where $m_1 \neq m_2$. If we eliminate t in (13), for $C \neq 0$ we do not get a rational relation between x and y.

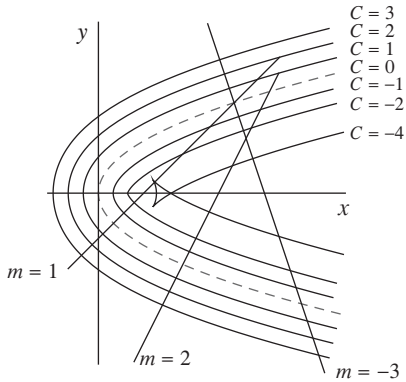


FIGURE 1: Depicting 7 members of the family of curves (13) orthogonal to the family of lines (1) ($m = 1,\ 2,\ -3$). Only the dashed curve is a parabola, the others are all non-parabolic.

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10.1017/mag.2021.68

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105.26 Areas above and below a curve

In [1] the author considers a strictly increasing, continuous, function $f(x)$ for $x \geqslant 0$ with $f(0) = 0$. The rectangle with base $[0, x]$ and height $f(x)$ is divided into two parts by the curve $y = f(x)$, and the areas of the parts in the rectangle above, and below, the curve are denoted by $A(x)$ and $B(x)$, respectively. The author then asks for an explanation based on 'dynamic geometry' of the fact that if $f(x) = x^n$ then $A(x)/B(x) = n$. In this Note we shed light on this topic by discussing the case of a general function f and, in particular, we ask whether, given a positive function ψ,

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there is always a function f such that $A(x)/B(x) = \psi(x)$. The discussion is elementary, and suggests examples that readers can explore for themselves.

We start with a function f, as described above (so that $f(x) > 0$ when $x > 0$), and for $x > 0$ we define $A(x)$ and $B(x)$ as above (note that $A(x)$ is only defined if f is increasing, and $B(x) > 0$ because f is strictly increasing). Clearly,

$$B(x) = \int_0^x f(t)\,dt, \qquad A(x) = xf(x) - B(x) = xf(x) - \int_0^x f(t)\,dt.$$

The discussion in [1] is about the ratio $A(x)/B(x)$ but here we prefer to consider the (equivalent) function Φ_f defined for $x > 0$ by

$$\Phi_f(x) = 1 + \frac{A(x)}{B(x)} = \frac{A(x) + B(x)}{B(x)}.$$

With this we see that f satisfies the integral equation

$$\Phi_f(x) \int_0^x f(t)\,dt = xf(x). \tag{1}$$

This gives a convenient way to find $A(x)/B(x)$ from f, and it confirms the obvious fact that Φ_f does not change if we replace f by a constant multiple of f.

Let us assume for the moment that given any positive, continuous, function Φ, we can solve the *integral equation*

$$\Phi(x) \int_0^x f(t)\,dt = xf(x) \tag{2}$$

to obtain a positive, continuous solution f. Then, from (1) and (2), we see that $\Phi = \Phi_f$. This means that, given a positive function Φ, if we wish to find a function f such that $\dfrac{A(x) + B(x)}{B(x)} = \Phi(x)$, then it is sufficient to solve the integral equation (2). However, as $B'(x) = f(x)$, (2) can be written as the separable *differential equation*

$$\frac{B'(x)}{B(x)} = \frac{\Phi(x)}{x}$$

which can be solved. Indeed, if Φ_f is positive and continuous, and we adopt the normal convention that $\int_b^a = -\int_a^b$ when $a < b$, then the solution (up to a scalar mulitple) is

$$B(x) = \exp\left(\int_1^x \frac{\Phi(t)}{t}\,dt\right). \tag{3}$$

With this, we have

$$f(x) = B'(x) = \frac{B(x)\Phi(x)}{x} = \left(\frac{\Phi(x)}{x}\right) \exp\left(\int_1^x \frac{\Phi(t)}{t}\,dt\right), \tag{4}$$

and the following result summarises this discussion.

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Theorem: Suppose that Φ is a continuous function on $(0, +\infty)$ with $\Phi > 1$. Then the function f given by (4) is, up to a scalar multiple, the unique continuous function f with the property that $1 + \dfrac{A(x)}{B(x)} = \Phi(x)$.

For example, if we put $\dfrac{A(x)}{B(x)} = n$ we obtain the converse to the example in [1, 2], namely

$$f(x) = \frac{(1+n)}{x} \exp\left(\int_1^x \frac{1+n}{t}\, dt\right) = (n+1)x^n;$$

thus $\dfrac{A(x)}{B(x)} = n$ if, and only if, $f(x) = Kx^n$ for some constant K.

Finally, the analysis above can, of course, be carried out on any interval $[0, X]$, where $X > 0$; it is not necessary to insist that f is defined for all positive x. Readers may like to examine (1) with their own choices of f, and (4) with their own choices of Φ, and thereby discover other specific examples. For example, if $\Phi(x)$ is of the form $\dfrac{xg'(x)}{g(x)}$, then $B(x) = g(x)$ so that $f(x) = g'(x)$.

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10.1017/mag.2021.69

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105.27 A remark on $\lim_{n\to\infty} \sqrt[p_n]{p_1 p_2 \ldots p_n} = e$

In [1], S. M. Ruiz proved that if p_n denotes the n-th prime number, then

$$\lim_{n\to\infty} \sqrt[p_n]{p_1 p_2 \ldots p_n} = e. \tag{1}$$

In his paper, Ruiz wrote "In doing so, we shall need to make use of the Prime Number Theorem in the form $p_n \sim n \ln n$, which is equivalent to saying that there exist positive constants A and B such that, for $k > 1$, $A \ln k \leqslant \dfrac{p_k}{k} \leqslant B \ln k$."

This claim is wrong because they are not equivalent. In fact, for the Prime Number Theorem (PNT) in the form $p_n \sim n \ln n$, the result is a stronger one that A, B can be universally quantified. More precisely, A, B can

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