by Darboux and Goursat in their research on the theory of surfaces.[1] In both cases, the coefficients of operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{D}}$ are uniquely defined in terms of the coefficients of $\mathcal{L}$. The main property of Darboux-Laplace transformations is that the transformation operator $\mathcal{D}$ maps the kernel of the operator $\mathcal{L}$ into the kernel of $\hat{\mathcal{L}}$: if $\psi \in \mathrm{Ker}\mathcal{L}$, then relation (2.4) implies that $\mathcal{D}\psi \in \mathrm{Ker}\hat{\mathcal{L}}$. There exists a wide class of Darboux–Laplace transformations for linear hyperbolic operators different from the classical Laplace transformations (see [30, 31]), but we will consider only classical ones throughout this paper.

Proposition 1. *Laplace transformation $\mathcal{L} \mapsto \hat{\mathcal{L}}$ defined by the transformation operator $\mathcal{D} = \partial_x + b$ is invertible if $h \neq 0$. The inverse transformation $\hat{\mathcal{L}} \mapsto \mathcal{L}$ is given by the operator $-h^{-1}(\partial_y + a)$. Similarly, the transformation defined by the operator $\partial_y + a$ is invertible if $k \neq 0$, and the inverse transformation is given by $-k^{-1}(\partial_x + b)$.*

Operator relation (2.4) is equivalent to a system of PDEs for the coefficients of all four operators. Proof of Proposition 1 is based on the fact that operators $\hat{\mathcal{L}}$, $\mathcal{L}$ and $-h^{-1}(\partial_y + a)$ provide a solution to this system for the suitably chosen fourth operator.

Consider a hyperbolic operator

$$\mathcal{L}_0 = \partial_x\partial_y + a_0\partial_x + b_0\partial_y + c_0$$

and apply the Laplace transformation $\mathcal{D}_0 = \partial_x + b_0$:

$$\mathcal{L}_1\mathcal{D}_0 = \mathcal{D}_1\mathcal{L}_0 \quad \text{where} \quad \mathcal{L}_1 = \partial_x\partial_y + a_1\partial_x + b_1\partial_y + c_1.$$

Apply the Laplace transformation $\mathcal{D}_1 = \partial_x + b_1$ to the operator $\mathcal{L}_1$, etc. This procedure produces the sequence of hyperbolic operators starting from $\mathcal{L}_0$ and such that any two neighbouring operators are related by a Laplace transformation. The second Laplace transformation $\mathcal{D}_0' = \partial_y + a_0$ allows to continue

[1] These Laplace transformations have nothing in common with the Laplace integral transform.

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the sequence to the left:

$$\mathcal{L}_{-1}\mathcal{D}'_0 = \mathcal{D}'_1\mathcal{L}_0,$$

etc. This leads to the sequence

$$\ldots \xleftarrow{\mathcal{D}'_j} \mathcal{L}_{-j} \xleftarrow{\mathcal{D}'_{j-1}} \mathcal{L}_{-j+1} \xleftarrow{\mathcal{D}'_{j-2}} \ldots \xleftarrow{\mathcal{D}'_1} \mathcal{L}_{-1} \xleftarrow{\mathcal{D}'_0} \mathcal{L}_0 \xrightarrow{\mathcal{D}_0} \mathcal{L}_1 \xrightarrow{\mathcal{D}_1} \ldots \xrightarrow{\mathcal{D}_{j-1}} \mathcal{L}_j \xrightarrow{\mathcal{D}_j} \mathcal{L}_{j+1} \xrightarrow{\mathcal{D}_{j+1}} \ldots, \tag{2.5}$$

which is called *the Laplace series* of the operator $\mathcal{L}_0$. If some operator $\mathcal{L}_r$ in its positive part has zero h-invariant, then the sequence terminates at $\mathcal{L}_r$ since this operator is factorisable. Similarly, if $\mathcal{L}_{-s}$ has zero k-invariant for some $s \geqslant 0$, then sequence (2.5) terminates from the left. The Laplace series is said to be finite if it terminates from both sides.

Laplace invariants of any three consecutive operators in the Laplace series satisfy the relations

$$k_{j+1} = h_j, \quad h_{j+1} = 2h_j - h_{j-1} + (\ln h_j)_{xy}.$$

The second of these relations is one of the forms of the *two-dimensional Toda lattice*.

The theorem below generalises formulae (2.2) (2.3). It follows from the fact that each operator in the Laplace series is obtained from the previous one by a Laplace transformation, which maps the kernel of the initial operator into the kernel of the next one.

Theorem 1. *If the Laplace series of $\mathcal{L}_0$ terminates in the term $\mathcal{L}_r$ for some $r \in \mathbb{N}$, then the general solution of the equation $\mathcal{L}_0\psi = 0$ is given by*

$$\psi_0(x,y) = -\frac{1}{h_0}(\partial_y + a_0)\left(-\frac{1}{h_1}(\partial_y + a_1)\left(\ldots\left(-\frac{1}{h_{r-1}}(\partial_y + a_{r-1})\right)\ldots\right)\right)\psi_r(x,y), \tag{2.6}$$

where ψ_r has the form

$$\psi_r(x,y) = \left(\int \left(X(x)e^{\int b_r(x,y)dx - \int a_r(x,y)dy}\right)dx + Y(y)\right)e^{-\int b_r(x,y)dx} \tag{2.7}$$

and X and Y are arbitrary functions of one variable. If the Laplace series of $\mathcal{L}_0$ terminates in the term $\mathcal{L}_{-s}$ for some $s \in \mathbb{N}$, then the general solution is given by

$$\psi_0(x,y) = -\frac{1}{k_0}(\partial_x + b_0)\left(-\frac{1}{k_{-1}}(\partial_x + b_{-1})\left(\ldots\left(-\frac{1}{k_{-s+1}}(\partial_x + b_{-s+1})\right)\ldots\right)\right)\psi_{-s}(x,y), \tag{2.8}$$

where ψ_{-s} is defined as follows:

$$\psi_{-s}(x,y) = \left(\int \left(Y(y)e^{\int a_{-s}(x,y)dy - \int b_{-s}(x,y)dx}\right)dy + X(x)\right)e^{-\int a_{-s}(x,y)dy}.$$

Remark 1. *The possibility to obtain the general solution in the form (2.6) or (2.8) is a consequence of the fact that an operator with zero Laplace invariant can be factorised. The method of consecutive application of Laplace transformations to a hyperbolic operator with the Laplace series that terminates at least in one direction is called the Laplace cascade method.*

Remark 2. *We define Laplace transformations in an algebraic way as mappings between hyperbolic operators. Originally, Darboux introduced such transformations in geometrical terms in his study of line congruences; see [1]. Within the frame of this approach, any conjugate net (i.e., a parameterised two-dimensional surface $\mathbf{r}(u,v)$ in three-dimensional Euclidean space such that the families of coordinate lines are conjugate to each other) admits two different transformations defined in the following way. Fix a coordinate line $u = \text{const}$ and consider all coordinate lines of the other family that intersect it. They are known to form a developable surface. Edges of regression of such developable surfaces corresponding to different coordinate lines $u = \text{const}$ form a new surface, which appears to be the Laplace image of the initial one. The second transformation is defined in a similar way using the other family of coordinate lines. This construction is described in detail also in [32], but in a rather archaic way, or very briefly*

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in [8]. We will not use the geometrical nature of the Laplace transformations; we only highlight their significance in classical differential geometry. Note also that the mixed derivative $\mathbf{r}(u,v)_{uv}$ *is orthogonal to the normal vector to the surface for any conjugate net* $\mathbf{r}(u,v)$*. Hence the relation*

$$\mathbf{r}_{uv} + a\mathbf{r}_u + b\mathbf{r}_v = 0$$

holds for some scalar functions $a = a(u,v)$ *and* $b = b(u,v)$*. Therefore, the components of the vector* $\mathbf{r}(u,v)$ *that parameterises the surface satisfy a linear hyperbolic equation. In a projective setting, it takes the form*

$$\mathbf{r}_{uv} + a\mathbf{r}_u + b\mathbf{r}_v + c\mathbf{r} = 0.$$

This relates our algebraic construction with the original geometric approach developed by Darboux.

3. Darboux formulae for general solution

Formulae (2.6) and (2.8) from the previous section allow one to obtain the general solution of a particular hyperbolic equation in explicit form given that its Laplace series is finite at least in one direction. But on the one hand, they include the integration of an arbitrary function, and on the other hand, they do not provide a suitable description of all hyperbolic equations with finite Laplace series, which would be interesting from a geometrical point of view. In this section, we review the Darboux formulae from [1, 21] that give the general solution in a more suitable form.

Theorem 2. *If the Laplace series of a hyperbolic operator* $\mathcal{L}_0$ *is finite,* $h_r = k_{-s} = 0$ *for some* $r, s \geqslant 0$*, then there exist coefficients* $A_i(x,y)$ *and* $B_j(x,y)$ *such that the general solution of the equation* $\mathcal{L}_0\psi = 0$ *is given by*

$$\psi(x,y) = A_0X + A_1X' + \ldots + A_sX^{(s)} + B_0Y + B_1Y' + \ldots + B_rY^{(r)}, \tag{3.1}$$

where $X(x)$ *and* $Y(y)$ *are arbitrary functions of one variable (primes and superscripts denote derivatives).*

If a hyperbolic equation $\mathcal{L}_0\psi = 0$ *admits a solution of the form (3.1) for some* $A_i(x,y)$ *and* $B_j(x,y)$*, then its Laplace series is finite:* $h_r = k_{-s} = 0$*.*

Remark 3. *Not every expression of the form (3.1) defines the general solution for some hyperbolic equation with the finite Laplace series since there are restrictions on the coefficients* A_i*,* B_j*. Indeed, consider the simplest case of an operator with zero Laplace invariants* $h_0 = k_0 = 0$*. Hence, formula (3.1) takes the form* $\psi = AX + BY$*. Differentiate it with respect to x and y and substitute into the initial hyperbolic equation. Since X and Y are arbitrary, this yields*

$$a_0 = -(\ln A)_y, \quad b_0 = -(\ln B)_x, \quad (\ln A)_{xy} = (\ln B)_{xy}. \tag{3.2}$$

The third relation shows that functions A and B could not be chosen at random.

A complete description of all restrictions that should be imposed on the coefficients of (3.1) is given by the *Darboux determinant formula*.

Theorem 3. *Let* $\mathcal{L}_0$ *be a hyperbolic operator with the finite Laplace series, where* $h_r = k_{-s} = 0$ *for some* $r, s \geqslant 0$*. Then there exists a function* $\omega = \omega(x,y)$ *and functions of one variable*

$$\xi_1(x), \xi_2(x), \ldots, \xi_{r+s+1}(x), \eta_1(y), \eta_2(y), \ldots, \eta_{r+s+1}(y) \tag{3.3}$$

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such that the general solution of $\mathcal{L}_0\psi = 0$ *is given by*

$$\psi(x,y) = \omega(x,y)\cdot \begin{vmatrix} X & X' & \dots & X^{(s)} & Y & Y' & \dots & Y^{(r)} \\ \xi_1 & \xi_1' & \dots & \xi_1^{(s)} & \eta_1 & \eta_1' & \dots & \eta_1^{(r)} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \xi_{r+s+1} & \xi_{r+s+1}' & \dots & \xi_{r+s+1}^{(s)} & \eta_{r+s+1} & \eta_{r+s+1}' & \dots & \eta_{r+s+1}^{(r)} \end{vmatrix} \tag{3.4}$$

where $X(x)$ *and* $Y(y)$ *are arbitrary functions of one variable.*

For any two families of linearly independent functions of one variable (3.3) and for any function $\omega = \omega(x,y)$ *there exists a hyperbolic differential operator* $\mathcal{L}_0$ *with the finite Laplace series that terminates at the terms* r *and* $-s$*, such that formula (3.4) provides the general solution of* $\mathcal{L}_0\psi = 0$*.*

Example 1. *In the simplest case* $h_0 = k_0 = 0$ *formula (3.4) takes the form*

$$\psi(x,y) = \omega(x,y)\,(\eta(y)X(x) - \xi(x)Y(y))\,.$$

This means that for any choice of ξ*,* η *and* ω *the function* ψ *defines the general solution of the corresponding hyperbolic equation with vanishing Laplace invariants. Note that in this case* $a_0 = -(\ln(\omega\eta))_y$*,* $b_0 = -(\ln(\omega\xi))_x$*, and the compatibility condition from (3.2) is satisfied since* ξ *and* η *are functions of one variable. For simplicity, we may set* $\omega = 1$*: this gives the general solution of a gauge-equivalent hyperbolic equation with zero Laplace invariants.*

Remark 4. *Formula (3.1) describes the general solution of any hyperbolic equation with a finite Laplace series. The Darboux determinant formula (3.4) allows us to construct solutions of hyperbolic equations with finite Laplace series of prescribed length: any choice of functions (3.3),* $X(x)$*,* $Y(y)$ *and* $\omega(x,y)$ *provides a particular solution* ψ *of a certain hyperbolic equation* $\mathcal{L}\psi = 0$ *with a terminating Laplace series. From a geometrical point of view, determinant formula (3.4) provides a large stock of parameterised surfaces with conjugate coordinate nets and with finite Laplace series in explicit form.*

Remark 5. *It follows from (3.4) that if functions (3.3) are linearly independent, then* $\psi(x,y) \equiv 0$ *if and only if pairs*

$$(X,Y),\ (\xi_1,\eta_1),\dots,(\xi_{r+s+1},\eta_{r+s+1})$$

are linearly independent with constant coefficients (see [1]). This provides a procedure to construct non-zero solutions of hyperbolic operators with finite Laplace series.

Example 2. *An important particular case of conjugate nets is given by the so-called Königs nets: algebraically they are characterised by an additional constraint that the Laplace invariants of the corresponding hyperbolic operator are equal (see [3]). Any operator with equal Laplace invariants is gauge-equivalent to an operator of the form* $\partial_x\partial_y + c$*, so Königs nets are defined by solutions of the Moutard equation*

$$\psi_{xy} + c\psi = 0.$$

It follows that the Laplace series is symmetric for any Königs net. Darboux formula (3.4) provides a large stock of Königs nets with finite Laplace series: in this case one has to set $\xi_i = \eta_i$ *for all* $i = 1,2,\dots,r = s$*. In particular, this allows us to obtain explicit solutions for the corresponding Moutard equation. For example, if* $\xi_1 = 1$*,* $\xi_2 = x$*,* $\xi_3 = \frac{x^2}{2}$*, then (3.4) yields*

$$\psi(x,y) = X' + Y' + \frac{1}{x-y}(X-Y),$$

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where we choose $\omega = \frac{2}{(x-y)^2}$ *for simplicity. Hence we obtain the general solution for Moutard equation*

$$\psi_{xy} - \frac{2}{(x-y)^2}\psi = 0$$

and, therefore, a family of Königs nets with the Laplace series of the length 3. *Arbitrary choice of independent functions* ξ_1, ξ_2, ξ_3 *provides the general solution for some Moutard equation with the Laplace series of the length* 3 *(see [1]).*

Remark 6. *It is also possible to find effectively the coefficients of hyperbolic operator* $\mathcal{L}_0$ *that is constructed given families (3.3) of arbitrary functions. Indeed, the expansion of determinant (3.4) along the first row allows us to express the coefficients* $A_0, \dots, A_r, B_0, \dots, B_s$ *in (3.1) in terms of functions (3.3). The coefficients* a_0, b_0 *and* c_0, *in turn, can be expressed in terms of* A_i, B_j *in an explicit form: these formulae can be found in [1]. We will only give their analogue in the discrete case.*

4. Laplace invariants in discrete case

In this section, we aggregate basic facts on the Laplace invariants, Laplace transformations and the cascade method both in the purely discrete and in the semi-discrete cases. Although, to the best of our knowledge, some of the propositions below are not contained in the existing literature in explicit form, their proofs are either trivial or similar to the proofs of analogous propositions in the continuous case. Therefore, we give only statements without proofs in this section.

In the purely discrete case, consider a sequence of hyperbolic difference operators

$$\mathcal{L}_j = T_n T_m + a_j T_n + b_j T_m + c_j,$$

where functions a_j, b_j and c_j are defined on the (n, m)-lattice and T_n, T_m are the shift operators:

$$T_n \psi_{n,m} = \psi_{n+1,m} \quad T_m \psi_{n,m} = \psi_{n,m+1}.$$

Clearly, the operator $\mathcal{L}_j$ can be factorised in two different ways:

$$\begin{aligned}\mathcal{L}_j =& (T_n + b_{j,n,m})(T_m + a_{j,n-1,m}) + a_{j,n-1,m} b_{j,n,m} k_{j,n,m} = \\ &= (T_m + a_{j,n,m})(T_n + b_{j,n,m-1}) + a_{j,n,m} b_{j,n,m-1} h_{j,n,m},\end{aligned}$$

where

$$k_{j,n,m} = \frac{c_{j,n,m}}{a_{j,n-1,m} b_{j,n,m}} - 1 \quad \text{and} \quad h_{j,n,m} = \frac{c_{j,n,m}}{a_{j,n,m} b_{j,n,m-1}} - 1$$

are the *Laplace invariants* of difference operators $\mathcal{L}_j$.

In the entirely discrete case, Darboux–Laplace transformations are introduced similarly to the continuous case: difference operator $\mathcal{D}$ defines a transformation for a difference hyperbolic operator $\mathcal{L}$ if there exist a hyperbolic operator $\hat{\mathcal{L}}$ and operator $\hat{\mathcal{D}}$ such that operator relation (2.4) is satisfied. In this case, such a relation is equivalent to a system of partial difference equations on the coefficients of the operators. Even if $\mathcal{D}$ and $\hat{\mathcal{D}}$ are first-order operators, a complete description of all possible Darboux–Laplace transformations for a given difference hyperbolic operator $\mathcal{L}$ is still an open problem (see [33]). Nevertheless, this system of difference equations admits two simple particular solutions for any $\mathcal{L}$: corresponding transformations are given by operators $T_n + b_{j,n,m-1}$ and $T_m + a_{j,n-1,m}$. These transformations are discrete analogues of classical Laplace transformations, and we will call them *discrete Laplace transformations*.

Suppose any two neighbouring operators $\mathcal{L}_j$ and $\mathcal{L}_{j+1}$ are related by the Laplace transformation defined by $\mathcal{D}_j = T_n + b_{j,n,m-1}$. Then the Laplace invariants satisfy the *purely discrete two-dimensional Toda lattice* [6]:

$$k_{j+1,n,m} = h_{j,n,m}, \quad \frac{h_{j,n+1,m} h_{j,n,m-1}}{h_{j,n,m} h_{j,n+1,m-1}} = \frac{(1 + h_{j+1,n,m})(1 + h_{j-1,n+1,m-1})}{(1 + h_{j,n,m})(1 + h_{j,n+1,m-1})}.$$

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Discrete Darboux–Laplace transformations of a more general form were considered in [33].

Proposition 2. *Two hyperbolic second-order difference operators $\mathcal{L}$ and $\bar{\mathcal{L}}$ have the same Laplace invariants $\bar{k}=k$ and $\bar{h}=h$ if and only if they are related by a gauge transformation of the form $\bar{\mathcal{L}}=\omega_{n+1,m+1}^{-1}\mathcal{L}\omega_{n,m}$ for some function $\omega_{n,m}$.*

Remark 7. *Note that in the discrete case the Laplace invariants are defined not quite similarly to the continuous case: there is division by $a_{j,n,m}$ and by $b_{j,n,m}$. This comes from the fact that functions*

$$\tilde{k}_{j,n,m}=c_{j,n,m}-a_{j,n-1,m}b_{j,n,m},\quad \tilde{h}_{j,n,m}=c_{j,n,m}-a_{j,n,m}b_{j,n,m-1}$$

are not invariant with respect to gauge transformations.

Remark 8. *Throughout the whole paper, we will assume that coefficients $a_{0,n,m}$ and $b_{0,n,m}$ of the initial operator are non-zero on the whole lattice. As a consequence, coefficients of all operators in the series also appear to be non-zero (this follows from the relations on the coefficients of neighbouring operators that are deduced from (2.4)). Hence, the Laplace invariants are well-defined. The assumption we made is rather natural from a geometrical point of view: hyperbolic equations that we consider define Q-nets, that is, discrete surfaces made of quadrilaterals (see [3]), and if a coefficient of such a hyperbolic operator turns into zero, then the corresponding quadrilateral degenerates into a triangle.*

In the continuous case, we can use the cascade method for finding the general solution of a hyperbolic equation whenever its Laplace series terminates. But in order to do this, we need to find the general solution of the corresponding hyperbolic equation with zero Laplace invariant. In the discrete case, there is a problem that the apparatus of difference equations is less developed than the apparatus of differential equations: it is possible to express the general solution of a factorisable hyperbolic differential equation in compact form (2.2) or (2.3) using integration, although appropriate notation for a similar situation in the discrete case does not exist in mathematics. Indeed, using the representation

$$\mathcal{L}=(T_n+b_{n,m})(T_m+a_{n-1,m})$$

of a difference operator for which $k_{n,m}=0$, one can denote $\varphi_{n,m}=\psi_{n,m+1}+a_{n-1,m}\psi_{n,m}$ and consider non-homogeneous difference equation $\varphi_{n+1,m}+b_{n,m}\varphi_{n,m}=0$ for the new unknown function $\varphi_{n,m}$. For any integers n,m the value of $\varphi_{n,m}$ can be recursively expressed in terms of the coefficients and the initial data $\varphi_{0,m}$. Consecutively, given $\varphi_{n,m}$, one can solve the first order difference equation for $\psi_{n,m+1}$ and write down the general solution in terms of the initial data $\psi_{k,0}$ and $\psi_{0,k}$. For example, for positive values of m this expression has the form

$$\psi_{n,m}=\alpha_{n,m}\psi_{n,0}+\gamma_{n,m}^{0}\psi_{0,0}+\gamma_{n,m}^{1}\psi_{0,1}+\gamma_{n,m}^{2}\psi_{0,2}+\ldots+\gamma_{n,m}^{m}\psi_{0,m},\tag{4.1}$$

where α, γ^k are some coefficients. Unfortunately, the particular form of this expression is rather ugly and depends on the signs of integers n and m. Therefore, we do not make an attempt to give this formula explicitly and only emphasise that there exists an effective way to express the general solution in terms of the initial data $N_n=\psi_{n,0}$ and $M_m=\psi_{0,m}$ once the corresponding difference operator is factorisable. Note that expression (4.1) is local in arbitrary function N_n and non-local in arbitrary function M_m (i.e., it depends on a variable number of its shifts). Similarly, if $h_{n,m}=0$, then the general solution can be expressed in the form

$$\psi_{n,m}=\beta_{n,m}\psi_{0,m}+\delta_{n,m}^{0}\psi_{0,0}+\delta_{n,m}^{1}\psi_{1,0}+\delta_{n,m}^{2}\psi_{2,0}+\ldots+\delta_{n,m}^{n}\psi_{n,0}.\tag{4.2}$$

Another way to represent the general solution of a hyperbolic difference equation with a factorisable operator is given by

Proposition 3. *If $\mathcal{L}=(T_n+b_{n,m})(T_m+a_{n-1,m})$, then the general solution of $\mathcal{L}\psi=0$ has the form $\theta_{n,m}+\theta_{n,m}^{0}$, where $\theta_{n,m}$ is the general solution of the homogeneous equation $(T_m+a_{n-1,m})\psi_{n,m}=0$, $\theta_{n,m}^{0}$ is a*

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particular solution of the non-homogeneous equation $(T_m + a_{n-1,m})\psi_{n,m} = \varphi_{n,m}$ *and* φ *ranges the space* $\mathrm{Ker}(T_n + b_{n,m})$.

If $\mathcal{L} = (T_m + a_{n,m})(T_n + b_{n,m-1})$, *then the general solution of* $\mathcal{L}\psi = 0$ *has the form* $\chi_{n,m} + \chi^0_{n,m}$, *where* $\chi_{n,m}$ *is the general solution of the homogeneous equation* $(T_n + b_{n,m-1})\psi_{n,m} = 0$, $\chi^0_{n,m}$ *is a particular solution of the non-homogeneous equation* $(T_n + b_{n,m-1})\psi_{n,m} = \tau_{n,m}$ *and* τ *ranges the space* $\mathrm{Ker}(T_m + a_{n,m})$.

Proposition 4. *The Laplace transformation* $\mathcal{L} \mapsto \hat{\mathcal{L}}$ *defined by the operator* $\mathcal{D} = T_n + b_{n,m-1}$ *is invertible if* $a_{n,m}b_{n,m-1}h_{n,m} \neq 0$. *The inverse transformation* $\hat{\mathcal{L}} \mapsto \mathcal{L}$ *is given by the transformation operator* $-(a_{n,m}b_{n,m-1}h_{n,m})^{-1}(T_m + a_{n,m})$. *Similarly, the transformation defined by the operator* $T_m + a_{n-1,m}$ *is invertible if* $a_{n-1,m}b_{n,m}k_{n,m} \neq 0$, *and the inverse transformation is given by* $-(a_{n-1,m}b_{n,m}k_{n,m})^{-1}(T_n + b_{n,m})$.

Remark 9. *The Laplace transformations* $T_n + b_{n,m-1}$ *and* $T_m + a_{n,m}$[2] *are not inverse to each other on the level of particular functions: we need to multiply by a coefficient in order to obtain the inverse transformation. But if everything is considered up to multiplication by a function* $\omega_{n,m}$, *then this difference is not essential. Therefore, in such a situation, we will not specify the particular form of this coefficient and will think of these transformations as being inverse to each other.*

Theorem 4. *If the Laplace series of* $\mathcal{L}_0$ *terminates in the term* $\mathcal{L}_r$ *for some* $r \in \mathbb{N}$, *then the general solution of the equation* $\mathcal{L}_0\psi = 0$ *is given by*

$$\psi_{0,n,m} = -\frac{1}{a_{0,n,m}b_{0,n-1,m}h_{0,n,m}}(T_m + a_{0,n,m})\left(-\frac{1}{a_{1,n,m}b_{1,n-1,m}h_{1,n,m}}(T_m + a_{1,n,m})\dots\right.$$
$$\left.\dots\left(-\frac{1}{a_{r-1,n,m}b_{r-1,n-1,m}h_{r-1,n,m}}(T_m + a_{r-1,n,m})\right)\dots\right)\psi_{r,n,m}, \tag{4.3}$$

where ψ_r *is the general solution of a factorisable equation* $\mathcal{L}_r\psi = 0$.

If the Laplace series of $\mathcal{L}_0$ *terminates in the term* $\mathcal{L}_{-s}$ *for some* $s \in \mathbb{N}$, *then the general solution is given by*

$$\psi_{0,n,m} = -\frac{1}{a_{0,n,m-1}b_{0,n,m}h_{0,n,m}}(T_n + b_{0,n,m})\left(-\frac{1}{a_{-1,n,m-1}b_{-1,n,m}h_{-1,n,m}}(T_n + b_{1,n,m})\dots\right.$$
$$\left.\dots\left(-\frac{1}{a_{-s+1,n,m-1}b_{-s+1,n,m}h_{-s+1,n,m}}(T_n + b_{-s+1,n,m})\right)\dots\right)\psi_{-s,n,m},$$

where ψ_{-s} *is the general solution of a factorisable equation* $\mathcal{L}_{-s}\psi = 0$.

Consider a sequence of hyperbolic differential-difference operators

$$\mathcal{L}_j = \partial_x T_n + a_{j,n}\partial_x + b_{j,n}T_n + c_{j,n}, \tag{4.4}$$

where $a_{j,n}$, $b_{j,n}$ and $c_{j,n}$ are functions depending on discrete variable $n \in \mathbb{Z}$ and on continuous variable $x \in \mathbb{R}$ and where T is the shift operator: $T_n\psi_n(x) = \psi_{n+1}(x)$. The operator $\mathcal{L}_j$ can be rewritten in two different ways,

$$\mathcal{L}_j = (\partial_x + b_{j,n})(T + a_{j,n}) + a_{j,n}k_{j,n} = (T + a_{j,n})(\partial_x + b_{j,n-1}) + a_{j,n}h_{j,n},$$

where

$$k_{j,n} = \frac{c_{j,n}}{a_{j,n}} - (\ln a_{j,n})'_x - b_{j,n} \quad \text{and} \quad h_{j,n} = \frac{c_{j,n}}{a_{j,n}} - b_{j,n-1}$$

[2]The lack of shift in the coordinate direction n in the coefficient of the second transformation operator does not contradict the definition of discrete Laplace transformations since this operator defines a transformation for hyperbolic operator $\mathcal{L}_{j+1}$ whose T_n-coefficient is equal to $a_{j,n+1,m}$ (this follows from difference equations on the coefficients of the operators).

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are the *Laplace invariants* of differential-difference operator $\mathcal{L}_j$. Similarly to the continuous case operator (4.4) can be factorised if and only if at least one of its Laplace invariants vanishes.

Suppose any two neighbouring operators $\mathcal{L}_j$ and $\mathcal{L}_{j+1}$ are related by a *Laplace transformation*, that is, they satisfy relation (2.4), where $\mathcal{D}_j = \partial_x + b_{j,n-1}$. This operator relation can be rewritten in terms of coefficients:

$$\begin{cases} k_{j+1,n} = h_{j,n} \\ \left(\ln \frac{h_{j,n}}{h_{j,n+1}}\right)'_x = h_{j+1,n+1} - h_{j,n+1} - h_{j,n} + h_{j-1,n} \end{cases}.$$

These equations are one of the forms of the semi-discrete two-dimensional Toda lattice [6]. Laplace transformations in the variable n are defined by difference operators $\mathcal{D}'_j = T_n + a_{j,n}$.

Remark 10. *In the semi-discrete case, the division by $a_{j,n}$ in the definition of the Laplace invariants is also necessary because the functions*

$$\tilde{k}_{j,n} = c_{j,n} - a_{j,n,x} - a_{j,n}b_{j,n}, \quad \tilde{h}_{j,n} = c_{j,n} - a_{j,n}b_{j,n-1}$$

are not invariant with respect to gauge transformations of the form $\mathcal{L} \to \omega_{n+1}^{-1}\mathcal{L}\omega_n$. Therefore, in this case, we also assume that the coefficients $a_{0,n}$ are non-zero. Since operator relation (2.4) yields the condition $a_{j,n} = a_{j+1,n}$, the coefficients of all operators in the lattice do not turn into zero, and hence, the Laplace invariants are well-defined.

Similarly to the continuous case, the following proposition holds.

Proposition 5. *Laplace transformation $\mathcal{L} \mapsto \hat{\mathcal{L}}$ defined by the operator $\mathcal{D} = \partial_x + b_{n-1}$ is invertible if $a_n h_n \neq 0$. The inverse transformation $\hat{\mathcal{L}} \mapsto \mathcal{L}$ is given by the operator $-(a_n h_n)^{-1}(T_n + a_n)$. Similarly, the transformation defined by the operator $T_n + a_n$ is invertible if $a_n k_n \neq 0$, and the inverse transformation is given by $-(a_n k_n)^{-1}(\partial_x + b_n)$.*

5. Darboux formulae in discrete case

In this section, we prove the analogues of the Darboux formulae for the general solution of a linear hyperbolic equation with the finite Laplace series in a purely discrete case. Our aim is to prove the following:

Theorem 5. *Let $\mathcal{L}_0$ be a hyperbolic difference operator with a finite Laplace series where $h_r = k_{-s} = 0$ for some $r, s \geqslant 0$. Then there exist functions*

$$\nu_{1,n}, \nu_{2,n}, \ldots, \nu_{r+s+1,n} \quad \text{and} \quad \mu_{1,m}, \mu_{2,m}, \ldots, \mu_{r+s+1,m} \tag{5.1}$$

of one discrete variable and a function ω_{nm} such that the general solution of $\mathcal{L}_0\psi = 0$ is given by the determinant formula

$$\psi_{0,n,m} = \omega_{n,m} \cdot \det \begin{pmatrix} M_m & M_{m+1} & \ldots & M_{m+r} & N_n & N_{n+1} & \ldots & N_{n+s} \\ \mu_{1,m} & \mu_{1,m+1} & \ldots & \mu_{1,m+r} & \nu_{1,n} & \nu_{1,n+1} & \ldots & \nu_{1,n+s} \\ \mu_{2,m} & \mu_{2,m+1} & \ldots & \mu_{2,m+r} & \nu_{2,n} & \nu_{2,n+1} & \ldots & \nu_{2,n+s} \\ \vdots & & & \vdots & & & & \vdots \\ \mu_{r+s+1,m} & \mu_{r+s+1,m+1} & \ldots & \mu_{r+s+1,m+r} & \nu_{r+s+1,n} & \nu_{r+s+1,n+1} & \ldots & \nu_{r+s+1,n+s} \end{pmatrix}, \tag{5.2}$$

where N_n and M_m are arbitrary functions of one variable.

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For any two families of linearly independent functions of one variable (5.1) and for any function $\omega_{n,m}$ there exists a hyperbolic difference operator $\mathcal{L}_0$ with a finite Laplace series that terminates at the terms r and $-s$ such that formula (5.2) provides the general solution of $\mathcal{L}_0\psi = 0$.

The proof of this general result is divided into a number of propositions, lemmas and theorems.

Lemma 1. *If the Laplace series of $\mathcal{L}_0$ terminates at the term $\mathcal{L}_r$ where $r > 0$, then the equation $\mathcal{L}_0\psi = 0$ admits a solution of the form*

$$\psi_{0,n,m} = B_{0,n,m}M_m + B_{1,n,m}M_{m+1} + \cdots + B_{r,n,m}M_{m+r}, \tag{5.3}$$

where M_m is an arbitrary function of one discrete variable. If the Laplace series terminates at the term $\mathcal{L}_{-s}$ where $s > 0$, then the equation $\mathcal{L}_0\psi = 0$ admits a solution of the form

$$\psi_{0,n,m} = A_{0,n,m}N_n + A_{1,n,m}N_{n+1} + \cdots + A_{s,n,m}N_{n+s}, \tag{5.4}$$

where N_n is an arbitrary function of one discrete variable.

Proof. If $h_r = 0$, then the general solution of the equation $\mathcal{L}_r\psi = 0$ has the form (4.2). Remove the nonlocal part by setting $\psi_{n,0} = 0$ for all $n \neq 0$, denote $M_m = \psi_{0,m}$ and use (4.3) to obtain solutions of $\mathcal{L}_0\psi = 0$. Since a shift operator of order r is applied to solutions of $\mathcal{L}_r\psi = 0$, we get (5.3). Formula (5.4) is obtained similarly. □

Remark 11. *Note that formulae (5.3) and (5.4) do not give the general solution of $\mathcal{L}_0\psi = 0$.*

Lemma 2. *If a difference hyperbolic equation $\mathcal{L}_0\psi = 0$ admits a solution of form (5.3) where M_m is an arbitrary function of one variable, then there exists a non-negative $p \leqslant r$ such that the Laplace series terminates: $h_p = 0$. If $\mathcal{L}_0\psi = 0$ admits a solution of form (5.4) where, N_n is an arbitrary function of one variable then there exists a non-negative $p \leqslant s$ such that the Laplace series terminates: $k_{-p} = 0$.*

Proof. Use induction in r to prove the first claim. If $r = 0$, then the equation $\mathcal{L}_0\psi = 0$ admits a solution of the form $\psi_{n,m} = B_{0,n,m}M_m$. Substitute it into the equation:

$$B_{0,n+1,m+1}M_{m+1} + a_{n,m}B_{0,n+1,m}M_m + b_{n,m}B_{0,n,m+1}M_{m+1} + c_{n,m}B_{0,n,m}M_m = 0.$$

Since M is arbitrary, this leads to equations

$$\begin{cases} B_{0,n+1,m+1} + b_{n,m}B_{0,n,m+1} = 0 \\ a_{n,m}B_{0,n+1,m} + c_{n,m}B_{0,n,m} = 0 \end{cases}.$$

Multiply the first equation by $a_{n,m+1}$, shift the second equation in m and consider the difference:

$$a_{n,m+1}b_{n,m}B_{0,n,m+1} - c_{n,m+1}B_{0,n,m+1} = 0$$

This relation has to be satisfied for all natural n, m. Hence $a_{n,m+1}b_{n,m} - c_{n,m+1} = 0$, which is equivalent to $h_{n,m+1} = 0$.

Assume the proposition holds for all equations admitting solutions (5.3) of order r. Consider equation $\mathcal{L}_0\psi = 0$ having such a solution of order $r + 1$. This means that the result of substitution of this solution into the equation, which is an expression of the form

$$C_{0,n,m}M_m + C_{1,n,m}M_{m+1} + \cdots + C_{r+2,n,m}M_{m+r+2}, \tag{5.5}$$

identically vanishes. Note that the leading coefficient in (5.5) has the form

$$C_{r+2,n,m} = B_{r+1,n+1,m+1} + b_{n,m}B_{r+1,n,m+1} = 0. \tag{5.6}$$

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If $h_0 = 0$, then the proposition is proved. Otherwise, apply the Laplace transformation $T_n + b_{n,m-1}$:

$$\psi_{1,n,m} = (T_n + b_{n,m-1})\left(B_{0,n,m}M_m + B_{1,n,m}M_{m+1} + \cdots + B_{r+1,n,m}M_{m+r+1}\right) = \\ = b_{n,m-1}B_{0,n,m}M_m + \cdots + (B_{r+1,n+1,m} + b_{n,m-1}B_{r+1,n,m})M_{m+r+1}$$

Compare the leading coefficient in the latter expression with (5.6) to conclude that ψ_1 has the form (5.3) of order r and hence by the assumption the Laplace series of the corresponding operator $\mathcal{L}_1$ terminates at the term $\mathcal{L}_p$, where $p \leqslant r$. But $\mathcal{L}_1$ is the next term in the Laplace series of the initial operator $\mathcal{L}_0$. Therefore, its Laplace series terminates not later than at the $(r+1)$-th term. The second claim is proved similarly. □

Lemma 3. *If the Laplace invariants of $\mathcal{L}_0$ are non-zero and the general solution of the hyperbolic equation $\mathcal{L}_0\psi = 0$ has the form*

$$\psi_{0,n,m} = A_{0,n,m}N_n + A_{1,n,m}N_{n+1} + \cdots + A_{s,n,m}N_{n+s} + B_{0,n,m}M_m + B_{1,n,m}M_{m+1} + \cdots + B_{r,n,m}M_{m+r}, \tag{5.7}$$

where M and N are arbitrary functions of one discrete variable and $A_0 \neq 0$, $A_s \neq 0$, $B_0 \neq 0$, $B_r \neq 0$, then the general solution of the equation $\mathcal{L}_1\psi = 0$, where $\mathcal{L}_1 = (T_n + b_{n,m-1})\mathcal{L}_0$, has the form

$$\psi_{1,n,m} = \hat{A}_{0,n,m}N_n + \hat{A}_{1,n,m}N_{n+1} + \cdots + \hat{A}_{s+1,n,m}N_{n+s+1} + \hat{B}_{0,n,m}M_m + \hat{B}_{1,n,m}M_{m+1} + \cdots + \hat{B}_{r-1,n,m}M_{m+r-1},$$

where $\hat{A}_0 \neq 0$, $\hat{A}_{s+1} \neq 0$, $\hat{B}_0 \neq 0$, $\hat{B}_{r-1} \neq 0$, and the general solution of the equation $\mathcal{L}_{-1}\psi = 0$, where $\mathcal{L}_{-1} = (T_m + a_{n-1,m})\mathcal{L}_0$, has the form

$$\psi_{-1,n,m} = \tilde{A}_{0,n,m}N_n + \tilde{A}_{1,n,m}N_{n+1} + \cdots + \tilde{A}_{s-1,n,m}N_{n+s-1} + \tilde{B}_{0,n,m}M_m + \tilde{B}_{1,n,m}M_{m+1} + \cdots + \tilde{B}_{r+1,n,m}M_{m+r+1},$$

where $\tilde{A}_0 \neq 0$, $\tilde{A}_{s-1} \neq 0$, $\tilde{B}_0 \neq 0$, $\tilde{B}_{r+1} \neq 0$.

Proof. Apply the Laplace transformation to (5.7) to get the general solution of $\mathcal{L}_1\psi = 0$. Obviously it has the form

$$\psi_{1,n,m} = \hat{A}_{0,n,m}N_n + \hat{A}_{1,n,m}N_{n+1} + \cdots + \hat{A}_{s+1,n,m}N_{n+s+1} + \hat{B}_{0,n,m}M_m + \hat{B}_{1,n,m}M_{m+1} + \cdots + \hat{B}_{r,n,m}M_{m+r},$$

and we need to prove that $\hat{B}_{r,n,m} = 0$. Note that

$$\hat{A}_{s+1,n,m} = A_{s,n+1,m}, \quad \hat{B}_{r,n,m} = B_{r,n+1,m} + b_{n,m-1}B_{r,n,m}, \quad \hat{B}_{r-1,n,m} = B_{r-1,n+1,m} + b_{n,m-1}B_{r-1,n,m}; \tag{5.8}$$

hence, $\hat{A}_{s+1} \neq 0$. Substitute (5.7) into hyperbolic equation $\mathcal{L}_0\psi = 0$. This gives an expression of the form

$$C_{0,n,m}N_n + C_{1,n,m}N_{n+1} + \cdots + C_{s+1,n,m}N_{n+s+1} + D_{0,n,m}M_m + D_{1,n,m}M_{m+1} + \cdots + D_{r+1,n,m}M_{m+r+1} = 0.$$

Since functions M and N are arbitrary, all coefficients vanish. In particular,

$$0 = D_{r+1,n,m} = B_{r,n+1,m+1} + b_{n,m}B_{r,n,m+1}.$$

Compare this with the second relation in (5.8) to verify that $\hat{B}_r = 0$. Besides this,

$$0 = D_{r,n,m} = B_{r-1,n+1,m+1} + a_{n,m}B_{r,n+1,m} + b_{n,m}B_{r-1,n,m+1} + c_{n,m}B_{r,n,m}. \tag{5.9}$$

Assume that $\hat{B}_{r-1} = 0$. Hence (5.8) yields $B_{r-1,n+1,m+1} = -b_{n,m}B_{r-1,n,m+1}$. Therefore, (5.9) can be rewritten as

$$0 = \left(c_{n,m} - a_{n,m}b_{n,m-1}\right)B_{r,n,m} = a_{n,m}b_{n,m-1}h_{n,m}B_{r,n,m},$$

which contradicts the assumption $B_r \neq 0$ since h is non-zero.

Note that

$$\hat{A}_{0,n,m} = b_{n,m-1}A_{0,n,m} \neq 0, \quad \hat{B}_{0,n,m} = B_{0,n+1,m} + b_{n,m-1}B_{0,n,m}.$$

If $\hat{B}_{0,n,m} = 0$, then express $B_{0,n+1,m} = -b_{n,m-1}B_{0,n,m}$ and substitute this into the equation $D_{0,n,m} = 0$:

$$0 = a_{n,m}B_{0,n+1,m} + c_{n,m}B_{n,m} = \left(c_{n,m} - a_{n,m}b_{n,m-1}\right)B_{0,n,m}.$$

Since $h \neq 0$, we arrive at a contradiction. The second claim is obtained similarly. □

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Theorem 6. *If the Laplace series of $\mathcal{L}_0$ is finite, $h_r = k_{-s} = 0$ for some $r, s \geqslant 0$ and the operators $\mathcal{L}_{-s+1}, \dots, \mathcal{L}_{r-1}$ have non-zero Laplace invariants, then the general solution of difference equation $\mathcal{L}_0\psi = 0$ has the form (5.7), where N_n and M_m are arbitrary functions of one variable, $A_{i,n,m}$ and $B_{j,n,m}$ are some particular functions depending on the coefficients of the initial equation, and $A_0 \neq 0$, $A_s \neq 0$, $B_0 \neq 0$, $B_r \neq 0$.*

If a hyperbolic equation $\mathcal{L}_0\psi = 0$ admits a solution of the form (5.7), where $A_0 \neq 0$, $A_s \neq 0$, $B_0 \neq 0$ and $B_r \neq 0$, then its Laplace series is finite: $h_r = k_{-s} = 0$ and the operators $\mathcal{L}_{-s+1}, \dots, \mathcal{L}_{r-1}$ have non-zero Laplace invariants.

Proof. The general solution of $\mathcal{L}_0 = 0$ can be obtained by the application of s Laplace transformations to the general solution of $\mathcal{L}_{-s} = 0$. We will construct such a solution in a special form. Since $h_{-s} = 0$, operator $\mathcal{L}_{-s}$ can be factorised, and therefore, according to Proposition 3, the general solution has the form $\psi_{-s,n,m} = \theta_{n,m} + \theta^0_{n,m}$. Note that $\theta_{n,m} = \alpha_{n,m}N_n$, where N_n is an arbitrary function of one variable and $\alpha_{n,m}$ depends on coefficients of $\mathcal{L}_{-s}$. Similarly, the general solution of the equation $\mathcal{L}_r = 0$ has the form $\psi_{r,n,m} = \chi_{n,m} + \chi^0_{n,m}$, where $\chi_{n,m} = \beta_{n,m}M_m$ and M_m is an arbitrary function. Set $\chi^0 = 0$ and apply the composition $\mathcal{M}$ of inverse Laplace transformations:

$$\mathcal{M}\chi_{n,m} = \omega_{n,m}\left(T_m + a_{-s+1,n-1,m}\right)\left(T_m + a_{-s+2,n-1,m}\right)\dots\left(T_m + a_{r,n-1,m}\right)\chi_{n,m} \in \mathrm{Ker}\mathcal{L}_{-s}$$

(see Remark 9). Since $\chi_{n,m} = \beta_{n,m}M_m$ depends on an arbitrary function and

$$\left(T_n + b_{-s,n,m}\right)\left(T_m + a_{-s,n-1,m}\right)\mathcal{M}\chi_{n,m} = 0,$$

the function $\varphi_{n,m} = \left(T_m + a_{-s,n-1,m}\right)\mathcal{M}\chi_{n,m}$ parameterises the general solution of the homogeneous equation $\left(T_n + b_{-s,n,m}\right)\psi_{n,m} = 0$. Hence, $\theta^0_{n,m} = \mathcal{M}\chi_{n,m}$ satisfies the non-homogeneous equation

$$\left(T_m + a_{-s,n-1,m}\right)\psi_{n,m} = \varphi_{n,m}.$$

Therefore the general solution of $\mathcal{L}_{-s}\psi = 0$ has the form

$$\psi_{-s,n,m} = \alpha_{n,m}N_n + \mathcal{M}\chi_{n,m}.$$

Apply the Laplace transformations $\mathcal{N} = \left(T_n + b_{-1,n,m-1}\right)\dots\left(T_n + b_{-s,n,m-1}\right)$ to obtain the general solution of $\mathcal{L}_0\psi = 0$:

$$\begin{aligned}\psi_{0,n,m} = \mathcal{N}\left(\alpha_{n,m}N_n + \mathcal{M}\chi_{n,m}\right) = \mathcal{N}\left(\alpha_{n,m}N_n\right) + \tilde{\omega}_{n,m}\left(T_m + a_{1,n-1,m}\right)\dots\left(T_m + a_{r,n-1,m}\right)\left(\beta_{n,m}M_m\right) = \\ = A_{0,n,m}N_n + A_{1,n,m}N_{n+1} + \dots + A_{s,n,m}N_{n+s} + B_{0,n,m}M_m + B_{1,n,m}M_{m+1} + \dots + B_{r,n,m}M_{m+r}\end{aligned}$$

since $\mathcal{N}$ and $\left(T_m + a_{1,n-1,m}\right)\dots\left(T_m + a_{r,n-1,m}\right)$ are difference operators in T_n and T_m of orders s and r respectively.

The second claim follows from Lemmas 2, 3. □

Suppose the Laplace series of $\mathcal{L}_0$ is finite, $h_r = k_{-s} = 0$ for some $r, s \geqslant 0$, and the operators $\mathcal{L}_{-s+1}, \dots, \mathcal{L}_{r-1}$ have non-zero Laplace invariants. Then the general solution of the equation $\mathcal{L}_r\psi = 0$ has the form

$$\psi_{r,n,m} = A_{0,n,m}N_n + A_{1,n,m}N_{n+1} + \dots + A_{r+s,n,m}N_{n+r+s} + B_{0,n,m}M_m.$$

Without loss of generality, one may assume that $B_0 = 1$ (apply an appropriate gauge transformation, which does not change the Laplace invariants). Hence, the hyperbolic equation $\mathcal{L}_r\psi = 0$ admits a solution of the form $\psi_{r,n,m} = M_m$. This yields $b_{r,n,m} = -1$, $a_{r,n,m} = -c_{r,n,m}$. Thus,

$$\mathcal{L}_r = \left(T_m + a_{r,n,m}\right)(T_n - 1) \quad \text{and} \quad (T_n - 1)\,\psi_{r,n,m} \in \mathrm{Ker}\left(T_m + a_{r,n,m}\right).$$

Therefore, $(T_n - 1)\,\psi_{r,n,m} = \alpha_{n,m}\tilde{N}_n$ for a certain $\tilde{N}$. This leads to the relation

$$\begin{aligned}-A_{0,n,m}N_n + (A_{0,n+1,m} - A_{1,n,m})N_{n+1} + \dots + \\ +(A_{r+s-1,n+1,m} - A_{r+s,n,m})N_{n+r+s} + A_{r+s,n+1,m}N_{n+r+s+1} = \alpha_{n,m}\tilde{N}_n.\end{aligned} \tag{5.10}$$

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Denote

$$\lambda_{0,n,m} = -\frac{A_{0,n,m}}{\alpha_{n,m}}, \quad \lambda_{1,n,m} = \frac{A_{0,n+1,m} - A_{1,n,m}}{\alpha_{n,m}}, \ldots, \lambda_{r+s+1,n,m} = \frac{A_{r+s,n+1,m}}{\alpha_{n,m}}.$$

Note that all coefficients $\lambda_0, \ldots, \lambda_{r+s+1}$ do not depend on m since N_n and $\tilde{N}_n$ are functions of one discrete variable n. Therefore we drop the second index in $\lambda_{0,n,m}, \ldots, \lambda_{r+s+1,n,m}$ for convenience.

Lemma 4. *Function $\alpha_{n,m}$ satisfies a linear difference equation of order $r+s+1$ in the variable n with coefficients depending only on n.*

Proof. Express the functions $A_{i,n,m}$ in terms of $\alpha_{n,m}$ and $\lambda_0, \ldots, \lambda_{r+s+1}$:

$$\begin{aligned}
A_{0,n,m} &= -\alpha_{n,m}\lambda_{0,n} \\
A_{1,n,m} &= A_{0,n+1,m} - \alpha_{n,m}\lambda_{1,n} = -T_n\left(\alpha_{n,m}\lambda_{0,n}\right) - \alpha_{n,m}\lambda_{1,n} \\
A_{2,n,m} &= A_{1,n+1,m} - \alpha_{n,m}\lambda_{2,n} = -T_n^2\left(\alpha_{n,m}\lambda_{0,n}\right) - T_n\left(\alpha_{n,m}\lambda_{1,n}\right) - \alpha_{n,m}\lambda_{2,n} \\
&\ldots \\
A_{r+s,n,m} &= A_{r+s-1,n+1,m} - \alpha_{n,m}\lambda_{r+s,n} = -\sum_{i=0}^{r+s} T_n^i\left(\alpha_{n,m}\lambda_{r+s-i,n}\right) \\
A_{r+s,n+1,m} &= \alpha_{n,m}\lambda_{r+s+1,n}.
\end{aligned}$$

Hence, function $\alpha_{n,m}$ satisfies the linear difference equation

$$\lambda_{0,n}T_n^{r+s+1}\alpha_{n,m} + \ldots + \lambda_{r+s}T_n\alpha_{n,m} + \lambda_{r+s+1}\alpha_{n,m} = 0. \tag{5.11}$$

□

Let $\nu_{1,n}, \nu_{2,n}, \ldots, \nu_{r+s+1,n}$ be a fundamental system of solutions of linear ordinary difference equation (5.11). Then $\tilde{N} = 0$ once $N = \nu_i$ is chosen in (5.10) for some $i = 1, 2, \ldots, r+s+1$. Hence

$$(T_n - 1)\left(A_{0,n,m}\nu_{i,n} + A_{1,n,m}\nu_{i,n+1} + \cdots + A_{r+s,n,m}\nu_{i,n+r+s} + M_m\right) = 0$$

for every $i = 1, 2, \ldots, r+s+1$. This means that

$$\mu_{i,m} := A_{0,n,m}\nu_{i,n} + A_{1,n,m}\nu_{i,n+1} + \cdots + A_{r+s,n,m}\nu_{i,n+r+s}, \quad i = 1, 2, \ldots, r+s+1,$$

are functions only of m.

Proposition 6. *In the above setting, the general solution of the equation $\mathcal{L}_r\psi = 0$ is given by $\psi_r = \frac{\Delta_1}{\Delta}$, where*

$$\Delta_{1,n,m} = \det\begin{pmatrix} M_m & N_n & N_{n+1} & \ldots & N_{n+r+s} \\ \mu_{1,m} & \nu_{1,n} & \nu_{1,n+1} & \ldots & \nu_{1,n+r+s} \\ \mu_{2,m} & \nu_{2,n} & \nu_{2,n+1} & \ldots & \nu_{2,n+r+s} \\ \vdots & & & \ddots & \vdots \\ \mu_{r+s+1,m} & \nu_{r+s+1,n} & \nu_{r+s+1,n+1} & \ldots & \nu_{n+s+1,n+r+s} \end{pmatrix}, \tag{5.12}$$

$$\Delta_{n,m} = \det\begin{pmatrix} \nu_{1,n} & \nu_{1,n+1} & \ldots & \nu_{1,n+r+s} \\ \nu_{2,n} & \nu_{2,n+1} & \ldots & \nu_{2,n+r+s} \\ \vdots & & \ddots & \vdots \\ \nu_{r+s+1,n} & \nu_{r+s+1,n+1} & \ldots & \nu_{n+s+1,n+r+s} \end{pmatrix}. \tag{5.13}$$

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Proof. Let ψ_r be the general solution of $\mathcal{L}_r\psi = 0$. Then there exist families of functions $\nu_1, \dots, \nu_{r+s+1}$ and $\mu_1, \dots, \mu_{r+s+1}$ depending on n and m respectively such that the following relations are satisfied:

$$\begin{cases} \psi_{r,n,m} - A_{0,n,m}N_n - A_{1,n,m}N_{n+1} - \dots - A_{r+s,n,m}N_{n+r+s} = M_m \\ \quad - A_{0,n,m}\nu_{i,n} - A_{1,n,m}\nu_{i,n+1} - \dots - A_{r+s,n,m}\nu_{i,n+r+s} = \mu_{i,m}, \quad i = 1, 2, \dots, r+s+1 \end{cases}. \tag{5.14}$$

Solve this system with respect to the unknowns $\psi_r, -A_0, -A_1, \dots, -A_{r+s}$ and apply Cramer's rule to obtain the formula for ψ_r. □

Consider the general solution of $\mathcal{L}_0\psi = 0$. On the one hand, it has the form (5.7); on the other hand, it is obtained from the general solution of equation $\mathcal{L}_r\psi = 0$ by application of Laplace transformations: $\psi_r = \mathcal{M}\psi_0$, where $\mathcal{M}$ is a difference operator of order r and ψ_r is the quotient of determinants (5.12) and (5.13). Hence $\psi_0 = 0$ whenever $N = \nu_i$ and $M = \mu_i$ for a certain $i = 1, 2, \dots, r+s+1$. Therefore, up to multiplication by an appropriate function $\omega_{n,m}$, the following equations are satisfied:

$$\begin{cases} \psi_{0,n,m} - B_{0,n,m}M_n - \dots - B_{r,n,m}M_{n+r} - A_{0,n,m}N_n - \dots - A_{s,n,m}N_{n+s} = 0 \\ \quad - B_{0,n,m}\mu_{i,n} - \dots - B_{r,n,m}\mu_{i,n+r} - A_{0,n,m}\nu_{i,n} - \dots - A_{s,n,m}\nu_{i,n+s} = 0, \quad i = 1, 2, \dots, r+s+1 \end{cases}.$$

Solve this system with respect to the unknowns $\psi_0, -B_0, \dots, -B_r, -A_0, \dots, A_s$ and apply the projective version of Cramer's rule to obtain (5.2). This proves the first part of Theorem 5.

In order to prove the second part of Theorem 5, choose arbitrary independent functions of one variable

$$\nu_{1,n}, \nu_{2,n}, \dots, \nu_{r+s+1,n} \quad \text{and} \quad \mu_{1,m}, \mu_{2,m}, \dots, \mu_{r+s+1,m} \tag{5.15}$$

and solve linear system (5.14) for $A_0, A_1, \dots, A_{r+s}$. This provides

$$\psi_{r,n,m} = A_{0,n,m}N_n + A_{1,n,m}N_{n+1} + \dots + A_{r+s,n,m}N_{n+r+s} + M_m. \tag{5.16}$$

Lemma 5. *Function (5.16) satisfies a second-order hyperbolic equation $\mathcal{L}_r\psi = 0$ such that $h_r = 0$ for the operator $\mathcal{L}_r$.*

Proof. Since $\mu_{i,m} = -A_{0,n,m}\nu_{i,n} - A_{1,n,m}\nu_{i,n+1} - \dots - A_{r+s,n,m}\nu_{i,n+r+s}$ is a function of m for every $i = 1, 2, \dots, r+s+1$, function ν_i satisfies the difference equation

$$(T_n - 1)\left(A_{0,n,m}\nu_{i,n} + A_{1,n,m}\nu_{i,n+1} + \dots + A_{r+s,n,m}\nu_{i,n+r+s}\right) = 0,$$

which has order $r+s+1$. Consider the linear ordinary difference equation

$$\lambda_{0,n}\nu_{n+r+s+1} + \dots + \lambda_{r+s,n}\nu_{n+1} + \lambda_{r+s+1,n}\nu_n = 0,$$

having $\nu_{1,n}, \nu_{2,n}, \dots, \nu_{r+s+1,n}$ as a basis in the space of solutions. Such an equation is unique up to multiplication by an arbitrary function $\alpha_{n,m}$. Therefore, the following relations are satisfied:

$$\alpha_{n,m} = -\frac{A_{0,n,m}}{\lambda_{0,n,m}} = \frac{A_{0,n+1,m} - A_{1,n,m}}{\lambda_{1,n}} = \dots = \frac{A_{r+s-1,n+1,m} - A_{r+s,n,m}}{\lambda_{r+s,n}} = \frac{A_{r+s,n+1,m}}{\lambda_{r+s+1,n}}.$$

This implies that $(T_n - 1)\,\psi_{r,n,m} = \alpha_{n,m}\tilde{N}_n$, where $\tilde{N}$ is a function of n. Hence

$$\left(T_m + \gamma_{n,m}\right)(T_n - 1)\,\psi_{r,n,m} = \left(T_m + \gamma_{n,m}\right)\left(\alpha_{n,m}\tilde{N}_n\right) = \left(\alpha_{n,m+1} + \gamma_{n,m}\alpha_{n,m}\right)\tilde{N}_n$$

for any $\gamma_{n,m}$. Therefore if $\gamma_{n,m} = -\frac{\alpha_{n,m+1}}{\alpha_{n,m}}$ for all $n, m \in \mathbb{Z}$, then $\mathcal{L}_r\psi_r = 0$, where

$$\mathcal{L}_r = \left(T_m + \gamma_{n,m}\right)(T_n - 1) = T_nT_m + \gamma_{n,m}T_n - T_m - \gamma_{n,m}.$$

Note that $h_r = 0$ for any operator of such kind. □

According to Theorem 6, if the hyperbolic equation $\mathcal{L}_r\psi = 0$ admits a solution of the form (5.16), then the Laplace series of $\mathcal{L}_r$ is finite. In particular, $h_{-s} = 0$. Apply Laplace transformations to $\mathcal{L}_r$ and

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to ψ_r in order to obtain the operator $\mathcal{L}_0$ in the Laplace series and the general solution of the equation $\mathcal{L}_0\psi = 0$. On the one hand it has the form

$$\psi_{0,n,m} = C_{0,n,m}N_n + C_{1,n,m}N_{n+1} + \ldots + C_{s,n,m}N_{n+s} + D_{0,n,m}M_m + D_{1,n,m}M_{m+1} + \ldots + D_{r,n,m}M_{m+r}.$$

On the other hand, consider the determinant in (5.2) constructed using functions (5.15) and expand it along its first row. This expression has the form

$$\tilde{\psi}_{0,n,m} = \tilde{A}_{0,n,m}N_n + \tilde{A}_{1,n,m}N_{n+1} + \cdots + \tilde{A}_{s,n,m}N_{n+s} + \tilde{B}_{0,n,m}M_m + \tilde{B}_{1,n,m}M_{m+1} + \cdots + \tilde{B}_{r,n,m}M_{m+r}.$$

Note that since ψ_0 is obtained from ψ_r, which was constructed earlier, the substitution $N = \nu_i$, $M = \mu_i$ implies that $\psi_r = 0$ for every $i = 1, 2, \ldots, r+s+1$; therefore, we get $\psi_0 = 0$ for such a choice of N and M as well. This leads to a linear system

$$C_{0,n,m}\nu_{i,n} + C_{1,n,m}\nu_{i,n+1} + \cdots + C_{s,n,m}\nu_{i,n+s} + D_{0,n,m}\mu_{i,m} + D_{1,n,m}\mu_{i,m+1} + \cdots + D_{r,n,m}\mu_{i,m+r} = 0$$

for the unknowns $C_1, \ldots, C_{s,n,m}, D_1, \ldots, D_{r,n,m}$. Note also that such a choice of N and M yields $\tilde{\psi}_0 = 0$ and, therefore, also the relations

$$\tilde{A}_{0,n,m}\nu_{i,n} + \tilde{A}_{1,n,m}\nu_{i,n+1} + \cdots + \tilde{A}_{s,n,m}\nu_{i,n+s} + \tilde{B}_{0,n,m}\mu_{i,m} + \tilde{B}_{1,n,m}\mu_{i,m+1} + \cdots + \tilde{B}_{r,n,m}\mu_{i,m+r} = 0,$$

where $i = 1, 2, \ldots, r+s+1$. Hence, the unknowns

$$C_1, \ldots, C_{s,n,m}, D_1, \ldots, D_{r,n,m} \quad \text{and} \quad \tilde{A}_1, \ldots, \tilde{A}_{s,n,m}, \tilde{B}_1, \ldots, \tilde{B}_{r,n,m}$$

satisfy the same system of homogeneous linear equations, and these $(r+s+1)$-tuples are proportional since functions (5.15) are independent. Therefore there exists $\tilde{\omega}$ such that $\psi_{0,n,m} = \tilde{\omega}_{n,m}\tilde{\psi}_{0,n,m}$ defines the general solution of $\mathcal{L}_0\psi = 0$, where $\mathcal{L}_0$ has a finite Laplace series. For any other ω function $\omega_{n,m}\psi_{0,n,m}$ satisfies equation $\tilde{\mathcal{L}}_0\psi = 0$, where $\tilde{\mathcal{L}}_0 = \frac{1}{\omega_{n+1,m+1}}\mathcal{L}_0\omega_{n,m}$ is gauge-equivalent to $\mathcal{L}_0$ and therefore has the same series of Laplace invariants. This completes the proof of Theorem 5.

In the discrete case, it is also possible to express in terms of functions (5.1) not only the general solution (5.2), but also the coefficients of the operator $\mathcal{L}_0$. The functions $A_{0,n,m}$ and $B_{0,n,m}$ can be expressed in terms of (5.1) using the expansion of determinant (5.2) along its first row; see Remark 6. The coefficients of $\mathcal{L}_0$ can be expressed in terms of them.

Proposition 7. *If the general solution of the equation $L_0\psi = 0$ with the Laplace series terminating at the terms r and $-s$ is given by (5.2), then the coefficients of the operator $\mathcal{L}_0$ have the form*

$$a_{n,m} = -\frac{A_{s,n+1,m+1}}{A_{s,n+1,m}},\ b_{n,m} = -\frac{B_{r,n+1,m+1}}{B_{r,n,m+1}},\ c_{n,m} = \frac{\sum_{i=0}^{s} A_{i,n+1,m+1} + a_{nm}\sum_{i=0}^{s} A_{i,n+1,m} + b_{nm}\sum_{i=0}^{s} A_{i,n,m+1}}{\sum_{i=0}^{s} A_{i,n,m}}. \tag{5.17}$$

Proof. Consider the general solution of $\mathcal{L}_0\psi = 0$ in form (5.7) and apply shifts:

$$\begin{aligned}
\psi_{0,n+1,m} &= A_{0,n+1,m}N_{n+1} + \cdots + A_{s,n+1,m}N_{n+s+1} + B_{0,n+1,m}M_m + \cdots + B_{r,n+1,m}M_{m+r}\\
\psi_{0,n,m+1} &= A_{0,n,m+1}N_n + \cdots + A_{s,n,m+1}N_{n+s} + B_{0,n,m+1}M_{m+1} + \cdots + B_{r,n,m+1}M_{m+r+1}\\
\psi_{0,n+1,m+1} &= A_{0,n+1,m+1}N_{n+1} + \cdots + A_{s,n+1,m+1}N_{n+s+1} + B_{0,n+1,m+1}M_{m+1} + \cdots + B_{r,n+1,m+1}M_{m+r+1}.
\end{aligned}$$

Note that the expression

$$\tilde{\psi}_{0,n,m} := \psi_{0,n+1,m+1} - \frac{A_{s,n+1,m+1}}{A_{s,n+1,m}}\psi_{0,n+1,m} - \frac{B_{r,n+1,m+1}}{B_{r,n,m+1}}\psi_{0,n,m+1}$$

does not contain the highest shifts N_{n+s+1} and M_{m+r+1}, that is it has form (5.7). Moreover, the substitution $N = \nu_i$, $M = \mu_i$ for every $i = 1, 2, \ldots, r+s+1$ implies that $\psi_0 = 0$ and, hence, that $\tilde{\psi}_0 = 0$. Using the same argument as before, we deduce that there exists a function ω such that $\tilde{\psi}_{0,n,m} = \omega_{n,m}\psi_{0,n,m}$.

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Hence,

$$\psi_{0,n+1,m+1} - \frac{A_{s,n+1,m+1}}{A_{s,n+1,m}}\psi_{0,n+1,m} - \frac{B_{r,n+1,m+1}}{B_{r,n,m+1}}\psi_{0,n,m+1} = \omega_{n,m}\psi_{0,n,m}. \tag{5.18}$$

This means that we constructed a hyperbolic second-order difference operator with a unitary leading coefficient that is annihilated by the general solution ψ_0 of the initial equation. Hence, we only need to prove the formula for $c_{n,m} = -\omega_{n,m}$. Consider the particular solution that corresponds to $N_n \equiv 1, M_m \equiv 0$ and substitute it into (5.18) to find ω. □

Remark 12. *It is also possible to obtain a formula for $c_{n,m}$ in (5.17) in terms of coefficients $B_{j,n,m}$. This corresponds to another choice of a particular solution. Moreover, some other particular solution would give yet another formula for $c_{n,m}$. There is no contradiction in such an ambiguity since the coefficients $A_{i,n,m}$, $B_{j,n,m}$ satisfy strong restrictions imposed by formula (5.2).*

Remark 13. *A special class of discrete Königs nets with finite Laplace series has been studied recently [34]. Probably some special choice of functions (5.1) in the determinant formula (5.2) provides solutions that correspond to Königs nets of this kind. This needs further investigation.*

6. Semi-discrete case

In this section, we briefly discuss semi-discrete analogues of the Darboux formulae. All proofs are similar to the ones in the continuous and in the entirely discrete cases.

Theorem 7. *If the Laplace series of a semi-discrete hyperbolic operator $\mathcal{L}_0$ is finite, $h_r = k_{-s} = 0$ for some $r, s \geqslant 0$ and the operators $\mathcal{L}_{-s+1}, \dots, \mathcal{L}_{r-1}$ have non-zero Laplace invariants, then the general solution of difference equation $\mathcal{L}_0\psi = 0$ has the form*

$$\psi_{0,n} = A_{0,n}N_n + A_{1,n}N_{n+1} + \dots + A_{r,n}N_{n+r} + B_{0,n}X + B_{1,n}X' + \dots + B_{s,n}X^{(s)},$$

where N_n and X are arbitrary functions of variables n and x respectively, $A_{i,n}$ and $B_{j,n}$ are some particular functions depending on the coefficients of the initial equation, and $A_0 \neq 0$, $A_r \neq 0$, $B_0 \neq 0$, $B_s \neq 0$.

If a hyperbolic equation $\mathcal{L}_0\psi = 0$ admits a solution of the form (5.7), where $A_0 \neq 0$, $A_r \neq 0$, $B_0 \neq 0$ and $B_s \neq 0$, then its Laplace series is finite, $h_r = k_{-s} = 0$, and the operators $\mathcal{L}_{-s+1}, \dots, \mathcal{L}_{r-1}$ have non-zero Laplace invariants.

Theorem 8. *Let $\mathcal{L}_0$ be a hyperbolic differential-difference operator with a finite Laplace series where $h_r = k_{-s} = 0$ for some $r, s \geqslant 0$. Then there exist functions*

$$\nu_{1,n}, \nu_{2,n}, \dots, \nu_{r+s+1,n} \quad \text{and} \quad \xi_1(x), \xi_2(x), \dots, \xi_{r+s+1}(x) \tag{6.1}$$

of one discrete variable and a function $\omega_n = \omega_n(x)$ such that the general solution of $\mathcal{L}_0\psi = 0$ is given by the determinant formula

$$\psi_{0,n} = \omega_n \cdot \det \begin{pmatrix} X & X' & \dots & X^{(s)} & N_n & N_{n+1} & \dots & N_{n+r} \\ \xi_1 & \xi_1' & \dots & \xi_1^{(s)} & \nu_{1,n} & \nu_{1,n+1} & \dots & \nu_{1,n+r} \\ \xi_2 & \xi_2' & \dots & \xi_2^{(s)} & \nu_{2,n} & \nu_{2,n+1} & \dots & \nu_{2,n+r} \\ \vdots & & & \vdots & & & & \vdots \\ \xi_{r+s+1} & \xi_{r+s+1}' & \dots & \xi_{r+s+1}^{(s)} & \nu_{r+s+1,n} & \nu_{r+s+1,n+1} & \dots & \nu_{r+s+1,n+r} \end{pmatrix}, \tag{6.2}$$

where N_n and $X(x)$ are arbitrary functions of one variable.

For any two families of linearly independent functions of one variable (6.1) and for any function $\omega_n = \omega_n(x)$, there exists a hyperbolic differential-difference operator $\mathcal{L}_0$ with a finite Laplace series that terminates at the terms r and $-s$ such that formula (6.2) provides the general solution of $\mathcal{L}_0\psi = 0$.

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